# 1 *Frege (1991)*

Gottlob Frege (1848–1925) lived his life in relative obscurity. He corresponded with some of the great mathematicians of the age, but excepting Russell was largely unappreciated by them. He also corresponded with Wittgenstein, and was heard in lecture by Carnap. These three, particularly Russell, expanded his public and kept his reputation alive. His philosophical importance came to be widely appreciated, however, only in the middle of this century. He is now regarded as the father of analytic philosophy.

Frege founded modern logic. In 1879 he published *Begriffsschrift* which constituted the first fundamental advance in logic since Aristotle. In this work, Frege stated the syntax and semantics for the propositional calculus and first- and second-order quantificational logic. The standard of rigor that he brought to this work was unprecedented.

Frege's work in logic is his greatest contribution. But it was conceived primarily as a means to a further end. He wanted to establish *logicism*, the view that the mathematics of number is reducible to logic. This view derives from Leibniz; but before Frege no one had a sufficiently rigorous or powerful logic to argue for it in a systematic way.

In *Die Grundlagen der Arithmetik*, 1884, Frege set out to define the primary expressions of arithmetic in purely logical terms. The key expressions were '0', 'the successor of', and 'natural number'. Using these definitions he hoped to prove within his logic the theorems of arithmetic. This latter task he reserved to *Die Grundgesetze der Arithmetik* (first volume, 1893; second, 1903).

In *Grundlagen*, one of the most brilliant of all works in philosophy, Frege criticized rival views in the philosophy of mathematics—particularly empiricist, psychologistic, and Kantian views—as a way of motivating the definitions that he proposed. Although his project was conceived as a mathematical one, his genius lay in the deep philosophical motivations that he developed for it. Nearly all of Frege's criticisms of the views he discusses, in the particular forms that he discusses them, are regarded as devastating.

Frege argues that numbers, though abstract and causally inert, are objective. It is disputed whether Frege held the Platonist view that numbers are abstract (not in space and time) and completely independent of minds for their existence and character. But the preponderance of evidence, which grows as his career unfolds, suggests that he was an ontological Platonist, not only about numbers but about functions, thought contents, and various other abstract entities. Frege did not, however, maintain a Platonist epistemology: He did not hold that we have a special intuitive faculty for apprehending abstract objects like numbers. Rather he developed the rudiments of a modern "pragmatic" epistemology, one of his most distinctive philosophical achievements.

The key to Frege's pragmatic epistemology lies in his "context principles", which are stated in various non-equivalent ways in *Grundlagen*. Simplifying, the idea is that one's conception of reference should be derivative from the analysis of the role of expressions (particularly singular expressions) in true propositions. One determines the true propositions, in the usual way, within successful cognitive practice. One identifies successful cognitive practice by seeing what enterprises produce successful communication and reasoned agreement. The content of this doctrine can be seen more clearly in its application to mathematics. Mathematics counts as successful cognitive practice because it yields successful communication, and agreement according to rational, checkable procedures. So its fundamental theorems should be counted true. Given an analysis of the logical form and semantics of truths of mathematics, which Frege carries out, reference to mathematical objects is required for the truth of mathematical theorems. Combined with various arguments that mathematical objects are abstract and mind-independent, the pragmatic epistemology yields a defense of ontological Platonism.

Frege's epistemology rivals a view that would begin by putting constraints on the notion of reference or knowledge (such as the constraint that they have to be accompanied by a causal relation, or be explained in some favored way). Such a view might argue from the claim that mathematical reference or knowledge cannot meet those constraints, to the view that mathematical theorems are not literally true or to the view that mathematics cannot be committed to abstract, mind-independent objects. Frege would regard such a procedure as backwards.

Frege's definitions in *Grundlagen* of the key mathematical terms are very close to those that would be given today. But they rely on the notion of an extension of a concept. In a footnote in section 68 and in section 107 Frege exhibits some unclarity about this notion. Much of his work between 1884 and 1893 was an attempt to clarify the notion, and to justify the key axiom that made use of the notion. This axiom states that all and only F's are G's if and only if the extension of F is identical with the extension of G. This axiom was a key to Frege's logicism. Frege found it less obvious than his other axioms. (Cf. a remark in the Introduction to *Grundgesetze*.)

Again, some of Frege's greatest contributions came as means to a further end. His ground-breaking theory of language in "Function and Concept" (1891), "Concept and Object" (1892), and "On Sense and Denotation"

(1892) was motivated by the desire to clarify the key notion and justify the key axiom. In these articles, Frege proposed to analyze language in such a way that the semantical value of complex expressions would be shown to be a function of the semantical values of their parts. To this end, he took predicates to denote functions. The functions, which he called "first-level concepts", take objects as arguments and yield truth or falsity as values. Higher-level concepts take functions as arguments and again yield truth-values as values. This analysis was the first systematic statement of a compositional, truth-conditional semantics. Such an approach has dominated philosophy of language in this century.

Frege also developed a distinction between sense and denotation (*Sinn und Bedeutung*). The distinction was introduced by an example. In a true sentence of the form "a = b", the denotations or referents of the two proper names are the same. So at the level of denotation, true sentences of that form do not differ from sentences of the form "a = a". But identities of such forms typically differ in what they express; their cognitive values typically differ. Frege proposed that the senses that their respective parts express differ, even though the referents or denotations are the same.

Frege produced parallel compositional theories of sense and denotation. The denotation of a (declarative) sentence was held to be a truth-value. The sense of such a sentence was held to be an abstract thought. The theory of sense enters in an elegant and plausible way into Frege's account of intensional contexts. (An intensional context is a linguistic context in which exchange of expressions that are ordinarily co-denotational does not appear to preserve the denotation of expressions within which the exchange is carried out.) Simplifying slightly, Frege held that in such contexts, expresssions denote their ordinary senses rather than their ordinary denotations, and that substitution of co-denotational expressions *in the context* preserves the denotations of the containing expressions. For example, in "Al believes 2 + 2 = 4", the expression "2 + 2 = 4" denotes not a truth-value, but a thought. Exchange of sentences with the same truth-value as "2 + 2 = 4" will not necessarily preserve the denotation (truth-value) of the whole belief sentence; only exchange of sentences that ordinarily express the same thought will do so—since in the context, "2 + 2 = 4" denotes a thought, not a truth-value. Thus Frege identified some of the primary problems in modern semantics and produced a fruitful and arguably correct strategy for dealing with them.

Frege's notion of sense is less familiar than it may at first seem to be. Although he did associate senses with expressions of natural languages, he did not (or did not in general) identify senses with what moderns would count conventional linguistic meanings. His primary notion for understanding senses was that of a cognitive value, not what is conventionally or normally understood by an expression in a community. He thought that the senses of demonstratives vary with almost each occasion of use, though the conventional linguistic meanings of demonstratives do not thus vary. The idea is that the user's perspective on the world varies with each use. Frege did not believe that sense varies for non-demonstrative expressions to that extent. But he did identify sense with a more idealized conception of cognitive value than would be common today. In fact, he tended to think of senses of non-context-dependent expressions in natural languages as what would be understood by speakers of the language if the speakers had perfected the language for the purposes of knowing about the world (including the world of mathematics) and of expressing that knowledge in an ideally perspicuous way. Thus it was coherent to suppose, from Frege's point of view, that no one could fully and correctly explicate the sense of some expression that is in common use. The sense of an expression might depend on a rationale for its use that is implicit in that use, but that no one has yet come to understand. Thus the sense of number expressions would be fully explicated only when logicism is fully established and articulated. Frege thought that fully understanding (in the sense of being able to explicate) the sense of an expression in a language is not in general separable from understanding the reality that the language is used in knowing.

The sense–denotation distinction remains important in theories of language and cognition. But Frege marshalled it to justify his ill-fated axiom. He developed an intricate argument for claiming that the two sides of the main biconditional in the axiom had the same sense. If this were true, the axiom would clearly be true. But Russell's paradox, which Frege learned of in a letter from Russell in 1902, showed that Frege's axiom is false. This result undermined Frege's version of logicism. Frege's notion of the extension of a concept was never fully clarified. Frege's primary ends were thwarted. But his contributions to logic from 1879 were independent of the axiom. And many of his contributions to philosophy of mathematics and language and to epistemology are of permanent value.

The bibliography of this article is retained as an integral part of the article, unlike the bibliographies in the original publications of the other articles in the collection, whose entries are collected in a bibliography at the end of the collection. The intent is to retain the "introductory" style of this article.

## BIBLIOGRAPHY

Burge, Tyler, "Frege on Extensions of Concepts, From 1884 to 1903" *The Philosophical Review*, 93 (1984), 3–34.

—— "Frege on Truth", in L. Haaparanta and J. Hintikka (eds.), *Frege Synthesized* (Dordrecht: D. Reidel, 1986).

—— "Frege on Sense and Linguistic Meaning", in David Bell and Neil Cooper (eds.), *The Analytic Tradition* (Oxford: Blackwell, 1990), 30–60.

Church, Alonzo, *Introduction to Mathematical Logic*, i (Princeton: Princeton University Press, 1956).

Dummett, Michael, "Gottlob Frege", in P. Edwards (ed.), *Encyclopedia of Philosophy* (New York: MacMillan Co. and The Free Press, 1967).

——*Frege: Philosophy of Language* (New York: Harper and Row, 1973).

——*The Interpretation of Frege's Philosophy* (Cambridge, Mass.: Harvard University Press, 1981).

Frege, Gottlob, *The Basic Laws of Arithmetic*, ed. M. Furth (Berkeley: University of California Press, 1964).

——*Collected Papers* (Oxford: Basil Blackwell, 1984).

——*Conceptual Notation* (Oxford: Oxford University Press, 1972).

——*The Foundations of Arithmetic*, trans. J. L. Austin (Oxford: Basil Blackwell, 1980).

——*Philosophical and Mathematical Correspondence* (Chicago: University of Chicago Press, 1980).

——*Posthumous Writings* (Chicago: University of Chicago Press, 1979).

Parsons, Charles, "Frege's Theory of Number", in M. Black (ed.), *Philosophy in America* (Ithaca, NY: Cornell University Press, 1965).

Sluga, Hans, *Gottlob Frege* (London, Boston, and Henley: Routledge and Kegan Paul, 1980).