# *A Theory of Aggregates*

TYLER BURGE

UNIVERSITY OF CALIFORNIA, LOS ANGELES

Introductions to the mathematical notion of set standardly explicate that notion by distinguishing it from a vaguer, more ordinary notion of aggregation, collection, or combination. Since Cantor and Frege, students have been warned that sets are not located in space-time and cannot be perceived or acted upon, and that a set of, say, fifty-two playing cards is not to be confused with the set of the four suits. The severity of this warning—and the usefulness and power of set theory—have led to the neglect of the ordinary notion of aggregation. The purpose of this paper is to refine that notion and give it a place in semantical and philosophical analysis.[1]

Refinement is worthwhile for two reasons. First, the notion is useful in giving semantically intelligible form to some of our descriptions involving plural constructions and mass terms. Second, the notion will serve to hone the distinction between concrete and abstract objects.

## I

(1) The stars that presently make up the Pleaides galactic cluster occupy an area that measures 700 cubic light years.

The term 'occupy' in (1) seems intuitively to relate the stars to an area of space. But how are we to give formal representation to the subject term? We cannot reasonably utilize a quantified analysis to the effect that every star in the cluster occupies the area. 'Occupy' does not apply to the stars taken individually—

NOÛS 11 (1977)

but somehow collectively. We might change the logic underlying our semantical theory so as to admit plural subjects and multigrade relations. But although this alternative is doubtless worth exploring, it seems likely to breed undesirable complication. The most natural tack is to treat the subject term in (1) as a singular term and give the relational predicate the reading 'occupies'. We have precedent (for example, in our dealings with quantified sentences) for counting the difference between singular and plural forms of natural-language predicate expressions irrelevant as far as our formal representation of those expressions is concerned, and for locating the effect of the plural elsewhere in the sentence. Thus, the expression 'occupy' in 'All men occupy space' is formally represented by the same predicate as is 'occupies' in 'Burge occupies space'. But if we are to treat the subject term in (1) as a singular term, what does it denote?[2]

A first impulse is to invoke the set of the relevant stars. But such an invocation fails to represent (1) under a plausibly intended interpretation. For sets never occupy space. There are several possible defenses against this objection, but none seems satisfactory. One defense denies that the view that (1) predicates 'occupies' of a set and an area commits one to saying that a set occupies the area. (Cf. Grandy [8].) Another claims that any semantical account of (1) will have to introduce notions that are unfamiliar to many native speakers and that there is nothing wrong with introducing a set to interpret the subject term and a notion of set-occupation, which is easily explained, to interpret the predicate. The first defense is *ad hoc* and undermotivated. The second stands only if one cannot give an illuminating account of (1) which is more satisfying to the intuitions—and more adequate in representing the intentions—of natural-language users. Both defenses are made less persuasive by considering sentences like, 'The Pleiades galactic cluster occupies the area'. In such cases it is clear that the predicate which represents 'occupies' applies to a physical object (in this case, the cluster.)[3] There is no intuitive basis for thinking that the stars occupy the area in a different sense ("set-occupies") from the sense in which the cluster occupies it.

An alternative in representing (1) is to look to the calculus of individuals and appeal to a sum (cf. [10] and [5], esp. pp. 51-2) of the relevant stars. This interpretation rejects the view that the subject term in (1) applies to an abstract object. But it would nevertheless misrepresent our method of individuating concrete pluralities. The relevant sum of stars would be individuated in terms of all objects having common (say, spatio-temporal) parts with the stars. Such parts—and hence ultimately the sum—would be "constructed from" and individuated by reference to objects (such as occupied space-time points) that have no other objects as parts. It is, I think, uncontroversial that in using ordinary discourse we do not take predicates true of partless objects as the primitive basis for any such construction or individuation. It is equally clear that we do not individuate objects denoted by plural descriptions like 'the stars in the cluster' by reference to *all* objects that share some spatio-temporal region with any of the stars. We count pluralities of stars the same just in case they involve the same stars.

The question of what method we employ to individuate concrete pluralities should be distinguished from the question of whether the two methods we have mentioned always pick out the same objects. The first question bears on the formal representation of natural and scientific discourse and is the only one at issue here. I shall remark on the second in Section IV.

Problems similar to those developed with reference to (1) also arise in interpreting sentences like:

(2) Stone Mountain is constituted of a meteor, which is in turn constituted of its atoms.
(3) Bengal tigers are distributed over south-central Asia.
(4) The molecules in the flask are in equilibrium.
(5) The weight of the marbles is 55 pounds.

Of course, some of these sentences may be roughly "paraphrased" in such a way as to avoid reference to an aggregate-like entity. But such "paraphrases" are sufficiently complicated to be disqualified as candidates for representing the logical form of the original sentences. Moreover, there does not seem to be any uniform way of coming up with such paraphrases. Our notion of aggregate is meant to aid in interpreting plausible, formal representations of pluralized descriptions in sentences like (1)-(5).

## II

From a purely formal point of view, the principles of aggregation that I shall employ are familiar. They may be obtained by two steps: first, limit set theory to first-order sets (sets with only individuals as members), where there is no null set and where each singleton set is identified with its member; second, replace the primitive 'is a member of' by 'is a member-component of' (abbreviated '$\alpha$'). We then define '$x$ is an aggregate', '$A(x)$', as follows:

$$A(x) \leftrightarrow_{\mathrm{df}} (\exists z)\,(\exists y)\,(z\alpha x \;\&\; y\alpha x \;\&\; z \neq y).$$

Before further articulating the principles of aggregation, we must say a word about the underlying logic. I shall be assuming a logic with definite descriptions and a restriction on instantiation:

(6) $(x)A \;\&\; (\exists y)\,(y = t) \rightarrow A(x/t)$,

where '$A$' ranges over well-formed formulae, '$t$' over singular terms, and '$A(x/t)$' is read, 'the result of substituting $t$ for all occurrences of the variable $x$, rewriting bound variables where necessary'. The chief principle for descriptions is

(7) $(x)\,(x = (\iota y)A \leftrightarrow (y)\,(A \leftrightarrow y = x))$,

where variable $x \neq$ variable $y$, and $x$ is not free in $A$. The effect of failure of reference on truth-value is summarized in the principle

(8) $At \rightarrow (\exists y)\,(y = t)$,

where $A$ is any atomic predicate, including identity, $t$ is any argument expression of $A$, and $y$ is not free in $t$. Thus (roughly), if a singular term in an atomic sentence does not denote, the containing sentence is untrue. The semantics supplies no denotation for intuitively non-denoting singular terms. Aside from (6)-(8) our underlying logic contains no surprises. (For details, see [2].)

We now turn to axioms for aggregation. To simplify formulation, I introduce the following definition schema for an analog to set-abstraction:

$$[x\!: \phi(x)] =_{\mathrm{df}} (\iota y)((x)(x\alpha y \leftrightarrow (\exists z)(\phi(z) \;\&\; x\alpha z))).$$

The definiendum literally reads, 'the totality of objects that are member-components of some $\phi$'. Analogous to comprehension we have:

(A1) $y\alpha\ [x: \phi(x)] \leftrightarrow (\exists z)(\phi(z)\ \&\ y\alpha z)$.[4]

Lacking unrestricted existential generalization, the existence of a null aggregate cannot be proved from (A1). Since we want to deny such existence, we should be explicit about it:

(A2) $-(\exists y)(y = [x:\ x \neq x])$.

Identity conditions for aggregates are determined by this principle of extensionality:

(A3) $A(z)\ \&\ A(y) \rightarrow (y = z \leftrightarrow (x)(x\alpha y \leftrightarrow x\alpha z))$.

The antecedent is motivated by the regional character of our theory. Since we are not assuming that all objects are aggregates, the antecedent is necessary to avoid collapsing distinctions among non-aggregates. On our view, only individuals are member-components of aggregates:

(A4) $x\alpha y \rightarrow I(x)$.

It will be convenient to take individuals as their own unique member-components:

(A5) $I(x) \rightarrow (x\alpha x\ \&\ (z)(z\alpha x \rightarrow z = x))$.

The analogy to Quine's treatment of individuals will be evident. (Cf. [14].)

Our axioms yield existence assertions about aggregates only if they are supplemented with existence assertions about individuals. Supplementation must come from the rest of our empirical theory. For example, we need to know that there are tigers (and that tigers are individuals) before we can derive that there is an aggregate of all tigers. Conditionalized analogs of some of the weaker methods of generating sets are derivable from ($A$1). For future reference, we note the principle of union:

($U$) $(\exists u)(u\ \alpha\ y) \vee (\exists v)(v\ \alpha\ z) \leftrightarrow (\exists w)(w = [x : x\ \alpha\ y \vee x\ \alpha\ z])$.

The motivation for developing a theory of aggregates on an analogy with set theory is that the resulting principles seem the most natural way of reconstructing our ability to form and apply natural-language plural constructions like those in (1)-(5). Such constructions apply to pluralities which are individuated purely by reference to the objects specified. The constructions do not commit one to assumptions about the nature of the spatio-temporal configurations in which the individuals arrange themselves[5] or, more particularly, to a view about their spatio-temporal proximity. The etymology of our technical term 'aggregate' may suggest such assumptions; but we shall ignore the suggestion, as ordinary speakers often do.

The formal analogies between aggregates and first-order sets should not obscure the differences. Aggregates, like their member-components and unlike sets, are physical objects having spatio-temporal location and the capability of exerting forces and undergoing changes. Like their member-components, aggregates come into and pass out of existence. Member-components of aggregates are spatio-temporal parts of the aggregate, although not all parts of an aggregate are member-components of it. For example, each tiger in the aggregate of all tigers is part of that aggregate; but whereas proper parts of tigers (tiger hearts) are parts of that aggregate, they are not member-components of it.

One might grant the usefulness of aggregates in interpreting sentences of science and ordinary discourse, yet still wonder whether aggregates are eliminable in favor of other kinds of entities. Since my interest here is interpretative, I need not argue the issue of reduction. But a few remarks are in order.

The two most obvious reductions would be to individuals and to sets. Quite apart from general questions about the ontological relevance of reductions of this sort, neither seems immediately compelling as an argument for elimination. Take the reduction to individuals. It would require eliminating occurrences of predicates like 'is scattered', 'is in equilibrium', and so on, in favor of complex descriptions of relations between individuals. The complexity of carrying out the task for all aggregates under all circumstances becomes discouraging when one considers the complexity of micro-macro explanations (reductions?) in statistical mechanics, where certain simple aggregates are considered only under the most idealized conditions. Moreover, reductions of aggregates to individuals would have to be argued for one by one, since there seems to be no general method of carrying them out.

The reduction of aggregates to sets would be simpler from a purely formal point of view. In addition to the easy mapping between such aggregates and first-order sets, we would also need to introduce new predicates ("set-scattered") to replace predicates peculiarly applicable to aggregates; and we would require a cumbersome duplication of predicates (adding "set-visible" and "set-occupies space" to go along with 'visible' and 'occupies space') in cases where we apply a predicate both to aggregates and to individuals. The details would be tedious, but perhaps not theoretically difficult. What is questionable is whether the procedure explains anything. It promises the same explanatory vacuity and intuitive implausibility that "eliminating" physical-object individuals by identifying them with their singleton sets would yield. Our feeling is that certain physical-object individuals play an epistemological role in our theorizing that would be distorted or obscured if they were eliminated in favor of sets. In Section III, I shall argue that certain aggregates play a similar role.

Our motivation for restricting principles for aggregates to those analogous to principles for *first-order* sets is what we want a system that is plausibly nominalistic: whatever else besides physical objects there may be, aggregates themselves are taken to be physical. Our "first-order" principles will not allow us to distinguish aggregates except by reference to physical-object individuals. In particular, for $n$ individuals, there will be $(2^n - (n + 1))$ aggregates. Suppose that we were to allow plurality aggregates as member-components. Then in addition to tigers $a$ and $b$ and the aggregate $[a, b]$, we could have the aggregate $[a, [a, b]]$. This situation would be out of keeping with the intuition that aggregates are physical objects, because it is *inconceivable* and *impossible* that, for any individuals $a$ and $b$, $[a, b]$ and $[a, [a, b]]$ could have different spatio-temporal locations, or that they could exert different forces or undergo different changes. Similar considerations count against admitting singleton aggregates over and above individuals.[6]

The notion of aggregate, like that of first-order set, presupposes a notion of individual. Given our interpretative purpose, we shall expect the individuals of the theory to be provided by our antecedently-developed empirical discourse. More specifically, we assume a finite list of syntactically simple, one-place predicates which apply to physical *things* (see n. 3) and which are derived from count-nouns. For purposes of interpreting our ordinary discourse, these will be roughly the primitive, sortal thing predicates of English. The list is understood to exclude count-noun predicates, such as 'is an aggregate', which are true of aggregates or which obey the principles of inclusive reference or cumulative reference to be stated in Section III. Intuitively, the relevant predicates will apply to objects individuated at least partly by reference to their structure or function, rather than merely to their composition.

Initially, we define '$x$ is an individual', abbreviated '$I(x)$', in terms of a disjunction of all the predicates on the list. As a rough illustration:

(*I*1) $I(x) \leftrightarrow_{df} \text{Aardvark}(x) \vee \text{Atom}(x) \vee \ldots \vee \text{Electron}(x) \vee \ldots \vee \text{Mountain}(x) \vee \ldots \vee \text{Zither}(x)$.

The specificity of a definition like (*I*1) renders the notion of individual extremely sensitive to changes in language or theory. For addition of predicates with extensions not included in the extensions of the old predicates will change the extension of 'individual'. I do not regard this sensitivity as itself a defect. The notion of individual is language-relative in something like the way (*I*1) suggests. But the method of (*I*1) will raise problems when we come to mass terms. A less informative, but more stable, definition is

(*I*2) $I(x) \leftrightarrow_{df} (\exists w)(x\alpha w)$.

I might say here that the enterprise of *defining* the notion of individual is not in itself a very important one. But making clear what may and what may not be taken as an individual is crucial to the philosophical interest of a theory of aggregates.

Our focus in (*I*1) on physical *things* was guided by a desire to simplify exposition. We could have allowed physical events as individuals without extensive changes in viewpoint. Indeed, doing so may help in understanding certain natural phenomena, such as light and electricity.

The premium placed on primitive one-place predicates in (*I*1) is important. Since we are not admitting non-extensional temporal operators into our logic, objects satisfying such predicates will not at any time fail to satisfy them. Thus, if 'atom' is placed on the list, it will *not* be understood as satisfied by an aggregate of elementary particles when and only when those particles are configurated in an atomic structure. On such an understanding, what *was* (not simply "made up") an atom would cease to be one if the atomic structure were broken by the dispersal of particles. Such a construal would take atomic structure as a temporally relative property of the aggregate, rather than as constitutive of an individual. And 'atom' would be a two-place phaseal predicate like 'banker' or 'sapling'. (What once was a sapling might cease to be one without ceasing to exist. Cf. [3].) To treat 'atom' as primitively one-place, on the other hand, is to view an atom as an entity numerically distinct from the aggregate of particles that constitute it. (A reason to view atoms in this way is that we often speak of them as continuing to exist, undispersed, even though they exchange some of their constituent particles.) Of course, the theory of aggregates is neutral as to what predicates are used to determine the individuals. But I shall be making the natural assumption that numerous structurally complex physical objects are individuals.

## III

The use of our theory in representing sentences (1)-(5) is straightforward. For example, we can read (1) as 'The aggregate of stars that presently makes up the Pleiades galactic cluster occupies an area that measures 700 cubic light years'. The aggregation operator defined in Section II can be invoked in all these cases.

Moreover, we can also handle sentences like

(9) The aggregates of stars presently in the Pleiades occupy an area that measures 700 cubic light years.

Sentences like (9) should probably be regarded as true. Each of the $2^n - (n + 1)$ aggregates (for $n$ stars in the cluster) occupies space. Collectively, the aggregates of stars should occupy the same space that the stars collectively occupy. Indeed, the aggregates of stars "taken collectively" are just the individuals in them "taken collectively". This can be seen by substituting 'aggregate of stars in the Pleiades' for '$\phi$' in the aggregation description.[7]

Grammatical parallels between mass terms and the plural forms of common nouns have often been noted. Semantical analogies also obtain. The principles of "taking objects collectively" that we have discussed in relation to concrete pluralities carry over to at least some stuffs. Analogs to sentences (1)-(5) are easily constructed. For example, to (3) and (4):

(10) Water is widespread.
(11) The hydrogen in the flask is in equilibrium.

Some of these analogs (e.g., (11)) have readings that may be represented with demonstrative-governed definite descriptions. But (11), like (10), may be taken as specifying a totality and thus is served by the aggregation operator.[8]

If a mass term is true of aggregates at all, it is ordinarily true of aggregates composed of other aggregates that also satisfy the predicate. To sharpen this point, we define a predicate for aggregate-inclusion:

$$y \subseteq x \leftrightarrow_{\mathrm{df}} (\exists w)\ (w\alpha y)\ \&(z)(z\alpha y \rightarrow z\alpha x).$$

One-place physical-object sortals are true of individuals but not ordinarily of aggregates of those individuals. A one-place mass-term predicate, on the other hand, is true of every aggregate included in the aggregate of all individuals which it is true of. For example, 'is some water' is true not only of the totality of water, but of all smaller quantities of water down to the molecules. Formally, one-place mass-term predicates that apply to aggregates will substitute for '$\varphi$' in the schema

(*IR*) $y \subseteq [x\colon \varphi(x)] \leftrightarrow \varphi(y).$

We shall call (*IR*) the *principle of inclusive reference*.

We must now ask how to define 'individual' in our extended theory. One might be inclined to do so by simply adding mass-term predicates to the disjunction in (*I*1), thereby including among the individuals quantities of stuff of any size. There are, however, two strong objections to defining individuals of stuff aggregates in this way. One is that doing so would prevent us from ever identifying aggregates determined by mass terms with aggregates determined by sortals. For example, a given quantity of water could not be identified with the aggregate of its constituent water molecules because the former would have member-components (non-minimal quantities of water) which the latter would not have.

A deeper objection is that such a definition forces a choice between giving up the theory's nominalism and altering its most central and natural principles. For by the definition, the principle of inclusive reference (*IR*), and the natural comprehension and extensionality axioms (*A*1) and (*A*3), we commit ourselves to multiplying entities in a way analogous to the way set theory does. To illustrate: take three water molecules (or any three quantities of water) named '*a*', '*b*', and '*c*'. In addition to the three molecules, there are three pair aggregates, each having two molecules as its member-components, and one triple aggregate, consisting of all three of the molecules. So far, so nominalistic. But since these aggregates are included in the largest water aggregate, we have by (*IR*) that the aggregates [*a*, *c*] and [*a*, *b*] satisfy the predicate 'is some water'. By the proposed definition, they are individuals. (This contradicts (*A*5). But suppose we give up (*A*5) and adjust our principles in some natural way.) Then, by the comprehension axiom (*A*1), we take the aggregate-union (cf. (*U*)) of these "individuals", thereby giving us the aggregate with these two pairs as its member-components. To preserve extensionality (*A*3), this aggregate must be distinguished from the previous four. The procedure could continue indefinitely to evolve from three water molecules a veritable overflow of unnaturally selected entities. Any alteration of (*IR*), (*A*1), or (*A*3) radical enough to block the argument would either deprive our theory of explanatory interest or change it into a theory (more like the calculus of individuals) where concrete totalities are not individuated by reference to member-components.

Given these alternatives, we should forego the proffered disjunctive definition of 'individual' and hold that not everything that satisfies a mass-term predicate is an individual of the theory. Whereas there is an aggregate of all minimal water units, there is no aggregate of all quantities of water (cf. (*A*4)).

As long as we focused on sortal predicates, (*I*1) and (*I*2) were equally acceptable definitions of 'individual'. This is because (*IR*) does not apply for sortals. Extending (*I*1) to include mass terms led to difficulties, so we take (*I*2) as the theory's definition of 'individual'. But we should not stop here. (*I*1) connected aggregate theory to the rest of our empirical discourse. Having dropped that definition in favor of a less informative one, we are left with the need to re-establish a connection. The gap can be spanned by adding postulates. We add one to the (metalinguistic) effect that if anything satisfies one of the sortal predicates in (*I*1), then it is an individual. More important, we utilize the findings of the various sciences to add postulates specifying what count as individuals of a given kind of stuff. For example, the discovery that water is $H_2O$ enabled us to postulate that water individuals are $H_2O$ molecules. Such statements as 'water is $H_2O$', which are usually discussed as identifications of properties, seem to me to be profitably viewed in this light. The statement may be seen either as an identification of aggregates or as a universally quantified biconditional. But the *point* of such statements is to specify the individuals of the macro-stuff aggregate.

This way of fitting aggregate theory into developing science provides an insight into the epistemological role of mass terms. Many such terms apply to aggregates (or phases of them, see below). Sortal predicates apply only to individuals. Since in learning to use sortals we learn what individuals satisfy them, we tend to think of ourselves as "constructing" aggregates out of more basic units. But where mass terms are concerned, our first exposure is normally to the aggregates. In talking of stuffs, we are sometimes talking of aggregates whose individuals we cannot yet specify. (This is, of course, why one should not take (*I*1) as it stands as the definition of 'individual' in a theory that contains mass terms.) We use our experience of stuff aggregates to seek out the relevant constituents. Stuffs are goad and guide to the discovery of new individuals.

Formalizing mass-term constructions with notation from aggregate theory does not commit one to attributing to the users of those constructions a belief that the referents are aggregates, much less a knowledge of what the relevant individuals are. The ordinary notion of the totality of water, which might be invoked in paraphrasing (10) under one of its interpretations, is probably essentially vague. The aggregate-abstraction notation used in representing this interpretation might be given the vague "totality" reading insofar as we are capturing the ordinary man's sense of the term. But the referent of the term is not similarly vague. Our formal semantics for the notation specifies what the referent actually is. Analogously, the rules of thumb used by ordinary speakers to individuate and reidentify portions of stuff are no clearer than practical purposes demand. But the rules that *are* used approximate—and suggest, given scientific background information—the principles of our theory. For example, we could expect an ordinary speaker to reidentify some gold as exactly the same by the rule that portion *a* is identical with portion *b* if and only if all and only relevant parts (i.e., gold parts) of *a* are parts of *b*. But the speaker need not be able to give a general explication of the notion of a relevant part in terms of sub-aggregates. Nor need he know what the minimal relevant parts are (gold atoms) or even that gold *has* minimal relevant parts. Speakers are often ignorant of the nature of stuffs and things they refer to.

Ordinary reference to member-components of stuff-aggregates is infrequent because such components are often not easily distinguishable. Sometimes this is merely a practical difficulty, and sometimes, as was once the case with water, the criterion is a matter for future discovery. But sometimes, as in the case of mixtures, the criterion may seem unobvious, even given the relevant scientific information. Some will take these cases as indicating that stuffs other than the pure substances (substances like water, gold) are not aggregates. But I think that the theory can be extended.

Let us consider mixtures. What, for example, are the individuals relevant to identifying a quantity of lemonade? If we halve a quantity of lemonade and continue the process, we soon reach a quantity that is evidently not lemonade. This suggests that there may be minimal lemonade units. Of course, we are pretty indifferent about just where to draw the minimality line; but we might note that we are vague about what the lemonade individuals are, in something like the way we are vague about individuals determined by count nouns like 'mountain'.

There remains, however, a serious difficulty not met by this suggestion. Suppose we proclaim that a minimal lemonade unit in a particular quantity of lemonade consists of 350 water molecules, one sugar molecule, and one citric-acid molecule. (To simplify matters, we assume that lemon juice is citric acid.) We could make analogous decisions about how to apportion additions to the mixture and how to partition arbitrary mixtures of lemonade. Still, we have the problem of saying which $H_2O$ molecules in the solution go with which sugar crystals and which citric acid molecules to make up the "lemonade individuals". And surely there is no non-arbitrary principle of segregation. The very nature of mixtures is such that no structural restrictions are placed on the mixed substances—other than that they be mixed to within certain (rough) proportions.

In thinking about how to respond to this situation, one question must remain uppermost: 'Under what conditions (of scatter, addition, substraction of components) would we say that we have the very same lemonade?'. I shall assume the following points. If one spills the lemonade on the linoleum, the spattered lemonade might still be reidentified as the same as was once in the container. If one dilutes some lemonade in a container with water, without making it so weak as not to count as lemonade any more, one has more (though weaker) lemonade. (I leave it open whether the original lemonade is still in the container, though I think it is.) If one dilutes lemonade with water to a point where the result is no longer lemonade, then ordinarily there will be no lemonade in the container. In such a case, the original lemonade may in principle be recovered by recovering the original components and mixing them appropriately. If one removes some water (*strictly* speaking, I think, even one molecule) from a container of lemonade, without disturbing the sugar and citric acid, there is less (though stronger) lemonade in the container; and we would not say that the original lemonade is scattered—partly in the container, partly elsewhere. Instead, there is less lemonade in the world. These points may be generalized.

How, then, are we to apply aggregate theory to mixtures? We can explicate these points by denying that a quantity of lemonade is an aggregate of individuals that are themselves lemonade. The relevant individuals are the water molecules, sugar molecules, and citric-acid molecules. Of course, any aggregate of such molecules will pre-exist the mixing that made them into lemonade. So if some lemonade is taken to "be" an aggregate of the relevant individuals, it must be such an aggregate in roughly the same sense in which some ice is some water ($H_2O$). The point is that the lemonade mixture is a temporal phase of the aggregate: a phase during which the relevant molecules are integrated within certain proportions. One is dealing with the same lemonade as long as one is dealing with the same aggregate and that aggregate and its components do not violate the appropriate mixture conditions.

We have inclined toward treating mixtures as phases of aggregates of individuals, while we have leaned toward taking atoms, molecules, mountains, stars, tigers, and marbles as individuals (rather than as phases of the aggregates of some of their constituents). What considerations underlie these inclinations? Some individuals can lose or exchange their immediate constituents yet remain the same object. This may be true of certain molecules; it is certainly true of atoms, mountains, stars, tigers, and marbles. On the other hand, a necessary condition for sameness of aggregate-phase is identity of member-components. Thus, although our lemonade could give or take an electron, it is individuated partly by reference to identity of its component molecules. This point suggests, I think, a more fundamental one. By and large, we incline more to treat a physical object (other than an elementary particle) as an individual the more its individuation is dependent on reference to a constant, "interesting" physical structure among its constituents (even if the constituents change) or to a constant function of the whole. Quantities of substances (like water) and other aggregates are individuated purely by reference to individual constituents. Mixtures are the intermediate case. The "structure", or relationship among the constituents, that is relevant to their individuation is minimal. Only spatial juxtaposition in certain proportions, and in certain minimal portions, is required; the rest is composition. This relative lack of a constant, "interesting" structure motivates our inclination not to treat mixtures as individuals. It also motivates not taking them as aggregates of all permutations of minimal, proportional groupings of their constituents. For the inconstancy of structure of such groupings makes them less fit for the explanatory purposes to which we put individuals. Or perhaps rather, their inconstancy of structure signals their secondary explanatory role.

I would not care to lean on these points too hard. The notion of "interesting structure" is vague and subject to further explication. Relative to one or another purpose, one might widen or narrow the class of individuals. My aim has been to characterize those inclinations that seem natural.

The formal effect of taking mixtures as temporal phases of aggregates is worth emphasizing. For most purposes, we can treat mass-term predicates like 'is some lemonade' as having one argument place (just as we can with 'is some ice' or 'is a sapling'). But for a context-free representation, mixture predicates should be taken to be covertly relative to a time. (For more on the representation of discourse about phases, see [3].) Thus, to formalize 'The lemonade that fills the glass is wet', we would write:

$$\text{Wet}((\imath y)(\text{Lemonade}(y, \text{now})\ \&\ \text{Fills}(y, \text{the glass}, \text{now})), \text{now})$$

or

$$\text{Wet}([x : \text{Lemonade}(x, \text{now})\ \&\ \text{Fills}(x, \text{the glass}, \text{now})], \text{now}).$$

(Observe that the aggregation operator does not commit us to the view that the individuals of the lemonade are themselves lemonade.)

Although the principle of inclusive reference does not apply for two-place, phaseal mass terms, a *principle of cumulative reference* does. A stuff predicate that applies to aggregates or aggregate phases applies to any aggregate-union of quantities of stuff to which it applies. For one-place mass-term predicates subbing for '$\phi$':

(*CR*1) $\phi(x)\ \&\ \phi(y) \rightarrow \phi([z : z\ \alpha\ x \vee z\ \alpha\ y])$.

For two-place ones:

(*CR*2) $\phi(x,t)\ \&\ \phi(y,t) \rightarrow \phi([z : z\ \alpha\ x \vee z\ \alpha\ y], t)$,

where '$t$' ranges over times.[9] We should regard science as revealing that a mass-term predicate has one or two argument places depending (partly) on whether it applies to a substance or a mixture. It would not be the first time that discoveries in science affected our analyses of logical form.

Mixtures are not the only challenge to the application of aggregate theory to stuffs. Organic tissues pose less tractable problems. But adequate consideration of these matters is out of bounds here. What I wanted to suggest was that the theory is a useful instrument in clarifying our methods of individuating quantities of stuff.

## IV

A certain notion of physical object has motivated several decisions in constructing our theory. In the space that remains I shall outline the rudiments of the notion I have been presupposing. (Cf. n. 3.)

A time-honored principle for individuating physical objects, usually taken to be necessary, is

(*P*) Physical objects are the same if and only if they occupy the same space-time.

The calculus of individuals in effect assumes (*P*) in its specification of complex physical objects. For example, the sum of all the water molecules (past, present, future) is said to be the *unique* object that exactly coincides with all the water molecules. On the other hand, from the viewpoint of the theory of aggregates as we have developed it, plural physical objects may coincide spatio-temporally. An aggregate of water molecules coincides with itself. But it is conceivable and in some sense possible that the hydrogen and oxygen atoms that constitute those molecules never exist apart from the molecules, so that the aggregate of those atoms would also coincide spatio-temporally with the aggregate of water molecules. But since aggregates are individuated by their member-components and the member-components differ, the aggregate of molecules and the aggregate of atoms would not be the same.[10]

The argument as stated rests on a thought experiment that is wildly implausible from a physical point of view. Hydrogen and oxygen atoms do not come into existence the moment they form a water molecule, nor do they often pass out of existence when the molecule decomposes. But there are rare cases in nature that satisfy our argument. When gamma rays in an electric field pass through certain nuclei, there is a significant probability in low-energy systems that an electron and a positron will be formed in the bonded state as positronium. Since positronium is unstable, the probability is high that its constituents will in a (very) short time collide and annihilate to produce two gamma rays (or sometimes just one). Thus, as far as physical theory can determine, many positronium atoms occupy the same space-time that the respective aggregates of their constituents do.[11] On fairly natural assumptions about what count as individuals (atoms as well as elementary particles), aggregate theory distinguishes these positronium atoms from the aggregates of their constituents. On these assumptions, (*P*) is not only not necessary; it is false.

Clearly, this argument against (*P*) depends on the view that non-minimally-sized constituents of the universe are individuals and on axioms ($A1$) and ($A3$). I assume that natural language and scientific theory commonly accept complex entities as individuals. If aggregate theory is a reasonable tool in explicating some natural and scientific discourse, then the view of physical objects presupposed in such discourse does not include ($P$). What might this view be?

In Sections II and III, I argued that allowing plurality aggregates as member-components would controvert nominalist intuitions because we would thereby be able to distinguish objects that could not possibly or conceivably differ in their spatio-temporal location or in the forces they exert or undergo. I have just argued that given natural assumptions about individuals, aggregate theory, though nominalistic, distinguishes objects which do not differ in spatio-temporal location. The key reconciling the arguments lies in the modal adverbs. Any positron-electron aggregate that coincides spatio-temporally with a positronium atom could conceivably or possibly have failed to constitute it. The positron and electron might never have been bonded (in which case the aggregate would have existed, but the atom would not have). Or the aggregate could have predated the formation of the atom or postdated its demise. There are also dispositional differences. For example, the aggregate is scatterable, whereas the atom is not.

It would be a mistake, of course, to think that the only differences between the individual and the coincident aggregate are modal. The two objects differ in kind, one being a positronium atom, the other, an aggregate. It is plausible to view the modal differences as inductively projected from categorical, non-modal, differences between other members of the respective kinds. (For the classical development of this theme, see Goodman [7].) For example, we project from the fact that most positron-electron pair aggregates have scattered, unbonded member-components. Positronium atoms, on the other hand, are identified by reference to the positron-electron bond.

Kind predicates themselves depend for their application at least partly on our inductions from, or reactions to, other categorical differences among objects. On the view of physical objects we have been elaborating, particular physical objects may on occasion satisfy different modal and kind predicates without differing in their spatio-temporal histories. The applications of kind predicates and modal predicates that we have developed by inductive projection from other cases commit us on these occasions to distinguishing coincidentally coincident objects. Thus, on the present view, individuation depends essentially on inductive (or reactional) projection. For proponents of (*P*), spatio-temporal location is ultimately all that counts.

On a natural selection of individuals, occasions in which different objects spatio-temporally coincide are relatively rare in nature—and perhaps for good reason. If modal predicates are applied on the basis of projections from statements using kind predicates and historical predicates, then two objects that satisfy the same historical predicates but different modal predicates will belong to different kinds most of whose respective members differ from one another historically. The assumption is that we usefully (and truly) distinguish historically indistinguishable physical objects by assigning them to different kinds only if most members of one kind have different spatio-temporal histories from all members of the other.

Another possible restriction on counterexamples to (*P*) should be mentioned. Of the two spatio-temporally coincident objects in our example, only one is a physicaindividual. Perhaps we never distinguish spatio-temporally coincident individuals. (Indeed, some philosophers would intend (*P*) only in this sense; cf. n. 3.) Of course, exploring this conjecture would necessitate further specification of the notion of individual. I have suggested that the individuals of a natural language or scientific theory are informatively identified by reference to its primitive, one-place physical-object, sortal predicates.[12] We acquire such predicates to facilitate induction and simplify empirical descriptions and generalizations. Thus, our notion of physical object beds down with our methods of induction and our intuitions about simplicity. Unfortunately, in philosophy, to uncover the bedfellows is not to reveal the form of their relationship.

## References

[1] Burge, Tyler, "Truth and Mass Terms," *Journal of Philosophy* 69(1972): 263-82.
[2] ———, "Truth and Singular Terms," NOÛS 8(1974): 309-25.
[3] ———, "Mass Terms, Count Nouns, and Change," *Synthese* 31(1975): 459-78.
[4] Frege, Gottlob, *The Foundations of Arithmetic* (Evanston, Ill.: Northwestern University Press, 1968). First published 1884.

[5] Goodman, Nelson, *The Structure of Appearance* (Cambridge, Mass.: Harvard University Press, 2nd ed., 1966).
[6] ———, "A World of Individuals," reprinted in *Philosophy of Mathematics*, ed. by P. Benacerraf and H. Putnam (Englewood Cliffs, N.J.: Prentice-Hall, 1964).
[7] ———, *Fact, Fiction, and Forecast* (Indianapolis: Bobbs-Merrill, 2nd ed., 1965).
[8] Grandy, R. E., "Comments," in *Approaches to Natural Language*, ed. by J. M. E. Moravcsik and P. Suppes (Dordrecht: D. Reidel, 1973).
[9] Heitler, W., *The Quantum Theory of Radiation* (Oxford: Clarendon Press, 3rd ed., 1954).
[10] Leonard, Henry, and Nelson Goodman, "The Calculus of Individuals and Its Uses," *Journal of Symbolic Logic* 5(1940): 45-55.
[11] Lewis, David, "Nominalistic Set Theory," NOÛS 4(1970): 225-40.
[12] Parsons, Terence, "An Analysis of Mass Terms and Amount Terms," *Foundations of Language* 6(1970): 362-88.
[13] Quine, W. V., *Word and Object* (Cambridge, Mass.: MIT Press, 1960).
[14] ———, *Set Theory and Its Logic* (Cambridge, Mass.: Harvard University Press, 1969).

## Notes

[1] I am indebted to David Kaplan, Adam Morton, and Wiley Gillmor for comments on successive phases of this paper. An abbreviated version was read at the Eastern Division of the APA, December 1973.

[2] Of course, the galactic cluster itself occupies the space. But the subject term in (1) does not denote the cluster. The cluster is not identical with the stars that presently make it up, since the same cluster can undergo changes in its constituent stars and since the stars that presently constitute it will eventually disperse, thus "outliving" it.

[3] I apply the word 'object' to non-compact, relatively amorphous pluralities and quantities of stuff, as well as to what the ordinary man (and I) would call "things". Objects, in this roughly Fregean sense, are the referents of the singular terms, or the values of the variables of a language or theory. If 'object' so used sticks in the reader's throat, perhaps 'entity' would go down more smoothly.

[4] It is possible to define an abstraction operator even closer to that of set theory. Thus let $\ulcorner \{x: \phi(x)\} \urcorner$ be defined as $\ulcorner (\iota y)\ ((x)(x\alpha y \leftrightarrow \underline{I}(x)\ \vee \phi(x)) \urcorner$. The analog to comprehension would be $\ulcorner y\alpha\ \{x: \phi(x)\} \leftrightarrow \phi(y) \vee I(y) \urcorner$. The other axioms could be left unchanged. This formulation is somewhat simpler in dealing with sentences like (1)-(5) that contain only sortals. But sentences like (9)-(11) and the issues about mixtures raised below are more smoothly handled with our present notation. I am indebted on this point to Wiley Gillmor. Our chosen formulation of ($A$1) makes ($A$5) more important than the alternative formulation would. But ($A$5) is far more natural for aggregate theory than its analog seems for set theory.

[5] In [11], David Lewis uses the conceptual tools of the calculus of individuals to define a number of pseudo-membership notions that in some respects serve our interpretative purposes better than traditional formulations do. Still, all these notions involve specifically spatial restrictions on the combinations that can be specified. There is no reason to think that natural language places such restrictions on plural constructions. (I should note that Lewis does not claim that it does.)

[6] The importance of these sorts of considerations to the abstract-concrete distinction has been brought out in Goodman [6]. The question of whether our theory meets Goodman's formal criterion for a nominalistic theory is a complicated one that I shall not pursue here. Instead of discussing whether the theory meets the criterion, I shall be arguing directly that the theory is nominalistic.

[7] Sentence (9) suggests a number of interesting problems whose complexity forbids adequate treatment here. For instance, as we know from Frege [4], the strategy just applied to (9) should not be mechanically reapplied to 'The aggregates in the cluster number $2^n - (n + 1)$, but the stars in the cluster number $n$'. Here we must relate the numbers to the general expressions 'aggregate in the cluster' and 'star in the cluster' (or to properties or concepts expressed by these general expressions) rather than merely to the referent of the definite descriptions. Similarly for 'The aggregate of all the stars is one in number, but the stars number 5,000'. Analogous considerations might be brought to bear on at least one interpretation of the wonderfully ambiguous sentence, 'The animal kinds are not scattered, but the animals are'.

[8] This remark makes good a promise to improve the analysis of sentences like (10) presented in Burge [1]: 279-80, esp. n. 18. I should note, however, that (10) is subject to yet other interpretations.

[9] These schemata are the aggregate analogs of Quine's schema (in terms of the calculus of individuals) for characterizing the cumulative reference of mass terms. (Quine [13]: 91.)

[10] The argument is derived from one given by Terence Parsons in [12]: 376-8. The important difference is that Parsons concludes—without justification, I think—that substances like water are abstract objects. He draws this conclusion because he does not distinguish between sums (in the sense of the calculus of individuals) individuated by spatio-temporal location, and aggregates (sums or collections) individuated in terms of member-components.

[11] For details, see Heitler [9]: 110-2, 256-75. I am indebted for the example to Dr. Leona Libby, Professor of Mechanics, UCLA.

[12] This suggestion is to be understood in the light of n. 3, and may be construed so as to apply to events. Moreover, just as 'object' is a fairly elastic term, so is 'physical'. The key adjectives, in my view, are 'one-place' and 'sortal'.