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Author(s): Tyler Burge
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Frege on Knowing the Foundation

TYLER BURGE

The paper scrutinizes Frege's Euclideanism—his view of arithmetic and geometry as resting on a small number of self-evident axioms from which non-self-evident theorems can be proved. Frege's notions of self-evidence and axiom are discussed in some detail. Elements in Frege's position that are in apparent tension with his Euclideanism are considered—his introduction of axioms in *The Basic Laws of Arithmetic* through argument, his fallibilism about mathematical understanding, and his view that understanding is closely associated with inferential abilities. The resolution of the tensions indicates that Frege maintained a sophisticated and challenging form of rationalism, one relevant to current epistemology and parts of the philosophy of mathematics.

From the start of his career Frege motivated his logicism epistemologically. He saw arithmetical judgments as resting on a foundation of logical principles, and he saw the discovery of this foundation as a discovery of the nature and structure of the *justification* of arithmetical truths and judgments. Frege provides no focused and sustained account of the foundation, or of our epistemic relation to it. It is clear from numerous remarks, however, that he saw the foundation as consisting of primitive logical truths, which may be used as axioms and which are self-evident. He thought that they are in need of no justification from any other principles. Logical and arithmetical principles other than the self-evident ones are justified by being provable from self-evident axioms together with self-evident definitions and self-evident rules of inference.

At first glance, Frege may seem to be a prototypical representative of the Euclidean, rationalist tradition in the epistemology of logic and mathematics. But further scrutiny reveals a more complex situation. Although Frege regards the axioms as self-evident, he expressed a sophisticated modern awareness of the fact that what can seem obvious may turn out not even to be true. (The distinction between self-evidence and obviousness will be discussed.) Ironically, this awareness led to at least dim anticipations of the problem with his fifth axiom, a problem that decimated his logicist project. Frege was aware that principles that he put forward as axiomatic—even some that, unlike Axiom V, have endured as basic principles of logic—were not found to be obvious by his peers. In arguing for his logic he made use of methods that were explicitly pragmatic and contextualist, in senses that will be discussed. The relations between these

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aspects of Frege's position and his views on self-evidence are worth understanding.

Moreover, whereas Frege maintained with the rationalist tradition that understanding certain truths justifies one in believing them, he developed an original conception of what is necessary for understanding. In particular, understanding presupposes inferential abilities, even where those inferences are not needed for epistemic support of belief in the thought that is understood.

The purpose of this essay is to investigate Frege's conception of the knowledge and justification of the primitive logical truths, from which he intended to take his axioms. Frege did maintain the primary tenets of traditional rationalism. But the nature of his work and his place in history gave his philosophical genius materials for supplementing the traditional view and developing neglected aspects of it, in such a way as to make it less vulnerable to some of the traditional objections. In fact, I think that the view that he developed is of current philosophical interest.

I shall begin in §I by outlining the basics of Frege's rationalist position—his Euclideanism about his axioms. In §II, I discuss how he introduces his axioms in his two main works of logic. I discuss puzzles about the relation between these modes of introduction and his belief that the axioms are self-evident. In §III, I expound "pragmatic" elements in his work that do not seem to fit with the traditional view. §IV is devoted to a further development of Frege's rationalism, with special attention to the notion of self-evidence and to how the Euclidean and "pragmatic" tendencies relate to one another.

I

Frege opens *Begriffsschrift* (1879) with the following statement

> In apprehending a scientific truth we pass, as a rule, through various degrees of certitude. Perhaps first conjectured on the basis of an insufficient number of particular cases, a general proposition comes to be more and more securely established by being connected with other truths through chains of inferences, whether consequences are derived from it that are confirmed in some other way or whether, conversely, it is seen to be a consequence of propositions already established. Hence we can inquire, on the one hand, how we have gradually arrived at a given proposition and, on the other, how it is finally to be most securely grounded [or founded: *begrundet*]. The first question may have to be answered differently for different persons; the second is more definite, and the answer to it is connected with the inner nature of

> the proposition considered. The most reliable way of carrying out a proof, obviously, is to follow pure logic, a way that disregarding the particular characteristics of objects depends solely on those laws upon which all knowledge rests. Accordingly, we divide all truths that require grounding (*Begrundung*) into two kinds, those for which the proof can be carried out purely by means of logic and those for which it must be supported by facts of experience. But that a proposition is of the first kind is surely compatible with the fact that it could nevertheless not have come to consciousness in a human mind without any activity of the senses. Hence it is not the psychological genesis but the best method of proof that is at the basis of the classification. (B Preface)[1]

This passage contains a number of recurrent themes. Frege draws a sharp distinction between psychological genesis and grounding. Grounding is firmly associated for Frege with justification—epistemic warrant or "support". "*Grund*" in German means not only ground, but also reason. So a grounding or founding is naturally associated with reasons. Frege sees proof as the primary relevant form of justification involved in grounding or founding a truth. The nature of the justification of a proposition is "connected with the inner nature of the proposition". The most reliable justification is one that "can be carried out purely by means of logic", one that rests solely on laws of logic "upon which all knowledge rests".

According to Frege's implicit conception of justification, justification or foundation is associated with propositions, eventually in Frege's mature work with thoughts (thought contents), not—or at least not explicitly—with beliefs of individuals.[2] The justification for a proposition consists in the best method of proving it—in an actual abstract proof structure which constitutes the *possibility* of carrying out a certain sort of proof.

[1] References to corresponding passages in English and German are separated by a slash mark. The following works of Frege will be cited with these abbreviations: *The Basic Laws of Arithmetic* (BL); *Begriffsschrift* (B); *Collected Papers on Mathematics, Logic, and Philosophy* (CP); *The Foundations of Arithmetic* (FA); *Grundgesetze Der Arithmetik* (GG); *Kleine Schriften* (KS); *Nachgelassene Schriften* (NS); *Philosophical and Mathematical Correspondence* (PMC); *Posthumous Writings* (PW); *Translations from the Philosophical Writings of Gottlob Frege* (GB); *Wissenschaftlicher Briefwechsel* (WB). Full references are given at the end of the paper.

[2] Later—in B (§13)—he associates it with "judgments of pure thought", which are seen as abstract, possible acts of judgment associated with ideal logical thinking.

It is unclear how Frege regarded inductive arguments. Sometimes (FA, §2) he contrasts inductive arguments with proofs. In other places, he seems to see them as ultimately deductive arguments: as starting with singular statements about particulars, together with a general law of induction, and yielding empirical laws as conclusions. He does not say how the conclusions of such "proofs" should be seen. That is, he does not indicate whether the laws are less than conclusively established, or whether the non-conclusive nature of the argument is built into the statement of the conclusion in some way. (Cf. FA, §3.)

Frege regards a justification as a structure which may be understood or psychologically mastered in different ways by different people—or perhaps not understood at all. The structure is associated with the "inner nature of the proposition", not with individual abilities or states of thinkers.[3]

As Frege implies, not all propositions require a grounding. Some stand on their own. In B §13, Frege continues:

> It seems natural to derive the more complex of these judgments from simpler ones, not in order to make them more certain, which would be unnecessary in most cases, but in order to let the relations of the judgments to one another emerge. Merely to know the laws is obviously not the same as to know them together with the connections that some have to others. In this way we arrive at a small number of laws in which, if we add those contained in the rules, the content of all the laws is included, albeit in an undeveloped state. And that the deductive mode of presentation makes us acquainted with that core is another of its advantages. Since in view of the boundless multitude of laws that can be enunciated we cannot list them all, we cannot achieve completeness except by searching out those that, *by their power*, contain all of them. Now it must be admitted, certainly, that the way followed here is not the only one in which the reduction can be done. That is why not all relations between the laws of thought are elucidated by means of the present mode of presentation. There is perhaps another set of judgments from which, when those contained in the rules are added, all laws of thought could likewise be deduced.

Frege here elaborates his view of a justificational structure. He sees simpler judgments as tending to be more basic than complex ones (NS 6,36/PW 6,36; KS 165/CP 180). More complex rules of inference are similarly to be resolved into simple, basic ones (FA §90; cf, GG vi–vii/BL 2–3).

There is an analogous structure among parts of thoughts that would undergo definition. Frege writes

> In the case of any definition whatever we must presuppose as known something by means of which we explain what we want understood by this name or sign … . To be sure, that on which we base our definitions may itself have been defined previously; however, when we retrace our steps further, we shall always come upon something which, being a simple, is indefinable, and must be admitted to be incapable of further analysis. And the properties belonging to these ultimate building blocks of a discipline contain, as it were in a nutshell, its whole contents. (KS 104/CP 113; cf. also KS 289–90/CP 302)

[3] For another early association of logical principles with justification, see NS 4,6/PW 4,5.

Both of the last two quoted passages show that Frege sees the structure as residing in the nature of the laws or contents. Basic, foundational laws have—by their content, or nature—the power of proving or entailing the others. As we shall see shortly, Frege thought that such laws are *unprovable*. Basic thought components are those that by their content, or nature, have the power of defining others. Such "simples" are *indefinable*.

The *Begriffsschrift* §13 passage indicates that Frege thinks that there are other laws than those he *takes* as basic *for the purpose of formalizing a system* of logic, which also have the power of yielding all the other laws as logical consequences. These laws also are basic *in themselves*. They do not need proof. Thus there appear to be more "basic" laws in logical reality than are needed to provide a simple and elegant formal presentation of logic (cf. KS 391/CP 404; NS 221–2/PW 205). Studying these alternative ways of deriving the non-basic theorems would yield deeper insight into the various relations among propositions.

Frege says little more about basic laws in *Begriffschrift*. His concern is to provide the logical tools to enable one to determine the exact nature of proof or justification. More particularly, he is concerned that failure to formalize the steps in a proof tends to lead to one's making inferences that inarticulately amalgamate several steps into one.[4]

The reason that this tendency is a source of concern for him is that it may lead one to think that the inference depends on "intuition", on a non-logical cognitive capacity, when in fact it is purely logical. The logical character of the inference may be obscured by the fact that too many logical steps are lumped together, without a clear account of any of them. The result would be a tendency to package ill-understood inferences under the ill-understood category "intuition".[5] A tendency to count logical inferences non-logical is, of course, an obstacle to Frege's *logicist thesis*: the thesis that arithmetic can be reduced to logic together with definitions of arithmetical terms in logical terms.

Frege's concern throughout is with understanding the nature of the justification or grounding of arithmetic. Both at the beginning and at the end of the Preface to *Begriffschrift* Frege announces his logicist aims. In both places, he associates understanding logical and arithmetical structures

[4] Cf. Preface of B and §23. Cf. also FA §§1–3; "On Mr. Peano's Conceptual Notation and My Own" (1897) KS 221ff./CP 235ff.

[5] The notion of intuition carries the special technical understanding of the term associated with Kant. Pure intuition (roughly the spatio-temporal structure of perceptual capacities) was supposed by Kant to be a source of a priori knowledge that is not logical. Frege agreed with this view as applied to geometry. But he thought appeal to it obscured the logical foundation underlying arithmetic. No doubt Frege believed that appeals to a non-technical notion of mathematical intuition (insight, sense of the obvious) also obscured the logical character of arithmetic axioms and proof.

with understanding the justification or ground of logical and arithmetical propositions. Understanding proof structure was for Frege equivalent to understanding the structure of justification (cf. also FA, §3). Frege's project was a project in the theory of knowledge.

The same picture of the justification of arithmetic and logic as resting on a small group of logical laws appears again in *The Foundations of Arithmetic* (1884). In §1, Frege commends "the old Euclidean standards of rigour". He writes, "The aim of proof is ... not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another" (§2). This is equivalent to affording us insight into the nature of justification: "After we have convinced ourselves that a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely?". Again, Frege invokes the task of finding the primitive truths to which everything else can be reduced. In §3, Frege indicates that whether arithmetic is "analytic"—whether it can be reduced to logic together with definitions—is a question about

> the ultimate ground upon which rests the justification (*Berechtigung*) for holding [a proposition of arithmetic] true .... The problem becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one .... (FA §64; also KS 104/CP 113–4; KS 289–90/CP 302)

In discussing the crucial definitions later in the book, he manifests clear concern that the definitions respect conceptual priority—which is ultimately justificatory priority.

Frege offers no large systematic discussion of knowledge of the foundations, of the primitive truths on which the justificational structure rests. In fact, he neglects to formulate his notions of analyticity and apriority so as to either include or rule out the foundations of logic.[6] But as we shall see, Frege makes many remarks about knowledge of the foundations, which enable one to put together an account of his view.

Frege assumes that the foundations are "general" propositions—not propositions about particulars (FA, §§3, 5). He thinks primitive truths of logic must not have concepts that are peculiar or special to any non-universal subject matter (as the concepts of geometry do—in their relevance to space). And he thinks that one of the aims of his project is, by showing how the truths of arithmetic rest on primitive truths, to place the truth of

[6] Michael Dummett (1991, Ch. 3) notices this. I think Dummett is right that this is an oversight on Frege's part—or at any rate that Frege intended no epistemic slight to the foundations. For he repeatedly connects foundations to justification, and repeatedly calls the foundations self-evident, as I shall discuss below.

the propositions of arithmetic beyond all doubt—on the firmest possible foundation (FA, §2). There is much more in *Foundations* that gives one insight into Frege's view of the epistemic status of the axioms. But I want to postpone discussion of some of this material until §IV.

In the two decades after *Foundations* there are further remarks that indicate Frege's Euclidean views about the basic truths. In 1897 he writes

> I became aware of the need for a conceptual notation when I was looking for the fundamental principles or axioms upon which the whole of mathematics rests. Only after this question is answered can it be hoped to trace successfully the *springs of knowledge* [my emphasis] upon which this science thrives. Even if this question belongs largely to philosophy, it must still be regarded as mathematical. The question is an old one: apparently it was already being asked by Euclid. (KS 221/CP 235)

Frege requires that for something to count as an axiom, it must be true, certain, and unprovable: "Traditionally, what is called an axiom is a thought whose truth is certain without, however, being provable by a chain of logical inferences. The laws of logic, too, are of this nature" (KS 262/ CP 273). Frege sometimes counts these three features—truth, certainty, and unprovability—as constituting the Euclidean *meaning* of "axiom". (KS 283/CP 295; cf. KS 313/CP 328; NS 183/PW 168; NS/266–7/PW 247).

This conception is indeed traditional. But it has been obscured by subsequent developments. Many use the term "axiom" now in a way that would allow for no incoherence or inappropriateness in talking of false axioms. On this usage, axioms are basic principles of a theory, something proposed by human beings and capable of being found to be mistaken. We naturally speak of Frege's Axiom V, which turned out to be inconsistent, but certainly was an "axiom" (in our less traditional sense) of his system of logic.

Frege's Euclidean conception of axioms takes them to be true first principles—basic truths—which might or might not be discovered or proposed by human beings. Something's being an axiom is not primarily a matter of being part of a theory. Frege does think that whether a basic truth is an axiom depends on its being used as an axiom, as starting point in a system of derivation. (This differentiates the notion *axiom* from the notion *basic truth*.) But for Frege a necessary condition on something's being an axiom is its being a basic truth—in particular, a foundational part of a mathematical or logical structure which it is the purpose of logicians and mathematicians to discern and express. If a proposition is not *true*, then it cannot possibly stand at the foundation of these structures. So it cannot be an axiom. Being an axiom, on Frege's conception, is necessarily being part of the foundation of these structures.

The sense in which Frege thought that the primitive truths are certain is more complex. The notion of certainty figures primarily in Frege's generalized motivational sections. He thought that basic truths' certainty grounds the certainty of theorems derived from them. It is clear that he did not regard this certainty in a purely psychological sense. He did not think that just anyone who had thought about axioms had to be maximally confident that they are true. The reason this is clear is that he knew that many mathematicians who had thought about the axioms of Euclidean geometry did not think of them (or in some cases, any other mathematical propositions) as true. He was also aware that many logicians did not accept his system of logic (KS 262ff./CP 273ff.; NS 183–4/PW 168–9).[7] He seems to talk of certainty as a property of the logical and mathematical truths. For example, in *Foundations* he associates certainty with the immovability of a boulder, and with being beyond reasonable doubt (FA, §2). There is some reason to think that Frege held that empirical propositions lacked this feature. (Compare KS 115/CP 125 with NS 286–8/PW 267–9.)[8]

Frege regarded not only axioms but most of the *theorems* of logic and at least the more ordinary theorems of arithmetic as being certain (B §13; FA, §2). He had almost no sceptical impulse, and he seemed to have regarded quite a lot of non-basic truths—as well as basic ones—as certain. Certainty appears to mean something like *beyond a reasonable doubt by someone who fully understands the relevant propositions*.

In addition to requiring that axioms be true and certain, Frege required that axioms be unprovable. Now this term "unprovable" seems to have two different but compatible interpretations for Frege. In his mature work Frege is very explicit that axioms are abstract thoughts—thought contents—not sentences. What are proved are, similarly, thoughts (*Gedanken*) not sentences. But on one interpretation of "unprovable", axioms are unprovable relative to a system—an ordering marked out by human beings—of true, certain thoughts. That is, they are starting points in a certain system of derivation. We noted earlier that a thought is an axiom only if it is used as an axiom. Only true and certain thoughts can be axioms in Frege's sense of "axiom". Truth and certainty are not relative to a system. But on this interpretation of "unprovable", unprovability is relative to a system. Whether a truth is unprovable depends partly on whether it is taken to be an axiom in a given logical theory. (One step "proofs" are not allowed.) In one system a thought may be an axiom, and in another system the same thought may be a theorem. The point is at least suggested in

[7] Frege's awareness of doubts about his logical system dates from the reception of his *Begriffsschrift*. Cf. his articles against the Booleans NS 9–59/PW 9–52. It also appears in the preface to BL. Cf. esp. GG xii/BL 9.

[8] Some passages in "Thoughts" (1918) (KS 342–61/CP 351–72) suggest that he thought that the existence of the physical world was certain.

Begriffsschrift §13. But Frege makes this interpretation of "unprovable" fully explicit only late in his writings (NS 221–2/PW 205). It is, however, clearly available to him throughout, and he may have had it in mind in other passages. In this sense, unprovability is not an intrinsic feature of a truth, but is rather a status accorded to it by virtue of its place in a given logical system. The same truth may have another place (as theorem) in another system.

I think, however, that this interpretation cannot provide all that Frege meant by "unprovable". In any case, being true, certain, and unprovable in the sense just adumbrated is not sufficient for being an axiom. Those notions cannot jointly constitute the Euclidean meaning of "axiom" for Frege—as he says truth, certainty and "unprovability" do. (Cf. again (KS 283/CP 295; cf. KS 313/CP 328; NS 183/PW 168; NS/266–7/PW 247.) In some passages where Frege calls axioms "unprovable", he clearly assumes that they are "*basic laws*" (e.g. KS 262/CP 273). Indeed, the connection between being an axiom and being a basic law is constant throughout Frege's work. Not just any true and certain thought would be a basic law, and appropriately taken as an axiom, merely by virtue of being *taken* as a contingently "unprovable" starting point for a system of proof. Theorems of arithmetic were regarded by Frege as true and certain, and they could be arbitrarily taken as starting points in a system of arithmetic. But Frege would not regard them as axioms.

In *Foundations* §5 Frege argues against those who regarded numerical equalities about particular numbers as unprovable. The argument takes the issue to be about a matter of fact, not one of choice. Even if his opponents took some or all such equalities as starting points for their systems of derivation, Frege would not concede that they were "unprovable" in the sense that he is using the term. He argues that all such mathematical propositions are provable, and this argument is meant not just to advertise his own intention not to take them as axioms, but to indicate that logicism is true about the nature of arithmetic: they are provable from basic laws of logic. No arithmetical statement is an axiom, because all are provable. Frege was seeking basic laws in a sense that transcended sociological facts about what (true, certain) propositions were used as starting points in actual systems of derivation.

The assumption that axioms are basic laws or basic truths can be explicated in terms of Frege's characterization of primitive general laws as being "neither capable nor in need of proof" (FA §3). This phrase comes directly from Leibniz, from whom Frege probably got it (Leibniz, IV, ix, 2). The passage suggests a second interpretation of "unprovable". Leibniz certainly thought of basic truths as unprovable in a sense that goes beyond their being *taken* as starting points in a system of derivation. Leibniz thought them unprovable in the sense that they could not be justified by being derived from epistemically prior truths. In the relevant passage (FA §3), Frege like Leibniz is discussing a natural order of justification among truths. Frege thought that basic truths have to stand on their own in a natural structure of proof (justification, grounding). Again, he thought of this structure as independent of theories put forward by human beings.

Frege sees proof not fundamentally as just any structure of logical derivation, but as a form of justification or grounding. From this perspective, basic truths are unprovable in the sense that they cannot be grounded or given a justification by being derived from other truths. They can be derived, according to logical rules, from other truths within certain systems. But the derivations would not be justifications, groundings, or proofs in this epistemically fundamental sense. So from this perspective, basic truths are unprovable in the sense that they cannot be grounded or justified by being proved from other truths.

Moreover, basic truths do not need proof; they do not need justification or grounding through derivation from other truths. This epistemic feature of basicness is also fundamental to being an axiom. Frege writes, "it is part of the concept of an axiom that it can be recognized as true independently of other truths" (NS 183/PW 168). Axioms have to be basic truths. To be basic, a truth must not need or admit of proof. To be basic, a truth cannot be justified by being derived from epistemically more fundamental truths, and yet does not need such justification because it can be recognized as true independent of it.

Basic truths are, of course, certain. But Frege's notion of axiom still does not collapse into his notion of basic truth. As is evident from *Begriffsschrift* §13, quoted above, Frege thought that different principles from the ones he proposed could provide an adequate basis for a formal logical system and could suffice to derive all the theorems of logic or arithmetic. In the context of the passage and his later work, Frege is obviously conceiving of a natural epistemic order of derivation. So he must have thought that there are more basic logical truths than are needed to derive and justify all the (non-basic) truths of logic and arithmetic. The epistemically basic truths overdetermine the whole system of logical-arithmetic truths. Thus although some basic truths might be expressed as theorems in a formal system, they are not, from the point of view of the natural order of justification or proof, essentially derivative. They are essentially basic. But in the relevant system, they would not be axioms. Thus not all basic truths that are candidates for being axioms are, relative to a given system, in fact axioms. They would not be "unprovable" in the first of the two senses that we distinguished, even though they are "unprovable" in the second sense.

I conjecture that according to Frege's fully elaborated notion of axiom, an axiom is a thought that is true, certain, basic, and unprovable in the first of our senses. Or equivalently, an axiom is a thought that is true, certain, unprovable in both of our senses, and not in epistemic need of proof.

As I have noted, Frege's point that axioms (and basic truths) do not need proof is associated with, and probably equivalent to, his claim (properly understood) that an axiom—and a basic truth—"can be recognized as true independently of other truths" (NS 183/PW 168). He clearly thinks that the axioms of geometry and the axioms of logic have this feature. And at least at the end of his career—probably throughout it—he thought that sense experience statements about the physical world lack it (NS 286–8/PW 267–9). The requirement that axioms do not need proof is closely related to his requirement that axioms be *self-evident* (FA §§5, 90; GG v2, §60/ GB 164 ("*selbst-verstandlich*"); GG 253/BL 127 ("*einleuchtend*")). Indeed, self-evidence must partly be understood in terms of recognizeability as true independently of recognition of other truths. Sufficient evidence to make believing them rational is carried in these individual truths themselves.

The meaning of the various modal notions that Frege uses in the phrases "unprovable" (in our second sense) and "can be recognized … independently" in his requirement that axioms and basic truths have a sort of self-justification is complex. Understanding the modal notions is closely related to understanding Frege's notion of self-evidence. Developing a deeper understanding of this latter notion will occupy us in §IV. For now, it is enough to see the general shape of Frege's rationalism.

Justification resides in a proof structure that is independent of language and theory, but has an objectivity and reality that waits to be discovered. The proof structure involves basic truths which are justified in themselves, without need of proof. They contain simple indefinable concepts and are self-evident. Similarly, basic inference rules (FA 90) and definitions ((WB 62/PM 36; KS 263/CP 274; KS 289–90/CP 302) are required to be self-evident. The certainty, or rational unassailability, of theorems which they entail is derivative from the certainty and self-evidence of the basic truths.

So far, we have a fairly familiar picture of Frege's indebtedness to the Euclidean rationalist tradition. I want to turn now to elements in Frege's views that are in apparent discord with his Euclidean epistemology. I will try to show wherein these further elements are compatible with and indeed enrich his rationalism.

II

Frege's practice in introducing his axioms in *The Basic Laws of Arithmetic* is at first glance at odds with his Euclidean commitments. Although Frege requires axioms to be unprovable, self-evident, and not in need of proof, he seems to provide *arguments* for at least some of them when he introduces them in *Basic Laws*.

For example, in §12 Frege introduces the material conditional as a function with two arguments, whose value is the False if the True be taken as the first argument and any object other than the True be taken as the second argument; and whose value is the True in all other cases. Then in §18 he introduces Axiom I. He writes

> By §12,
>
> $(\Gamma \rightarrow (\Delta \rightarrow \Gamma))$
>
> could be the False only if both Γ and Δ were the True while Γ was not the True. This is impossible; therefore
>
> $\vdash (a \rightarrow (b \rightarrow a))$. (BL, §18)

Here Frege argues from the way he introduced the relevant logical function to the truth of the relevant axiom. So after introducing the material conditional as a function in terms of its truth table, he argues from the truth table to the truth of the axiom. In every case, except that of Axiom V (whose introduction is non-standard because it is defective), he appeals to versions of recognizeably standard reasoning from truth conditions to the axioms.[9]

Now this practice may seem odd. It raises at least two puzzles. Frege seems to be *arguing* for axioms, which are supposed to be unprovable, self-evident, and not in need of proof. And he seems to be arguing for the axioms from *semantic claims*, whereas what purport to be the basic truths of logic are not about linguistic expressions or reference at all.

Let us begin with the first puzzle. It is clear that in a recognizable sense, Frege is giving arguments or demonstrations that are semantical—though we will have to qualify the sense. It seems equally clear that the arguments

[9] I am grateful to Christopher Peacocke for drawing my attention to the interest of these arguments, and to Richard Heck for pointing out a mistake I had made about two of the individual axioms. Frege's lack of confidence in his official claim of self-evidence for Axiom V leads to a special treatment of its key primitive expression in §31. There he attempts to demonstrate that the course-of-values operator, when given an appropriate grammatical completion, always produces denotations. §31 raises numerous difficult interpretative questions that I pass over here.

are arguments for the truth of the axioms. But it is less clear what the purposes of the arguments are. Moreover, they differ in interesting ways from one another.

In understanding the argument that Frege gives in §18 of *Basic Laws*, it is helpful to look at the counterpart passage in §14 of *Begriffsschrift*. There Frege writes,

> $\vdash (a \rightarrow (b \rightarrow a))$
>
> says "The case in which a is denied, b is affirmed, and a is affirmed is excluded". This is evident, since a cannot at the same time be denied and affirmed. We can also express the judgment in words thus, "If a proposition a holds, then it also holds in case an arbitrary proposition b holds."

This passage holds the key to the first puzzle. Frege clearly regards his argument as an elaboration of what is contained as evident in the axiom itself. It is an elaboration of an understanding of the thought, which is a basic truth. Frege does not see himself as starting with more basic truths—such as the principle of non-contradiction together with truths about the way the material conditional maps truth values onto truth values—and then justifying the axiom by reasoning to it from these resources. He sees himself as articulating in argument form what is contained in the very content of the basic truth he is arguing for. The truth is epistemically basic. Understanding it suffices for recognition of its truth.

Anyone who understands the truth can give the argument through understanding the material conditional. But the truth is not a conclusion of a proof, a structure of justification. For there is no justificational structure with truths more basic than the conclusion of the argument. The axiom is supposed to be unprovable in the second sense we have elucidated. Moreover, it does not need a proof. It is self-evident. It is evidence for its own truth. Epistemically, it is the truth's content, not the discursive argument, that is basic. The argument serves to articulate understanding of the thought content. It does so in a way that enables one to recognize that its truth is guaranteed by its content. The argument is not a derivation that justifies or grounds the thought in more basic truths.

In the *Begriffsschrift* passage, Frege indicates what the expression says, and he explains wherein what it says is evident. In the *Basic Laws* passage, he appeals to truth conditions associated with the relevant function—which he elsewhere identifies with the sense or content of the relevant proposition—and explains the truth of the axiom in the way he does in *Begriffsschrift*. The basic procedure is the same. Frege certainly saw the two cases as on a par. Frege is showing that the axiom's truth is evident from its content. The same point applies to all the introductory *arguments* for the axioms.

Let us broaden our perspective to include the second puzzle as well. In what sense does Frege argue from semantical principles in his introduction of Axiom I?

It should be noted that in the introduction of Axiom I (and of IV), Frege is not reasoning about symbols. So he does not take up a semantical perspective on his logical language at all. "Affirmed", in the *Begriffsschrift* formulation, converts in *Basic Laws* into "is the True", which is what the horizontal comes to express in Frege's mature work. The horizontal, or "is the True", is part of the expression of the axiom. "Is the true" is not a predicate of expressions or of axioms, truths, or thoughts. It denotes a concept of the truth value the True—a function that takes only the True into the True. The predicate occurs in Frege's logic, along with the material conditional, negation, the universal quantifier, and so on. So Frege is not here using a meta-logical perspective in the modern sense.

The introduction of the material conditional in §12 of *Basic Laws* is, as I have noted, not a discussion of a symbol, but an explication of the material conditional as a function. And the argument introducing Axiom I in §18, which involves "is the True", also does not mention symbols at any point.[10] In that strict sense it is not semantical. Although the argument is rigorous, it is not epistemically fundamental. What is fundamental is the content of a single thought—the axiom. The argument simply articulates the self-evidence of the thought by expanding on what is involved in understanding it.

Two of the axioms (V and VI) contain *singular terms*—non-sentential terms formed from operations on function variables. The explications of the relevant singular terms in *Basic Laws* (§§8, 9, 11) utilize "reference" or "denotation" (*Bedeutung*). These explications do mention expressions, and are unqualifiedly semantical. "Denotes" is a predicate that applies to linguistic expressions, and relates them to their denotations or referents. "Denotes" does not occur as a primitive logical term in any of the axioms.

Two other axioms (IIa and III) involve terms in places that we would normally reserve for not only variables but also singular terms in the sense explained in the previous paragraph. In these cases too, Frege's explication of the key expressions (the object-denoting term in universal instantiation and the identity predicate) invokes denotation. So again the explications are semantical.

In these four cases, Frege does not carry out any argument at all when he introduces the corresponding axioms (*Basic Laws*, §§18, 20). He simply cites the semantical explication of the expressions for the relevant

[10] The modal expressions in the BL §18 passage need further investigation. I am inclined to think that they would be regarded by Frege, controversially of course, as inessential to arguments that he gives. Again, the comparison to the B passage is useful.

functions (the universal quantifier, the identity sign, the definite description operator, and the course of values operator) and takes that appeal to make evident the truth of the corresponding axiom.[11] At most in introducing the axiom he reads through the axiom in its own terms, making sure that its key terms are understood.

So "denotes" occurs only in preliminary explications of logical symbols, not in any explicit argumentation that takes the axioms as conclusions. This may be significant. For there is, as noted, no argumentation for those axioms that we would nowadays read as containing singular terms. And *none* of the argumentation for the other axioms is semantical, in the sense that none of those arguments mention symbols. All of the arguments could be carried out within the language of the logic of *Begriffsschrift* (by avoiding the modal terminology). So *none* of the arguments that have the axioms as conclusions (as opposed to the preliminary explications of logical symbols or logical functions) are strictly semantical. They utilize only expressions (*modulo* the modal expressions) which could be formulated purely in Frege's logic. It is not clear (or crucial to my purposes) what the significance of all this is. But I will return to these points.

I have noted that the explication of the material conditional is, in the first instance, an introduction of a function, not an explication of the logical symbol that denotes the function. That explication mentions no symbols. After that introduction, also in §12, Frege speaks of the denotation of the symbol for the material conditional. But it is the introduction of the function (without reference to expressions) that Frege cites in his argument.

The explications of negation (§6) and the definite description operator (§11) are like that of the material conditional. The function is introduced first, and the denotation of the sign is later noted in a meta-remark on the already established introduction of the function.

By contrast, the explication of identity (§7), which figures in the argument for axioms IV and III (§§18–9), does mention the symbol. And as I have noted, all the explications involving the singular terms (the free variables, terms formed from the definite description operator, and terms formed from the course-of-values operator) are full-blooded semantical explications.

This switching among methods shows, of course, that Frege was not *systematically* doing semantics of a language in the modern sense. He

[11] It is true, of course, that Frege goes on to try to prove in §31 that singular terms involving the course-of-values operator have a reference. The proof is in some respects non-standard, and of course it fails. I take it that the attempt at a proof, which he would surely not have thought necessary for the other singular terms, was a sign of unease over the status and even truth of the proposed axioms involving the course of values operator.

moves easily from using a denotation predicate that occurs outside his logical system to using a truth predicate that occurs in it. He moves easily from explications that introduce expressions to those that introduce the logical functions directly. In all cases, Frege regarded himself as both explaining his symbols and introducing functions that play the key role in the corresponding axioms.

Frege gives relatively rigorous, if not entirely systematic, semantical explications of his logical expressions. His explications systematically track semantical, if not necessarily model-theoretic, exposition. (I think Frege usually saw his language not as an uninterpreted symbolism, but as a perfect language carrying definite sense, differing from natural languages in that the sense and structure of the language serves rational inference ideally. For certain limited mathematical purposes, however, he does treat his language as a reinterpretable syntax in something like the model-theoretic fashion.) He expects such explications to aid in recognizing the truth of the axioms.

So Frege is doing several things at once. He is explaining the intended sense of his formulae by giving their truth conditions. He is implicitly justifying his logical language and the formulae that he uses by showing that they express logical truths and valid inferences, which are antecedently understood to be self-evident. And he is, in the arguments that derive the axioms, eliciting the self-evidence of the thoughts expressed by the axioms by bringing one to think through their content. But there is no sense in which he is justifying the axioms, the thoughts expressed by that language, through semantical argumentation. In fact, the argumentation he gives is within his logic (again, *modulo* the modal elements). Its function is not to justify the axioms by deriving them from prior truths, but to elicit understanding of them, an understanding that is supposed to suffice to enable one to recognize their self-evident truth.

Frege's introduction of his *methods of inference*—in contrast to the axioms—*is* systematically semantical. Using quotes, he introduces modus ponens this way:

> From the propositions [sentences, *Satze*] $\vdash$ $(\Gamma \rightarrow \Delta)$" and "$\vdash \Gamma$" we may infer "$\vdash \Delta$"; for if Δ were not the True, then since Γ is the true $(\Gamma \rightarrow \Delta)$ would be the False. (GG/BL §14).

Frege is doing something different here from what he does when he gives the arguments for the axioms (which are thought contents—not formulae). What he calls a "method of inference" is a rule for moving from *sentences* (albeit fully meaningful sentences) to *sentences*. The argument that he gives is for the legitimacy of such rules. The argument is carried out within his logic (again assuming that the modal locutions are dispensable). But it is an argument about the legitimate use of his symbols.

Thus the arguments for the "methods of inference" differ from the "arguments" for the axioms. The arguments for the axioms in some cases rely on a preliminary explication of the logical notation. But they are not arguments that certain *expressions* (those that express the axioms) express logical truths. They are arguments whose conclusions are the axioms themselves, carried out in what we would call the object-language—using but not mentioning logical expressions. The axioms themselves are language-independent truths. The arguments for the methods of inference, by contrast, are arguments that certain transformations among meaningful sentences (not among what the sentences express) are truth-preserving. This contrast is important for understanding the epistemic function of Frege's argumentation for the axioms and methods of inference.

Later in *Basic Laws*, in criticizing formalist arithmetic, Frege implies a need to "justify" or "ground" "rules of inference", by appeal to the reference of the signs (§§90, 91, 94). Inevitably, the actual practice of proof must be formulated in terms of the permissible transitions among symbols expressing *Gedanken*. Frege is here writing of justifying or grounding methods of inference understood as ways of moving from symbols to symbols. He is justifying his introduction of his logical symbolism, not the language-independent logical principles or rules of inference that are expressed by the symbolism.[12]

Frege did not think of rules or methods of inference, in so far as they are transitions from symbols to symbols, as epistemically basic. It is not *these* methods of inference, rules about permissible transitions from expression to expression, that he refers to as self-evident when he discusses the epistemology of logic. For elsewhere Frege frequently indi-

[12] Heck (forthcoming) and Stanley (1996, pp. 45–70), both cite these passages—§§90–4. The key thing to remember in reading these passages is that they concern rules about formulae, not rules concerning language-independent abstract thoughts. The passages oppose a formalist understanding of the *language* of arithmetic. In the context of §§90–4, Frege is claiming against the formalists that his symbolism is not arbitrary. He is writing there about arithmetical truths (which certainly do, according to Frege, need justification—they are not basic truths), about the use of formulae, and about rules of inference as methods of moving from one formula (understood as having a sense) to another. Frege did not think of the axioms or language-independent rules as formulae at all. I can find no place where he speaks of justifying them. The language-independent axioms and rules are not in need of justification. Frege wants to justify his use of symbols, not the principles of inference underlying that use. Stanley makes substantially this point (p. 63). I do not, however, accept his claim that "Frege is treating his theory as an uninterpreted set of syntactic operations on strings of symbols". A semantics for meaningful language is still a semantics. Heck and Stanley are concerned to bring out the large role of semantical reasoning in BL. I think that Heck, in particular, is right in maintaining that all of the elucidations that lead to the introduction of the axioms are in effect part of semantical explanations of Frege's symbolism—even if they are equally explications of the relevant functions mentioned in the axioms.

cates that rules of inference, in the strictest, most fundamental sense, have true thoughts, not expressions, as premises (WB 35/PMC 22; KS 318–9/ CP 334–5). So logical inference is fundamentally a transition from abstract, language-independent thought contents (the eternal entities, *Gedanken*) to abstract thought contents. Rules of inference are fundamentally rules about such transitions. When Frege calls rules of inference "self-evident" (FA 90), he has in mind nothing about sentences, but logical principles of inference that the methods of inference—as principles about symbols—express. He assumes that modus ponens, understood as a method for moving from thoughts or judgments to thoughts or judgments, is self-evidently sound. There is no justification for modus ponens understood that way.

Although Frege indexes sentences and speaks of them as used in proof, the signs simply express and make formally perspicuous a proof structure of language- and mind-independent thought contents. Frege is very explicit that axioms are *Gedanken*, not symbolic expressions (KS 318-9/ CP 334–5; NS 221–2/PW 205–6). The thought contents are epistemically fundamental. He intends to provide no justification for rules of inference understood as transitions among *thoughts*. Those transitions are self-evident.

So Frege never contemplates justifying axioms or rules of inference (in the fundamental sense of these terms, which apply to language-independent thought contents or rules), much less justifying them by semantical arguments that make reference to symbols. The idea of justifying the truth of axiomatic *Gedanken* by appeal to premises that refer to symbols, which are language- and mind-dependent, would have seemed absurd to him. So although the introduction of modus ponens in §14 is justified by a semantical soundness argument, the justification is of an operation on his language, not of the underlying rule of inference among *Gedanken*. Although the introductions of the axioms are preceded, in some cases, by semantical explications of the terms in the expressions for the axioms, the axioms are not considered as true sentences, but as *Gedanken*. And the arguments for them are non-semantical arguments expressible within the logic. These arguments, as I have claimed, are articulations of the self-evidence of the axioms, which are not provable from more basic thoughts, and are not in need of proof. Both the rules of inference, as applying to thoughts, and the axioms (also thoughts) are taken to be self-evident. Only the language needs justification.

The issues over the sense in which Frege was doing semantics are complex and subtle. I think that Frege was clearly engaging in substantial and ineliminable semantical reasoning in various parts of *Basic Laws*. But I do not wish to discuss these issues further here. What is important for my purposes is that from an epistemic point of view, Frege was not taking the semantics of expressions to be more basic than his axioms, despite the fact that his work is clearly an early version of semantical reasoning. For the axioms whose formulations contain singular terms, he provides no argumentation at all, only an immediate appeal to an understanding of the axioms through the symbols that he has explicated. In the other cases, he does provide an argument with the axiom as conclusion. But these arguments are not semantical in the sense that they contain no steps that refer to symbols. And they do not argue from truths more basic than the conclusion. They do not justify, ground, or prove the conclusion in Frege's epistemically honorific sense of "proof". The arguments function, as the *Begriffsschrift* version of them clearly indicates, to provide an explication or articulation of the content of the axiom: they are discursive representations of an understanding of the axiom. The axioms, the basic true thoughts, are fundamental.

Let us look at this point more closely by returning to the argument for the first axiom in §18 of *Basic Laws*. Frege appeals to a line of the truth table for the conditional, a line that might be expressed (roughly) by this formula:

$$-(\Gamma \rightarrow \Delta) \rightarrow (\Gamma \,\&\, -\Delta).$$

Transforming this into a sentence relevant to the form of the axiom, we get:

$$-(a \rightarrow (b \rightarrow a)) \rightarrow (a \,\&\, (b \,\&\, -a))$$

By non-contradiction, modus tollens, and double negation removal, we get:

$$(a \rightarrow (b \rightarrow a)).$$

This is one formalization of Frege's argument, but the formalization would not be the one Frege would give in his Logic. For he explains "and" (§12) in terms of negation and the conditional. Thus

$$(\Gamma \text{ and } \Delta)$$

is explained as

$$-(\Gamma \rightarrow -\Delta).$$

Frege elsewhere clearly regards double negations as having *the same sense* as the result of dropping the double negations. It is not clear whether he regards the explication of "and" in terms of the negated conditional as giving the *sense* of sentences containing "and". But he may. He writes, "We see from these examples how the 'and' of ordinary language ... [is] to be rendered" (§12).

Suppose that he *does* regard the "rendering" as giving the sense of the ordinary language sentence involving "and". Then the initial formula in the argument in §18 would have been conceived by Frege as having *the same sense* as

$$\neg(\Gamma \rightarrow \Delta) \rightarrow \neg(\Gamma \rightarrow \Delta).$$

The second step would have the sense of an analogously trivial truth, with its antecedent and consequent being identical. (Again we assume that double negation removal does not alter sense.) So the reasoning that Frege is articulating would not, on this view, be moving from one truth to another, but simply thinking through and expressing in different ways, via ordinary language, the character of the axiom which is the apparent conclusion of the reasoning. This seems to be his procedure in *Begriffsschrift* §14, though of course there he had not developed the sense-reference distinction.

This view of the sense of conjunctions would depend on a fairly (and I think implausibly) coarse-grained conception of the senses of logical expressions in ordinary language. But it is certainly not obvious that it was not Frege's view.

There is some circumstantial evidence for thinking that this might have been Frege's view of those arguments. In his writing on Axiom V outside of *Basic Laws*, he maintains that the two sides of the biconditional in the axiom *have the same sense* (KS 130–1; GB 26–7). Let us concentrate on instances of the axiom where the relevant function-expressions are predicates. The two sides are

$$(x)(F(x) \leftrightarrow G(x))$$

and

$$\grave{\epsilon}F(\epsilon) = \grave{\alpha}G(\alpha).$$

Frege seems to have considered an argument that was supposed to bring out informally the supposed sameness of sense of these two sides, at least for the case in which "F" and "G" are predicates or concept expressions.

This argument, if it had been successful, would certainly have elicited the self-evidence of Axiom V, since the Axiom would have had the same sense as

$$(x)(F(x) \leftrightarrow G(x)) \leftrightarrow (x)(F(x) \leftrightarrow G(x)).$$

The argument would have justified Axiom V as a logical law, since the left side of Axiom V is certainly a proposition of pure logic; and the biconditional is a logical function.

Frege considered the case in which "F" and "G" are predicates or concept expressions. He maintained that "*x* is F" has the same sense as "*x* falls under the concept F". There is evidence that he thought that "*x* falls under the concept F" has the same sense as "*x* falls in the extension of the concept F". For "the concept F" *canonically* refers to an object—presumably an extension or course of values of the relevant predicate.[13] This sort of informal argument is something Frege appears to have been experimenting with in his writings on philosophy of language just before publication of *Basic Laws*. The unintuitive consequences of his view that "the concept F" denotes an object, not a concept, probably prevented him from articulating the argument in *Basic Laws*.

The argument would have carried out the same strategy that I have conjectured underlies the argument in §18 (for Axiom I). It is a manipulation of different ordinary expressions that have the same sense, but which purportedly brings out the self-evidence of a logical axiom.

It is not obvious, however, that Frege regarded his argument for Axiom I in §18 in the hyper-tautological way I have outlined. And it need not have been his view, for purposes of maintaining his position on the epistemic role of his arguments introducing the axioms. He indicates, as we have seen, that there are more basic truths in logical reality than are needed to axiomatize logic and arithmetic (B, §13). So the reasoning in the semantical arguments in *Begriffsschrift* and *Basic Laws* might be seen as articulating the character of the self-evident axiom by appealing to an argument that one would have to be able to give in order to understand the axiom, even though the premises of the argument are no more basic than the axiom, and thus can provide no justification for it, or proof of it in Frege's epistemically freighted sense of "proof".

Understanding one basic truth may demand that one be able to understand its logical connections to others—even though understanding any one basic truth would suffice to recognize its truth and provide one with sufficient "evidence" to recognize its truth. (I will discuss this point further in §IV.) What the argument does is to bring out vividly the content of the axiom, whose understanding renders its truth evident.

It seems to me plausible that if one understands Axiom I, one realizes that it is true. One can by considering the condition under which it would be false realize that it cannot be false. Or one can simply recognize that it is true because it indicates that if a is true, it is true whether or not some arbitrary proposition is true. It seems to me that this understanding may necessarily require an ability to accept the argument that I outlined involving conjunction. Perhaps to understand the conditional one must understand conjunction (and vice versa). But the argument adds no justificatory force to a belief in the axiom. The belief is justified by the understanding involved in the belief. The argument provides no additional justification not already available from understanding the axiom. It does not proceed from steps more basic than the axiom. For to understand those steps, one

[13] I discuss this argument at greater length and provide citations (1984, pp. 24–30).

must have an understanding of the material conditional sufficient to recognize the truth of the axiom independently of that particular argument.

These points simply instantiate Frege's view that axioms are basic truths, and basic truths do not need proof. Basic truths can be (justifiably) recognized as true by understanding their content.

I think that there are two interesting philosophical questions associated with the position that I have outlined. One is whether there are methods for determining a truth to be basic in the sense Frege relies upon. Frege gives no general account of such methods, and many would doubt that there is such a category of basic truths. But Frege does provide a range of philosophical arguments, particularly in *Foundations*, which are meant to persuade one of the basicness or non-basicness of various truths. Perhaps by systematizing and deepening the sorts of arguments Frege gives in *Foundations*, we would develop better ways for isolating such a category. We certainly have intuitions that some logical truths are more basic than others—and even some that seem as basic as possible. I am not persuaded that there is no fruitful subject matter here.

The other question is whether one can develop in more depth the relation between the point that understanding a thought requires inferential abilities (so that the understanding is articulable through inference), and the point that justification of belief in basic truths derives from understanding a single truth's content. The distinction is certainly tenable, but a fuller account of what goes into justification and what goes into understanding would be desirable.

Frege famously realized that understanding a thought requires understanding its inferential connections to other thoughts. So although a basic truth may carry its "evidence", its justification, in its content, an articulate understanding of that content might require connecting it to other contents, including truths that may be equally or less (but not more) basic. Frege's arguments "for" his axioms elicit understanding of the axioms by bringing out these connections.

Whether or not Frege understood his argument in §18 in the hyper-tautological way, the arguments that take his axioms as conclusions are compatible with his view that the axioms are basic truths that do not need or admit proof. The axioms are unprovable in that no genuine proof—no justification from more basic truths—is possible. Since they are self-evident, they do not need proof. They are self-evident because their justification is carried in their own contents. Understanding the content of an axiom suffices to warrant one in believing it. The point of the arguments is to articulate the content of the axioms and to elicit a firm understanding of them that resides in an understanding of their constituent senses.

III

Frege rarely supports his logical system or his logicism by invoking the traditional features of the axioms that we have discussed. Except for his insistence on rigour of formulation, much that is striking and original about his methodology bears little obvious relation to the Euclidean tradition. Although he alludes to self-evidence frequently, he almost never appeals to it in justifying his own logical theory or logical axioms. He never says or implies that convictions about self-evidence are infallible. I think that Frege believed that there is no infallible guarantee that one's commitments on logical or geometric truth are correct.

There is abundant evidence that Frege had some sympathy with modern caution about the reliability of appeals to what is obvious. He praised the refusal to be satisfied with even Euclidean standards of rigour, which refusal led to questioning the Parallel Postulate, and eventually to non-Euclidean geometries (FA, §2). Although he thought that Euclidean geometries are true, he knew that he had eminent opponents. His belief that Euclidean geometry is true (indeed true of space) is surely based primarily on his sense that Euclidean geometry is more obvious than non-Euclidean geometries. But he bolsters this belief with argument: he argues that the geometries are incompatible, that Euclidean and non-Euclidean geometry cannot both be true, that a false system must be banned from the sciences, and that in view of its *longevity* one can hardly regard the Euclidean system as on a par with astrology. Frege uses these meta-considerations to support putting forward the axioms: "It is only if we do not dare to do this [treat Euclidean geometry on a par with astrology] that we can put Euclid's axioms forward as propositions that are neither false nor doubtful" (NS 184/PW 169). Although less than convincing, this is hardly the argument of a dogmatic rationalist.

Frege was not always conservative in his attitudes about traditional mathematical intuitions. He begins *Foundations of Arithmetic* with a litany of cases in which attempts to provide a foundation of proof had led to a sharper grasp of concepts, new mathematical theories, and deeper grounding of mathematical practice. He notes that epistemic standards in mathematics, especially in view of the advent of analysis, had been lax. He continues,

> Later developments, however, have shown more and more clearly that in mathematics a mere moral conviction, supported by a mass of successful applications, is not good enough. Proof is now demanded of many things that formerly passed as self-evident ("*selbstverstandlich*"). (FA §1).

He goes on to challenge the view that various arithmetic equations involving numbers in the thousands are self-evident ("*einleuchtend*") (FA §5).

Moreover, Frege backs Cantor's introduction of actual infinities, even though they were thought by many contemporary mathematicians to be deeply contrary to the limits of mathematical intuition (FA §85; KS 163ff./CP 178ff.). He bases belief in the actual infinite not, of course, on direct mathematical intuitive powers, but on the role of the infinite in arithmetic and on his confidence that he could derive claims about it from arithmetical, and ultimately logical principles. But he clearly recognized that common mathematical beliefs about what is self-evident or intuitive or obvious could be flat out mistaken.

As I have noted, Frege's logic was not well-received by the dominant mathematicians of the day. They found not only his notation, but some of his principles misconceived. Frege responds not by insisting on the self-evidence of his principles, but by arguing that the only way to get a true logic is by providing a deeper analysis of judgments and inferential patterns than his Boolean opponents had provided (e.g. NS 37/PW 33). In *Basic Laws* we find Frege recommending to those who are sceptical of his logical system that they get to know it from the inside. He thinks that familiarity with the proofs themselves will engender more confidence in his basic principles (GG xii/BL 9: FA section 90). In the Introduction to *Basic Laws*. Frege repeatedly appeals to advantages, to simplicity, and to the power of his axioms in producing proofs of widely recognized mathematical principles, as recommendations of his logical axioms.[14] He was aware that what people find intuitive or obvious is no safe guide to accepting or rejecting his own logical theory.

I think that we can assume that Frege thought that mathematical and logical intuition and judgment, even in outstanding mathematicians and logicians, is thoroughly fallible. Let me codify this point in two principles. He thought (a) that the fact that a mathematical or logical proposition is found obvious by competent professionals at a given time provides no infallible guarantee that it is true, much less a basic truth. He thought (b) that there is no guarantee that true mathematical or logical principles (including basic truths) will be found to be obvious by competent professionals at a given time.

The evidence for (b) is Frege's recognition of contemporary attitudes toward the axioms of Euclidean geometry and his awareness of scepticism about his own logical principles. The evidence for (a) is Frege's method of argument for accepting Euclidean geometry, his repeated criticism of

[14] This is the sort of argument usually associated with Zermelo (1908) in his defense of the axiom of choice. Frege's use of the argument form antedates Zermelo's.

"instinct" and "intuition" as ways of founding mathematics, his experience in struggling to find an acceptable logic against what was regarded by other (Boolean and Kantian) logicians as already constituting an acceptable logic, and perhaps his own uncertainty about Axiom V.[15]

This awareness of the fallibility of mathematicians' sense of what is obvious was part of the advanced spirit of the age. It did not constitute any sort of scepticism about mathematical knowledge, or even a concession that mathematical principles are less than "certain". Frege had a deep confidence in the ability of mathematical practice eventually to arrive at truth. And he maintained the traditional rationalist view that mathematics and logic are "certain" and epistemically more solid than empirical science. But these traditional views were tempered with a historical awareness of changes in these disciplines, and with an original thinker's awareness of how crooked the road to discovery could be. The nature of Frege's fallibilism will become clearer as we proceed.

Frege's method is non-Euclidean not only in his relative neglect of appeals to self-evidence when he is arguing for his logical theory, but also in his original way of developing that theory. As I have noted, in analyzing inferences Frege is concerned that appeals to self-evidence not be allowed to obscure the formal character of the inferences, which can be found only by rigorous logical analysis. This analysis is arrived at not primarily by consulting unaided intuition, but by surveying inferential patterns in actual scientific-mathematical reasoning. Frege ridicules the idea that one will find the appropriate logical concepts and logical structures ready-made by consulting intuition. He writes

> All these concepts have been developed in science and have proved their fruitfulness. For this reason what we may discover in them has a far higher claim on our attention than anything that our everyday trains of thought might offer. For fruitfulness is the acid test of concepts, and scientific workshops the true field of study for logic. (NS 37/PW 33)

As is well known, Frege's method was to reason to logical structure by observing patterns of judgments and patterns of inferences—and then

[15] I have elsewhere discussed the bends and turns in Frege's changes of mind about Axiom V, and his attempts to persuade himself of its truth and its status as a basic law of logic (1984, pp. 24–30). It is clear, before as well as after his recognition of Russell's paradox, that Frege had and expected doubts about using the principle as an axiom, and seemingly even about its truth. Yet Frege did commit himself to Axiom V's being an axiom in the traditional sense, which would require that it be true, certain, unprovable, and self-evident—or not in need of proof. Any reflection on this situation at all would have enabled him to distinguish the objective property of self-evidence required of an axiom and his psychological state of finding the axiom less than completely obvious. He seemed to have hoped that the axiom would become more obvious with greater familiarity with the notion of an extension of a concept.

postulate formal structures that would account for these patterns. While this method makes use of intuitions about deductive validity, it has at least as much kinship to theory construction as to intuitive mathematical reflection.

Frege's methods of analysis are closely associated with his famous contextualist methodological pronouncements. He holds that one can understand the "meaning" (later, sense and reference) of individual words only in the context of propositions. And he thought that one understood such semantical infrastructure only by understanding patterns of inferences—not by simple reflection. But understanding was traditionally supposed to be the basis for recognition of the truth of self-evident propositions. So if understanding requires such "discursive" procedures as logical analysis and theory construction—or at least the tacit abilities that such conscious construction codifies—it would seem that Frege's method constitutes a substantial qualification on traditional rationalist conceptions of reflection.

Frege's contextualism extends beyond methodology. He uses a contextualist argument for defending the existence of abstract objects in *Foundations of Arithmetic* (§§56–68.). He thinks that we are justified in believing in the existence of numbers as objects if we are justified in accepting mathematical propositions whose analysis shows number expressions to be singular terms.

On Frege's view, justification for accepting mathematical propositions seems to take three forms. Justification derives from what Frege calls "actual applications" (FA §§1–2). It derives from considerations of simplicity, duration, fruitfulness, and power in pure mathematical practice. It derives from understanding the self-evident foundations (axioms, definitions, inference rules) and from carrying out proofs.

Frege does not develop his notion of "actual applications" in detail. But the notion seems to attach to successful. applied mathematical practice. He appeals to the role of arithmetic or mathematical principles (like the associative law) in inductively supported applications within natural science or ordinary counting as one sort of application (FA §§2, 26). Here there seems to be an inductive confirmation of arithmetic through its success in application to non-mathematical domains.

The invocation of actual applications should be seen in the context of a broader conception of "pragmatic" justification within mathematical practice. Frege emphasizes that pure mathematical practice *works*. It produces a community of agreement through finding some systems "better", "simpler", "more enduring". It is this practice that Frege appealed to in defending Euclidean geometry, in the argument we discussed above.

Pragmatic considerations also enter into Frege's conception of the justification of definitions.[16] He thought that definitions are confirmed by their fruitfulness, by their ability to further mathematical practice (FA §88; KS 245/CP 255). This view is nowhere more strikingly expressed than at the point of the key definition in *Foundations of Arithmetic*. Immediately after defining "the Number which belongs to the concept F", he writes, "That this definition is correct will perhaps be little evident ('*wenig einleuchten*') at first" (FA §69). He then goes on to argue that certain ordinary language objections to the definitions can be laid aside, because there is "basic agreement" between definiens and definiendum on our "basic assertions" about numbers and because discrepancies in ordinary usage do not raise serious rational objections to the definitions—only objections of habit and usage.

Frege entitles the next section "Completion and Proving-Good of our Definition". He writes, "Definitions prove good through their fruitfulness" (FA §70). Proving fruitful consists in aiding in a chain of proofs. Thus although as we have seen (KS 263/CP 274; KS 289–90/CP 302) Frege held that definitions are self-evident (*selbst-verstandlich*), and even, sometimes, that they preserve the sense of the definiendum (so that the senses of definitions are of the form "$a = a$"), he says here (FA §70) that a correct definition may be "little evident at first". They are self-evident in themselves, but not evident "at first" to us. Their intrinsic self-evidence might become more obvious to us over time, through their role in proofs; and we may receive some confirmation of their worth (self-evidence) through this role. Once the definitions are fully mastered and thoroughly used, presumably this external form of confirmation would be overdetermined and unneeded.

Instead of reflection, Frege appeals to mathematical practice—to observing the role of the definition in facilitating proof—as a way of confirming the worth, and seemingly the correctness, of the definition. Frege seems to have thought that one arrives at good definitions partly through the process of logical analysis, dependent on theory construction, that we have just been discussing.[17]

Frege thought that success in proving principles that are independently regarded as valid provides some justification for the principles (as well as

[16] I hope that it is clear that by calling epistemic considerations "pragmatic" I am in no way implying that Frege thought them any less able to put us on to truth about a reality that is independent of our practice.

[17] Frege's account of non-stipulative definition, and perhaps his conception of it, changes over the course of his career. But as far as I can see, these variations do not affect the primary points I am making. For a discussion of these matters which I think overrates the changes, but with which I am in basic agreement, see Dummett (1991, Chs. 3 and 12).

definitions) used in carrying out the proof. Comparisons of simplicity and success in carrying out such proofs are "tests" of a system. He writes,

> The whole of the second part [the Proofs of the Basic Laws of Number] is really a test of my logical convictions. It is prima facie improbable that such a structure could be erected on a base that was uncertain or defective. Anyone who holds other convictions has only to try to erect a similar structure upon them, and I think he will perceive that it does not work, or at least does not work so well. As a refutation in this I can only recognize someone's actually demonstrating either that a better, more durable edifice can be erected upon other fundamental convictions, or else that my principles lead to manifestly false conclusions. (GG xx–vi).

Here durability and working well, or working better, are tests of the basic principles and methods of his theory. It appears that Frege regarded these considerations as indicative of a kind of justificational support for his theory.

Frege saw these sorts of justification through "applications" and through "pragmatic" or other "methodological" considerations as insufficiently satisfying. They provide only "inductive", or only "prima facie", "probable" support. He shows little interest in justification through applications at all. Such justification might suffice to silence scepticism—in that one would find that the "boulder" is in fact "immovable" (FA §2). But it does not show what is holding the boulder so securely in place. Justification through consideration of pure mathematical practice seems to produce, for Frege, a secondary test, or a basis for prima facie perhaps inductive, justification. But Frege thought that the deeper justification lies in the structure of proof, which eventually leads back to logical axioms. He thought that the full "certainty", the rational unassailability, of mathematics would not be understood unless this proof structure was laid bare. Frege holds that whether he is right about his views about the logicist nature of this proof structure is a matter that can be determined only through the carrying out and the checking of the proofs (FA §90).

But assessing his logicism would require more than this. Knowing the order of reasons could not derive simply from checking proofs or soundness. It would also require finding that the proofs produce insight into justificational priority. The proofs in Frege's logical theory must match the proof structure that constitutes the justificational ordering among mathematical propositions—from first principles, including basic truths, to derivative principles. Such insight must derive from discursive reasoning both within and about the system. Frege thinks that recognizing that such a match has been attained is dependent on becoming familiar with his system, which is not just a matter of immediate reflection or insight. This view emerges repeatedly in the Introduction to *Basic Laws*.

Frege writes, "Because there are no gaps in the chains of inference, every 'axiom' ... upon which a proof is based is brought to light; and in this way we gain a basis upon which to judge the epistemological nature of the law that is proved" (GG vii/BL 3). He continues

> I have drawn together everything that can facilitate a judgment as to whether the chains of inference are cohesive and the buttresses solid. If anyone should find anything defective, he must be able to state precisely where, according to him the error lies: in the Basic Laws, in the Definitions, in the Rules, or in the application of the Rules at a definite point. If we find everything in order, then we have accurate knowledge of the grounds upon which each individual theorem is based. (GG vii/ BL 3)

Here again we encounter the view that one might find proposed axioms and inference rules in the foundations to be defective. The language clearly suggests that defect might in principle lie not only in the ordering or in proposed axioms not being logical, but even in proposed basic principles not being true or sound. "Finding everything in order" (which includes not only freedom from defect but being in the right justificational order) seems to require thorough familiarity with the system, and considerable discursive reasoning within it.

In recommending his new system Frege says that the introduction of courses-of-values of functions provides "far greater flexibility" and cannot be dispensed with (GG ix–x/BL 6). He recommends the introduction of truth values (in terms of which extensions of concepts are explained) by saying

> How much simpler and sharper everything becomes by the introduction of truth-values, only detailed acquaintance with this book can show. These advantages alone put a great weight in the balance in favor of my own conception, which may seem strange at first sight. (GG x/BL 7)

Frege does not appeal to immediate insight or obviousness. He appeals to "advantages" which can be appreciated only by a detailed mastery of his theory, only through discursive reasoning.

Frege is aware of the unobviousness of his proposals:

> I have moved farther away from the accepted conceptions, and have thereby stamped my views with an impress of paradox.... I myself can estimate to some extent the resistance with which my innovations will be met, because I had first to overcome something similar in myself in order to make them. (GG xi/BL 7).

Frege's view that acceptance of his proposals depends on detailed mastery of the system extends even to the acceptance of his *basic principles*, the basic axioms and rules of inference:

> After one has reached the end in this way, he may reread the Exposition of the *Begriffsschrift* as a connected wholes … . In this way, I believe, the suspicion that may at first be aroused by my innovations will gradually be dispelled. The reader will recognize that my *basic principles* [my emphasis] at no point lead to consequences that he is not himself forced to acknowledge as correct. (GG xii/BL 9).

The basic principles gain something from our seeing what obviously correct consequences they have and from recognizing "advantages" of simplicity, sharpness, and the like. Here again, Frege is defending not merely the logicality but the truth of his basic proposed principles, through pragmatic modes of reasoning.

What do the basic principles gain from our seeing their consequences and our realizing their various "advantages"? If they are indeed axioms, they can be recognized as true "independently of other truths". The sort of justification that derives from understanding them and recognizing their truth through this understanding needs no further justificatory help from reflecting on their consequences or the advantages of the system in which they are embedded. The recognition of advantages seems to provide a prima facie, probabilistic justification that applies to the whole system, but derivatively to elements in it. Such recognition may provide indirect grounds for believing that the axioms are indeed basic and indeed true. But the supposed self-evidence of the axioms is ideally the primary source of their justification. They do not need or admit of proof. They gain a secondary, broadly inductive justification. And we gain greater sharpness of our understanding of them. I shall develop these points in §IV.

Let me summarize what I have been saying about Frege's epistemology. Frege thought that we are fallible in our convictions even on matters of self-evidence. He thought that we have no direct intuitive access to numbers as objects. Instead, we are justified in believing in them only through their role in making true mathematical propositions that we are justified in believing. Our justification for believing in these propositions is partly pragmatic—we find their place in mathematical practice secure through long usage, through advantages of simplicity, plausibility, and fruitfulness, and through applications to non-mathematical domains. A deeper justification for believing in these propositions lies in finding their place in a logicist proof structure, by understanding the grounds within this structure that support them (if they are non-basic) or by understanding the self-evidently true basic principles.

Understanding this structure requires some of the discursive reasoning that plays a role in secondary, "pragmatic" justification. Understanding requires not only the logical analysis involving theory construction that I noted above, but also the production of proofs and the recognition that these proofs capture an antecedent order of justificational priority. Understanding the axioms requires, in some cases, reasoning from them in producing proofs, and even reasoning to them from reflection on their content. Frege knew, from painful experience, that other competent mathematicians would not immediately recognize his success (if he were indeed successful). He says that such recognition would depend on working through the proofs and acquiring increasing familiarity with the "advantages" that the conceptions that he had introduced offered. Thus whatever role self-evidence plays in his epistemology seems to be qualified by pragmatic considerations that result from reasoning within and about his system of proofs over time. I want to go into these qualifications in more detail in the next section.

IV

How does Frege's rationalist appeal to self-evidence accord with the fallibilist, pragmatic elements in his position? How could he appeal to pragmatic and philosophical considerations in persuading others of the analyticity of arithmetic and of the soundness of his logical system, when he held that justification ultimately comes down to self-evidence? Are the two tendencies simply ill-matched, ill-thought-out philosophical strands in the thinking of a mathematician?

This last question suggests serious underestimation of Frege's philosophical depth. I think that the integration of the two strands is one of his finest philosophical achievements. To make progress on our questions, we must scrutinize what Frege meant by "self-evident".[18]

Frege probably did not regard himself as using a well-honed technical term. Although no one has remarked it, as far as I know, the term "self-evident" that appears in the standard English translations does not translate a single German counterpart. Sometimes Frege uses "*einleuchtend*" (and grammatical variants); sometimes he uses "*selbst-verstandlich*"; and occasionally he uses "*evident*" and "*unmittelbar klar*". There are differences of meaning among these terms in colloquial German, but I have not found consistent differences in Frege's usage.[19]

I shall investigate the meaning of these terms for Frege by considering how he used them. Frege regarded both axioms of geometry (FA, combin-

[18] I have benefited in this section from correspondence in 1991 with Robin Jeshion, who had thought independently about Frege's Euclideanism, and who emphasized the importance of not neglecting subjective elements in Frege's remarks about self-evidence.

[19] I do believe that "*selbst-verstandlich*" is never used with any mentalistic overtones, whereas the others occasionally are. I will develop the matter of mentalistic overtones below.

ing §§13 and 90 ("*einleuchtend*"); cf. NS 183/PW 168) and logical axioms (FA §90 ("*einleuchtend*"); GG v2, §60/ GB 164 ("*selbst-verstandlich*"); GG 253/BL 127 ("*einleuchtend*")) as self-evident. At KS 393/CP 405, Frege writes, "the truth of a logical law is immediately evident ('*einleuchtet*') from itself, from the sense of its expression". By "logical law" he means a basic truth (or an axiom) of logic. (Cf. KS 262/CP 273.) Frege also regards rules of inference (FA §90 "*einleuchtend*") and thoughts expressed by propositions formed from correct definitions (KS 263/CP 274; KS 289–90/CP 302 "*selbst-verstandlich*") as self-evident. Axioms and thoughts expressed by propositions formed from correct definitions could be used as primitive steps in proofs; and rules of inference could be used as primitive modes of transition in proofs.

It is not clear to me whether Frege regarded *any* truths of arithmetic as self-evident.[20] Very likely he did not. In *Foundations of Arithmetic* §5 he criticizes those who take propositions involving addition of larger numbers as self-evident and demands proof of such propositions. Although he thinks that propositions involving addition of smaller numbers, like 1 + 1 = 2, are *provable* and do not depend for their justification on Kantian intuition, he neither denies nor affirms that they are immediately self-evident ("*unmittelbar einleuchtend*" or "*unmittelbar klar*"). In so far as Frege was thinking of self-evidence in terms of recognizability as true independently of recognition of other truths (in the deepest proof-theoretic order of reasons), it is not surprising that he would not count arithmetic truths, even simple ones, as self-evident. (Cf. KS 393/CP 405.) It would seem that he might reserve the notion for basic truths, basic rules of inference, and definitions. At any rate, I know of no place where Frege counts arithmetical truths self-evident. There is reason to believe that Frege may have reserved the notion of self-evidence for truths that are "not in need of proof".

Frege does, however, regard arithmetical truths as *certain*, that is, beyond reasonable doubt given understanding of the proposition (B §13; FA §2). And in §14 of *Foundations* Frege says that denying any of the fundamental propositions of arithmetic leads to complete confusion: "Even to think at all seems no longer possible". This point serves the view that arithmetic has a wider domain than geometry—the domain of everything thinkable. But he regards this argument for the analyticity of arithmetic as non-demonstrative (FA §90). So he regarded his claim that denying fundamental arithmetical principles would throw thought into complete confusion as fallible.

[20] One could imagine such exceptions as 1 = 1, which are themselves obvious logical truths. I mean arithmetical truths whose truth depends essentially on arithmetical notions.

Our discussion in §I shows that self-evidence is not the same as obviousness, or immediate psychological certainty. In Frege's primary usage, self-evidence appears to be compatible with lack of obviousness to individuals. Frege thought that something could be an axiom and yet be found by professional mathematicians or logicians to be unobvious. He speaks of propositions that "formerly passed as self-evident" ("*selbst-verstandlich*") (FA §1). Here he implies that something might seem to be self-evident but not be so.

Frege himself was uncertain about at least one of the thoughts that he proposed as an axiom (hence as self-evident). The Introduction to *Basic Laws* concludes with a statement that fatefully leaves open the possibility that one of the basic principles, one of the proposed axioms, is defective; he even identified the faulty "axiom" as the only likely source of difficulty. Since he thought it was an axiom, he must have, at least sometimes, thought that *it* was certain, but because of insufficient analysis or incomplete understanding, *he* was not.[21] Moreover, he found that other basic

[21] Richard Heck (forthcoming) holds that Frege was in doubt only about the status of Axiom V as a logical truth, not about its truth. Although this is not obviously false, I see no clear evidence for this view, and some evidence to the contrary. Frege probably did worry about its logical status independently of worrying about its truth. But I believe that he was uneasy about its truth as well. It was clearly less obvious than the other axioms, because the notion of *course of values* was relatively new to mathematics, despite its connection to the notion of extension and the graph of a function. The dispute between Frege and the iterative set theorists suggests that there was fundamental doubt on both sides about the viability of the notions (respectively) of set and of extension. Cantor was not committed to, and probably would not have accepted, Axiom V. Frege, who had read Cantor, would have been aware of this. (Cf. Burge 1984.) Moreover, Frege's concerns about Axiom V in the introduction of BL explicitly focus on its truth—as well as its status: He invites the reader to find "error" or something "defective" in (among other places) BL (GG vii/BL 3), and he explicitly indicates that a dispute can arise only with respect to Axiom V. He associates this invitation with dispelling suspicion of his principles by having the reader see that they lead to no mistakes:

> In this way, I believe, the suspicion that may at first be aroused by my innovations will gradually be dispelled. The reader will recognize that my *basic principles* [my emphasis] at no point lead to consequences that he is not himself forced to acknowledge as correct." (GG xii/BL 9)

And he ends the introduction by again raising the possibility of error, falsehood—as well as the possibility that different axioms would produce a more durable structure. He declares (with fatefully exaggerated bravado) that no one will be able to find such error (GG xxvi/BL25). As Heck points out, Frege took himself to have provided a meta-theoretic "justification" of Axiom V in §31. But Frege could not have thought that a semantical proof had any more certainty than the axiom that it explicated or justified, since if Axiom V was an axiom in Frege's sense, it was self-evident and did not need proof. Thus any psychological uncertainty about the axiom (uncertainty expressed *after* the giving of the proof) would have to transfer to uncertainty about the proof as well. Given that the proof used new methods (and in fact turned out to be defective), it is not surprising that Frege would have retained doubts even after giving the proof.

principles of his logic were not universally accepted by opposing logicians. But he maintained the views, which he several times expresses in the pre-paradox period, that the basic principles that he proposed are genuine axioms and that axioms are self-evident.

The distinction between self-evidence and psychological certainty, or felt obviousness, is a corollary of the distinction between justification and discovery that Frege draws at the opening of *Begriffsschrift* (quoted in §I). Self-evidence on one primary construal is meant to be bound up with the account of the justification of logical and mathematical truths, not with psychological means of discovering their truth.

The foundations of logic and mathematics were supposed to be "unprovable"—not justifiable by derivation from other thought contents. But Frege thought that one was nevertheless justified in holding them to be true. Their justification rests not only on their being unprovable, but on their not being "in need of proof". That is, they are rationally acceptable in themselves. This is part of the literal etymological meaning of "self-evident": they carry their "evidence", their rational support, in themselves and are dependent on none from other propositions.

Frege states this point quite directly: "... it is part of the concept of an axiom that it can be recognized as true independently of other truths" (NS 183/PW 168). This remark is virtually echoed in terms of self-evidence: "... the truth of a logical law [a basic truth of logic] is immediately evident from itself, from the sense of its expression" (KS 393/CP 405). Frege does not mean that it is psychologically possible to recognize it as true independently of other thoughts. For the concepts of an axiom and of a basic law are explicitly (in many places) intended to be independent of psychological considerations. But the appeal to recognizability obviously involves some implicit reference to mind, or a recognizing capacity. I think that this reference is to an ideal rational mind with full understanding.

The implicit reference to mind is also contained in Frege's basic epistemic concerns. Frege is interested in the sources or springs of knowledge. He announces his interest in knowledge in the opening passage from *Begriffsschrift* that I quoted at the outset of this paper. The basic laws are laws on which all *knowledge* rests. In the middle of his career (KS221/CP235) and at the end (NS 286–94/PW 267–74), he remains interested in the springs of knowledge—empirical, logical, geometrical. Knowledge involves a contribution by mind—belief or judgment. To understand knowledge, one must conceive of some relation between the purely abstract proof-structure, the propositional system of grounds or reasons, and mind.

Similarly, in framing his account of his basic epistemic categories—apriority and analyticity—Frege has in mind some conception of justification for judgment (holding-true), not simply justification or grounding for an abstract proposition or thought content:

> Now these distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgment but the justification for making the judgment .... When a proposition is called a posteriori or analytic in my sense, this is not a judgment about the conditions, psychological, physiological and physical, which have made it possible to form the content of the proposition in our consciousness; nor is it a judgment about the way in which some other person has come, perhaps erroneously, to believe it true; rather, it is a judgment about the ultimate ground upon which rests the justification (*Berechtigung*) for holding it to be true. (FA §3)

The ultimate justificatory ground (the ground or the justification) is independent of minds. The content of any mathematical justification is independent of minds. The problem of the foundation of arithmetic is basically a problem in mathematics. But that ground or content is justification for *belief*, or holding-true, or recognition of truth.

The relevant notion of mind here though is abstract and ideal. There is no reference to individual minds or to the psychology of recognition, belief, or judgment. When Frege writes, "... it is part of the concept of an axiom that it can be recognized as true independently of other truths" (NS 183/PW 168), he means that the truth can be rationally and correctly recognized as true by a rational mind independently of resting the rationality of this recognition on derivation of the truth from other recognized truths.[22]

The basic truths are laws at the foundation of a justificational structure. The other truths receive their justification by being logically derivative from the basic ones. And the basic ones carry their justification intrinsically, in that their truth can be justifiably recognized from the nature of those truths, in justificational independence of consideration of other truths. On this conception self-evidence is an intrinsic property of the basic truths, rules, and thoughts expressed by definitions. It is intrinsic in that it is independent of relations to actual individuals. It does involve implicit relation to an ideal mind. But ideal minds are abstractions, themselves understood in terms of rational capacities, which are characterized

[22] Cf. also GG II §147/GB 181. These references to an ideal rational mind are not psychologistic, since there is no reference to actual minds, and since the notion of a rational mind is understood in terms of a capacity to recognize the truth of and logical relations between elements in the Platonic logical structure that Frege is discussing. Cf. Burge (1992).

in terms of the abstract justificational structures. These structures are for Frege independent of individual minds, or indeed any individuals.

Although Frege thought that axioms (basic truths) *can* be recognized as true and basic by *actual human minds*, it is certain that he did not think that it is part of the concept of an axiom (or basic truth) that they can be. In so far as it is a necessary truth that axioms are recognizable as true by human minds, it would be a truth that derived from necessary conditions on any possible mind (qua mind), not from conditions placed on axioms or basic truths by the notions of actual mind or human mind. In most of the formulations containing such notions as "unprovability", "certainty", "recognizability", Frege intends implicit reference to some sort of ideally rational mind. His notion of an ideally rational mind is, as I have emphasized, constitutively dependent on the abstract structures that define the norms for rationality.

I have outlined a rational reconstruction of what might be called Frege's objective conception of self-evidence, one that accords with much in the rationalist tradition. To give a rough summary: A truth is self-evident in this sense if (i) an ideally rational mind would be rational in believing it; (ii) this rationality in believing it need not depend for its rationality on inferring it from other truths—or reasoning about its relation to other truths; it derives merely from understanding it; and (iii) belief in it is unavoidable for an ideally rational mind that fully and deeply understands it. I think that this is the main thrust of what Frege intended in requiring that axioms, rules of inference, and definitions be self-evident.

But this cannot be a complete account of Frege's conceptions of self-evidence. Even if justification is supposed to derive somehow from an abstract structure that is independent of minds and waiting to be discovered, it is clear that Frege is not indifferent to the fact that *individuals* are justified in accepting truths in this structure. He does not ignore actual knowledge or actual justifications for individuals. He is quite aware that *we* have mathematical and logical knowledge, and that *we* make judgments. In order to discuss how his abstract normative principles bear on actual mathematical practice and knowledge, Frege had to relate the abstract structure of thoughts (or propositions) to minds. Through most of his career, Frege has little to say about such a relation. But some points can be extracted for discussion here.

Frege's terms that translate "self-evident" usually make no explicit reference to actual minds. But there are some uses that explicitly or implicitly presuppose such reference. The word "*einleuchtend*" translates normally as obvious, clear, or evident. The verb from which the adjective derives takes the dative in many standard constructions. Frege typically uses the term without the dative (often intending implicit relation to some ideal rational mind), but at least once he includes the dative (FA §90): "Often ... the correctness of such a transition is immediately self-evident *to us*" [my emphasis]. As we have seen (in FA 69) he notes that his primary definition will be "little evident at first"—clearly intending that it will be little evident to ordinary individual reasoners, at first.

Moreover, (in FA §5) Frege argues from the fact that certain thoughts are not immediately clear (*klar*) or immediately evident (*einleuchtend*) to the conclusion that they are not axioms but are provable. He explicitly regards this not as a demonstrative argument, but as one that adds probability (FA §90). If he meant "self-evident" purely in the sense of "self-justifying", he would be arguing that since they are in need of justification by reference to other propositions, they are not axioms and thus are in need of proof (and since they are justified, they are provable). But this *would* be a demonstrative argument, and its premise would beg the question against his opponents.[23] Clearly, Frege's premise must be taken to be a remark about the fact that it is not immediately (non-inferentially) obvious to our minds that the relevant thought contents are true.

Similarly, when in 1903 Frege learns that Axiom V is false, he remarks that he had always recognized that it is not as self-evident (*einleuchtend*) as the others (GG II, 253/BL 127). The comparative form suggests obviousness to actual minds, which admits of degrees. Frege also says that he always recognized that Axiom V lacks the self-evidence that must be required of logical laws. Here too he must be saying that it had not achieved the obviousness to actual minds (his mind) that should underwrite the postulation of thought contents as logical laws. It would have been obviously untrue for him to have said that he always recognized that Axiom V did not meet the objective requirement of ideal self-evidence that is a requirement on logical laws. For then he would have been involved, before the discovery of the paradox, in an obvious and easily discovered inconsistency.

If he had always recognized Axiom V not to be self-evident in the more objective sense that axioms are required to be self-evident, he would never have proposed it as an axiom, as a first principle at the foundation of the

[23] Frege could have been arguing from the assumption that it is not obvious to an ideally rational and understanding mind, to the conclusion that it is not an axiom. And he might have regarded the assumption as "merely probable". This argument seems enthymemic. It requires the additional premise that we are (probably) ideally rational and that we have ideal understanding of the arithmetic propositions. But there are two reasons for thinking that this is not Frege's argument. In the first place, Frege clearly thought that before him no one had full understanding of the arithmetic propositions, since no one had uncovered their logical natures. In the second, Frege appears to be appealing to immediate intellectual experience of our not finding large addition problems immediately obvious—not to some circuitous argument in terms of ideal rationality.

structure of logic and mathematics. For doing so would be obviously inconsistent with his own principles. Indeed, in *Basic Laws* he says "it must be demanded that every assertion that is not completely self-evident should have a real proof" (GG vol II, §60; GB 164). Axiom V is certainly asserted without "real proof"—a deductive argument from more basic truths. Frege was surely not involved in such a simple-minded inconsistency before the discovery of the paradox. His use of "self-evident" ("*einleuchtend*") in the appendix seems to have a somewhat different meaning from the meaning it has in his official pronouncements on the self-evidence of axioms. Perhaps Frege is sliding between more and less objective uses of "self-evident". But the issue is very subtle and requires further elaboration.

Before discovering the paradox, Frege recognized that Axiom V was less obvious to him than the other axioms. But he must have thought—at least as the public position that he committed himself to—that it *was in fact* self-evident in a sense required of axioms, a sense that is compatible with not being fully obvious or self-evident to him or other individuals. He must have thought that it is compelling for a fully rational mind that fully understood the principle. In finding it less than fully obvious, he must have regarded himself as having a less than ideal insight into the principle. After discovery of the paradox he realized that the lack of obviousness had turned out to be a sign of trouble with the proposition, not merely insufficiency in his insight.

How are we to take Frege's remark in the appendix to *Basic Laws* that self-evidence, in a sense that includes a relatively subjective component, is required of axioms? In that remark he seems to say that axioms must be required to be self-evident in a sense that would include (in addition to other things) a large degree of obviousness to individuals, whereas Axiom V always lacked the requisite degree of obviousness to individuals.

Now it is possible that one should regard Frege's remark as autobiographically important, but insignificant for casting light on his reflective views. For it was written in a hurry and at a time of anguish, immediately on being informed by Russell of the contradiction in his system.

But the same tension between objective requirements of self-evidence and subjective experiences of obviousness for axioms emerged in the Euclidean tradition. For example, some regarded the axiom of parallels as insufficiently (subjectively) obvious to be an axiom. Others took this lesser obviousness as a historical fact or a human failing which did not bear on the objective foundational status of the axiom—its self-evidence in the sense of being recognizable as true without needing or admitting proof.

There is good reason for inquiry to require that proposed axioms be obvious in a subjective sense. For if a proposition is not immediately obvious, then one will want either further justification of it—some sort of proof or further argument—or additional clarification of one's understanding of it. And this state might lead one reasonably not to take it as a starting point in justification.

On the other hand, it seems that a requirement that axioms be obvious to individuals, though having a subjective component, must retain some element of objectivity. It cannot be a final requirement on axioms or basic truths, only a deeply significant prima facie requirement. It must allow for individuals' learning histories. Every sophisticated rationalist allowed that truths that are self-evident could be foggily grasped or apprehended. So the fact that an individual does not find some proposition obvious cannot in general show that the proposition is not self-evident, or that it cannot be taken as the foundation for a justificational system. So subjective obviousness cannot be an absolute requirement on an objectively correct system. At any time an individual can be unsatisfied with a principle as an axiom because of its lack of obviousness. Whether this feature is a final objection to the principle's being an axiom depends on whether the lack lies in the principle or in the individual. Still, inevitably, one must make use of what is obvious in trying to determine what is (ultimately) objectively self-evident. One must place some reliance on one's own contingent and perhaps limited rationality and understanding.

How then are we to understand the relation between the intrinsic self-justifying character of the axioms and the clarity or obviousness of their truth to *us*?

Answering this question depends on understanding the enormous degree to which Frege thought we tend to understand concepts, or thought components, incompletely. He thought that in arithmetic and logic, we think with thought components whose correct explication or real inferential structure we do not fully understand. His argument for his logic presupposes, and often makes explicit, that most logical categories are obscured to us by the non-logical functions of language. His argument for his definitions of arithmetical terms, in *The Foundations of Arithmetic*, is prefaced with and saturated by an emphasis on the idea that only through centuries of labour (culminating in his own work) had mankind come to a true understanding of the concept of number and of various other arithmetical notions (FA vii).[24] Frege recognizes that the correctness of his critical definition "will hardly be evident at first" (FA §69—another psychologically tinged usage). And he goes on to try to make it "evident" by improving our understanding of the critical notions involved in it. Def-

[24] For extensive discussion of this issue in Frege, see Burge (1990, 1984).

initions that are self-evident may not *seem* to be self-evident, but they may help us become more clearly aware of the content (sense) of a word that we have already been using.

In *Basic Laws* Frege indicates that the introduction of truth-values (which are basic for his explanation of terms in Axiom V) makes everything "sharper" in a way that only detailed acquaintance with the book can show (GG x/BL 7—quoted in §II). Sharpness is Frege's term for complete grasp or understanding. (Cf. footnotes 24.) Understanding the basic principles is enhanced by seeing their role in deriving theorems of the system (GG xii/BL 9).

What is it to be self-evident but not seem so? Descartes' appeals to self-evidence were not meant to receive immediate approbation from anyone who would listen. Self-evident propositions are self-evident to anyone who adequately *understands* them. Descartes and other traditional rationalists did not assume that understanding would be immediate or common to all mankind, or even common to all socially accepted experts. Understanding was something more than mere mastery of the words or concepts to a communal or conventional standard. It involved mastering a deeper rational and explanatory order. Acknowledgment of the truth of self-evident propositions would be immediate and non-inferential, only given full understanding in the relevant deeper sense.

Frege is relying on this tradition. This reliance explains his belief that our acceptance of thoughts as basic principles is fallible. It explains his view that familiarity with the details of his system will enhance acceptance of his basic principles. Full understanding is necessary for the self-evidence of basic principles to be psychologically obvious (evident, clear).

What is original about his position is not his view that a thought might be self-evident but not seem self-evident—self-evident but not obvious to an individual. It is not his idea that subjective obviousness or subjective unobviousness might submit to reversal through deeper conceptual development and understanding. What is original is his integration of these traditional views with his deep conception of what goes into adequate understanding. This conception rests on his method of finding logical structure through studying patterns of inference. Coming to an understanding of logical structure is necessary to full understanding of a thought. And understanding logical structure derives from seeing what structures are most fruitful in accounting for the patterns of inference that we reflectively engage in.

Thus Frege's "pragmatic" claim, that one will see the correctness and lose the sense of strangeness of his first principles by reasoning within his system and seeing the various advantages yielded by his proposals, is compatible with his appeal to self-evidence. It serves not to justify the first principles (except in a secondary, inductive way which will be overshadowed, given full understanding) but to engender full understanding of them. One might recognize the truth of the axioms independently of other truths only in so far as one fully understands the axioms. But understanding them depends not only on understanding Frege's elucidatory remarks about the interpretation of his symbols, but also on understanding their logical structure—their power to entail other truths, and their reason-giving priority. This latter understanding is *not independent* of reasoning that connects them to other truths. All full understanding involves discursive elements, even if recognition of the truth of axioms is, given sufficient understanding, "immediate".

Frege's view that incomplete understanding might impede obviousness of self-evident principles also explains why he thought that philosophical argumentation was worthwhile in helping people see that his axiomatic theory matches a proof structure that constitutes the order of justification in logic and mathematics. Like mathematical practice, philosophical argumentation deepens understanding. By associating arithmetical principles with general features of logic—chiefly universality of subject matter and relevance to understanding norms of thought and judgment—Frege hoped to improve our understanding of the kinship between logic and arithmetic, and sharpen our sense of logical, conceptual, epistemic priority. By distinguishing between sense and tone, sense and reference, function and object, he hoped to clarify our understanding of first principles.

As I have noted, I think that Frege's pragmatism and contextualism play another, secondary, role in his epistemology. They not only play a role in accounting for understanding. They provide a secondary, fallible, non-demonstrative justification. Frege seems to think that reflection on mathematical practice—which is hardly separable from much of what he counted philosophy—provides supplementary strong grounds to accept his definitions and his exposition of logical priority, even though a foundational proof structure, when fully understood, provides deeper grounds. This justification, which dominates *The Foundations of Arithmetic*, seems to be independent of the carrying out of proofs that display proof-theoretic justifications. It is hard to see coming to understand justificational priority of a truth, as opposed to mere mathematical acceptability of a proof, apart from such reflection on mathematical practice. Although Frege regarded justification implicit in the shape, stability, and fruitfulness of mathematical practice as secondary (and presumably he did not think of it as *strengthening* proof-theoretic justifications), it seems to have been for him a first defence against scepticism, and a sufficient one. I am inclined to think that his confidence on this score is the main reason why he never developed his conception of the relation between human understanding and self-evidence.

Department of Philosophy TYLER BURGE
University of California, Los Angeles
Los Angeles
California, CA 90024
USA

REFERENCES

Burge, Tyler 1984: "Frege on Extensions of Concepts, From 1884 to 1903". *The Philosophical Review*, 93, pp. 3–34.

——1990: "Frege on Sense and Linguistic Meaning", in D. Bell and N. Cooper (eds.) *The Analytic Tradition*, Oxford: Basil Blackwell.

——1992: "Frege on Knowing the Third Realm". *Mind*, 101, pp. 633–49.

Dummett, Michael 1991: *Frege—Philosophy of Mathematics*. Cambridge, MA: Harvard University Press.

Frege, Gottlob 1879: *Begriffsschrift* (B), I. Angelelli (ed.), Hildesheim; Georg Olms, 1964.

——1884: *The Foundations of Arithmetic* (FA), J. L. Austin (trans.), Evanston, Illinois; Northwestern University Press, 1968.

——1893, 1903: *Grundgesetze Der Arithmetik* (GG), Hildesheim: Georg Olms, 1962.

——1893, 1903: *The Basic Laws of Arithmetic* (BL), M. Furth (trans.), Berkeley: University of California Press, 1967

——1966: *Translations from the Philosophical Writings of Gottlob Frege* (GB), P. Geach and M. Black (eds.), Oxford: Basil Blackwell,..

——1967: *Kleine Schriften* (KS), I. Angelleli (ed.), Hidesheim: Georg Olms.

——1969: *Nachgelassene Schriften* (NS), H. Hermes, F. Kambartel, F. Kaulbach, (eds.), Hamburg: Felix Meiner.

——1976: *Wissenschaftlicher Briefwechsel* (WB), H. Hermes et al., (eds.), Hamburg: Felix Meiner.

——1979: *Posthumous Writings* (PW), P. Long and R. White (trans.), Chicago: University of Chicago Press.

——1980: *Philosophical and Mathematical Correspondence* (PMC), H. Kaal (trans.), Chicago: University of Chicago Press.

——1984: *Collected Papers on Mathematics, Logic, and Philosophy* (CP), McGuinness (ed.), Oxford: Basil Blackwell..

Heck, Richard, forthcoming, "Frege and Semantics", in Ricketts (ed.) *The Cambridge Companion to Frege*, Cambridge: Cambridge University Press.

Leibniz, Gottfried (written 1705; published 1765): *New Essays on Human Understanding*, P. Remnant and J. Bennett (trans.), Cambridge: Cambridge University Press.

Stanley, Jason 1996: "Truth and Meta-Theory in Frege". *Pacific Philosophical Quarterly*, 77, pp. 45–70.

Zermelo, Ernst 1908: "A New Proof of the Possibility of a Well-Ordering", in J. van Heijenoort (ed.), *From Frege to Gödel*, 1981, Harvard University Press: Cambridge, MA.