UNIVERSITY OF BUCHAREST
FACULTY OF PHILOSOPHY

PHD THESIS

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THE NATURE AND LOGIC OF VAGUENESS

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I owe my basic understanding of vagueness and theories of vagueness to Timothy Williamson's classic "Vagueness". Other heavy influences include the works of Kit Fine and Elia Zardini. Although I propose a different solution to the conundrum of vagueness, a good part of this PhD thesis can be read as a commentary on the work of these three great authors.

I am grateful for the trust of Professor Mircea Dumitru. His intellectual rigor and constant support led to this work being finished. I have much benefited from discussions with Radu Anghel, Mihai Bădiță, Dragoș Bîgu, Constantin Brîncuș, Ruxandra Călborean, Alexandru Dragomir, Matti Eklund, Laurențiu Leuștean, Lavinia Marin, Emilian Mihailov, Daniel Nica, Gheorghe Ștefanov, Paula Tomi, Constantin Vică, and Elia Zardini. My special thanks go to Tudor Glodeanu, with whom I discussed most of the arguments of this work, as I was writing it.

Contents

Introduction

I

Is a man of 180 cm tall? Is there such a thing as the southernmost rock of a mountain, say the Everest? If I remove one by one the grains making a heap of sand on the beach, is it possible to pinpoint the moment when there would no longer be a heap to speak of? These questions seem to be related. They seem to involve a quality common to *tall*, *mountain* and *heap* which we will call *vagueness*. This quality leads us to refuse judgment on their bearers' correct application. It also seems to append to them as bearers of meaning. Whatever underlies the uses of *tall* in English will (at least partially) underlie those of *alto* in Spanish, i.e. the property of vagueness seems to be borne by the same kind of entities which bear meaning. Thus, vagueness opens a window into theories of meaning and the relation between language and thought, topics pertaining to philosophy of language.

Were we to answer 'yes' to the first question, a reply would run along the lines of: 'Suppose another man was just a cm less tall. You surely would accept that this second man would also be tall. Then we could imagine a third man in the series and so on, up to the point where you would accept a 150 cm man as tall'. This paradox is known as the sorites. Each soritical step (e.g. from 180 cm to 179 cm, from 179 cm to 178 cm, etc.) can be formalized in uncontroversial classical logic, of the sort used in science and mathematics. Should we reject some of the premises of the paradox or should we alter the formal system used to derive the conclusion? It is the task of logic to answer this question, as to provide a model for reasoning with vague language.

Finally, since Everest surely stands for something real, philosophers asked whether reality is such that mountains have clear starting and ending points or whether any particle of air is in or out of a certain cloud. It is controversial whether we can call the real world vague. We would go against a long tradition holding that the ideal for language is to be precise in relation to reality i.e. a reflection, and I argue that the tradition is correct. However, vagueness thus bears also on metaphysics.

Our three questions have led us to topics pertaining to philosophy of language, logic and metaphysics. These are central fields of analytic philosophy, the tradition started by Gottlob Frege and Bertrand Russell. It seems that the main issue raised by vagueness is that common language, reasoning and experience of reality belie the classical theories developed for understanding science and mathematics.

II

To introduce the approach of this work, let us start with a simple statement of natural language: ‘An Englishman speaks English better than a Frenchman’. It compares two kinds of people, just as a soritical step for *tall* compares a man of 180 cm with a man of 179 cm. The statement seems quite true. The trouble is that my friend Jacques speaks English better than many, perhaps all, Englishmen. One possibility is then to say that the statement is false, but speakers take it as true because of their ignorance of Jacques. As applied to vagueness, since it seems that there is no Jacques-like counterexample to separate numbers of cm corresponding to short people from those corresponding to tall people, that ignorance should be *necessary*, in order to explain why vague soritical steps seem true. This is meant to illustrate the *epistemicist* approach, which maintains classical logic while blaming our impression of truth when confronted with falsities on ignorance. Another, more intuitive, possibility is to say that the statement is *partially* true. It is not completely true but it may be 99.2-point true on a scale from 0 to 100 points, where 0 and 100 stand for complete falsity and complete truth, respectively. This is meant to illustrate the *plurivaluationist* approach to vagueness, which abandons the classical dichotomy between true and false statements called *bivalence*. A third possibility is to say that speakers have not yet bothered to decide who is precisely English and French. Then, if forced to decide on their terms, the statement will be true if true under all possible (and reasonable) such interpretations. The case of Jacques proves that the statement is indeterminate, and even if it might be true under many interpretations of *Englishman* and *Frenchman*, it will still be indeterminate, as there are some other interpretations (including Jacques) under which it is false. This is meant to illustrate the *supervaluationist* approach to vagueness, which also abandons bivalence globally, while keeping it under complete interpretations. A fourth possibility is to deny that our statement has meaning: the fact that it seems so true, yet there are counterexamples to it, would prove that our language is incoherent. This is meant to illustrate the *nihilist* or *incoherentist* approach to vagueness. Finally, the argument of this work is to say that the statement has a hidden qualifier, taken as implicit in common language. In fact, it should read ‘In most cases, an Englishman speaks English better than a Frenchman’. Then, it is simply true, because within more than 50%, or even 90%, of all available pairs of Englishmen and Frenchmen, the former speak English better. On this model, vague soritical steps should be read statistically, as holding for 50%, 90% or 99% of cases. And vagueness is otherwise classical: each man is either tall or short, bivalence holds and there is no ignorance involved. Moreover, vagueness has some structural properties, to which we now turn.

III

The main thesis of this work is that vagueness is dispersion. Imagine that there are one million people at each number of cm of height. So one million of 190 cm, one million of 189 cm and so on. I hold that for 191, 190, 189 cm, all persons of that height are tall. But then, among the one million of 188 cm, there is a first short – not tall – person, most likely suffering of kyphosis. As we descend the ladder of cm of height, the proportion of short people increases. Up until, say, 170 cm and below where each million people are all short. We can say that the relation of *having an equal or smaller number of cm of height than* vaguifies the predicate of *being tall*. Thus, vagueness has a classical definition. It is there where a monadic predicate is distributed across the ranks of a total preorder such that, in brief, (1) there is an initial chain of ranks containing only negative cases of the predicate and a final chain of ranks containing only positive cases, (2) there are intermediate ranks of the relation such that negative and positive cases are intermingled, and (3) most elements are inside a *safe zone*, inside the initial negative chain or the final positive chain, such that most of the time tolerance holds. Higher-order vagueness can also be defined as dispersion at some smoothing of successive ranks in the ordering. As long as there are enough ranks of persons by height, you can have as many concentric rings as satisfy your intuition that a person may be n^{th}-determinately tall without being $n+1^{\text{th}}$-determinately tall.

The transition from the rank of 1,000,000 tall people to the rank of 999,999 tall people is as gradual as possible in a well-defined sense – because there are 1,000,000 in total at that rank. And it is less abrupt than the fuzzy solution of having all one million people of 188 cm fall by 0.01 to a degree of 0.99 tallness or the supervaluationist way of making the tallness of all 1,000,000 of 188 cm indeterminate.

There are three groups of arguments which support the main thesis. Firstly, an analysis of the sorites paradox reveals that it is a second-order paradox of two main premises, namely *divisibility* (e.g. '*large difference* can be split into finite *small differences*') and *predicate tolerance* (some predicate is hereditary in the *small difference* relation e.g. 'If a person of *n* cm is tall, so is a person of *n*-1 cm'). It can be expressed in classical first-order logic (*FOL*) if a finite upper bound on the number of elements e.g. of *humans*, is put. Secondly, FOL can express our definition of a vague predicate, using the same technique of bounding the number of humans to be able to speak of the proportion of negative to positive cases. Thirdly, a purely notational extension of FOL will allow us to speak of the *broad* predicate, that is, the union of the predicate with the *dispersion zone*, defined as the elements at intermediary ranks in the relation. And the *strict* predicate is their difference. Then, we can say that a weak *non-contradiction* (NC) fails (e.g. 'Some persons are both broadly tall and broadly short'), as does a weak *law of excluded middle* (LEM) (e.g. 'Some persons are neither strictly tall nor strictly short'), a weak *predicate tolerance* always holds ('If a person of *n* cm is strictly tall, a person of *n*-1 cm is broadly tall'), as does what I will call a weak *predicate-ordering monotony* (e.g. 'If a person of *n* cm is tall, a person of *n* cm or more is broadly tall'). They are defined classically, without any Kripke or fuzzy semantics.

A natural objection is that the connection between the predicate *tall* and the relation *to have as many cm of height or more* is severed under the theory of this work. Someone can have an equal or higher number of cm than some tall person without being tall themselves. I think that is because *tall*, like all natural language predicates, is multidimensional: height in cm, kyphosis, stoops, hair arrangement, and domicile go into in. Saying that a predicate is multidimensional, as vague predicates are often taken to be, is saying precisely that not a single dimension makes the difference in all cases. Without specifying any class of comparison, there is a good sense that there are now some tall persons domiciled in Asia of less cm than some short people domiciled in Northern Europe. Therefore, it is the vagueness of the common concept of *tall* which we need to study, and I aim to provide a framework of handling multidimensionality, without classes of comparison.

IV

This work is divided into three parts. The first part is an analysis of the sorites paradox, i.e. the paradox of the heap. The treatment of this ancient paradox is still the benchmark for theories of vagueness. I discuss its forms and how the arbitrary parameters in its popular formulations can be eliminated to get its general logical form. This general logical form is second-order, but expressible in FOL under a finite upper bound on the number of objects. Thus, FOL is justified as logic of vagueness. This part contains a distinction between soritical form and effect and, to my knowledge, the only classification of cases for which soritical form does not guarantee soritical effect. A large part of the discussion is then dedicated to separating metaphysical soritical arguments, which are not convincing, from semantic arguments which are paradigmatic for vague predicates. As stated, the general form of the paradox involves divisibility and predicate tolerance. An analysis of the former shows that it cannot be rejected reasonably, so it is understandable that philosophers focused on the latter. This first part engages with the work of Jonathan Barnes, Michael Dummett, Kit Fine, Gottlob Frege, P.T. Geach, Delia Graff, Dominic Hyde, Graham Priest, Nicholas J. J. Smith, Peter Unger, Timothy

Williamson, Crispin Wright, Elia Zardini, Pablo Cobreros, Paul Egré, David Ripley and Robert van Rooij and others.

The second part navigates the philosophical literature on vagueness. For a neutral definition of vagueness, I take it to be the study of the divergence of natural language reasoning from classical logic. An analysis of Frege's position on logic, natural language and vagueness serves as an introduction. Then, I compare philosophical theories which have been proposed for vagueness. They raise specific objections and three general issues. Can a non-classical logic for vagueness have a classical meta-language? Can it replace FOL as universal logic and, if not, express our natural language reasoning? Do such logics have more intuitive appeal than notational variants of FOL? To the list of cited authors, this second part adds Dorothy Edgington, Joseph Goguen, Rosanna Keefe, Kenton Machina, Francis Jeffry Pelletier, Brian Weatherson, and others.

The third part of this work argues for FOL as logic of vagueness. It first discusses what measurements are and how they can be captured by total preorders, e.g. the set of men plus an ordering of them by height in cm. FOL expressions to be used in defining vagueness are formulated, then I propose a definition for *vague predicate, gradually vague predicate* and for the so-called relatives of tall: broadly tall, strictly tall, ideally tall, probably tall, etc. These definitions help express the gradual, measurable and classical nature of vagueness. Finally, a discussion of higher-order vagueness and multidimensionality helps clarify the present approach to vagueness. To the list of cited authors, this third part adds Matti Eklund and Patrick Greenough.

Treating vagueness as dispersion has the apparent disadvantage of making some, even many, people of more cm than some tall people short. But it also has advantages. It partially coincides with standard epistemicism, being classical, but without claiming that there is a single number of cm separating tall and short people. It partially coincides with plurivaluationism in its capacity to express percentage-wise intermediate ranks, i.e. heights corresponding to both tall and short people, but without introducing intermediate truth values. It partially coincides with supervaluationism in expressing inter-definable *strict* and *broad* qualifiers, but without any alethic propositional operator such as *definitely*.

Part I

The sorites paradox

The sorites paradox takes its name from *sōrós*, the Greek name for heap, especially heap of corn[1]. The Ancient Greek paradox known as the Heap starts by stating that one grain of corn does not make a heap. And no difference of one grain can transform a non-heap into a heap. Therefore no number of grains can make a heap. But Greek philosophers discussed other paradoxes as well, for example one about numbers from Diogenes Laertius: “It is not the case that two are few and three are not also. It is not the case that these are and four are not also (and so on up to ten thousand). But two are few: therefore ten thousand are also”[2]. Let us analyze why these two arguments and others like them are sorites paradoxes and why other arguments are not.

Chapter 1

Common characteristics of the sorites

The sorites paradox is commonly[3] classified as having two main forms, the propositional and the mathematical induction form.

(CS) The conditional sorites:

(CS.m) A man of 200 cm is tall.	A
(CS.M.1) If a man of 200 cm is tall, so is one of 199 cm.	$A \supset M_1$[4]
(CS.M.2) If a man of 199 cm is tall, so is one of 198 cm.	$M_1 \supset M_2$
…	...
(CS.M.100) If a man of 101cm is tall, so is one of 100 cm.	$M_{99} \supset B$

[1] Henry George Liddell, and Robert Scott, *A Greek-English Lexicon, revised and augmented throughout by Sir Henry Stuart Jones with the assistance of Roderick McKenzie* (Oxford, Clarendon Press, 1940). Accessed Oct 14, 2019. http://www.perseus.tufts.edu/hopper/text?doc=Perseus:text:1999.04.0057:entry=swro/s.

[2] Diogenes Laertius, "Lives of the philosophers" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997), 58.

[3] Rosanna Keefe and Peter Smith, "Introduction: theories of vagueness" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997), 10-11.

[4] Propositional letters A, B and M_1, …, M_{99} are of the same type. Subscripts have no special significance. The same convention applies below. For the logical notation used throughout this work, see Appendix 2.

(CS.C) A man of 100 cm is tall.	B *Explanation:* From (CS.m) and (CS.M.1) by modus ponens we derive M_1. We then similarly use M_1 with (CS.M.2) to get M_2 and so on up to *B* from (CS.M.100).

(IS) The inductive sorites:

(IS.m) A man of 200 cm is tall. (IS.M) For any number of cm, if a man of that number is tall, so is a man of that number minus one.	Ta_{200} $\forall n, n \in \mathbb{N} . Ta_n \supset Ta_{n-1}$
(IS.C) A man of 100 cm is tall.	Ta_{100} *Explanation:* Mathematical induction on natural number *n*, with a_n indicating a male height of *n* cm.

There is, first, a *minor premise, indicated by* '.m', which states a certainty: that 200 cm make a tall man, that one million grains of corn make a heap, etc. Then a single *major premise* (in the case of the inductive sorites) or a finite number of premises called *steps* playing its role. Namely to express *tolerance.* They are indicated by '.*M*' above. *Tolerance* means that a small constant difference does not seem compatible with the switch from a negative to a positive case of the predicate in question[5]. The name 'tolerance' comes from the work of Crispin Wright who wrote:

> "… we encounter the feature of a certain *tolerance* in the concepts respectively involved, a notion of a degree of change too small to make any difference, as it were. There are degrees of change in point of size, maturity and colour which are insufficient to alter the justice with which some specific predicate of size, maturity or colour is applied"[6]

Then, we only need to understand *predicate* (here *tall* or *being tall*) as that unity of meaning corresponding to a property, that has the formal characteristic of forming a sentence together with

[5] In this work, when a transition from negative to positive cases is discussed, it is to be read as covering a symmetric transition from positive to negative cases as well, and the reverse.
[6] In this work I take it that the relation of difference is tolerant towards the monadic predicate, not the converse. Crispin Wright, "On the Coherence of Vague Predicates." *Synthese* 30, no. 3/4 (1975), 333.

an object, sentence which may be at least true or false[7]. Let us call the small constant difference from one step to another the *soritical step* (here *1 cm*) and the relation embedding it the *soritical relation* (i.e. *being taller by one cm*). Finally, the *conclusion*, extending the same treatment as that from the minor premise to a very divergent case is meant to be what is paradoxical about the argument.

Sorites paradoxes are classified as *descending*, when the minor premise has a larger quantity than the conclusion (as above) or *ascending,* when the converse holds. For example, a sorites for *short* started from 'A man of 100 cm is short' and ending with 'A man of 200 cm is short'.

Note that the sorites can be converted from ascending to descending by using the *complement* of the predicate in question. *Tall-short, bald-hirsute, heap-non-heap* are standard pairs of a predicate and its complement, which I will call *doublets*. The ease of their reciprocal replacement in the sorites had been often remarked in the literature[8]. At the very least, this characteristic guarantees that soritical predicates (*tall*, *bald,* etc.) have both positive and negative cases (those covered by their complement) in the natural world, either past or present. We will say that they are *naturally distributed*.

Therefore, a *soritical predicate* is any naturally distributed predicate from which a convincing sorites can be built. And we already have a first understanding of what it is to be a *vague context,* namely to be an utterance containing a soritical predicate.

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[7] Since the advent of modern model theory, predicates or properties are defined as the set of their positive cases see for example Tarski "[…]in the opinion of numerous logicians, it is unnecessary to distinguish at all between the concept of a class and that of a property". Alfred Tarski, *Introduction to Logic and to the Methodology of Deductive Sciences* (New York. Dover Publications. 1995), 67. I use here the least theory-laden notion that is compatible with modern logic. That is, going back to a pre-Frege reading of *predicate* as 'that which is spoken of the subject' would be pseudo-scientific today. See also Richard Heck and Robert May, "The Function is Unsaturated" (2013). Forthcoming in *The Oxford Handbook of Analytical Philosophy*, ed. Michael Beaney. Accessed Oct 14, 2019. https://pdfs.semanticscholar.org/9bde/a4a9212009070bc9617673517b44a9180ff7.pdf, 10-11.

[8] E.g. Sainsbury: "Boundaryless concepts tend to come in systems of contraries: opposed pairs like child/adult, hot/cold, weak/strong, true/false, and the more complex systems exemplified by our color terms". R. M. Sainsbury, "Concepts without boundaries" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997), 258.

Chapter 2

Propositional and inductive sorites

2.1. Four rules of inference

The propositional form above can be further classified according to the rule of inference it employs:

(cs) Conditional, uses *modus ponens:*

If A_1 then A_2; $A_1 \therefore A_2$.

If A_2 then A_3; $A_2 \therefore A_3$.

etc.

(js) Conjunctive, uses *modus ponendo tollens*

Not both A_1 and not A_2; $A_1 \therefore A_2$.

Not both A_2 and not A_3; $A_2 \therefore A_3$.

etc.

(ds) Disjunctive, uses *modus tollendo ponens*

not A_1 or A_2; $A_1 \therefore A_2$.

not A_2 or A_3; $A_2 \therefore A_3$.

etc.

(ns) Negative, uses *modus tollens*

If A_2 then A_3; not $A_3 \therefore$ not A_2.

If A_1 then A_2; not $A_2 \therefore$ not A_1.

etc.

These Medieval Latin names can be translated as 'affirming by affirming', 'denying by affirming', 'affirming by denying' and 'denying by denying'. For example:

a) Modus (ponendo) ponens. *If a then b; a ∴ b:*

If you are honest, you are poor; You are honest ∴ You are poor.

b) Modus ponendo tollens. *Not both a and b*; *a ∴ not b:*

It is false that you are honest and rich; You are honest ∴ You are poor.

c) Modus tollendo ponens. *a or b; not a ∴ b:*

You are not honest or you are poor; You are honest ∴ You are poor.

d) Modus (tollendo) tollens. *If a then b; Not b ∴ Not a:*

If you are honest, you are poor; You are not poor∴ You are not honest.

These rules of inference were recognized long before the rise of modern logic in the 19th century[9]. Under the classical truth-functional definition of ⊃, ∧, ∨, they are equivalent. This can be written in the object language using the material biconditional:

(ModeEquiv) $(A_1 \supset A_2) \wedge A_1 \leftrightarrow \neg(A_1 \wedge \neg A_2) \wedge A_1 \leftrightarrow (\neg A_1 \vee A_2) \wedge A_1 \leftrightarrow (\neg A_2 \supset \neg A_1) \wedge A_1$

However, taking this equivalence as definitive would be prejudging the issue. Is classical logic the appropriate logic for vague contexts? Suffice it to say that as long as each such rule is intuitively valid, we need to take it into consideration.

2.2. The negative form and double negation elimination

The negative form (i.e. modus tollens) is missing from the current philosophical debate, although it is a simple way of converting an ascending to a descending sorites or the converse. For example you can create (CS.2) from (CS) above, while only replacing (CS.m) with a secondary minor premise:

(CS.2.m) A man of 100 cm is not tall.

Using modus tollens as rule of inference, it will lead to the paradoxical:

(CS.2.C) A man of 200 cm is not tall.

One reason for this neglect is that philosophers have focused on the forms of the major premise(s), which the negative form shares with the popular conditional formulation. But the negative form also helps convert a descending sorites for one soritical predicate to an ascending sorites for its complement. You can create (CS.3) from (CS) above, while replacing (CS.m) with:

(CS.3.m) A man of 100 cm is short.

and replacing 'tall' with 'not short' in all of (CS.M.1- CS.M.100). Then by modus tollens you get:

(CS.3.C) A man of 200 cm is short.

This shows that we can understand all forms of the sorites without the elimination of double negation. It has been claimed[10], for example, that in (js) above, the first intermediate conclusion under modus ponendo tollens is in fact $\neg\neg A_2$, where double negation needs to be eliminated to get A_2

[9] E. J. Lemmon, *Beginning Logic* (Chapman and Hall/CRC, 1971), 61.
[10] Kit Fine, "The possibility of vagueness", *Synthese* 194 (10) (2017), doi: 10.1007/s11229-014-0625-9, 3711.

for reuse with the next step. However, as the modus ponendo tollens example at 2.1.b) above shows, we have the alternative to use the complement of the predicate in question. So the elimination of double negation not is necessary if 'not poor' is interchangeable with 'rich' and 'not rich' with 'poor', inside each step of the inference. And we use in common reasoning such a replacement rule, viz. the popularity of doublets[11].

2.3. Doubting the conditional form

Of the four propositional variants above, the conditional and the conjunctive sorites have always been standard. Ancient Greek philosophers gave a preference to the conjunctive form, because it comes closest to expressing tolerance: no difference one soritical step apart. The Ancients treated the conditional form ambivalently. In a Philonian reading of *if then*, namely what is now known as the material conditional (⊃), they recognized its equivalence with the conjunctive form. But the common reading of the conditional was the Chryssipan, which Williamson sees as our strict conditional: it is impossible to have the second, given the former[12].

However, in modern times, the vast part of the debate concentrated on the conditional form. This may be due to the central status of the material conditional in modern logic. It was taken as primitive – together with negation – in Frege's "Begriffsschrift"[13]. And all modern natural deduction systems since Gentzen introduce an early elimination rule for material implication that corresponds to modus ponens[14]. In contrast, the other three rules of inference are proved as theorems in such systems.

A recent argument by Kit Fine denies the plausibility of the conditional form of the major premise(s), thus of both the conditional and negative forms. Fine's argument is based on comparing what he takes the conditional and conjunctive forms to express, conditions he calls, respectively, '*Tolerance*' (a different notion that our tolerance introduced in Chapter 1) and '*Cut-Off*':

[11] Dominic Hyde, "The Sorites Paradox", in *Vagueness, A Guide*, ed. Giuseppina Ronzitti (Springer Verlag, 2011), 5.
[12] For this paragraph, see Timothy Williamson, *Vagueness* (London, Routledge, 1994), 24. For strict conditionals see Graham Priest, *An Introduction to Non-Classical Logic: From If to Is* (Cambridge University Press, 2008), 72.
[13] Gottlob Frege, "Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens"; English translation by Stefan Bauer-Mengelberg, in *From Frege to Gödel, A Source Book in Mathematical Logic, 1879-1931*, ed. Jean Van Heijenoort (Harvard University Press, 2002), 13-20.
[14] See E. J. Lemmon, *Beginning Logic*, 9. Also Graeme Forbes, *Modern logic: A text in elementary symbolic logic* (Oxford University Press, 1994), 92.

> "Tolerance permits us to make the transition from the truth of the antecedent claim (that Left is bald) to the truth of the consequent claim (that Right is not bald)[sic][15], while Cut-Off forbids us from making the transition from the truth of the antecedent claim to the falsity of the consequent claim. But to say that we are forbidden (given the truth of the antecedent claim) to assert the falsity of the consequent claim is not to say that we are permitted to assert the truth of the consequent claim unless we also take for granted that the consequent claim is either true or false".[16]

That Fine makes a play here on a non-material reading of the conditional is obvious. Asserting the conditional major premise(s), he says, is saying that the consequent can be asserted when the antecedent is true. While the conjunctive major premise(s) says only that the consequent must not be denied in that situation. After limiting his discussion to the formulation of major premise(s) themselves, he will go on to distinguish their rules of inference, seeing modus ponendo tollens fail while modus ponens keeps its validity. It is a strange logic that in which inferences which are weaker on their intended reading fail while stronger ones do not[17]. However, Fine's argument can be shown to be incompatible with the interchangeability of soritical doublets. Reformulate his quote above as:

(KF.C.1) It is true that we have tallness of Right, given tallness of Left. It is false that we have falsity of tallness for Right, given tallness of Left.

Truth being eliminable and replacing falsity of tallness with shortness:

(KF.C.2) We have tallness of Right, given tallness of Left. It is false that we have shortness of Right, given tallness of Left.

Replacing 'it is false that …' with 'the falsity of … is true':

(KF.C.3) We have tallness of Right, given tallness of Left. It is true that we have falsity of shortness of Right, given tallness of Left.

Truth being eliminable and replacing falsity of shortness with tallness:

(KF.C.4) We have tallness of Right, given tallness of Left. We have tallness of Right, given tallness of Left.

[15] This seems to be a typo, it is of course to be read as 'that Right is bald'.
[16] Kit Fine, "The possibility of vagueness", 3713.
[17] A discussion of Fine's recent compatibilist semantics is in Chapter 12.

Replacing falsity of tallness with shortness and falsity of shortness with tallness is the crucial move. This is stronger than our previous replacement of 'not tall' with 'short' and 'not short' with 'tall'. But it can be justified by the fact that Fine equated outer negation with a metalinguistic interdiction while inner negation was taken as falsity. The form of premise is '*Not(A and Not B)',* and the reason Fine has to read the two '*Not'* particles differently must also extend reading '*Not(tall)'* as '*short'* if we were reading '*Not tall'* as '*short'* already. Fine's intention is to affirm neither tallness nor non-tallness for Right. But affirming tallness for Left forces him to eliminate the first half for Right, getting something like: 'Given Left is tall, I will deny shortness for Right, but I keep the possibility that Right is not tall'. But by doublet interchangeability, that would be tantamount to contradiction[18].

In conclusion, if interchangeability of a soritical predicate with the negation of its complement is accepted, as we do all the time, the conditional form of the sorites is equivalent to the conjunctive form.

2.4. The disjunctive form, predicate-ordering monotony and LEM

There is also an argument against the disjunctive form. Fine proceeds in four steps[19]:

(KF.D.1) The sorites is about a kind of indeterminateness of predicates.

(KF.D.2) Indeterminateness means at least that the Law of Excluded Middle (LEM) is not plausible for such predicates.

(KF.D.3) Saying 'Either a man of 200 cm is not tall or one of 199 cm is so' is an even stronger statement than LEM for the first step predicate, that is 'Either a man of 200 cm is not tall or a man of 200 cm is tall'. The reason is that the first disjunct is identical and the second disjunct of the former statement is stronger than the second disjunct of the latter, because by the principle that a man of more cm will be tall provided another of less cm is, it implies it.

(KF.D.C) Therefore, the disjunctive soritical major premise(s) is not plausible.

[18] This interchangeability is of course a way to reject 'A is not B and not not-B' as intelligible otherwise than as a contradiction. For an opposing view, see Nicholas J. J. Smith, *Vagueness and Degrees of Truth* (Oxford University Press, 2008), 260-261.
[19] Kit Fine, "The possibility of vagueness", 3712.

First of all, the third step assumes that all tall people have more cm than all short (not tall) people. This is an extremely common assumption of philosophers of vagueness, traceable to the *internal penumbral connections* in Fine's 1975 first approach to the topic[20]. I will call this *predicate-ordering monotony,* as it says that the distribution of the property among the objects is monotonous with the ordering: as the ranks of cm increase, there can be no short person at the same or previous rank with a tall person. But no formal rule requires it: *tall* is a monadic predicate and *having more cm*[21] is a relation i.e. binary predicate, they are assigned independently. I find it obvious that, as things stand in the real world, there are now many short persons (say some men in New York) of more cm than some tall persons (say some women of Seoul), i.e. without specifying a class of comparison. Or that someone with an impressively high hair arrangement would be tall even if they have lower scalp-to-toe measurements than a stooped short person[22]. I will discuss this in the third part of this work. Be that as it may be, the correct principle at work in step c) above is 'No difference one cm apart', either ascending or descending, what Fine calls 'Cutoff'. Therefore the disjunctive statement may be statistically stronger, but it is deductively as strong as LEM.

More importantly, LEM is not especially counter-intuitive at soritical predicates. Near the ends, LEM should not cause any trouble to Fine. Since men of 200 cm are tall, we can infer that they are either tall or short. May LEM be then a problem at doubtful cases of in-between heights, as it has been often claimed[23]? First, suppose John believes in Fine's Cutoff principle, expressed as:

(NoCutoff) There is no way that being one cm apart from a tall person is compatible with shortness.[24]

And suppose Mary believes in:

(TallRich) There is no way that being tall on Wall Street is compatible with poverty.

Persons in Mary's situation often reason disjunctively with soritical predicates:

(TallRich.M) On Wall Street, you are either not tall or you are rich.

[20] "If Herbert is to be bald, then so is the man with fewer hairs on his head." Kit Fine, "Vagueness, Truth and Logic", *Synthese*, Vol. 30, No. 3/4 (1975), 276.

[21] The same argument applies to *tall* and *taller*, the grammatical appearance is not a logical rule. See Chapter 16.

[22] "Now 'taller' does indeed seem more precise than 'tall'. But it does not seem perfectly precise; stoops and curly scalps may produce borderline cases even for it." Timothy Williamson, *Vagueness*, 156.

[23] "The failure of excluded middle may seem natural enough in borderline cases" Timothy Williamson, *Vagueness*, 118.

[24] Fine's formulation is "If two cases are sufficiently alike then it is not the case that the first is bald and the second is not". Kit Fine, "The possibility of vagueness", 3713. This will be argued in Chapter 17 to be either circular or vacuous.

And if (TallRich.M) is a natural expression of belief in (TallRich), a belief in (NoCutoff) leads to:

(NoCutoff.M) Of persons one cm apart, either one is not tall, or the other is tall as well.

I do not deny that LEM is indeed rare in common speech. People do not say unprompted that right angles are over 90 degrees or not, be the predicate determinate or not. But it is precisely when principles force them to choose in doubtful cases i.e. when common language reasoning gets going, that it gets used. Love is a very doubtful notion, but 'You either love me or you do not' is the commonest LEM utterance there is[25]. Since both LEM and disjunctive syllogisms are commonly used with soritical predicates in doubtful cases, what basis would someone have to not accept (NoCutoff.M) given (NoCutoff)? None, it seems.

To conclude, a logic for vague reasoning needs to block the sorites for all four rules of inference.

2.5. The inductive sorites and tolerance

Since its reported invention by Eubulides in 4th century BC Greece to the present time, the propositional form has been the standard way of presenting the sorites, shortly followed by the moral: no one centimeter can make the difference. That moral is a (negated) variant of (IS.M) from the inductive form. For example Galen in his "On medical experience", after going through the individual steps, states: "And I know of nothing worse and more absurd than that the being and not-being of a heap is determined by a grain of corn"[26]. So we only go through the individual steps in order to affirm this moral, which expresses tolerance.

But using the inductive first-order version was argued to be unnecessary. This the import of Michael Dummett's classical "Wang's paradox" from 1975, that deals with a sorites for *small number* with a soritical step of *1* and '0 is small' as minor premise[27]. If anyone attacks the inductive form i.e.

[25] For a critical viewpoint of such statements, see J. A. Goguen, "The Logic of Inexact Concepts", *Synthese*, Vol. 19, No. 3/4 (Apr., 1969), 360.

[26] Galen, "On Medical Experience" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997), 58.

[27] The conclusion in Dummett's presentation is stronger: "every number is small", seen as a long conjunction of the conclusions of many standard soritical constructions. Michael Dummett, "Wang's paradox" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997), 101. For historical context, see Crispin Wright, *Wang's Paradox*, Draft of paper for Dummett LLP volume, accessed Oct 15, 2019 https://as.nyu.edu/content/dam/nyu-as/philosophy/documents/faculty-documents/wright/Wright-Crispin-dummett.pdf.

induction for such soritical predicates, it can be replaced by a succession of premises such as (CS.M.1-100), with no appeal to 'induction as a principle of inference'[28].

But there are some particularities to our (CS) argument above. The propositional form, such as a long succession of modus ponens, is shared with many other sound and unsound arguments[29]. Also, it does not express the logical relationship between the arbitrary parameters. Those are the precise initial and final quantity (here *200* and *100*), the precise soritical step (here *1*) and the precise number of steps to get between them (here *100*).

The inductive form does away with the last of these. The first-order transcription of the inductive form makes use of first-order arithmetic. It assumes the standard definition of natural numbers and it states the inductive property for any $n \in \mathbb{N}$. This means that 'a_n' in the 'Ta_n' of (IS.M) is referring to a height, not to persons[30]. It is assumed that there is such a height for each natural number such that the successor relationship holds. Then, the soritical property is hereditary in the successor relation[31]. This means that if a height of a number has it, so does a height of the subsequent number. This dispenses with the need of going through the one hundred steps.

It is illuminating to reflect on an inductive form for a non-integer soritical step, such as π/4 cm[32]. Since induction is made on $\mathbb{N}$, the definition of 'a_n' would need to be modified to include multiplying π/4 by *n*. The starting and ending point would change too. If started at 64 π (201.0619…) cm, the end might be, for example, 32 π (100.5309…) cm. If started at 200 cm as before, the end might be 200-32 π (99.4690…) cm.

We see that the size of the soritical step determines the number of steps and the end point, given the starting point. But not even the inductive form captures logically this relation. In contrast, it says too

[28] Michael Dummett, "Wang's paradox", 102.
[29] See 6.1 below.
[30] You cannot say that *n* ranges over $\mathbb{N}$ and then say that a_n in (IS) above refers to a person without specifying how each person is connected with $\mathbb{N}$, as for example postulating an ordered set in which each person corresponds to a certain number. To wit, Hyde speaks of the inductive successor relation as the "addition of one hair" to a person, but this runs directly into the problem of what other properties of the respective person are affected, which we'll discuss in 5.3.4 below. Dominic Hyde, "The Sorites Paradox", 13.
[31] For the first definition of hereditary property, see #24 formula 69 in Gottlob Frege, "Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens", 55. He uses it in #27, formula 81, at page 62. The latter is called by Michael Beaney "the key step of mathematical induction" in Gottlob Frege, "Begriffsschrift" in *The Frege Reader*, ed. Michael Beaney, 76. Incidentally, Frege's discussion of formula 81 is one of the first modern discussions of the sorites. See presentation at 11.3 below.
[32] While measurements are rational numbers and the rational/irrational distinction does not affect my argument, you can give a sense to walking π meters by saying you walked half the circumference of a circle drawn on asphalt with a radius of one meter. Walking 10 π meters would be walking that circumference five times and so on.

much: the principle that π/4 cm cannot make a difference is not the same as the principle that the number one cannot make a difference if multiplied by π/4. The former has wide acceptability, the latter less so. Since the inductive form is both too weak – it does not capture all logical relations – and too strong – it uses a powerful technique on heights instead of people, for relatively little benefit, there may be a good kernel that needs to be retained.

That kernel is *predicate tolerance*: two people, not heights, one descending cm apart from each other, cannot be one a negative and the other a positive case of the predicate. We keep the hereditary nature of *tall* 'T', but note by 'S' the relation of being one cm taller, not the successor relation:

(PredTol) $\forall xy \,.\, Sxy \supset \neg(Tx \wedge \neg Ty)$[33]

The same logical form can be given to predicate-ordering monotony, with 'R' being the relation of having the same number of cm or less:

(PredOrdMonotony) $\forall xy \,.\, Rxy \supset \neg(Tx \wedge \neg Ty)$

Thus the difference between predicate tolerance and predicate-ordering monotony is that while predicate tolerance is *weakening* the predicate application i.e. the relation is from more cm to less cm for *tall* or from less hairs to more hairs for *bald*, predicate-ordering monotony is *strengthening*, i.e. from less or equal number of cm to equal or more cm for *tall* and from more or equal number of hairs to equal or less hairs for *bald*. The former creates the sorites paradox, while the latter is usually taken as obvious, despite clear counter-examples e.g. if you have a premature aging syndrome, you will be old at an age where most others are young. For now, the difference between them shows itself at the step of formalization (i.e. saying that 'S' and 'R' stand for positive or equal-to-negative difference in cm), not logical form. However, taking any of them as a total preorder, such as ≤ (read 'to be of lower or equal number of cm') will allow having an unitary treatment for both, as I will do in Chapters 14-16.

The succession of conditional steps in propositional (CS) expresses this predicate tolerance – at each difference of one cm – in propositional logic. Since all cases from 200 cm to 100 cm are covered by the relation between each antecedent and consequent, why work on an above zero-order version of the paradox? As Fine writes: "Let us focus on the sentential rather than the quantificational versions of the soritic arguments, since I doubt that the quantificational versions give rise to any essentially

[33] It can be checked that this is classically equivalent with Frege's formulation: $\forall xy \,.\, Sxy \wedge Tx \supset Ty$.

new issues"[34]. There are two problems with that. Firstly, we will see below that the argument can be formulated in different first-order versions which are subtly but importantly different. Saying that if some man of 187 cm is tall, there is a man of 186 cm which is also tall is different from saying that if all men of 187 cm are tall, all men of 186 cm are tall. The two readings correspond to different theories of how soritical predicates work. Secondly, (PredTol) states a relation between predicates i.e. sets of elements or units of meaning, while (CS.M.1) states a relation between the truth values of two sentences. The philosopher working on the propositional sorites gives up expressing relations such as that between the start point, end point and numerical step, and (PredTol) itself in the object language, hence the debate is confined to the metatheory.

A similar objection applies to Graham Priest's "Inclosures, Vagueness, and Self-Reference", where an analysis of the sorites paradox in terms of the Inclosure Schema[35] is proposed. Priest does not express the relation of *being 1 cm less tall*, making do with the predicate P, a set A for objects called 'a_0', ..., 'a_n' with Pa_0 and $\neg Pa_n$ true, and a set Ω of all P objects. He then gets the paradox by speaking of "a first member of A not in [Ω]". But A is a set, so not ordered under his definition, so he cannot speak of the "first member" in any case. For the ordering to come into the picture, relations (dyadic predicates) would need to be introduced in the object language and vagueness would be revealed as the interplay of a predicate and a relation, as I propose in this work.

In conclusion, the mathematical induction form expresses better the essence of the paradox, but the ease of working with the propositional form has made the latter by far the most popular. However, that essence i.e. predicate tolerance has a proper expression in classical first-order logic, being the same with the notion of a predicate being hereditary in a relation.

[34] Kit Fine, "The possibility of vagueness", 3710.

[35] The Inclosure Schema is "1. There is a set Ω such that $\Omega = \{x : \phi(x)\}$, and $\theta(\Omega)$ (*Existence*); 2. If $X \subseteq \Omega$ and $\theta(X)$, (a) $\delta(X) \notin X$ (*Transcendence*); (b) $\delta(X) \in \Omega$ (*Closure*);". For the sorites, Priest takes $\phi(x)$ to be Px, $\theta(X)$ to be a vacuous condition (self-identity), Ω to be a subset of a soritical series $A = \{a_0, ..., a_n\}$ of Pa_0 and $\neg Pa_n$, $\delta(X)$ to be the first member of A not in X. Graham Priest, "Inclosures, Vagueness, and Self-Reference". *Notre Dame Journal of Formal Logic* (Volume 51, Number 1, 2010), 70-71.

Chapter 3

Chaining

The sorites is a finite *chained* paradoxical argument. Chaining means that each successive step produces an intermediate result that is used in the next step, up until the final paradoxical conclusion. An illustration for our propositional (CS) derivation would be:

$$
\begin{array}{llllll}
\underline{A, A \supset M_1} & & & & & \\
\hline
M_{1}, & \underline{M_1 \supset M_2} & & & & \\
\cline{2-2}
 & M_{2}, & \underline{M_2 \supset M_3} & & & \\
\cline{3-3}
 & & \dots & \dots & & \\
\cline{4-4}
 & & & M_{99}, & \underline{M_{99} \supset B} & \\
\cline{5-5}
 & & & & \mathit{B} &
\end{array}
$$

Premises are underlined. The single conclusion is in *italic typeface*. The other formulas are derived by modus ponens to be reused in a further modus ponens. What is important to note is that premises only include *A* and the conditional statements i.e. only they need to command the acceptance of proponents. The rules of inference then take over and deliver the conclusion. The structural meta-rule for any logic to allow chaining is known as the *Cut rule*. It says that if formula A is a consequence of a set (including empty set) Γ of formulas, and formula B is a consequence of A and a set Δ taken together, then formula B is a consequence of sets Γ and Δ taken together[36].

If you refuse to chain this argument and do it as a set of one hundred separate arguments, you need to command acceptance for each intermediate premise, including something like M_{49} viz. 'A man of 150 cm is tall'. Such acceptance seems at first implausible, because one cannot accept M_{49} independently. Then chaining would be essential for the sorites. However, Williamson raises the following challenge: suppose someone has already accepted arguments concluding in such a statement as M_{49}. Is not there a metalinguistic rule that, if one accepts arguments leading to some conclusion, that conclusion can be reused as a premise for further arguments?[37] That is, if you lead

[36] Timothy Williamson, *Alternative Logics and Applied Mathematics.* Draft of 5 March 2018. Accessed Oct 14, 2019 http://media.philosophy.ox.ac.uk/docs/people/williamson/appliedmaths.pdf, 13.
[37] Timothy Williamson, *Alternative Logics and Applied Mathematics*, 15.

an interlocutor to a conclusion, he should accept it not only as it is, but also its handling in further argument[38]?

Suppose there is no such metalinguistic rule and natural language reasoning can be captured by a number of single classical arguments. In Chapter 12 I will discuss Elia Zardini's non-transitivism, a substructural solution to block the sorites by restricting the Cut rule for such arguments. It accommodates a limited span of inference, just not all one hundred (or more, or less) steps. The disadvantage is that it cannot accommodate the precise nature of the divisibility relation between the span length and the soritical step, because it blocks too great a part of reasoning with soritical predicates. To see why, note that since the Middle Ages 'sorites' has also been the name for chained Barbara syllogisms of the form: *All A are B, all B are C, all C are D, all D are E ∴ All A are E*[39]*.* These are not sorites arguments in the sense of this work because of the lack of a numerical soritical step and the accompanying major premise(s). But we can require the logic that blocks the sorites paradox not to block any such simple chained argument containing soritical predicates. For example, a modification to logic that blocks (CS) must not block:

Men of 188 cm are tall; Tall men are athletic; Athletic men are fast ∴ Men of 188 cm are fast.

And neither longer Barbara polysyllogisms. If the rules of inference are to be modified, it should be done in a principled limited way, the principle being that only problematic i.e. contradictory instances of the sorites should be blocked and nothing else. This is a test substructuralists fail, because they cannot distinguish between a reasoning from *tall* to *athletic,* which is not paradoxical, from a reasoning from *tall* to *tall*, which might be. This point makes any alteration of chaining unpalatable.

As Dummett puts it: '*...to deny that, in the presence of vague predicates, an argument each step of which is valid is necessarily itself valid [...] seems, however, in turn, to undermine the whole notion of proof (=chain of valid arguments), and, indeed, to violate the concept of valid argument itself'*[40] .

[38] Such a rule is assumed by the *forced march* variant of the sorites, which is only alluded to in Chapter 8.
[39] Timothy Williamson, *Vagueness*, 31.
[40] Michael Dummett, "Wang's paradox", 103.

Chapter 4

The zero case and threshold arguments

Some[41] cite Genesis 18 as containing the first sorites. Abraham asks God about Sodom:

> "'24 Will you really sweep it away and not spare the place for the sake of the fifty righteous people in it? 25 Far be it from you to do such a thing—to kill the righteous with the wicked, treating the righteous and the wicked alike. Far be it from you! Will not the Judge of all the earth do right?'
>
> 26 The Lord said, 'If I find fifty righteous people in the city of Sodom, I will spare the whole place for their sake.'
>
> 27 Then Abraham spoke up again: 'Now that I have been so bold as to speak to the Lord, though I am nothing but dust and ashes, 28 what if the number of the righteous is five less than fifty? Will you destroy the whole city for lack of five people?'
>
> 'If I find forty-five there,' he said, 'I will not destroy it.'[...][42]
>
> 32 Then he said, 'May the Lord not be angry, but let me speak just once more. What if only ten can be found there?'
>
> He answered, 'For the sake of ten, I will not destroy it.'
>
> 33 When the Lord had finished speaking with Abraham, he left, and Abraham returned home."[43]

This is a good example of enjambment of a number of paradoxical forms. First, we seem to have an argument about the justice of God: "Will not the Judge of all the earth do right?" It suggests Euthyphro's dilemma in Plato's eponymous dialogue[44] i.e. should God respect justice. Less problematically, it can be read as stating that justice is not compatible with fifty righteous people being destroyed, thus acting as a soritical minor premise along the lines of (CS.m) above.

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[41] Timothy Williamson, *Vagueness*, 276, following Barnes.
[42] The skipped verses include three further downwards steps, one of five, two of ten persons each.
[43] Genesis 18 (New International Version).
[44] Socrates: "The point which I should first wish to understand is whether the pious or holy is beloved by the gods because it is holy, or holy because it is beloved of the gods". Plato, *Euthyphro*, 10a, The Project Gutenberg EBook Accessed Oct 14, 2019. http://www.gutenberg.org/files/1642/1642-h/1642-h.htm

Under this reading, it was proposed that there is a sorites for 'not just to be destroyed' with a soritical step of five, then ten people. But the argument is not properly soritical, and not only because of the unequal soritical step. It is not paradoxical. It stops before getting to a clearly unacceptable conclusion. That is, Abraham's questions do not go beyond ten people. And the subsequent verses suggest that there was only one righteous person, namely Lot. Moreover, he and his family are saved.

There are however authors[45] for which accepting soritical judgement means accepting the application of the soritical predicate even in the *zero case*. Let us call a *zero case* a man of zero cm (for *tall*) or a head of zero hairs (for *hirsute*). These two zero cases are clearly different. While a zero-hair man is not hirsute, there are such men. But there cannot be zero cm men. Therefore, in the case of the zero case for *tall* the paradoxical character of:

(TallZero.C) A man of zero cm is tall.

is due to the fact that such a description is vacuous. That is, its negative form:

(TallZero.m.2) A man of zero cm is not tall.

would be false as well, under a theory of general descriptions similar to the analysis given by Bertrand Russell to indefinite and definite descriptions[46]. In common speech, saying that men of zero cm are not tall would be saying that such people exist and that they are all not tall. But the first conjunct is false. If both (TallZero.C) and (TallZero.m.2) are false, they are not contradictory and we can live with this. Such a reading of indefinite descriptions is intuitive for natural language[47], and there is not much benefit in making common reasoning work with vacuous predicates. Just as we have required of soritical predicates to be naturally distributed, we can require of each step in a sorites to use only predicates which are naturally distributed, i.e. have both positive and negative cases in the natural world, either past or present.

Of course, this applies not only to the zero case, but also to cases that go beyond zero, call them *minus cases*. Negative-centimeter or negative-hair men are unacceptable for soritical form, unlike negative-

[45] E.g. Crispin Wright, "Language-mastery and the sorites paradox" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997), 151.

[46] Bertrand Russell, "On Denoting", *Mind*, New Series, Vol. 14, No. 56: 479-493 (Oct., 1905).

[47] Peter Ludlow and Stephen Neale, "Indefinite Descriptions: In Defense of Russell", *Linguistics and Philosophy* Vol. 14, No. 2 (Apr., 1991).

fortune people, namely people in debt[48]. To return to the argument between Abraham and God, the zero or minus case would fail, were it invoked, because zero or negative people cannot be saved.

The third, and best, way to look at the Genesis fragment above is as what we can call a *threshold argument*. There is some number of righteous people that tips the balance of justice towards mercy. It is lower than fifty. The balance of justice is not tolerant i.e. there may be grave punishments inflicted for behavior marginally worse than what goes unpunished. As a last Biblical reference, note that Lot's wife is to be saved just to then be turned into a pillar of salt for looking back towards Sodom, a crime that might not seem objectionable. The point is that there is a threshold between the number too high to destroy and that low enough to destroy and such a threshold is not unacceptable.

The classical Greek threshold paradox is Zeno's *grain of millet* paradox: one grain of millet does not make a sound, two grains of millet do not make a sound, and therefore ten thousand grains of millet fail to make a sound[49]. There are authors who cite this as a sorites and authors who do not[50]. It is empirically not a sorites. A sonograph can detect the sound made by a single grain and it seems like the only reason for the lack of sound of few grains is the existence of a sensorial threshold: any sound pressure below some number of micropascals cannot be heard by humans. That threshold may be person-specific, but it is there. In the case of sound, we accept a universal standard of measurement and the idea of a sensorial minimum. They lead us to reject any type of major premise(s) along the lines of:

(Millet.M) If n grains do not make a sound, $n+1$ grains do not make a sound.[51]

It has been claimed that *all* sorites paradoxes are what I called *threshold arguments*. Epistemicists such as Williamson and Roy Sorensen claim there is a switching point or *cut-off*, albeit unknowable, between the negative and positive cases of *tall*[52]. But that does not mean that all threshold arguments are soritical as well. For example, this argument:

[48] See also *(Material)* in Chapter 5 for an acceptable negative number, which fails of soriticality for a different reason.

[49] Jonathan Barnes, *The Presocratic Philosophers* (Routledge. 1982), 203.

[50] Barnes himself first took the millet paradox to be the same as the sorites, but then changed his mind in Jonathan Barnes, "Medicine, Experience, Logic", in *Science and Speculation*, ed. J.Barnes, J. Brunschwig, M.F.Burnyeat, and M.Schofield (Cambridge, 1982). See his revision in the "Preface to the Revised Edition" of Jonathan Barnes, *The Presocratic Philosophers.*

[51] An analysis in depth of this argument will be given in 7.4 below.

[52] "As J. L, King emphasizes, proponents of the epistemic approach must say that a millimeter difference can make the difference between a runner starting from New York being far from San Francisco and his not being far from San Francisco" Roy A. Sorensen, "Vagueness, measurement, and blurriness", *Synthese* 75 (1) (1988), 63.

(Precise.m) A man of 200 cm is over 180 cm.

(Precise.M) If a man of *n* cm is over 180 cm, so is a man of *n-1* cm.

∴

(Precise.C) A man of 170 cm is over 180 cm.

It has a soritical formal appearance, but it is much less acceptable then Zeno's paradox or our (CS). That is because it includes the numerical threshold in its very formulation. Let us call such a number simply *standard of separation*, because it separates the negative from the positive cases. Even for common soritical predicates, the revelation that such a standard exists is enough for the argument to lose its force. Imagine for example, a sorites for *young* started at 17 years with a soritical step of one month, up to 60 years. Citing the fact that there is a law[53] limiting youth to 35 years would stop any such argument, at the price of saying: 'oh, I did not know we are talking about *young* in the legal sense'.

In conclusion, we need to add one condition for being a sorites, namely that all step predicates be naturally distributed. Secondly, not only should the argument lead to a contradiction, but no accepted standard of separation – one number separating negative from positive cases – should exist. The standard of separation can be either explicit – in the argument itself – or it can be readily available, either in law or science. For example, suppose an argument for *to be solid water* started at -100 centigrade with a soritical step of one centigrade. Since water turns liquid at zero, such a standard of separation exists.

[53] For example Romanian Law of Youth no. 350/2006 states at art 2.a) that all citizens with ages between 14 and 35 are young.

Chapter 5

Soritical effect and soritical form

The sorites is a *paradox*, meaning that its premises and rules of inference seem highly acceptable while its conclusion is unacceptable[54]. I take acceptability as reasonability for the ideal speaker, based on general linguistic ability. This may be a phenomenal, not completely reliable guide to reality, but philosophy means also fighting against such deviations by criticism. This is the reason why, when discussing the paradox, I assume as little prior theory as possible. Many standard discussions of the sorites are full from the get-go of such concepts as precision and vagueness or borders and boundarilessness[55]. These should be introduced only by definition and only when they are unavoidable i.e. when they express a distinction between studied phenomena.

At the end of the previous chapter I introduced two new restrictions: that step predicates be naturally distributed as well and that no standard of separation should be available. This cries for a clarification of what they are restrictions of. To clear things out, let us now distinguish *soritical form* from *soritical effect*. The soritical effect is the paradoxical character, which requires, for now, just that no standard of separation be available. I will better define it. Also, some arguments may have common characteristics with soritical arguments without having soritical effect. Let us say that they have soritical form if they have the minimal formal characteristics of the sorites: one naturally distributed soritical predicate, naturally distributed step predicates chained in one of the inference forms listed in Chapter 2, and a numerical step generating a soritical relation. And we will see that the precise quantified reading we choose makes a difference to what goes through or not.

[54] See Sainsbury’s understanding of *paradox*: “an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises” R.M. Sainsbury, *Paradoxes 3rd edition* (Cambridge University Press, 2009), 1.
[55] Dominic Hyde and Diana Raffman, "Sorites Paradox", *The Stanford Encyclopedia of Philosophy (Summer 2018 Edition)*, ed. Edward N. Zalta, https://plato.stanford.edu/archives/sum2018/entries/sorites-paradox/.

5.1. Soritical effect

5.1.1. Getting a contradiction

Philosophers since Aristotle have held that contradictions are abhorrent to human reason[56]. To get a contradiction for the sorites, just add the negation of the conclusion as a secondary minor premise:

(CS.m.2) A man of 100 cm is not tall.

Surely, (CS.m.2) has the same acceptability as the initial minor premise (CS.m) viz. 'A man of 200 cm is tall'. So it seems that we have an explicit contradiction between (CS.m.2) and our conclusion (CS.C) viz. 'A man of 100 cm is tall'.

Let us now redefine *soritical effect* as the derivability of such a contradiction. There are two reasons to do so. The first is heuristic. An explicit contradiction is an indication that we deal not only with unpleasant, but with a highly damaging issue. The second is that the sense in which a contradiction occurs is significant to the interrogative form of the sorites, discussed in Chapter 8.

5.1.2. Classification

There are many failures of obtaining a contradiction. I will classify them by their *point of failure* and by their *source.* The point of failure is either the minor premise, the major premise(s) or the conclusion, when and only when one of the first two is not acceptable or the conclusion is acceptable. The source is one of the formal characteristics: soritical predicate (e.g. *tall*), soritical relation (e.g. *to be one cm apart*), or the chaining of step predicates expressing the transition from negative to positive cases. Identifying the source requires stating how it makes the point of failure different from what would have let the argument go through, i.e. get a contradiction.

[56] Aristotle: "[…] the most certain of all beliefs is that opposite statements are not both true at the same time" *Metaphysics*, Accessed Oct 14, 2019 http://www.perseus.tufts.edu/hopper/text?doc=urn:cts:greekLit:tlg0086.tlg025.perseus-eng1:4.1011b.

We get:

Failing argument	*Point of failure* *Source*	*Comment*
(Blond.m) A man of one million hairs is blond. (Blond.M) If a man of *n* hairs is blond, so is one of *n-1* hairs. ∴ (Blond.C) A man of 0 hairs is blond.	Minor premise Predicate	Blondness is not guaranteed by number of hairs. But the major is acceptable and the conclusion unacceptable.
(Beat.m) A person of 3 million heartbeats is a child. (Beat.M) If a person of *n* heartbeats is a child, so is one of *n+1*. ∴ (Beat.C) A person of 90 million heartbeats is a child.	Minor premise Relation	If infancy exists, because of varying heart rhythms[57], there is no number of heartbeats at which all persons are children.[58]
(Tall-.m) A man of 150 cm is tall. (Tall-.M) If a man of *n* cm is tall, so is one of *n-1* cm. ∴ (Tall-.C) A man of 130 cm is tall.	Minor premise Chaining	The starting point of chaining is too low, thus both ends are negative cases of the predicate.
(Precise.m) A man of 200 cm is over 180 cm. (Precise.M) If a man of *n* cm is over 180 cm, so is a man of *n-1*. ∴ (Precise.C) A man of 170 cm is over 180 cm.	Major premise(s) Predicate	The predicate has a standard of separation, incidentally in the predicate formulation.

[57] Heart rates are very variable. A human of 250bpm may be an infant at the same number of heartbeats as an adolescent of 30bpm. This soritical argument is given by Wright following Esenin-Volpin. Crispin Wright "Language-mastery and the sorites paradox", 155-156. See the extended discussion at 7.5 below.

[58] The difference with (Blond) above is that here the relation (*being one heartbeat away*) is a close cousin of the preferred soritical relation for *childhood*, namely time. If Wright's argument has used *being one second away*, it would have gone through. In (Blond) the relation has no such connection.

(FewChild.m) A family with one child has few children. (FewChild.m) If a family with *n* children has few children, so does a family with *n+3* children. ∴ (FewChild.C) A family with ten children has few children.	Major premise(s) Relation	Three extra children can make all the difference for having few children or not, even though there is no standard of separation[59].
(Material.m) An object of 200 grams is material. (Material.M) If an object of *n* grams is material, so is one of *n-1*. ∴ (Material.C) An object of -10 grams is material.	Conclusion Predicate	The predicate *material* is compatible with zero or negative mass[60], so the conclusion is not unacceptable.
(Wrinkle.m) A person of one million wrinkles is aged. (Wrinkle.M) If a person of *n* wrinkles is aged, so is one of *n-1*. ∴ (Wrinkle.C) A person of zero wrinkles is aged.	Conclusion Relation	Having wrinkles is only a cousin of the preferred soritical relation, which is time. It is acceptable to be aged with no wrinkle.
(Tall+.m) A man of 200 cm is tall. (Tall+.M) If a man of *n* cm is tall, so is one of *n+1*. ∴ (Tall+.C) A man of 230 cm is tall.	Conclusion Chaining	Since the soritical relation goes in the wrong direction, both ends are positive cases of the predicate

The eight cases can be grouped in five reasons of failure:

[59] This is classified as having its source in the relation, because replacing three with one would have seen the paradox through, obtaining Weatherson's paradox of discrete terms. Brian Weatherson, *Vagueness as Indeterminacy,* October 19, 2006, Accessed Oct 14, 2019 https://pdfs.semanticscholar.org/57a3/66e8b9ba754001e6bb2f5b02d861d80e5539.pdf, 4. See also Chapter 17.
[60] Zero and negative mass have been theorized. H. Bondi, "Negative mass in general relativity". *Reviews of Modern Physics*. 29 (1957), 423-424.

a) The chaining interval is not adequate

The paradox fails when the ends are not dissimilar (i.e. one negative, the other positive). (Tall-) and (Tall+) have badly chosen starting or end points. This failure is attributable to construction, not to the predicate or relation. Since construction cannot undermine major premise(s) of acceptable soritical form, there is no *chaining* source at a *major premise(s)* point of failure in the table above.

b) The argument uses a poor cousin of the preferred ordering of the predicate

Soritical predicates have generally one *preferred ordering*[61], meaning a soritical relation which is tolerant towards the predicate. Replacing it with something else can stop the paradox. *Tall* is soritical for height ordering, *child* for age ordering, *heap* for ordering by the number of grains. When these are replaced with partial correlates, such as having larger clothes for *tall*, having wrinklier skin for *aged* or having a larger perimeter for *heap,* failures may happen, as in (Wrinkle) and (Beat) above. I will reserve for Chapter 17 the question of how such orderings combine to determine a single vague predicate.

c) The predicate is not distributed across the soritical relation

(Blond) and (Material) fail in the same way, namely that while the relation seems to have a connection of meaning with the predicate, the latter is not distributed across it. In (Blond), there is no number of hairs guaranteeing that anyone is blond and in (Material) there is no number of grams guaranteeing an object is not material. The predicate *blond* has a preferred soritical ordering by hair color, therefore an argument with soritical effect can be built upon it. But because of its scientific meaning, *material* seems to have no such relation, being not soritical.

d) The predicate has a coarse soritical step

(FewChild) plays on the fact that there is no single number such that families of up to that number of children have few children, as remarked by Weatherson[62]. That is, maybe one child is few, maybe two are few, and so on. But it fails because of the large soritical step: a difference of three children seems

[61] Hyde and Raffman name such preferred orderings “dimensions decisive of the predicate’s application”. If we understand by *dimension* being a measurable quality, as seems natural, it can be reduced to an ordering, as we will see in Chapter 14. Dominic Hyde and Diana Raffman, "Sorites Paradox".

[62] Brian Weatherson, *Vagueness as Indeterminacy*, 4-5.

to make a difference in any case between few and not few children. Let us call such a soritical step *coarse*, that is, too rough to guarantee tolerance. For example, imagine our initial (IS) with the major premise replaced by:

(IS.M.2) For any number of cm, if a man of so many cm is tall, so is a man of so many minus twenty.

If the soritical step is coarse, we do not need a threshold to deny soritical effect. Incidentally, this shows that the discussion about borders has been overdone. There is no such limit here, but it is not acceptable that three children apart still mean few children, no matter the starting point. It is too coarse a relation and we cannot built a sorites for it[63].

e) The predicate has a standard of separation

In (Precise) there is a clear *standard of separation* as defined in the previous chapter. 180 cm is the numerical figure where the predicate *to be over 180 cm* stops applying (in a descending sorites).

5.2. Basic criteria for soritical form

Let us now list the criteria for soritical form we have announced, using our usual (CS).

5.2.1. Chaining of naturally distributed step predicates

The paradox links applications of step predicates: *to be a man of 200 cm, to be a man of 199 cm, … , to be a man of 100 cm.* These are all naturally distributed i.e. there exist both positive and negative cases for each. Their chaining is in one of the inference forms listed in Chapter 2, here modus ponens.

The requirement of natural distribution is a formal one, because (1) it constrains the semantics and any interpretation that will model a vague context (2) it avoids possible objections based on a different understanding of descriptions. As discussed, if someone understands ‘A man of zero cm is tall’ as ‘There are men of zero cm and all such men are tall’, the logical form would be different. And the paradoxical character would be lost in favor of non-problematic rejection.

[63] Hyde gives a similar example from *small* in the series 1, 100, 200, 300, … , 10.000. Dominic Hyde, "The Sorites Paradox", 4.

5.2.2. A naturally distributed soritical predicate

To be a tall man is the soritical predicate. It is naturally distributed i.e. it has both positive and negative cases.

5.2.3. A numerical soritical step defining a soritical relation between the step predicates

The 1 cm of difference between the step predicates is the numerical soritical step. Say the soritical relation is that holding between ordered pairs of 200 and 199, 199 and 198, 198 and 197 and so on. Let us call it S:

(S.S.Def) $S = \{\langle 200, 199\rangle, \langle 199, 198\rangle, \ldots, \langle 101, 100\rangle\}$

It is easy to prove that if $D = \{n \in \mathbb{N} \mid 100 \leq n \leq 200\}$, S holds only and at all elements of D and their antecessor, except for the minimum of D by ≤ i.e. 100:

(S.S.Num) $\forall xy\ .\ Sxy \leftrightarrow y = x + 1$

But there is a problem. We do not have a domain of numbers. In the inductive sorites above, *male heights* themselves were taken as domain, since it seems acceptable to say that a 200 cm height is the 199 cm height plus one, generating a paradox when asking whether all such heights come under the predicate *to be a tall height.* Since our predicate is now *to be a tall man*, we cannot define the soritical relation as (S.S.Def) or (S.S.Num), but need to inquire what kind of logical relationship the step predicates have. And that relationship should have a connection with the numerical step, i.e. *-1*.

5.3. Quantified readings

Let us first give three forms which might be sensitive to the precise reading of the sorites:

(House.m) A man with a nice house and one million euros is rich.

(House.M) If a man with *n* euros is rich, so is a man with *n*-1 euros.

∴

(House.C) A man with a nice house and a debt of ten billion euros is rich.

And the case above at 5.1.2:

(Blond.m) A man of one million hairs is blond.

(Blond.M) If a man of *n* hairs is blond, so is one of *n-1* hairs.

∴

(Blond.C) A man of 0 hairs is blond.

And the paradox about *few* that is attributed to Diogenes Laertius, similar to Wang's paradox discussed in 2.5:

(Few.m) Two are few.

(Few.M) If *n* are few, so are *n+1*.

∴

(Few.C) Ten thousand are few.

Can the argument in (House) be continued from the minor premise up to the conclusion? Is there a reading under which (Blond.m) avoids the objection above that blondeness is not determined by the number of hairs? What is (Few) really talking about? Let us list five ways of reading the premises quantificationally, call them the coherence, existential, universal, identity and statistical readings.

5.3.1. The coherence reading

This is the strongest and least intuitive, requiring that if there is at least one tall man of *n* cm, so are all men of *n-1* cm. In predicate logic, that would be:

(m.Coherence) $\exists x \,.\, C_n x \wedge Tx$

(M.Coherence) $\exists x\, (C_n x \wedge Tx) \supset \forall x\, (C_{n-1} x \supset Tx)$[64]

(C.Coherence) $\forall x \,.\, C_{100} x \supset Tx$

[64] Each schematic major premise containing *n* is to be replaced by a number of single steps not containing *n.*

However, (M.Coherence) is threatened by free logic proponents, because at each step you would need the derivation of $\exists x\ \phi x$ from $\forall x\ \phi x$. Secondly, it is dubious philosophically. To use Williamson's suggestion, if someone is taller not only because of height, but partly because of an especially tall-looking hair arrangement[65], why would all men solely shorter in scalp-to-toe measurements be tall? And it would conflict with the usual logical transcription of 'An element of a first property has another property', namely $\forall x . Fx \supset Ax$.

5.3.2. The existential reading

It is a weaker reading, requiring that if at least one man of *n* cm is tall, so is at least one man of *n-1* cm. The minor premise stays as above, while the major one changes to:

(M.Existential) $\exists x\ (C_n x \wedge Tx) \supset \exists x\ (C_{n-1} x \wedge Tx)$

(C.Existential) $\exists x . C_{100} x \wedge Tx$

This would require each step to be changed along the lines of 'If there are tall men of 200 cm, there are tall men of 199 cm', which is not how the paradox is usually formulated. And it would be compatible with soritical predicates having a tolerance chain of one element at each step, leaving the vast majority of cases incoherently distributed across the step predicates. Suppose such a predicate *rich₂* with one positive case of a person at each natural number of euros – as personal fortune – between zero and one million. Suppose all other positive cases of *rich₂* were only and all men with a fortune of less than one thousand euros. But if the sorites is to be read according to (M.Existential), *rich₂* is soritical, just as our common *rich*. This would belie the central role the sorites has been taken to play in displaying the behavior of natural-language predicates and our expectations of them[66].

As under the coherence reading, (House) would get going, because you can infer $\exists x . Mx \wedge Rx$ from $\exists x . Hx \wedge Mx \wedge Rx$[67]. But the conclusion cannot be derived, as the converse does not hold.

[65] See already cited Timothy Williamson *Vagueness,* 156, also Elia Zardini, "A Model of Tolerance", *Studia Logica* 82 (2006), note 1, 1.

[66] Andrew Bacon remarks that we have the same expectations of *rich* in English and its Russian translation, so we have deep expectations from vague concepts. Andrew Bacon, *Vagueness and Thought* (Oxford University Press, 2018), 5.

[67] H standing for *to have a nice house* and M for *to have a million euros* in (House.m) above.

5.3.3. The universal reading

It states that, if for all objects, being a man of *n* cm guarantees tallness, then for all objects, being a man of *n-1* cm guarantees tallness. Keeping the conclusion from (C.Coherence), we have:

(m.Universal) $\forall x \,.\, C_n x \supset Tx$

(M.Universal) $\forall x\,(C_n x \supset Tx) \supset \forall x\,(C_{n-1} x \supset Tx)$

This is the most natural formulation of the sorites in predicate logic. It requires all objects of a step predicate have the soritical predicate in order to put tolerance in motion i.e. affirm that all objects of the neighboring step predicate have it too. This means that the sorites embeds what Frege called concept subordination[68] throughout: in its minor premises, major premises and conclusion. The predicate of *having 200 cm* is subordinated under that of *tall*. If this holds, so is subordinated the predicate of *having 199 cm* and so on up to a conclusion where we can derive the paradox: that the predicate of *having 100 cm* is also subordinated under *tall*. Concept subordination is a simple idea of wide application.

Remark that (House) above still does not go through. Because you cannot derive $\forall x.\, Mx \supset Tx$ from $\forall x.\, Hx \wedge Mx \supset Tx$. If you want it to go through, pay the small price of reformulating it, either by removing the predicate of having a nice house from the minor premise, or by adding it to the major. As for (Blond), while it has soritical form, this reading supports dismissing its soritical effect for the known cause: not all million-haired men are blond, as the minor premise requires.

Finally, the universal formulation allows two natural readings of (Few), each to be used in appropriate cases. The first option is to take (Few) and other such arguments about number as talking of classes of each size and use normal predicate letters for being a group of the respective size. This is natural here, because Diogenes Laertius was supposedly saying that two apples are few, two pillows are few and so on, as already indicated by the plural verbal form chosen by the translator. It would give a reading like (m.Universal) above, with 'C_n' standing for *being a group of n objects* and

[68] Frege: "The relation of subordination of a concept under a concept is quite different from that of an individual falling under a concept [...]. In general I represent the falling of an individual under a concept by F(x)[...]. The subordination of a concept Ψ( ) under $\boldsymbol{\Phi}$ ( ) is expressed by $\forall x.\, \Psi x \supset \boldsymbol{\Phi} x$" [the final formula was translated from Frege's to modern notation by the author]. Gottlob Frege, "Letter to Marty" in *The Frege Reader*, ed. Michael Beaney, 81.

'T' for *being a group of few objects*. This would also conform to Frege's foundational concern that number statements are what he calls second-order properties[69].

The second option, call it the *individual reading* is to equate identity with a step predicate and read 'T' as 'being a small number', resembling Wang's paradox cited by Dummett:

(m.Individual) $\forall x \,.\, x = 1 \supset Tx$

(M.Individual) $\forall x\, (x = n \supset Tx) \supset \forall x\, (x = n+1 \supset Tx)$

(C.Individual) $\forall x \,.\, x = 10000 \supset Tx$

This option has real objects such as numbers linked soritically, fulfilling the condition that the step predicates be naturally distributed, e.g. *identity with 1* has 1 as positive case and all other elements as negative cases. So, while in our case of (Few), the normal universal reading seems fine, some metaphysical arguments may turn on the individual reading, as we will see in Chapter 7.

An important misconception on which the universal reading sheds light is that the denial of tolerance implies the existence of a general threshold such that all objects situated above are positive cases of the predicate and all those below are negative[70]. For example, denying (CS.30) would mean that all men of 171 cm are tall and all men of 170 cm are not. That is false. For the negation of tolerance to be true, it is enough for one man to have 170 cm and not be tall. The inference from the statement that one man of 170 cm is tall to the statement that all men of 170 cm are tall has nothing to do with soritical form. It is a separate thesis, called in this work *predicate-ordering monotony* (e.g. 'If a man of n hairs is bald, so is any many of n hairs or less'), as explained in 2.4 and 2.5 above. I think this thesis is incorrect and its rejection is the key to understanding the logic of vagueness in the third part of this work.

5.3.4. The identity reading

It states: 'For each man with *n* hairs that is tall, all men with the same characteristics as him but *n-1* hairs are tall'. This would express logically the standard disclaimer added by philosophers of

[69] "'The number of planets is 7' does not mean '7' is a property of planets and a conceptual mark of 'number of planets', but a second-level property of 'number of planets'". Gottlob Frege, "On concept and object" in *The Frege Reader*, ed. Michael Beaney, 184.

[70] Two examples of taking such unique *cut-offs* as an obvious consequence of denying tolerance are Steven Schiffer, *The Things We Mean* (Oxford University Press, 2003), 189-193 and Andrew Bacon, *Vagueness and Thought* (Oxford University Press, 2018), 6. But the list can go on.

vagueness that all other properties are presumed unchanged, but it is hard to put formally. In second-order logic, we could try:

(M.Identity) $\forall x \,.\, C_n x \land Tx \supset \forall y\ (\ \forall Q\ (Q \neq C_n \land Q \neq T \land (Qx \leftrightarrow Qy)) \land C_{n-1}y \supset Ty)$

saying that for any tall object of *n* cm, any object that has *n-1* cm and all properties besides *having n cm* and *being tall* in common with the first element is tall itself.

But of course, one cannot simply replace the property of having 200 cm with that of having 199 cm while keeping all other properties unchanged. Some organs change size, the skin wrinkles change shape, the man leaves the membership of the class of people whose height is a multiple of twenty and so on. Many critics of second-order logic doubt that the last one is really a property[71]. But that is precisely the point: since properties are inter-connected, one needs higher-order logic to express their putative connection. Asking whether *tallness* itself is kept as a property forces us to exclude all properties that used to determine it jointly with the removed property e.g. *being kyphotic, having 200 cm*, etc. This suggests a hierarchy of properties of various patterns of inter-determination.

Suppose that when removing the property of *tall* from an object, all correlated properties should also be removed. But there exist real-life correlations such as that, for example, height is strongly associated with wealth and educational achievement[72]? It then seems reasonable that, were you to take the height from at least some rich people, they would no longer be rich, a result which goes against the intuitive import of the identity reading. Namely, that all basic properties except height should be kept the same. The tentative conclusion is that the identity reading is metaphysically dubious: you may not have two real persons differing only by number of cm.

(House) would fail under the identity reading just as it fails above. But we have seen that it is easy to reformulate. (Blond) is another matter. This reading is the only one under which we could understand it as paradoxical. The minor premise would need to be read along the lines of 'Suppose a man of one million hairs is blond' and the major along the lines of 'Suppose such a man loses one hair

[71] "What are we to say about negative properties? Is it a property of the planet Saturn that it is not equal to the integer 17? In that case, although there are only a finite number of planets, our second-order quantifiers must range over infinitely many properties" William Ewald, "The Emergence of First-Order Logic", *The Stanford Encyclopedia of Philosophy* (Spring 2019 Edition), ed. Edward N. Zalta https://plato.stanford.edu/archives/spr2019/entries/logic-firstorder-emergence. Also see W.V. Quine, *Philosophy of Logic. Second Edition* (Harvard University Press, 1986), 66-70.

[72] Angus S. Deaton and Raksha Arora, *Life at the Top: The Benefits of Height* (June 2009). NBER Working Paper No. w15090. Available at SSRN: https://ssrn.com/abstract=1422968.

while keeping all other properties and accept that the new man stays blond'. Then the argument seems to go through.

Thus, there is a connection between the sorites and such metaphysical problems as fission or Theseus's ship. This is the source of the so-called *identity sorites*[73], proposed by Graham Priest. He argues that a standard sorites as (CS) can be reanalyzed into a Theseus's-ship-sorites: "Consider, again, the segmented color spectrum. The colour of the first segment is red. Any two adjacent segments have the same colour. Hence, the colour of the last segment is red (which is false)."[74]. For *tall*, we could take the *stature of a man of 190 cm* and the *stature of a man of 189 cm* to correspond to the same object, then through a series of such identity statements get to the paradoxical conclusion that even the stature of a man of 150 cm is tall. This is taking the sorites to be a problem of identity. However, there are three arguments against this transformation.

First, even assuming that such objects as statures exist, their identity is logically dubious. As Williamson writes

> "the indiscriminability of x_i and x_{i+1} is supposed to be a sufficient condition for the identity of their looks; it is certainly a necessary condition. But indiscriminability of this sort cannot be a necessary and sufficient condition for the identity of anything, for, unlike identity, it is not a transitive relation"[75].

Thus, it is certain that two indiscriminable reddish colors can be discriminated through a third, so they do not meet the definition of identity as indistinguishability, just as statures of a different number of cm be discriminated through a third, and so on. Relatedly, in his recent introduction of compatibilist semantics for vagueness, Fine distinguishes *weak identity* from *strict identity*, while limiting the transitivity of the former. Weak identity is the denial that adjacent objects in a soritical series are distinct. Strict identity is "weak identity without competitors", in the sense that not only should the objects be non-distinct, but there must also not exist any other object which is non-distinct from one of them but distinct from the other[76]. We will discuss Fine's semantics which models such

[73] E.g. for the name Dominic Hyde, "The Sorites Paradox", 12.
[74] Graham Priest, "Sorites and Identity", *Logique & Analyse*, 135-136 (1991), 294.
[75] Timothy Williamson's *Vagueness*, 179. For an argument for the transitivity of *looking as*, but solving the phenomenal sorites as premises conjointly incoherent, see Delia Graff, "Phenomenal Continua and the Sorites", Mind, New Series, Vol. 110, No. 440 (Oct., 2001), 934.
[76] Kit Fine, *Indeterminate Identity, Personal Identity and Fission* (2015). Accessed Oct 14, 2019, https://www.academia.edu/15116676/Indeterminate_identity_personal_identity_and_fission, 12.

a possibility of simultaneous agreement and disagreement in Chapter 12, along with the objections it raises. However, the point is the same: identity is not transitive across the entire soritical series.

Secondly, talk of *statures* for *tall* or of *hairiness of head* for *bald* is metaphysical dubious. Are there corresponding objects for *small* as applied to numbers? *Smallness* would surely seem circular, while *size* is certainly different between numbers 1 and 2. Moreover, since statures cannot exist without people and hairinesses without heads, we would commit ourselves to an ontology of first-order objects, second-order objects and so on, rediscovering the problem above of formulating the causal link between properties as between objects. It seems much to ask of an allegedly intuitive paradoxical argument.

Thirdly, the logical form of the arguments are different. Superficially, the sorites seems a propositional argument, but, as we will see in Chapter 6, its general specific form is second-order, expressible in FOL with a finite upper bound. While metaphysical arguments, such as *the problem of the many*, have a different logical form, as we will see in Chapter 7. All three points tell against the identity sorites as a sorites.

5.3.5. The statistical reading

Finally, the weakest formulation is the statistical, which would prefix 'In most cases' to what we thought was the major premise(s) under the universal reading. This is the natural reading of much of what goes on in natural language reasoning. See common statements: 'An Englishman speaks English better than a Frenchman', 'The plane is faster than the train' or 'Seas are different from lakes'. They should not be formalized through an existential or universal sentence: they hold in the overwhelming majority of cases, but they all have exceptions. We need a 'for most objects' quantifier:

(M. Statistical) $\exists_{>1/2(H,\,m)} x \forall y \,.\, C_n x \wedge Tx \supset (C_{n-1} y \supset Ty)$

Start from the minor premise as in the universal reading. Then, we can introduce a *majority quantifier* within the limits of FOL with two conditions. First, we assume speaking only of the positive and negative cases of the soritical predicate that have also another predicate, such as *human*. That is, we only speak of humans when talking of tall men and non-tall men. Secondly, we impose a finite upper bound on the number of elements having this predicate, such as 10^{12} for humans[77]. Let 'H' be the *human* predicate and *m* be the bound. Predicate logic with identity can express that there is a

[77] Based on the current figures Wikipedia cites for the total number of people to have ever lived.

maximum number of humans. Then we can descend from *m* to *1*, one at a time, affirming at each step that if there are so many humans, the bounded formula holds for at least that intermediate number divided by two. A long disjunction of all such steps is equivalent to saying that the bounded formula holds for most humans. Then $\exists_{>1/2(H, m)}$ signifies ‘for at least half of humans of which there are no more than *m*’. Also, ½ is just an example, we can apply the technique to yield 90% and so on.

Formally, using $Floor_{n/2}$ for *n* divided by 2 and rounded-down to the previous integer, and *n* any finite integer:

(Eq) $\exists_{=n}x\ Hx \stackrel{def}{=} \exists v_1 \ldots v_n \,.\, Hv_1 \wedge \ldots \wedge Hv_n \wedge v_1 \neq v_2 \wedge v_1 \neq v_{3\ \ldots}\ v_1 \neq v_n \wedge \ldots \wedge v_{n-1} \neq v_1 \wedge v_{n-1} \neq v_2 \wedge \ldots \wedge v_{n-1} \neq v_n \wedge \forall x\ (Hx \supset x = v_1 \vee \ldots \vee x = v_n)$[78]

(Min) $\exists_{>n}x\ Hx \stackrel{def}{=} \exists v_1 \ldots v_n \,.\, Hv_1 \wedge \ldots \wedge Hv_n \wedge v_1 \neq v_2 \wedge v_1 \neq v_{3\ \ldots}\ v_1 \neq v_n \wedge \ldots \wedge v_{n-1} \neq v_1 \wedge v_{n-1} \neq v_2 \wedge \ldots \wedge v_{n-1} \neq v_n$

(Maj) $\exists_{>½(H,\, n)}x\ \phi \stackrel{def}{=} (\exists_{=n}x\ Hx \supset \exists_{>Floorn/2+1}x\ \phi) \vee (\exists_{=n-1}x\ Hx \supset \exists_{>Floor(n-1)/2+1}x\ \phi) \vee \ldots \vee (\exists_{=1}x\ Hx \supset \exists_{>Floor1/2+1}x\ \phi)$

The two limitations are philosophically reasonable. The first is a mildly stronger requirement of our natural distribution requirement: that some common grouping must be known to the speaker[79]. When reasoning with common language, we are speaking of *humans* that are tall men or not, are men of 200 cm or not, are men of 199 cm or not and so on[80]. As for the second, we do not normally reason in natural language about finitude, which is out of bounds for FOL[81].

This reading makes the paradox a case of the fallacy of division. The minor premise is true. We can accept the major. For a (very large) majority of cases, being one cm apart from a tall man guarantees tallness. But this is insufficient to derive the individual steps such as:

(M.Universal.30) $\forall x\ (C_{170}x \supset Tx) \supset \forall x\ (C_{169}x \supset Tx)$

Or even:

[78] The suspension points should be read as to exclude any affirmation of non-identity with itself for any variable. So theres no $v_m \neq v_m$ sub-clause for any $1 \leq m \leq n$. Same for (Min).
[79] It is easy to obtain the common predicate, just take the union of all predicates in the argument.
[80] Mutatis mutandis we could have certainly had the predicate *man* with a finite upper bound and ‘T’ stand for *tall*, not *tall man* as here.
[81] See the demonstration template using the compactness theorem in 6.3.2 below.

(M.Statistical.30.2) $\exists_{>1/2(C170[82], m)}\, x\, (C_{170}x \supset Tx) \supset \exists_{>1/2(C169, m)}\, x\, (C_{169}x \supset Tx)$

The statistical reading is the only one to abandon the parallelism between the first-order and propositional formulations, because 'for at least half of humans' binds the entire formula. Most importantly, we can avoid statistical quantifiers such as $\exists_{>1/2(H, m)}$ and save a variant of universally quantified predicate tolerance (PredTol), as formulated in 2.5. In Chapter 16, I will define a *vague predicate* through the interplay of the monadic predicate e.g. *tall* and its preferred ordering relation R e.g. *having less or equal cm of height than*. There will be an initial chain of negative cases, such as adjacent short people corresponding to small heights in cm and a final chain of positive cases, such as adjacent tall people corresponding to large heights in cm, leaving a central intermediary zone of dispersion where tall and short people are intermingled by height in cm. Then the union of the predicate, e.g. *tall* with the dispersion zone will be called the *broad predicate* under the relation, namely *broadly tall* (noted 'T^{R1}'). And their difference will be called the *strict* predicate under the relation, namely *strictly tall* (noted 'T^{R0}'). Therefore: 'If a man of *n* cm is tall, a man of *n+1* cm is broadly tall' and 'If a man of *n* cm is strictly tall, a man of *n-1* cm is broadly tall' will hold. This will be written, with 'R' for the relation of being at the previous rank in the total preorder (e.g. *having one less cm of height* if the precision of measurement is one cm):

(PredOrdMonotony.2) $\forall xy\ .\ Rxy \supset \neg(Tx \wedge \neg T^{R1}y)$

(PredTol.2) $\forall xy\ .\ Rxy \supset \neg(T^{R0}y \wedge \neg T^{R1}x)$

Where superscripted '1' indicates the broad predicate and '0' the strict predicate. Then, the definition of *vague predicate* will make (M.Statistical) hold for any such predicate and its preferred ordering relation. That is because our intuitions about vague predicates is that they are not completely chaotic: there are safe zones such that for most men as ordered by height in cm, tolerance holds. But (PredOrdMonotony.2) and (PredTol.2) will *always* hold at vague predicates, as their definition posits a dispersion zone. This means that a notational extension of FOL can handle vagueness.

The elaboration of this view can be found in the third part of this work. In any case, it would pull the rug from under most of the historical debate on the sorites, while staying a classical solution. It takes the source of the sorites to be a mistake of generalization, as in the following argument:

(Bite.m) Mike is a dog and Spot is a hamster.

[82] This supposing there are no C_{170} or C_{169} elements not being H.

(Bite.M) Hamsters do not bite dogs.

∴

(Bite.C) Spot does not bite Mike.

A contradiction can be derived with an objective report of the situation unfolding in my living room:

(Bite.m.2) Spot bites Mike.

The wrong step, of course, is that (Bite.M) is not true, it is true *only in a large proportion of cases.*

I conclude the examination of our five variants for reading the sorites with a qualified endorsement of the universal reading for the next few chapters. The coherence and existential readings are piecemeal. The identity reading, while illuminating, requires higher-order logic to formalize and is metaphysically dubious, as two real objects may not differ by a single property. I will elaborate the statistical reading in the third part of this work, but it is clear that it does not deliver a paradox, but only its semblance. The reading which is both compatible with a good part of the philosophical literature on the sorites and paradoxical is the universal one: the sorites as a chain of concept subordination. Let us now turn to studying the general logical form of the sorites, linking together the formal characteristics identified: naturally distributed step predicates chained in one of the inference forms listed above, one naturally distributed soritical predicate and a numerical step generating a soritical relation.

Chapter 6

The general form of the sorites

6.1. Divisibility

The sorites has been known since Antiquity as the '*little-by-little*' argument[83]. Since both its minor and major premise(s) deal in numbers, it may be called a *numerical paradox*. But I think it is best to call it a *divisibility* paradox. To see why this is so, we first imagine a little-by-little argument – chained argument with a little numerical difference at each step – that is not soritical:

(BigBang.m) Someone is born now.	A	$\exists x\, Bx$
(BigBang.M.1) If someone is born now, someone was born at least one year ago.	$A \supset M_1$	$\exists x\, Bx \supset \Box\exists x\, Bx$
(BigBang.M.2) If someone was born at least one year ago, someone was born at least two years ago.	$M_1 \supset M_2$	$\Box\exists x\, Bx \supset \Box\Box\exists x\, Bx$
...		
(BigBang.M.60B) If someone was born at least sixty billion years ago, someone was born at least sixty billion one years ago.	$M_{60B} \supset B$	$\Box^{60B}\exists x\, Bx \supset \Box^{60B+1}\exists x\, Bx$
(BigBang.C) Someone was born at least sixty billion one years ago.	B	$\Box^{60B+1}\exists x\, Bx$

I give the propositional transcription, which is essentially identical with the (CS) above, i.e. a chain of modus ponens. The third column contains a more general reading, using an iterated modal operator for 'one year ago'. The numerical step of the paradox i.e. *one year* is acceptable since there are no parents of age lower than one year[84].

If by *general logical form*, we mean a logical formalization that has no arbitrary parameters and is specific to the argument form that is being analyzed, the general logical form of (BigBang) is $\Box^n\exists x\, Bx \supset \Box^{n+1}\exists x\, Bx$ with $n \in \mathbb{N}$, a kind of modal inflation principle that states that before any birth, there was another birth in advance[85]. The one year becomes a part of the meaning of '□', having no logical

[83] Galen, "On Medical Experience", 58.
[84] Wikipedia names Lina Medina as the youngest documented human mother at five years and seven months.
[85] This is not to be read as a standard tense operator that applies to all past times e.g. *[P]* in Graham Priest, *An Introduction to Non-Classical Logic: From If to Is*, 49-51.

connection across steps essential for the argument. So the paradox must be solved by putting some limit on *n,* viz. the year when the first human appeared as vouched by anthropological science.

One may argue that this argument is soritical on such a predicate as 'to be a mother's child'. The best we can do is:

(BigBangSorites.m) Anyone born before 2010 is a mother's child.

(BigBangSorites.M) If anyone born before year *n* is a mother's child, so is anyone born before *n-1*.

(BigBangSorites.C) Anyone born before year sixty billion is a mother's child.

This fails of soritical form because the step predicates are not naturally distributed. Sixty billion years ago i.e. much before Big Bang, there was no object in regard to which to have the clear intuition that it does not count as a mother's child. And we cannot cite a past age for which it would be clearly false, that is, unacceptable, as intended. Not only is the conclusion not paradoxical, but the modal inflation principle above has not the acceptability we associate with soritical tolerance. It is well-advised when talking of past time to put a limit of how far back we can go i.e. *n* above needs a reasonable limit.

In contrast, at the sorites, there are hundreds of millions of people – and maybe more *potential* human heights[86] - between 200 and 100 cm. Hence the progression on this interval does not need a limit, as that between now and sixty billion years ago. The sorites is about the reachability of a credible positive end point from a credible negative start point through a credible small interval that is tolerant i.e. does not allow switches from negative to positive cases. Arbitrary parameters such as the one hundred steps, the one cm soritical step and the starting and ending points (here 200 cm and 100 cm) are eliminable. But it is essential that there exist a soritical step, to divide the gap between the ends, in a finite number of steps.

Does this soritical step need be numerical? Most likely not. There can be arguments dividing a span through other means, but they are hard to find nowadays. For illustration, Williamson cites Carneades's 2nd century B.C. river paradox:

[86] See an illustrative FOL theory distinguishing a finite number of actual objects having a property from an infinite number of possible objects having the property in 13.4.

> "If Zeus is a god [...] Poseidon too, being his brother, will be a god. But if Poseidon [the sea] is a god, the [river] Achelous too will be a god. And if the Achelous is, so is the Nile. If the Nile is, so are all rivers. If all rivers are, streams too would be gods.[...] But streams are not."[87]

Similarly, let us look at the large span of meaning between the solidity of iron and the gaseous state of hydrogen. Suppose there were intuitions along the lines of: 'all metals are solid if one is', 'as is lithium, so is carbon', 'as is carbon so is water', 'as is water so is hydrogen'. Then you would have bridged the span through non-numerical means. While the sorites paradoxes in the literature have numerical steps, this is not necessary.[88]

What is the precise formalization of the connection between the ends, the number of steps and the size of the soritical step? First let us give precise reasons for which the propositional formulation of the sorites such as (CS), i.e. a chain of modus ponens, is not the general logical form of the sorites:

(6.1.1) Its formulation in words is an example for one predicate, not a schema central to the paradox (e.g. as is: 'a: "a is false"' at the Liar);

(6.1.2) One could not say by looking at the chain of modus ponens whether the argument was a soritical argument or any other chained argument such as (BigBang) above[89].

(6.1.3) Some logical patterns are not expressed formally:

a) It has arbitrary but equally-distanced numbers in it;

b) Step predicates are logically connected, but their connection is not expressed. For example, in FOL we can say: if C_n is the predicate of having *n* cm and S is the relation of *having 1 more cm*, $\forall x . C_{199}x \supset \exists y\ (C_{200}y \wedge Syx)$, $\forall x . C_{200}x \supset \exists y\ (Sxy \wedge C_{199}y)$, $\forall xy . C_{200}x \wedge C_{199}y \supset Sxy$.

In brief, the propositional form is not the general form of the sorites, being unspecific to it. Therefore, weakening the rules of inference in 2.1. (mpp, mpt, mtp, mtt) is not necessary for solving the sorites and a logic for vagueness needs to express relations, i.e. be above zero-order.

[87] Timothy Williamson, *Vagueness*, 28.

[88] For example for *chair:* "One can imagine an exhibition in some unlikely museum of applied logic of a series of 'chairs' differing in quality by least noticeable amounts." Max Black, "Vagueness: an exercise in logical analysis" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997), 72. I would have to revise my definition of standard of separation and related definitions, to include non-numerical thresholds. It's easy to do and not salient, as standard sorites arguments are numerical.

[89] Akin to formalizing: *(A) Socrates is wise ∴ (B) Someone is wise as*: *A ∴ B*. Graeme Forbes, *Modern logic: A text in elementary symbolic logic*, 149.

6.2. First-order formulations of the sorites

6.2.1. First form

This form expresses the universal reading of the sorites:

(m) $\forall x\, (C_{200}x \supset Tx) \wedge \forall x\, (C_{100}x \supset \neg Tx)$

(M1) $\forall x\, (C_{200}x \supset Tx) \supset \forall x\, (C_{199}x \supset Tx)$

...

(M100) $\forall x\, (C_{101}x \supset Tx) \supset \forall x(C_{100}x \supset Tx)$

We can prove:

(C) $\nexists x\, C_{100}x$

That is, there are no 100 cm elements. i.e. it implies nihilism à la Peter Unger[90].

6.2.2. Second form (from 6.2.1)

By NK, we get one hundred tolerance principles and can express a weaker conclusion:

(m) $\forall x\, (C_{200}x \supset Tx) \wedge \forall x\, (C_{100}x \supset \neg Tx)$

(M1) $\forall xy\, .\, C_{200}x \wedge C_{199}y \supset \neg(Tx \wedge \neg Ty)$

...

(M100) $\forall xy\, .\, C_{101}x \wedge C_{100}y \supset \neg(Tx \wedge \neg Ty)$

We can prove:

(C) $\nexists xyv_1...v_{99}\, .\, C_{200}x \wedge C_{199}v_1 \wedge ... \wedge C_{101}v_{99} \wedge C_{100}y$

That is, there cannot be 101 elements each of a number of cm from 200 to 100.

6.2.3. Third form (from 6.2.2)

We define the relation of *having 1 cm more*, S, and the interdefinability of height predicates:

(DS) $\forall xy\, .\, Sxy \leftrightarrow (C_{200}x \wedge C_{199}y) \vee (C_{199}x \wedge C_{198}y) \vee ... \vee (C_{101}x \wedge C_{100}y)$

(DP$_{199}$) $\forall xy\, .\, C_{200}x \wedge Sxy \supset C_{199}y$

...

(DP$_{100}$) $\forall xy\, .\, C_{101}x \wedge Sxy \supset C_{100}y$

We can prove:

(C) $\nexists xyv_1...v_{99}\, .\, C_{200}x \wedge Sxv_1 \wedge Sv_1v_2 \wedge ... \wedge Sv_{99}y$

That is, there is no S-chain of 101 elements, from 200 to 100 cm.

6.2.4. Fourth form (from 6.2.3)

We observe in 6.2.3 that we can prove:

(S) $\forall xy\, .\, Sxy \supset \neg(Tx \wedge \neg Ty)$

To eliminate the arbitrary starting points, we define L by:

(DL) $\forall xy\, .\, C_{200}x \wedge C_{100}y \leftrightarrow Lxy$

(N) $\forall xy\, .\, Sxy \supset \neg Lxy$

We can prove:

(C) $\nexists xyv_1...v_{99}\, .\, Lxy \wedge Sxv_1 \wedge Sv_1v_2 \wedge ... \wedge Sv_{99}y$

That is, there is no S-chain of 99 intermediary elements between two elements linked by L.

[90] Peter Unger, "Why there are no people", *Midwest Studies in Philosophy* 4 (1) (1979).

6.2.5. Fifth form

We keep only the following:

(PredTol) *Small difference* is tolerant towards predicate T. In other words, T is S-hereditary:

$\forall xy . Sxy \supset \neg(Tx \wedge \neg Ty)$

(FOL.L) *Large difference* can be split in one hundred *small differences*:

$\forall xy . Lxy \supset \exists v_1 ... v_{99} (Sxv_1 \wedge Sv_1v_2 \wedge ... \wedge Sv_{99}y)$

(FOL.N) *Small difference* is not *large difference* and *large difference* is not *small difference*:

$\forall xy . Sxy \supset \neg Lxy$

Then we can prove:

(FOL.C) There are no two elements, one T, the other not T, having a *large difference*:

$\nexists xy . Tx \wedge \neg Ty \wedge Lxy$

If *small difference* is *having 1 cm more* and *large difference* is *having 100 cm more*, with 'T' standing for *being tall,* the conclusion says that there are no two men at a distance of 100 cm, one tall, the other not tall.

The five forms are not meant as steps of a proof[91], but as illustrations of how to bring a propositional soritical formulation to a more general first-order form. Nothing beyond definitions is assumed. The tolerant step relation, e.g. *having 1 cm more* needs to be identified and its tolerance stated. Secondly, instead of using some number of major premises, we formulate a single claim that the relation that holds between the ends of the original formulation can be split in as many S-connecting elements. Finally, (FOL.N) is benign, helping us avoid stating verbosely that the connecting elements must not be identical. The L (*large difference*) relation is divisible into one hundred S (*small difference*) relations, each tolerant towards predicate T.

[91] Though we could derive 6.2.5 as a long implication from the premises in 6.2.1, plus the (S), (L), (DS) and (DP_n) definitions.

The numerical step is expressed as the S relation, whose repeated application generates the step predicates. Tolerance towards predicate T is simply its S-hereditary nature, the (PredTol) of 2.5 above.

From 6.2.5 you can easily get a version of the so-called *no-sharp-boundary paradox*, which assumes the negation of our conclusion as a premise and concludes in the inductive form with *n* a number in the soritical series that $\exists_n(Ta_n \wedge \neg Ta_{n+1})$ [92]. However, the advantage of the present formulation is that it is strictly logical, without appeal to natural numbers.

Nevertheless, problem (6.1.3.) is still with us. We do not need the precise values of 100 cm and 1 cm for the general sorites (90 or 200 would have done as well). The logical form embedding them as the 101 variables of (FOL.L) is still not general enough. What if we want to state generally that S *divides* L, be it by 1, 2, 3, 100, 200 or any number of connecting elements?

6.3. Divisibility of relations

6.3.1. Divisibility and transitive closure

What we want is this:

(Def.Divisibility) For any x, y and relations L and S, L is divisible by S just in case whenever Lxy holds, there is some $n \in \mathbb{N} \geq 0$ such that there are n elements $v_1 \ldots v_n$ and Sxv_1, $Sv_1v_2, \ldots, Sv_ny$ hold.

We notice that (Def.Divisibility) is equivalent with the left-to-right half of the definition of transitive closure of a relation by a finite chain[93].

(Def.TC) Let R be a binary relation, we denote its transitive closure by R^+:

For any x, y, R^+xy just in case for some $n \in \mathbb{N} > 0$ there exist $e_0, e_1, \ldots, e_n$ such that $e_0=x$, $e_n=y$ and Re_0e_1, $Re_1e_2, \ldots, Re_{n-1}e_n$.

[92] Sven Rosenkranz, "Agnosticism and vagueness", in *Cuts and Clouds: Vagueness, its Nature, and Its Logic*, ed. Richard Dietz and Sebastiano Moruzzi (Oxford University Press, 2009), 167-168.

[93] Kenneth G. Lucey, “The ancestral relation without classes”. *Notre Dame J. Formal Logic* 20 (1979), no. 2. doi:10.1305/ndjfl/1093882533, 281-284. Cited in ProofWiki’s Definition:Transitive Closure (Relation Theory)/Finite Chain https://proofwiki.org/w/index.php?title=Definition:Transitive_Closure_(Relation_Theory)/Finite_Chain&oldid=389701 Revision as of 21:00, 25 January 2019.

Therefore we can adapt the standard[94] demonstration that FOL cannot, in general, express transitive closures.

6.3.2. Proof that first-order logic cannot express divisibility as defined

(Dem.Divisibility) Suppose:

a) There is a FOL formula ϕ such that it expresses divisibility of L by S.

b) Two constants: c_1, c_2 such as Lc_1c_2, $\neg Sc_1c_2$ and for each $n \in \mathbb{N}$, add false formula D_n where:

$D_1 \stackrel{\text{def}}{=} \exists v_1 . Sxv_1 \wedge Sv_1y$

$D_2 \stackrel{\text{def}}{=} \exists v_1v_2 . Sxv_1 \wedge Sv_1v_2 \wedge Sv_2y$

...

$D_n \stackrel{\text{def}}{=} \exists v_1 ... v_n . Sxv_1 \wedge Sv_1v_2 \wedge ... \wedge Sv_{n-1}v_n \wedge Sv_ny$

Now, define theory Y as $\{\phi, Lc_1c_2, \neg Sc_1c_2, \neg D_1, ..., \neg D_n\}$. Y is inconsistent, because of ϕ as defined above i.e. it follows from (Def.Divisibility) that at least one of $D_1 ... D_n$ holds. Y being inconsistent, it has no model. By the compactness theorem, Y has a model just in case any finite subset of it has a model. So it follows that there is at least one finite subset of it that does not have a model.

But we can prove that all finite subsets of Y have a model. To see how[95], for any finite *m* such that $Y_m = \{\phi, Lc_1c_2, \neg Sc_1c_2, \neg D_1, ..., \neg D_m\}$ suppose $\neg D_1, ..., \neg D_m$ hold and D_{m+1} holds. All clauses in Y_m then hold.

Therefore, since we cannot find any finite subset of Y without a model, we have a contradiction and we need to reject a) above. In conclusion, divisibility as defined cannot be formulated in FOL. ■

[94] Uwe Keller, *Some remarks on the definability of transitive closure in first-order logic and Datalog*. Internal report (Digital Enterprise Research Institute, University of Innsbruck, 2004), Accessed Oct 15 2019 https://pdfs.semanticscholar.org/48d2/cbb49876f126bfb3541037810c3b9243aa9d.pdf, 1-5.

[95] Skipping trivial cases such as the finite subsets not including one of $\{\varphi, Lc_1c_2, \neg Sc_1c_2\}$.

6.4. General form of the sorites

We can write divisibility in second-order logic[96]. We shorten the notion of a transitive relation containing another:

Trans(S,C) $\stackrel{\text{def}}{=} \forall xy(Sxy \supset Cxy) \land \forall xyz\,(Cxy \land Cyz \supset Cxz)$

We can now formulate what we've been searching for.

(SOL.L) Relation L, *large difference* is divisible by S, *small difference*, i.e. L is included in the transitive closure of S:

$\forall xy(Lxy \supset S^{+}xy) \land \mathit{Trans}(S,S^{+}) \land \forall R(\mathit{Trans}(S,R) \supset \forall xy(S^{+}xy \supset Rxy))$

(PredTol) *Small difference* is tolerant towards predicate T, i.e. T is S-hereditary:

$\forall xy\,.\,Sxy \supset \neg(Tx \land \neg Ty)$

The three challenges of (6.1.1-3) are now met. The sorites in its general form is about a relation L being included in the transitive closure of a relation S which is tolerant toward predicate T, that is, T is S-hereditary.

6.5. Finitary doubts

6.5.1. Divisibility and infinity

(Dem.Divisibility) above is similar to the proof that FOL cannot express finiteness. The reason is that (Divisibility) is satisfied when any appropriate natural number exists.

But vagueness concerns natural language and real physical entities such as people, mountains and clouds. As before, the estimates are that there have ever been fewer than 10^{12} humans and that 10^{81} is an upper bound on the number of baryons in the universe[97], both far from infinity. This means that

[96] I thank Derek Elkins of math.stackexchange.com for the formulation.

[97] See Roger Penrose, *The road to reality* (Jonathan Cape 2004), 718 or problem (6.8) in Barbara Ryden, *Introduction to Cosmology*. January 13, 2006 Accessed Oct 16, 2019 http://carina.fcaglp.unlp.edu.ar/extragalactica/Bibliografia/Ryden_IntroCosmo.pdf, 131. Both give 10^{80} as the number of baryons in the observable universe.

finite models may suffice for modelling natural language and the physical world. But if we adopt a finite model, the theorems of compactness and completeness fail[98].

On the other hand, with a finite upper bound on the domain, say *m*, FOL can express transitive closures and, more specifically, the divisibility of L by S, as a long disjunction of $D_1, \ldots, D_m$. See the consequent below:

(5.1.1) $\forall xy\ .\ Lxy \supset Sxy \vee \exists v_1(Sxv_1 \wedge Sv_1y) \ \vee \ldots \vee \exists v_1 \ldots v_m\ (Sxv_1 \wedge Sv_1v_2 \wedge \ldots \wedge Sv_{m-1}v_m \wedge Sv_my)$

As argued in 5.3.5 above, there is a more palatable option, call it the *bounded submodel* variant. We keep the general infinite model of standard FOL but add what we know about a single predicate, i.e. that there are at most 10^{12} people.

(5.1.2) There are at most 10^{12} humans:

$\exists_{<10\char`^12}\ Hx$

where $\exists_{<n}\phi x$ is short for the FOL sentence that says that there are at most *n* elements such that ϕx.

Then we can define a relation Z from S, and R from our L, by saying that the new relations hold only between the bounded elements:

(5.1.3) $\forall xy\ .\ Sxy \wedge Hx \wedge Hy \ \leftrightarrow Zxy$

(5.1.4) $\forall xy\ .\ Lxy \wedge Hx \wedge Hy \ \leftrightarrow Rxy$

Now, we can state the divisibility of R by Z by replacing 'S' with 'Z' and 'L' with 'R' in (5.1.1) above. That is because in the (Dem.Divisibility) above we would now find that Y_m does not have a model, D_{m+1} being false by (5.1.2), with $m=10^{12}$.

6.5.2. The general first-order form of the sorites

We can add modified (5.1.1) as well as (5.1.2), (5.1.3), (5.1.4) to (N) and (S) from 4.1 to get:

(FOL.B.A) There are at most 10^{12} humans and for them relation Z coresponds to S (*small difference*) and R corresponds to L (*large difference*):

$\exists_{<10\char`^12}\ Hx \wedge \forall xy\ (Sxy \wedge Hx \wedge Hy \ \leftrightarrow Zxy) \wedge \forall xy\ (Lxy \wedge Hx \wedge Hy \ \leftrightarrow Rxy)$

---

[98] Gurevich, Yuri. *On Finite Model Theory*. Final version in: "Feasible Mathematics, Workshop, Cornell University, June, 1989" (ed. S.R. Buss andP.J. Scott), Perspectives in Computer Science, Birkhauser, 1990 Accessed Oct 16, 2019 https://pdfs.semanticscholar.org/79ad/5d698821b5b05b2b817b1fed56dc8138f3bd.pdf, 2.

(FOL.B.L) Relation R is in the transitive closure of Z (see 5.1.1):

$\forall xy \,.\, Rxy \supset Zxy \lor \exists v_1(Zxv_1 \land Zv_1y) \lor ... \lor \exists v_1...v_m\,(Zxv_1 \land Zv_1v_2 \land ... \land Zv_{m-1}v_m \land Zv_my)$, with $m=10^{12}$

(PredTol) *Small difference* S is tolerant towards predicate T, i.e. T is S-hereditary:

$\forall xy \,.\, Sxy \supset \neg(Tx \land \neg Ty)$

(FOL.B.m) There are two cases, one T, the other not-T, connected through a *large difference*:

$\exists xy \,.\, Lxy \land Tx \land \neg Ty$

Of course, the minor premise (FOL.B.m) is added just to get a contradiction. We can prove

(FOL.B.C) There is some element that is both T and not-T:

$\exists x \,.\, Tx \land \neg Tx$

The moral is that as long as we only want to speak of people, mountains and clouds, we do not need second-order logic. We can specify a high finite upper bound on their number. Since the notation will be extremely long, we might need appropriate shortened forms to comprehend what FOL can achieve. This will what Chapter 15 will be dedicated to.

This first-order form of the sorites is similar to that given by Delia Graff in her "Shifting Sands: An Interest-Relative Theory of Vagueness"[99]. The difference is that she uses an *n* when writing the analogue of (FOL.B.L) below, which cannot be stated in FOL to be finite, as proven above at 6.3.2. Therefore, her *n* is just an analogue of my *99* in the fifth form of 6.2.5 above. But it is essential for the sorites to be expressible using smaller and smaller predicate steps, as long as their number is finite. That is why it is a second-order paradox. The present proposal of bounding a single predicate such as *human* is metaphysically correct and at the same time, allows us to use FOL to fully express the

[99] "I will say that we have an instance of the sorites paradox when we are confronted with a group of sentences having the following form, each of which seems individually plausible: (A) Fa (B) $\forall xy \,.\, Fx \land Rxy \supset Fy$ (C) $\neg Fz$ (D) $\exists b_1 \dots b_n \,.\, Rab_1 \land Rb_1b_2 \land \dots \land Rb_{n-1}bn \land R_{bn}z$ [translated to the logical notation of this work by me]" Delia Graff, *Shifting Sands: An Interest-Relative Theory of Vagueness* (Forthcoming in Philosophical Topics, 2000) Accessed on Oct 15, 2019 https://semanticsarchive.net/Archive/GExYWY3N/shifting.pdf, 5.

sorites. These two facts taken together justify FOL as logic of vagueness, as I will argue in Part 3 of this work.

Moreover, the general form of the sorites brings together the propositional and the mathematical induction forms of Chapter 2. That is because the analysis of induction from Frege's "Begriffsschrift" on is done using the notion of a property being hereditary in a relation and that of the ancestral of a relation, which is a different definition for the transitive closure of the relation[100].

6.6. Doubting divisibility

The sorites is generated by two assumptions: *divisibility,* expressed in second-order logic by (SOL.L) and in FOL with a bounded submodel by (FOL.B.L) and *predicate tolerance* (or S-hereditary nature), expressed by (PredTol). The two combine to deny *the existence of a negative start point and a positive end point*, leading to the contradiction of (FOL.B.C). Thus, the argument causes no trouble for predicates which are universal e.g. *identical with oneself* or empty e.g. *non-identical with oneself*. It is paradoxical only for naturally distributed predicates, those having both negative and positive cases[101].

It then can be expected from solutions that tackle the paradox to undermine either divisibility or predicate tolerance. Most philosophical work and my own approach concentrate on the latter, as we will see in Chapter 12. But it is informative to attempt to attack divisibility and see why it fails.

6.6.1. Incommensurability

One way to deny divisibility is to accept that the soritical steps can add up, but to try to deny that their addition lands correctly on the ending point, so as to generate the paradox. You can start from 200 cm but if you cannot end the argument on 100 cm, there is no contradiction needed for soritical effect, given that the secondary minor premise (CS.m.2) in 5.1.1 above was 'A man of 100 cm is not tall'.

[100] Now called the proper ancestral, following Quine, cf. Gottlob Frege, "Begriffsschrift" in *The Frege Reader*, 75. See also Kenneth G. Lucey, "The ancestral relation without classes".

[101] As discussed at 5.1.2 points a-d, the soritical effect of this form depends on the positive and negative cases being grouped across soritical relation S, on the choice of starting and end points and on accepting the tolerance of S.

Suppose height measurements are indexed, for comparison purposes, to different values that their face value. Each *index value* can correspond to more than one face value. Suppose the index function is very simple: get the closest index value available. Say all heights are indexed at the closest 1.5 step over some basis number. If the basis is 100+*e* cm (that is 102.71828…), the indexing values would then be 104.21828…, 105.71828…, 107.21828… and so on. We would get in the propositional sorites:

(CS.3.M.1) If a man of 200 cm (Index: 199.71828…) is tall, so is one of 199cm (Index: 199.71828…)

(CS.3.M.2) If a man of 199 cm (Index: 199.71828…) is tall, so is one of 198 cm (Index: 198.21828…)

(CS.3.M.3) If a man of 198 cm (Index: 198.21828…) is tall, so is one of 197 cm (Index: 196.71828…)

If taken to refer to index values, predicate tolerance (PredTol) holds. For any two index values, if they are one cm apart, they cannot be one tall, the other not tall, because of the paradox of material implication: no index values are one cm apart. But some of (CS.3.M.1-100) and predicate tolerance when read as referring to face values fail. Cases of failure will include those face values indexed to a *short* index value which neighbor face values indexed to a *tall* index value.

There may be some naturalistic justification to the idea of indexing. Because of sensorial limitations, humans may collapse heights into some values which are disparate enough as to be easily discriminable either in person or by proxy i.e. by asking other people. While it is hard so get any evidence for any precise values here, that is not the decisive issue. What is decisive is that indexing or any other incommensurability approaches I can see will be hiding the paradox somewhat further away, not eliminating it. Saying that unknown index values exist as to make our simple 100 cm, 200 cm and 1 cm technically incorrect while well intended is not sufficient. Because there *are* face values which equal their index values under our index function, namely the index values themselves. Thus, the paradox can be restated with them. For our example, taking 199.71828… cm as starting point and 102.71828… cm as ending point, both connected by L (*large difference*), and *having 0.75 cm of height more* as S (*small difference*) would deliver the paradox: if *having 0.75 cm of height more* is tolerant, then there cannot be any two people, one tall, the other not tall at an *L* distance.

6.6.2. Infinitesimal differences and *small enough*

When debating the sorites, there is a tendency to make the soritical step smaller and smaller. Some works on the sorites now speak of tens of thousand of steps, which, since the ends are credible, make the soritical step very small[102]. Why not push the diminution to the end and adopt infinitesimal values – the reciprocal of hyperreal numbers – as numerical steps? The hyperreals are defined as numbers that are larger that anything of the form 1+1+1+...+1. Any addition series of their reciprocals is always smaller than 1.

If we only affirmed tolerance for such infinitesimal differences of height, divisibility would fail, because the infinitesimal steps would never add up to a natural number. But there are two powerful objections. First, two men can either have the same height, or the difference of their heights is a quantity in the normal sense. That is, it is comparable with a quantity of 1 cm, and such operations as multiplication may take it over 1 cm (or any number of cm). This seems obvious when talking of physical quantities and it is difficult to see any evidential basis for infinitesimal differences between real objects[103]. Secondly, even if there were infinitesimal differences between some physical measurements, people may legitimately intend non-infinitesimal i.e. real soritical steps. The point of choosing a very small but real numerical step may be not to approximate very roughly infinitesimal tolerance, but simply to find a value which our linguistic capacity guarantees as being of no import to the justice of using the predicate in question[104]. Taking Eubulides at his word, he chose the difference of one grain in the Heap for its own sake, not as an approximation of some even smaller difference between heaphood candidates.

This point is relevant for what can be called *small enough difference*. I cited Fine above writing "lf two cases are sufficiently alike then it is not the case that the first is bald and the second is not"[105]. Cobreros, Egré, Ripley and van Rooij have also discussed a semantics that extends the predicate *tall* to any object that is connected to a primarily tall object by a relation which is symmetric, reflexive but not necessarily transitive called *to be similar enough with.* What these theories of *sufficiently alike* or *similar enough* have in common is that they assume predicate tolerance holds and hypostatize a relation for such tolerance. Then, knowing that the respective relation is tolerant, they modify the semantics as to accommodate principles such as $\forall x \forall y(P(x) \land x \sim_P y \rightarrow P(y))$ [106], where ~ stands for

[102] For '100.000', see Elia Zardini, *Non-Transitivism*, author's draft, version of December 19, 2017, 2.
[103] Of course, people call *infinitesimal* those differences that fall below some threshold of detectability, but that is a different concept that would not block the paradox.
[104] Crispin Wright, "Further reflections on the sorites paradox" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997), 209.
[105] Kit Fine, "The possibility of vagueness", 3713.
[106] Pablo Cobreros, Paul Egré, David Ripley and Robert Van Rooij "Tolerant, Classical, Strict". April 2010. *Journal of Philosophical Logic* 41(2) DOI: 10.1007/s10992-010-9165-z-, 349.

the *similar enough* relation. Or, for Fine, the compatibilist semantics will see $\neg((A \vee \neg A) \wedge (B \vee \neg B))$ hold, together with $(A \vee \neg A)$ and $(B \vee \neg B)$ separately[107].

An objection can be raised: the tolerant relation (e.g. *being 1 cm apart*) either divides the *large difference* (e.g. *being 100 cm apart*) or does not. If it does not divide it, then we do not take Eubulides at his word of speaking of normal numerical differences of one grain, which can accumulate to difference of million grains, as discussed above. If it divides it, transitivity follows automatically, because the transitive closure of our relation contains the *large difference* relation. So, saying that a relation is *small enough to be tolerant* is either circular, because it assumes what must be shown i.e. namely that tolerance is compatible with divisibility or vacuous, because it does not meet the expectations of the paradox. The proposal by Cobreros, Egré, Ripley and van Rooij seems vacuous, since they also call the relation 'not visibly or relevantly taller or smaller' and 'indifference relation'[108]. It is hard to see how a non-transitive indifference relation such as theirs divides 100 cm.

That said, Cobreros, Egré, Ripley and van Rooij use the relation to define *tolerantly tall* as being either tall or in the respective relation with a tall person and *strictly tall* as being both short and not in it with a *short* (not-tall) person and they observe the failure of weakened NC and LEM: something can be neither "strictly tall nor strictly not tall, that is tolerantly tall and tolerantly not tall"[109] The intersection of them will be the borderline cases. This approach is similar with that given in the third part of this work, the difference being that I use classical FOL and use ranks of a total preorder (which is transitive) to define *broadly tall* (what they call *tolerantly)* and *strictly tall.*

In brief, divisibility of relations can be attacked by supposing tolerant differences are infinitesimal. However, this is counter-intuitive for what differences of measurements mean, i.e. that they are real quantities that can sum up to larger quantities. Philosophical attempts to bridge the gap and speak of *small enough* difference can be attacked as either circular, if they assume that divisibility is compatible with tolerance, or vacuous, if they deny divisibility.

In conclusion, the sorites is a second-order paradox that can be expressed in FOL if a finite upper bound on a predicate containing the soritical predicate and the soritical relation is specified. It relies on two premises: divisibility and predicate tolerance. Since divisibility seems impossible to attack, it

[107] See 12.5.4
[108] Pablo Cobreros, Paul Egré, David Ripley and Robert Van Rooij "Tolerant, Classical, Strict", 350.
[109] Pablo Cobreros, Paul Egré, David Ripley and Robert Van Rooij "Tolerant, Classical, Strict",385.

is understandable that most of its proposed solutions focus on predicate tolerance. We have seen that this is the same notion as the hereditary nature of some predicate such as *tall* in a weakening relation i.e. *having 1 cm more*.[110]

[110] See 2.5 for the distinction with predicate-ordering monotony which uses a *strengthening* relation.

Chapter 7

Semantic and metaphysical sorites

7.1. The sorites and the real

The sorites plays an outsize role in philosophical debate. There are more popular paradoxes, Zeno's paradoxes of motion among them. For example the Racetrack:

> "If a runner is to reach the end of the track, he must first complete an infinite number of different journeys: getting to the midpoint, then to the point midway between the midpoint and the end, then to the point midway between this one and the end, and so on. Since it is logically impossible for someone to complete an infinite series of journeys, the runner cannot reach the end of the track."[111]

Or so the proponent of the paradox claims. According to its most popular rejection, mathematical analysis proves the final sentence wrong: you can have an infinite series that sums up to 1 or any finite number[112]. Since they resist veridical formalization, this class of paradoxes do not occasion many logical developments nowadays.

There are, on the other hand, paradoxes which are impeccably univocal from a formal point of view, like set-theoretical paradoxes. For example Russell's paradox: *does the class that contains all classes that do not contain themselves contain itself?* Their existence has determined widely-adopted restrictions on set theory, so that contradictions are no longer constructible. Limited axiomatizations of set-theory, such as Zermelo–Fraenkel set theory with the axiom of choice (called *ZFC),* are the basis of mathematics and metalogic, including for alternative logics[113].

The sorites lies somewhere half way. It does not lack either popularity or logical expression. But its popular propositional formalization – a chain of modus ponens[114] – is neither specific to the paradox nor exhaustive for its workings. The philosopher has to add extra-logical qualifiers to make it work: supposing there is a predicate occurring twice at each step, supposing all other properties stay the same, etc. Its half-intuitive, half-technical character has been motivating the development of new

[111] R.M. Sainsbury, *Paradoxes 3rd edition*, 11.

[112] As Mark Sainsbury puts it in an exercise: "We can all agree that the series of numbers ½, ¼, 1/8, … sums to 1" R.M. Sainsbury, *Paradoxes 3rd edition*, 14.

[113] See Chapter 13 for what is to be a *ZFC logic* and some relevant philosophical issues for logics of vagueness.

[114] Or of one of the other three rules of inference in 2.1.

philosophical and logical approaches for the last hundred years. Those theories have remarkably wide scope: they redefine meaning (e.g. what is a predicate if some things seem neither to belong nor to not belong to it?), what is truth (e.g. are there other truth values beyond truth and falsity?), what is reality (e.g. are the notions of natural kinds and objecthood threatened by soritical arguments?).

The first two may seem natural, according to the general form which I proposed for the sorites. The meaning of common predicates seems to include both divisibility and tolerance. Therefore, a theory of the sorites must explain how natural language reasoning combines both exactitude and approximation. And, as Nicholas J. J. Smith argued, tolerance implies that a small difference in the justice of applying the predicate corresponds to a small difference in truth[115]. The truth, or *truths,* of attributing tallness cannot be much different between men of 171 cm and 170 cm[116]. But the third one, namely reality, is one step too far. If the question is: what does the sorites say about real objects, the answer for which I will argue is: not much, the sorites paradox does not apply to real objects.

A distinction must be made between what I will call *semantic* and *metaphysical* soritical forms. There are historically important metaphysical arguments discussed as sorites paradoxes, although they lack either soritical effect or soritical form. Let us cite three, in descending order of soriticality and ascending order of soundness. Wright followed Esenin-Volpin in building a sorites for *childhood* with a soritical relation of *having one more heartbeat* which we discussed (and rejected) as (Beat) in 5.1.1 above, seeming to prove that childhood never turns into adolescence[117]. Williamson argues against global nihilism i.e. the claim that the sorites proves that no common predicate has positive cases, by saying that some sorites paradoxes could be built on such predicates as *be an electron* (because of quantum indetermination) and *be a person* (because of variable spatio-temporal boundaries)[118]. Thirdly, Unger and Geach created *the problem of the many* by claiming that an object such as the cat

[115] He argues vagueness to be *closeness* which he defines as "If a and b are very similar in F-relevant respects, then 'Fa' and 'Fb' are very similar in respect of truth." Nicholas J. J. Smith, "Vagueness as Closeness", *Australasian Journal of Philosophy*, (Vol. 83, No. 2, June 2005), 164.

[116] By *truths* I mean here the positive cases of the predicate. That is, the logic of vagueness proposed in the third part of this work will argue that if 100% of persons having 190 cm are tall, almost 100% (say 99%) of persons having 189 cm will be tall too. That is, replace Smith's diminishing degrees of truths with statistical measurements over ranks of cm, which can expressed in FOL. See Chapter 16.

[117] Crispin Wright, "Language-mastery and the sorites paradox", 155-156. Wright's further claim i.e. that no child becomes an adolescent is as assailable.

[118] Timothy Williamson, *Vagueness,* 170.

Tibbles cannot be identified out of all the sets that share all its atoms except for those in one respective individual hair[119].

The analysis of the sorites developed until this chapter will provide criteria to isolate such metaphysical arguments. That is, soritical forms about real entities fail of soritical effect in one of two interconnected ways in which semantic arguments do not. Either because there are rare counter-examples to the major premise, such as a child being mechanically ventilated from 12 years to 20 years old, against Wright's argument. Or because their conclusion conflicts with scientific truth, such as quantum mechanics for Williamson's argument for *electron.* Finally, sounder metaphysically minded arguments abandon soritical form, as Unger and Geach's argument.

I thus intend to isolate metaphysical arguments wholly within the minimal methodology announced at the beginning of Chapter 5, without assuming any metaphysical or semantic theory myself. Here I'm only interested in semantic and metaphysical soritical forms, not in the distinction of semantic versus metaphysical causes of vagueness in the literature[120]. I rely on two theses which I think are rooted in our linguistic capacity: (1) to be an object is to have some separation and (2) in common reasoning, debate over meaning stops when scientific truth is invoked. They will be argued for.

7.2. Classes of separation and precision

Some arguments fail of paradoxicality, even if they have soritical form. In 5.1 soritical effect was defined as the capacity to derive a contradiction. Five situations in which the paradoxical character is missing were identified:

a) The chaining interval is not adequate.

b) The argument uses a poor cousin of the preferred ordering of the predicate.

c) The predicate is not distributed across the soritical relation.

d) The predicate has a coarse soritical step.

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[119] See Peter Unger, "The Problem of the Many", *Midwest Studies in Philosophy* 5 (1, 1980) and Geach, P. T., *Reference and Generality. 3rd edition*, (Cornell University Press, 1980), 215. 'Tibbles' is the name of Geach's cat, but my presentation of the argument is somewhat different from Geach's and Unger's.

[120] For a classification of these views, see Matti Eklund, "Metaphysical Vagueness and Metaphysical Indeterminacy", *M. Int Ontology Metaphysics* (2013) 14. https://doi.org/10.1007/s12133-013-0119-0.

e) The predicate has a standard of separation.

The first two are failures of construction. An adequate interval can be chosen and the preferred ordering can be used, to get both acceptable premises and an unacceptable conclusion. The last three reasons are due to peculiarities in the soritical predicate. *Material* may not seem too precise, that is, it casts doubt on the justice of equating non-soriticality with precision. But it has no apparently tolerant ordering under which some cases are positive and some are negative: there are objects of zero or negative weight. This peculiarity seems however easy to isolate by the standard proposed, i.e. distribution of the predicate across the soritical relation. Also easy to isolate are arguments which lack soritical effect because of coarse soritical steps. If you disagree that ten cm are tolerant for *tall* (that is, that *tall* is hereditary in the relation of having ten cm less), then restate the argument using 0.1 – one mm – as numerical step.

What is more difficult to isolate is the peculiarity of the predicate having a standard of separation. The definition which I have proposed equates it with a single numerical figure separating negative from positive cases. The figure is not a formal property of the paradox. You can think of it as a paradox solver's best tool: our task is to point out when soritical form lacks soritical effect. And our best tool is finding an acceptable numerical figure separating negative from positive cases. For example 0 centigrade for *to be solid water*[121] or 180 cm for *to be a man of 180 cm.* But suppose that the cases (both positive and negative i.e. the domain) can be split in relevant classes of varying size, each with its own such numerical figures. For example, the predicate *to be studying in a pre-graduation year* has one such numerical threshold for each graduating class: 2019 is the last such year for students graduating in 2020, while 2021 is the last such year for those graduating in 2022. The negative cases belong to the subclass corresponding to those elements that are not years or are associated with some class of students which graduated. Similarly, *to be a man of an even number of cm* has one such numerical threshold for each even number, that is, an infinite number of numerical thresholds. Since a soritical form for *to be a man of an even number of cm* or *to be studying in a pre-graduation year* has no soritical effect, we have a new tool. Let us call it *class of separation,* when there is a set of numerical figures such that soritical major premise(s) would fail for any subclass of our original soritical predicate. Then the argument is not paradoxical when such a class of separation exists.

[121] If this seems a definitional truth – Celsius scale being defined together with zero as switching point from liquid to solid – just replace that predicate example with *Is not water at its maximum density*. Since at four centigrade water has maximum density, that argument would not have soritical effect, that is, it has a standard of separation of 4.

This is a minimal way to exclude threshold arguments. If there are such numbers separating positive and negative cases, the predicate may be called *precise.* However, it is not an easy matter to hold fast to precision i.e. always have a readily available number in any situation. Even paradigmatically precise predicates, if put to common use, seem to lose it. See statements: 'The builders strived to achieve a right angle between the wall and the ceiling', 'I want this staircase to have an odd number of meters' and 'Since my height is 179.5 cm, I am a man of 179 cm'. The predicates thus seem capable of displaying soriticality, just as *tall* does, when used in everyday situations, especially under increasing use. What this shows is that the distinction precision-vagueness may not overlap with the distinction between bivalence-failure of bivalence. What precision means under my analysis is simply that a set of numbers is readily available (either through common knowledge, law or science) such that all situations are classified as either positive or negative cases under them as measurement (e.g. people of 190 cm are classified as not having 180 cm by the measurement of height which outputs the number 190). On the other hand, bivalence does not require that a number separating tall from not tall people exist, but only that each man be either tall or not, even chaotically so in terms of their height in cm.

The third part of this work will see multidimensionality illustrate this idea. Suppose tallness radically depends on a long list of natural dimensions: height in cm, degree of kyphosis, domicile, hair arrangement, date of classification as tall, etc. And, suppose than, as discussed in 5.3.4., no two people can have all such determining measurements in common except number of cm, i.e. that there is no class of comparison containing people only differing by number of cm and nothing else of relevance[122]. Then bivalence may hold – any person be either tall or short – although we cannot readily cite the numeric justification in cm i.e. *tall* is not precise. This is even compatible with the idea that, on inspection, such numbers may be found through a cumbersome procedure. Were a wise judge to know all the worldly facts about one person, he may have a justification for why they are tall or not. Since it would be an investigation about a single case, it would not make *tall* precise, even though bivalence held.

[122] By indiscernibility of identicals, different individuals need have different properties. If all vague predicates depend on all such properties uniquely combined, as discussed in Chapter 14, the argument that a common class of comparison can be found for more than one individual can be escaped. For such an argument, see Rosanna Keefe, "Vagueness without Context Change", *Mind*, New Series, Vol. 116, No. 462 (Apr., 2007). Keefe's argument was raised against the contextualism of Stewart Shapiro that uses varying extensions and anti-extensions (it may be called *context-related shifts*) in place of statistical expressions as the present work does. See Stewart Shapiro, *Vagueness in Context*. (Oxford University Press. 2006), vii.

7.3. Theory of separation

What is there to be done when a class of separation is not available, but we have good reasons to assert its existence?

Let us examine another version of (BigBang) of 6.1 above, abandoning references to motherhood:

(BigBangSimple.m) The Earth in 2010 was peopled by humans.

(BigBangSimple.M) If the Earth in year *n* was peopled by humans, so was the Earth in year *n-1*.

(BigBangSimple.C) The Earth in 4 billion B.C. was peopled by humans.

There was an Earth in 4 billion B.C., so the conclusion is paradoxical in the right way, being contradictory with an acceptable secondary minor premise:

(BigBangSimple.m.2) The Earth in 4 billion B.C. was not peopled by humans.

Let us read it along the lines of the (M.Individual) variant of (M.Universal), with constant 'e' for *Earth*.

(BigBangSimple.m.formal) $\forall x . x = e \wedge C_{2010}x \supset Tx$

(BigBangSimple.M.formal) $\forall x (x = e \wedge C_{n}x \supset Tx) \supset \forall x (x = e \wedge C_{n-1}x \supset Tx)$

The place of 'a man of 190 cm is tall' is taken by 'to be the Earth in 2010'. And read 'T' here as *to be peopled by humans*. The former would be more natural to formalize through a modal operator, but suppose we can give a sense to it. The predicate that subordinates *to be the Earth in 2009* is *to be peopled by humans*. Yet the argument does not seem to play on any specificity of this predicate, neither of its sub-components such as *human* or *population.* The conclusion is still paradoxical with *containing multicellular organisms* as soritical predicate:

(BigBangSimple.2.C) The Earth in 4 billion B.C. was containing multicellular organisms.

That happens while 'multicellular organism' is a scientific term little encountered in natural language reasoning, the domain of the sorites. There is no standard of separation here i.e. we cannot cite the year in which humans or multicellular organisms came to exist. And no class of separation either, as in any case, there are no relevant groupings of *to be the Earth,* so the class of separation would be here the singleton of the standard of separation. But we do have a true scientific theory of how these entities came into being: some mutations were selected and human DNA appeared. Multicellular life as defined by biologists began in the eukaryotes sometimes after 2 billion B.C. So it is sensible to reject the major premise(s), as we did for previous threshold arguments, but without being able to give the standard of separation.

Let us call *theory of separation* the scientific truth implying that a class of separation exists, i.e. that all cases can be divided exhaustively in some sub-classes such that all either (1) contain exclusively negative cases or (2) have one respective numerical threshold separating negative from positive cases. Here, we understand scientific truth in a wide sense, including cartography for the limits of oceans, resolutions of astronomy congresses for the definition of planets[123] and so on, as long as normal linguistic behavior is to pay deference to such truth. In Diana Raffman's term, such scientific truths are *legislative*[124]*.* For example, it is true that humans appeared after previous hominids. And it is true that atoms can collide with other non-overlapping atoms, so there is such a thing as an individual atom. And that multicellular organisms appeared on Earth. And that the Pacific Ocean stops at the Bering Strait where the Arctic Ocean begins.

Then an argument has no soritical effect when there is a theory of separation, because the major premise is unacceptable. As general as this is, it encodes usual linguistic practice. There is such a thing as a *public and dynamic* part of the meaning of *human* i.e. that which a person using the term must review and adopt to remain a competent speaker. That part is still modified by scientific theory such as paleontology and reasonable speakers pay deference to it. Nobody would find it convincing that the major premise (BigBangSimple.M) is sound and, hence, that there were humans in 4 billion B.C. That is because it is a scientific truth that humans appeared long after – and historical astronomy affirms that we can speak of *years* for the epoch of their advent – which means that the major premise is unacceptable.

A similar objection applies to Williamson's contention that a sorites could be built for 'there are electrons'[125]. Namely, that any major premise that would purport to extend the electron such that it is much larger than what it is, or that claimed that electrons have mass by a progression on mereological relations of quarks is not acceptable, as it conflicts with scientific truth. And Williamson cannot claim that such a sorites has intuitive basis. There is no conceptual meaning of common scientific vocabulary that can be used to invalidate scientific laws, otherwise science would not be

[123] Jenny Hogan, "Pluto loses planet status", *Nature*, Accessed on Oct 15, 2019 https://www.nature.com/news/2006/060821/full/060821-11.html.
[124] Diana Raffman, *Unruly Words. A Study of Vague Language* (Oxford University Press, 2014), 15.
[125] Timothy Williamson, *Vagueness*, 170.

empirical so as to apply the scientific method. It is true that the identity of some particles may be undetermined under the standard interpretation, but that does not mean that the existence of atoms or electrons is in doubt.

7.4. Rare counter-examples of separation

Let us now take the grain of millet paradox discussed in Chapter 4, having as major premise(s) and conclusion:

(Millet.M) If *n* grains do not make a sound, *n+1* grains do not make a sound.

(Millet.C) One million grains do not make a sound.

I argued that this premise is unacceptable because science holds that humans have a sensorial threshold. Any sound pressure below some person-specific number of micropascals cannot be heard by the respective person. But, as before, this is a *theory* of the existence of a numerical threshold, not a numerical threshold per se. Wikipedia cite a general figure of 20 micropascals for unimpaired young adults, which is surely a median and most likely conveniently adjusted to the closest round figure. For example, I may have 25.167 micropascals.

The situation may seem similar with (BigBangSimple) above. But there is a catch. The theory that humans have such hearing thresholds holds for some, even most humans, but not for all. There are completely deaf people. Therefore, we cannot affirm that (Millet) fails of soritical effect because of the existence of a class of separation i.e. that all situations can be so split as to have a respective numerical threshold. For a deaf person, no number of grains between one and one million makes a sound. The correct reason for which (Millet) fails of soritical effect is that, in most cases, thresholds exist and when they do not, the conclusion is not paradoxical. It is certainly acceptable that one million grains do not make a sound for deaf people.

To lay the argument, Zeno's paradox assumes *making a sound* is an observational predicate i.e. something makes a sound just in case someone can hear it. Objecting to this equivalence, as critics of Berkeley's immaterialism have done[126], would leave no basis for starting the paradox. Why would we

[126] Bertrand Russell, *The Problems of Philosophy* (Oxford University Press, 1912), 14-15.

accept the minor, that is, that one grain does not make a sound, if sound exists even in an empty forest? Let us formalize the equivalence as:

(Millet.2.Def) $\forall x \,.\, Tx \leftrightarrow \exists y\,(Hy \land Bxy)$

'T' stands for *to make a sound,* 'H' stands for *to be a hearer,* and 'B' for *to be hearable for*. It states that a thing makes a sound just in case there is a hearer such that the thing is hearable for them. By This makes the universal reading of (Millet.M) equivalent with a claim about the number of hearers:

(Millet.M.Univ) $\forall x\,(C_n x \supset \neg Tx) \supset \forall x\,(C_{n+1} x \supset \neg Tx)$

(Millet.2.M.Univ) $\nexists xy\,(C_n x \land Bxy \land Hy) \supset \nexists xy\,(C_{n+1} x \land Bxy \land Hy)$

Read 'C_n' as *to be a group of n grains.* The former is the original formulation: if *n* grains fail to make a sound, *n+1* grains will fail as well. The latter says that if nobody hears *n* grains, nobody hears *n+1* grains. The latter is much less acceptable than the former, once a single case of a person with an auditory threshold can be cited. This illustrates the illuminating role of formalization. There is no paradox in that for all people out there, one grain cannot be heard, but there is one person, say Mary, who can hear two grains. The negation of (Millet.2.M.Univ) is

$\nexists xy\,(C_n x \land Bxy \land Hy) \land \exists xy\,(C_{n+1} x \land Bxy \land Hy)$

The first conjunct is derived from previous chained steps and ultimately from the minor premise. And our Mary provides the second conjunct. The paradox is not saved by the coherence or existential reading above. The former is sensitive to the same counter-example while the latter delivers an acceptable conclusion i.e. that there are people that cannot hear million grains. Suppose the proponent of the paradox suggests that *not making a sound* should be true when at least one person cannot hear the grains. They can try something like:

(Millet.3.M) $\exists x \,.\, Hx \land (\forall y\,(C_n y \supset \neg Byx) \supset \forall y\,(C_{n+1} y \supset \neg Byx))$

Of course, this is short for:

(Millet.3.M.2) $\exists x \,.\, Hx \land (\forall y\,(C_1 y \supset \neg Bxy) \supset \forall y\,(C_2 y \supset \neg Bxy)) \land (\forall y\,(C_2 y \supset \neg Bxy) \supset \forall y\,(C_3 y \supset \neg Bxy)) \land \ldots \land (\forall y\,(C_{1M-1} y \supset \neg Bxy) \supset \forall y\,(C_{1M} y \supset \neg Bxy))$

This is the existential reading in disguise. It says that there is always a human such that if *n* grains are not hearable for them, *n+1* grains will not be hearable for them. But this is surely true as regarding deaf people, so the conclusion is not paradoxical.

Let us call our single counter-example, Mary, *a rare counter-example of separation.* The fact that she cannot hear one grain but can hear two grains defuses the paradox, so we have a counter-example. It is *rare* because we can concede that most cases are not like that, without conceding the argument. Deaf people cannot tell any difference of grains, and maybe there are people who experimentally are all over the map[127], but it is a scientific truth that people like Mary exist. And a single case is enough to make the major premise(s) unacceptable.

7.5. No real entity is soritical

Return to Wright's argument for *childhood* with a soritical step of one heartbeat. As argued above, as long as there exist both infancy and adolescence, there is no initial acceptable minor premise, because of varying heart rhythms. But under the assumption that there is no infancy and a replacement of childhood, say *childhood*$_2$ starts at one heartbeat, Wright's argument seems to go through:

(Child2Beat.m) A person whose heart beat one time is a child$_2$.

(Child2Beat.M) If a person whose heart beat *n* times is a child$_2$, so is one whose heart beat *n+1* times.

(Child2Beat.C) A person whose heart beat one million times is a child$_2$.

In this case the preferred ordering relation of *child*$_2$, namely that by time, seems to be reasonably captured by the soritical step. But it is not. Suppose someone's heart stops at the age of twelve and their blood is oxygenated mechanically for eight years. Then one heartbeat would surely make the difference between childhood and adulthood. The argument fails, once again, because it wandered in the realm of empiric discovery and medical science validates a rare counter-example to its major premise(s).

There is also an argument about cities, such as:

(Moscow.m) A place that is one meter away from St Basil's Cathedral is in Moscow.

(Moscow.M) If a place that is *n* meters away from St Basil's Cathedral is in Moscow, so is a place that is *n+1* meters away from St Basil's Cathedral.

[127] A cursory reading of the methodology of measuring individual auditory thresholds supports this. E.g. Kathleen Campbell, Tanisha Hammill, Michael Hoffer, Jonathan Kil, and Colleen Le Prell, "Guidelines for Auditory Threshold Measurement for Significant Threshold Shift", *Otol Neurotol*, 2016 Sep; 37(8): e263–e270. doi: 10.1097/MAO.0000000000001135.

(Moscow.C) Smolensk (that is, outside Moscow) is in Moscow.

But there are rare counter-examples to the major premise(s), for example the sign posts saying 'Leaving Moscow' on the road to Smolensk, so it does not go through.

The same happens for mountains*:*

(Mountain.m) A place that is at one meter from the K2 summit is in the Himalayas.

(Mountain.M) If a place that is at *n* meters from the K2 summit is in the Himalayas, so is a place at *n+1* meters from the K2 summit.

(Mountain.C) New Delhi (i.e. thousand km away from K2) is in the Himalayas.

This conflicts with cartographic, i.e. scientific, truth. And the major premise(s) have rare counter-examples of separation. In the South Iberian Peninsula, one meter can make a difference between being on the Rock of Gibraltar or rather in the eponymous Strait. Even with a step of one molecule, all air molecules in the direction of the sea are not part of the Rock.

Someone may say that there are neighborhoods, continents, regions or seas that simply fade into one another[128]. But this can be shown to be false. Be it either the Bronx, Europe, Cappadocia, the North Sea[129] or any other, I could not find an example that lacks at least a sign post saying 'Welcome to …', or some cartographic definition. When anchoring an oil rig, one meter makes a difference between being in one sea or another. As long as rare counter-examples exist, the major premise(s) can be rejected.

There is the possibility of asking the question, as they say, in the abstract[130]. Suppose there is a mountain that slowly recedes into a plain. You accept that being on the summit is being on the mountain. And that one meter cannot make the difference between being on the mountain and being on the plain. So we conclude that even at the lowest point on the plain we are still on the mountain. This is indeed a sorites in both form and effect, which we will call *semantic.* It uses the natural kinds of *mountain* and *plain* to build a soritical argument. Note that there is no real mountain for which we

[128] Timothy Williamson, *Vagueness*, 256.

[129] Gottlob Frege, *The Foundations of Arithmetic* (New York, Harper & Brothers, 1960), 34.

[130] See for example Peter Unger's argument about tables in general applying to real tables: "Having made our partitioning imaginatively, we then envision a physical process occurring to the putative table as follows. First, one fifth of the table is sliced off and ground to a find dust, perhaps even rendered into separated atoms. The dust, if not atoms, is then scattered to the winds, or sunk speck by speck into widely dispersed regions of the sea. Then a second fifth is sliced, ground and scattered, and so on. Now, when we are down to our last fifth, there is, quite clearly, no table present.", Peter Unger, "There Are No Ordinary Things", *Synthese*, Vol. 41, No. 2, (Jun., 1979), 132.

could build such a sorites, so no real objects of which to speak. For real mountains, maps show neighboring formations, sign posts have been installed, trekking guides have been sold. Even if you go on some unnamed hill and try to run the argument from there, to have your naming of the hill recognized, at least some of its boundaries must be defined and its distinction from other geographical forms affirmed. In which case it will stop being a matter of no concern to cartography, that is, to science.

Counter-examples can be constructed even for those semantic soritical arguments relying on natural kinds which let reality intrude. If the soritical step is *being one meter away from the summit*, suppose our mountain slowly fades in height but there is a single trench-like continuous line somewhere lower than half the distance from the summit to the lowest point. That would make the major premise(s) fail, because being on one side of the line or on the other looks like a good demarcation. Even the Heap is based on a natural kind. So take its minor premise that *one million grains make a heap.* And suppose the grains, instead of being collated by normal water droplets, are collated by some strongly adhesive substance that makes them into a lump. Then they would not count as a heap. This shows that assuming no empirical specificity is doing heavy lifting to ensure the acceptability of a semantic sorites based on natural kinds.

The moral is that it is unavoidable for mountains, cities, or clouds to have *some* limits, even if those limits are not the same in all directions, e.g. city areas are not circles. Being an object is being separated from other things at least in some place. Because of this, such objects cannot feature in soritical predicates for a convincing sorites. But the question of what precisely their borders are and why there are many divergent ways of specifying their full boundaries generates a veridical paradox, called *the problem of the many,* which I will present in the next subchapter.

Note that under a (M.Identity) reading, we would have the metaphysical tolerance we are looking for, but at the cost of the paradoxical character of the conclusion. That is, if it were possible to use a statement such as ‘A place that is identical with one in Moscow except one meter further away from St Basil’s Cathedral is in Moscow’. Because our sign post would then still be in Moscow even if moved, just as the Rock of Gibraltar’s slope facing the water would still be in the Rock of Gibraltar even if moved. But this cannot be readily formalized, as discussed in 5.3.4, and can be seen as metaphysically circular. The reason it works is precisely that we keep the two objects’ mereological relation with Moscow or Gibraltar, while modifying an accidental property such as distance. The last slope of the Rock of Gibraltar would be a part of the Rock of Gibraltar if moved a little further i.e. the Rock of Gibraltar would have grown itself by one meter. And the argument could thus stretch the Rock of

Gibraltar all the way to South Africa, by moving its southern slope further and further south. But this is not unacceptable, i.e. paradoxical. The soritical effect would be still lacking, as city expansion and geological change happen without contradiction.

Contrast this with our reading for (Blond) in 5.3.4. Under the identity reading, we saw the argument go through: if a blond man starts with one million hairs and loses one hair at a time, it is a paradoxical statement that he's still blond at zero hairs. Is Moscow lost all its meters, it would have simply disappeared. This shows that (Blond) also raises a semantic issue, unlike the problem of the many below.

7.6. The problem of the many

No sorites argument for real objects goes through. We cannot find acceptable major premise(s) extending Moscow to Smolensk. But suppose we moderate our aims. In the first case, what if we choose a non-numerical soritical step and only argue that no real mountain ever finishes even when its slope faces the sea? Which of the rare grasses on that slope are to be included in the Rock of Gibraltar or not? Similarly, the sign posts saying 'Leaving Moscow' were decisive in the latter argument. But are the sign posts themselves in Moscow or outside Moscow?[131] Or rather half the plastic is in and half is out? Timothy Williamson's *electron* argument discussed above is also amenable to this reading: how can we say that some object is an electron if we cannot determine its position and momentum? This is giving up soritical form and going over to the problem of the many.

For a formulation, suppose there are a thousand hairs close to the flesh of Tibbles the cat. Some are strongly attached, some are loosely attached, some are resting on others and some are already on the ground. Then define cats $cat_1 \ldots cat_n$, with $n = 2^{1000}$, where cat_m , $1 \le m \le n$, is the union of all the non-hair atoms in common with Tibbles and the other defined cats, with only one possible combination of the thousand hairs. That is, a hair combination is any member of the power set of the thousand hairs. Tibbles is clearly one of $cat_1 \ldots cat_n$, but many seem equally good candidates.

[131] Mehlberg is an example of conflating vagueness and the problem of the many in what regards cities: "The term 'Toronto' is vague because there are several methods of tracing the geographical limits of the city designated by this name, all of them compatible with the way the name is used. It may be interpreted, for instance, either as including some particular tree on the outskirts of the city or as not including it" Henryk Mehlberg, "Truth and vagueness" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith, 86.

Then, suppose we create a predicate for having the particular set membership of *cat_1*, say C_1, and argue that C_1 is sufficient for being Tibbles. And do the same up to C_n. That gives:

(Cat.m.1) There exists an object that is a C_1 and it is Tibbles.

(Cat.m.2) There exists an object that is a C_2 and it is Tibbles.

..

(Cat.m.n) There exists an object that is a C_n and it is Tibbles.

This does not look at all like our familiar (CS). The argument does not involve successive concept subordination between step predicates, is not a chained argument and there is no paradoxical conclusion. The problem is rather that *by definition* we have introduced a number of cats out of which we are not able to choose one, even though we need a single cat to mollify our strongest intuition: that there is only one Tibbles in the room. As put by Brian Weatherson: "the Problem of the Many is not a problem about change"[132]. A difference of two hairs makes cat_m and cat_{m-1} equally valid candidates to be the intuitively single cat in the room, while a difference of two hairs between steps in a sorites means that all elements with those two numbers are as bald. The first is a question of determining identity, the second is a question of determining applicability of predicates.

Let us see how the formalization of the problem of the many involves identity. A problem of the many can be derived from a soritical form in two steps:

a) Replace tolerance with equal justice to identify

Instead of a property such as *tall* being hereditary in a small-difference relation, replace it with the *justice to identify* being common to at least two entities connected by some relation. That is, there are at least two sets of atoms differing by the atoms in two hairs, such that both can be identified with Tibbles with equal justice. The difference is (1) limited tolerance, because not all sets differing by atoms in two hairs can equally be called Tibbles, e.g. Tibbles with a dog hair ten feet away would not be Tibbles (2) replacing the soritical predicate with a construction that is half-way between predicate and identity, namely *justice to identify as*. Reading 'T' as *identifiable with Tibble*s and 'S' as *differing by*

[132] Brian Weatherson, "The Problem of the Many", *The Stanford Encyclopedia of Philosophy* (Winter 2016 Edition), ed. Edward N. Zalta https://plato.stanford.edu/archives/win2016/entries/problem-of-many.

the atoms in two hairs, we can state this limited tolerance. There are two S-connected elements such as both are just to identify with Tibbles:

(POTM.m) $\exists xy \,.\, Sxy \wedge Tx \wedge Ty$

b) Replace divisibility with non-identity

State that the relation in question, i.e. *differing by the atoms in two hairs* guarantees non-identity between its arguments. This is very reasonable, since sets differing by any number of elements are not equal, by extensionality[133]:

(POTM.L) $\forall xy \,.\, Sxy \supset x \neq y$

Then, the conclusion to be derived is that justice to identify does not imply identity, i.e. there are distinct objects, both just to identify with Tibbles:

(POTM.C) $\exists xy \,.\, Tx \wedge Ty \wedge x \neq y$

In brief, the paradox of the many is a paradox about identity, which ensures its relevance for real objects. It is not about heredity in a relation i.e. predicate tolerance, so not a sorites.[134]

7.7. Classification

I now classify arguments that have been cited as sorites in the philosophical literature into four types:

a) *Gradable soritical arguments*: They use a gradable adjective (e.g. *tall, bald*) as soritical predicate. The paradox is generated by the transition from negative to positive cases through a tolerant soritical relation.

[133] See 13.4 for a discussion of extensionality and FOL.

[134] Contrast it with our individual universal reading for the major premise of (Few) in 5.3.4 above, namely 'If n are few, so is n+1': $\forall x\ (x = n \supset Tx) \supset \forall x\ (x = n{+}1 \supset Tx)$. This is not about identity in the same sense, because, as indicated, each antecedent can be read as a naturally distributed predicate having a single positive case.

b) *Natural soritical arguments:* Same as above, except they use a non-gradable adjective, a natural kind or a genus pertaining to real things (e.g. *heap*, *door,* etc.). As do their gradable counterparts, they are generally not topics of scientific investigation, reason for which there is no theory of separation or rare counter-examples to deprive them of soritical effect, at least under normal conditions.

c) *Metaphysical soritical arguments*: They use an entity: either abstract object, natural kind, or concrete object (e.g. *childhood*, *atom, Everest*), with a soritical step embedding a metaphysical relation, either generational, mereological, temporal, etc. The paradoxical character comes from going over the real boundaries of the entity. They fail of soritical effect because there is a theory of separation i.e. they conflict with scientific truth or there are rare counter-examples that falsify the major premise(s).

d) *Metaphysical problem-of-the-many arguments*: They use the same type of entity and metaphysical relation as the previous type. But they replace the soritical form with a different logical form, focused on the relation between justice to identify and identity. They have paradoxical character, originating in the variety of available ways to draw boundaries for an entity.

I say that the first two types are *semantic*, because there is no recourse to scientific truth with them. The difference is that those based on natural kinds require an assumption of normal empirical conditions, otherwise they would become metaphysical according to this classification. The final two types are *metaphysical* because (1) metaphysical soritical arguments that purport to be paradoxical are relevant to scientific truth, namely by conflicting with it (2) the metaphysical problem-of-the-many arguments are generable from metaphysical soritical arguments by keeping the metaphysical relation therein and replacing the soritical form, as we have seen.

Some things can be said of the deeper causes of this distinction, without wandering into metaphysics[135]. On the semantic side, soritical predicates such as *tall* or *heap* are common and observational, having a rich predictable meaning on which the paradox plays. Under our (M.Universal) reading, this facilitates concept subordination i.e. all objects that have each step predicate behave similarly towards the soritical predicate. But many metaphysical relations such as temporal or mereological ones are individual, uncommon and not observational. There is not much in common as to tolerance towards the predicate of *being in Moscow* between the sign post on the

[135] Issues close to those raised by the problem of the many are Theseus's ship, fission, and the notion of *substance.* For fission see Kit Fine, *Indeterminate Identity, Personal Identity and Fission*.

highway saying ‘Leaving Moscow’ and some sign post that is equally distanced from downtown but is safely inside a Muscovite neighborhood. If there were, i.e. if all places in Moscow were one-meter tolerant towards that predicate, we would be able to construct a semantic natural sorites. The universal reading is the key to rejecting the metaphysical sorites, proving that a rare counter-example suffices.

Finally, our appeal to scientific truth needs a defense. We have seen in Chapter 4 that any citation of a law defining *young* would stop the argument for this predicate, at least in what concerns the legal domain – which is not small. Science seems to achieve something as strong: in common language reasoning, it is enough to cite the fact that there is a contrary scientific truth, for any debate on the meaning of terms to stop. Having to choose between science and the meaning of common terms in areas that are prone to scientific research, there is no choice for reasonable speakers. But what about soritical arguments held between scientists while making science? Does not our criterion just move the paradox a little further, since scientists in this situation will not be able to make appeal to scientific truth? There is a distinction to be made. On one hand, from a sociology of science perspective, scientists do indeed use common language reasoning, at least in contexts less formal than scientific publishing. But in such contexts, there are still the theories which constrain them. The fact that a paleontologist runs (BigBangSimple) by his colleagues would not make it any less conflict with scientific truth. On the other hand, from a theoretical point of view, science abhors contradictions. If some definitions imply at the same time what we called divisibility, tolerance and adequate natural distribution of a predicate, the resulting contradiction would force the refinement of those definitions. But that would be a semantic paradox in the current sense, supported only by definitions that are unamenable, by their status as definitions, to empirical discovery. They will be changed as to better match it.

Chapter 8

Interrogative sorites

The forms of the sorites we discussed are all discursive arguments. When used in Antiquity, not only were they ended by a moral on the lines of Galen's in 2.5 above, but they were often expressed in an interrogative form. Mark Sainsbury and Timothy Williamson cite the Bald Man as '*Is a man with one hair on his head bald? Is a man with two hairs on his head bald? Is a man with ten thousand hairs on his head bald?* '[136].

Suppose we want to treat it as a discursive sorites in disguise, using deontic modality to express compulsion or interdiction of speech. We get:

(OS) Deontic sorites

(OS.m) It is not permissible to say that a 200 cm man is short.

(OS.M) If it is not permissible to say that a man of *n* cm is short, it is not permissible to say that a man of *n-1* cm is short.

(OS.C) It is not permissible to say that a 100 cm man is short.

Obviously, we could have expressed inductive step (OS.M) as a number (OS.M.1-DS.M.200) of propositional steps. The argument only depends on modus ponens – or any other rule of inference in 2.1. The minor premise is justified, as nobody should say a falsehood, according to the standard moral commandment that lying is wrong[137]. And since men of 200 cm are not short, nobody is allowed to say that they are. And (OS.C) is in contradiction with a secondary acceptable minor premise:

(OS.m.2) It is permissible to say that a 100 cm man is short.

This is justified by the equally plausible moral principle that truths are allowed to be spoken.

Is the major premise (OS.M) acceptable? Tolerance as used throughout this work concerns alethic situations, namely the relation between the tallness of persons. The issue here is whether this extends to modalities of truth. Such an extension clearly fails when the modalities are large temporal

[136] Mark Sainsbury and Timothy Williamson, "Sorites" in *A Companion to the Philosophy of Language*, ed. Bob Hale, Crispin Wright, Alexander Miller (Wiley Blackwell, 2017), 734.
[137] Immanuel Kant, *On a supposed right to lie because of philanthropic concerns*. Accessed Oct 14, 2019 https://pdfs.semanticscholar.org/aae1/988d5c2b465091316993bd1d1ecbddc26940.pdf.

differences e.g. '*If one year ago all men of n cm were short, one hundred years ago all men of n+1 cm were short*' or future modalities, e.g. *If in two millennia all men of n cm are short, in two millennia all men of n+1 cm will be short*'. In both cases the cause is the variation in what *short* meant for the past or will mean for the future. It also fails for belief, e.g. '*If it is believed that all men of n cm are short, it is believed that all men of n+1 cm were short*', because, of course, there may not be any opinion on men of *n+1* cm one way or the other. Williamson's variant of epistemicism uses a failure of epistemic modality in such cases[138], e.g. '*If it is known that all men of n cm are short, it is known that it is known that all men of n cm are short*' fails.

For our deontic modality, we justified (OS.m) on the basis of the falsehood of its propositional content. This cannot block denial of (OS.M). Suppose there was an *m*, 101 ≤ *m* ≤ 200, for which we affirm a straightforward negation of (OS.M):

(OS.n) It is not permissible to say that a man of *m* cm is short and it is permissible to say that a man of *m-1* cm is short.

Suppose *m* = 200, then (OS.n) is false, because a man of 199 cm is not short, so it is not permissible to say that he is short. But if *m* = 101, we do not have a reason to call (OS.n) false, since a man of 100 cm is short and it is permissible to say so. This means that (OS.M) cannot be justified by our interdiction of lying nor by tolerance in the sense used up to now.

If the counter-examples involving temporal operators are acceptable, so is the following. We have seen above that for some purposes, legislative activity restricts the tolerance towards some predicates. Suppose the military works according to a regulation classifying recruits as *short* below 165 cm and *tall* all the others, to ease the adaptation of military gear for body type. Such activity is plainly permissible (in real life) although forbidden by (OS.M), so (OS.M) should be rejected. But what if it is replied that such stipulations create different *precisified* predicates which we may call $tall_2$, $short_2$ and so on. Just as we used 'childhood$_2$' as distinct from 'childhood' when a different meaning was needed as an example, future speakers and current law may use 'tall' for $tall_2$, but (OS.M) still may hold for our common *tall* and *short*. However, institutional stipulation is not philosophical stipulation. Contrary to the latter which is only accepted under the reserve of being able to act as if it did not happen, stipulations performed by government, academies, great writers, etc. may become part of the meaning of our common terms such as *tall* and *short*[139]. If we accept, as above, that the

[138] See 12.4 below for a discussion of Williamson's epistemicism.
[139] I will argue in the third part of this work that common soritical predicates are non-monotonous with their main determination relation. For example, tallness may be determined mainly by scalp-to-toe measurements, but also

same predicate *short* had different boundaries for the past and for the future, as we did above, such change may happen. So not every kind of stipulation should be forbidden, viz. modifying rather than inventing meanings. Thus, (OS) is not acceptable.

Another attempt to transform the interrogative sorites into a discursive one leading to a contradiction is by using a *publicity* principle: ‘A is true just in case everybody must say that A’. Then you argue from everybody being forced to say that ‘A 200 cm man is tall’ to everybody being forced to say that ‘A 100 cm man is tall’. By the publicity principle, the argument leads to a straight contradiction. But the principle is too strong: why should everybody say that Romance languages include Ligurian?

If we cannot turn it into a discursive argument, we need to seek independent practical principles to support the interrogative sorites. The addition of two such principles along the lines of ‘One must answer questions with “Yes” for affirmation or “No” for denial’ and a replacement of the major premise(s) with ‘If someone affirmed that a man of *n* cm is tall, they must affirm that a man of *n-1* cm is tall’ are sufficient to turn the argument into what has been called a *forced march*[140]. The forced march, played according to these rules, will bring out a near-contradiction. That is, either two answers which are contradictory among themselves or one answer which conflicts with one of the minor premises assumed when starting the march. At each step, your previous answer induces an identical answer. If you start by ‘yes’ as you should in the case of adapting our common (CS) to an interrogative form, it seems like you will go on with ‘yes’s up to a clear case where you should have said ‘no’. While the forced march is not a discursive sorites, it has been considered a stronger form of paradox when compared to it[141]. I will not attempt to discuss it in this work.

by body constitution, hair arrangement, kyphosis, legislative stipulation etc. While the exact proportions are an empirical matter, it is determinate in each case whether a man is tall or short.

[140] Horgan, T. “Robust vagueness and the forced-march sorites paradox”. *Philosophical Perspectives, 8: Logic and Language*, ed. J. E. Tomberlin. Atascadero, Calif.: Ridgeview (1994), 159.

[141] See Rosanna Keefe, *Theories of Vagueness* (Cambridge University Press, 2003), 211 and Elia Zardini, "Living on the Slippery Slope: the Nature, Source and Logic of Vagueness." (PhD diss. University of St. Andrews, 2008), 311.

Chapter 9

Recapitulation. What is the sorites?

The sorites is a little-by-little, numerical, chained paradox that advances in one of the four medieval modes of inference. Its inductive form embeds the core problem it raises, namely predicate tolerance. Predicate tolerance is the same notion as that of a predicate being hereditary in a relation (here, of *small difference*), first advanced by Frege. That *small difference*, such as *having 1 cm more*, *having one grain of sand more* and so on is accepted as tolerant for the respective predicate by common speakers, i.e. that two elements being in this relation cannot be one a negative case and the other a positive case of the predicate. A distinction was drawn between predicate tolerance and predicate-ordering monotony. Predicate tolerance is weakening the predicate application i.e. the relation is from more cm to less cm for tall or from less hairs to more hairs for bald, while predicate-ordering monotony is strengthening it, i.e. from less or equal number of cm to more cm for tall and from more or equal number of hairs to less hairs for bald.

The general logical form of the paradox supplements tolerance with divisibility i.e. a *large difference* such as that between 200 cm and 100 cm is divisible by the *small difference* of 1 cm i.e. the soritical relation. Therefore, the sorites is a second-order paradox, which is expressible in first-order logic if a finite upper bound on the number of elements is put. I argued that this is reasonable to do in common reasoning, which does not deal with issues of finitude, and only concerns objects of the natural world, which are limited in number.

The question of why some arguments using this soritical form are paradoxical and others are not leads to a consideration of soritical effect, that is, the property of a soritical form to have acceptable premises and an unacceptable conclusion. A distinction is drawn with threshold arguments, those arguments where one or more thresholds can be cited, that makes the argument lack soritical effect i.e. not paradoxical. I hope that this list of reasons for soritical form not guaranteeing soritical effect is exhaustive:

a) The chaining interval is not adequate.

b) The argument uses a poor cousin of the preferred ordering of the predicate.

c) The predicate is not distributed across the soritical relation.

d) The predicate has a coarse soritical step.

e) The predicate has a standard of separation.

f) The predicate has a class of separation.

g) The predicate has a theory of separation.

h) The major premise(s) has rare counter-examples.

The last two cases correspond to metaphysical arguments which have been sometimes classified as sorites paradoxes. A distinction of the sorites is drawn with the problem of the many, which shows that the latter is concerned with identity and real objects, while the former is a semantic paradox.

Part II

Vagueness

Chapter 10

How to approach vagueness?

Semantic sorites arguments are *convincing* paradoxes. The two general premises are acceptable. Divisibility is a mathematical notion which seems unassailable: the difference between a man of 200 cm and a man of 100 cm can be split in pieces of 1 cm each. Predicate tolerance, on the other hand, is so much used in everyday situations that it seems precious. If a recipe asks for a tablespoon of sugar, understanding it either as heaped or level will not much affect the end product[142]. And I would be shocked if my Spanish teacher told me that 'alto' is only said of people of 181 cm or over and 'no alto' of those below.

The naturalness of tolerance taken together with the exactitude of divisibility delivers a contradiction. No wonder then, the sorites has been used by those taking the side of naturalness against those taking the side of exactitude, and the reverse. In Antiquity, the Stoic belief in objective knowledge was attacked with soritical arguments by the Skeptics[143]. Nowadays, it is classical logic that is being cast in the role of the Stoa. This is not a wholly undeserved twist of events, since its founders at the end of the 19th century – Frege and Russell – rejected the indeterminacy of natural language[144], even by appealing to the sorites, as we shall see. But it was during the 1960s, when the attention of analytic philosophy shifted from science to common language, that the philosophical debate on what the sorites says about the relation between language, logic and reality started in earnest. The position of Frege and Russell came to be seen as dated[145] and the tide turned. An

[142] This example is inspired by that of cooking a risotto with one more teaspoon of butter or not in Elia Zardini, *Non-Transitivism*, 5. His example is discussed also at 13.3 below.
[143] Timothy Williamson, *Vagueness*, 11.
[144] For Frege, see Chapter 11. For Russell e.g. "All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life, but only to an imagined celestial existence" Bertrand Russell, "Vagueness" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997), 65.
[145] E.g. Jean van Heijenoort: "... Frege's disregard of vagueness and other vagaries was, in a way, inevitable. But his logical laws have been formulated more than hundred years ago, and it is now perhaps time to look at the vagaries". J. van Heijenoort, "Frege and Vagueness" in *Frege Synthesized. Synthese Library (Studies in*

enormous number of articles, books, conferences, or pieces of software, not only in philosophy, but also in mathematics, computer science and linguistics, have created several intellectual traditions parallel with that of classical logic, each with its own dynamic. Such traditions include at a minimum trivaluationism, fuzzy logic, supervaluationism, subvaluationism, nihilism, incoherentism, dialetheism, contextualism, sub-structuralism, etc. In their turn, they helped develop the epistemicist tradition of defending the classical logic of Frege and Russell, but without dismissing natural language as indeterminate.

The field of inquiry is known as the study of vagueness, but 'vagueness', as used academically, is a term of art[146], removed from its common meaning[147]. The English 'vague' partially overlaps with 'ambiguous', 'general' and 'uninformative', while philosophical 'vagueness' is a technical term for the field of study developed around soriticality and the relation between classical logic and natural language reasoning.

Based on our understanding so far, I propose two principles for a theory-independent investigation of vagueness. The first is the organizing principle: it is a study of the supposed inadequacy of classical logic for natural language reasoning. The limiting principle is that predicate tolerance needs to be explained i.e. the way in which things stand such that the sorites is so convincing. This distinction helps navigate the literature on vagueness. Philosophical positions cluster around their points of difference with the classical viewpoint, while trying to make sense of the sorites. Vagueness may not be soriticality, but it needs to explain it[148].

The next chapter contains Frege's views on logic, natural language and the sorites, which can be read as an introduction to the issues. Then, I will present and compare the main non-classical theories and logics which have been proposed to deal with vagueness, together with *specific* objections that can be raised to them. Finally, three *general* objections against non-classical logics for vagueness will be discussed.

Epistemology, Logic, Methodology, and Philosophy of Science), vol 181, ed. L.Haaparanta, J.Hintikka. (Springer, Dordrecht. 1986), 45.

[146] Timothy Williamson, *Vagueness*, 36. Williamson goes on to talk of vagueness as blurred boundaries.

[147] Kit Fine, "The possibility of vagueness", 3702.

[148] "Vagueness is that phenomenon, whatever it is, that paradigmatically rears its head in sorites reasoning" Matti Eklund, *Metaphysical Vagueness and Metaphysical Indeterminacy*, 165.

Chapter 11

Frege and the origins of the debate on vagueness[149]

Let us take Frege's work as exemplary for the origins and purpose of classical logic. Since today the safest characterization of *classical logic* is classical first-order logic[150] (*FOL*), I will compare briefly Frege's conception of logic and FOL. I will then discuss the relation between logic and natural language in Frege and the roots of the debate on vagueness, as they can be found in his work.

11.1. Frege's Begriffsschrift and classical first-order logic

11.1.1 Valid inference

Frege's aim was to provide a single symbolic language for pure thought, which could later be applied to all sciences where validity is an issue, by way of special signs, so as to become a '*single formula language*'[151]. Modern FOL keeps most of that. It is an expression of valid inference, that is, truth-preservation. Its main component is a formal language in which inferences are expressed. As with Frege, the relation between natural language formulation and formal language formalization is an issue of greatest philosophical importance.[152] The formal language is made of symbols such as constant letters, variable letters, predicate letters, quantifiers and truth-functional operators – the last two represented, respectively, as cavities and strokes in Frege's symbolism. Among the operators, some – in Frege's case the material conditional and negation – may be taken as primitive and used to define the others. A proof system is then built through some axioms and inference rules. Natural deduction systems such as Gerhard Gentzen's NJ encode such rules of inference as *introduction and elimination rules*[153]*.* Both Frege's axiomatic treatment, now known as *Hilbert-style* and Gentzen-style systems allow proofs to take place. Premises e.g. definitions lead to conclusions. Conclusions obtained without premises are called *theorems.*

[149] This chapter includes fragments of Marian Călborean, *Frege and Vagueness*. July 24, 2018. Author's manuscript.
[150] Stewart Shapiro and Teresa Kouri Kissel, "Classical Logic". *The Stanford Encyclopedia of Philosophy* (Spring 2018 Edition), ed. Edward N. Zalta https://plato.stanford.edu/archives/spr2018/entries/logic-classical.
[151] Gottlob Frege, "Begriffsschrift" in *The Frege Reader*, ed. Michael Beaney, 50 (VI).
[152] For all three points see Graham Priest, *An Introduction to Non-Classical Logic: From If to Is*, 3. A great point of difference is that Priest and almost all introductions of FOL today treat the meta-language used to build interpretations as having the same importance as the language used for inferences.
[153] Gerhard Gentzen, "Investigations into Logical Deduction" in *The Collected Papers of Gerhard Gentzen*, ed. M.E. Szabo (Studies in Logic and the Foundations of Mathematics, Volume 55, North-Holland Pub. Co, 1969), 77.

As long as the axioms are reasonable, proofs should reasonably preserve truth, even without an interpretation to say what truth is[154]. Of course, there are counter-intuitive theorems in FOL. One propositional paradox cited by Priest states that "If John is in Paris he is in France, and if John is in London he is in England. Hence, it is the case either that if John is in Paris he is in England, or that if he is in London he is in France."[155] Relatedly, the first-order *drinker's paradox* states that "There is someone such that, if he is in the pub and drinking, then everyone in the pub is drinking."[156] Both are rooted in material implication – the reading of *if … then…* as solely denying that the antecedent can obtain without the consequent. However, these arguments, while motivating alternative readings of *if… then…* are not paradoxes in the same sense as our sorites, as an explicit contradiction is not easy to derive[157].

On the contrary, classical propositional logic (which we will call *CPL*), of which FOL is an extension, conserves in its axioms Ancient and Medieval logical laws. All contradictions are false, i.e. the law of non-contradiction which we call *NC*. Any disjunction of a sentence with its negation is true, i.e. the law of the excluded middle which we call *LEM*. That extends to inference modes. De Morgan's laws (transformation of conjunction to disjunction and the reverse, using negation) hold, as do the ones we've seen in Chapter 2 as featuring in sorites paradoxes: *modus ponendo ponens* (*mpp*) i.e. elimination of the material conditional, *modus tollendo tollens* (*mtt*) i.e. law of contrapositive, *modus ponendo tollens* (*mpt*) i.e. conjunctive syllogism and *modus tollendo ponens* (*mtp*) i.e. disjunctive syllogism.

On top of these fundamental axioms or inference rules, new axioms can be added, specific to the field or the intended application. Just as Frege intended, FOL expresses such supplementary collections of axioms, called *theories*[158].

11.1.2. Logicism

Frege's project and FOL diverge in what concerns the relation between logic and mathematics. Frege is a logicist. He accepts that his logical system depends on some ultimate axioms, but he believes them

154 John Etchemendy, *The Concept of Logical Consequence,* (Harvard University Press, 1990), 23-25.
155 Graham Priest, *An Introduction to Non-Classical Logic: From If to Is*, 15.
156 The first paradox has the form $(A \supset B) \wedge (C \supset D) \vdash (A \supset D) \vee (C \supset B)$. The second has the form $\vdash \exists x \,.\, Px \wedge Dx \supset \forall y(Px \supset Dy)$.
157 That is, the secondary premise which needs to be added for contradiction is not as clear-cut as 'a man of 100 cm is not tall' used in 5.1.1 above.
158 Some authors require theories be deductively closed. If T the theory and $\vdash$ the relation of entailment, T is deductively closed just in case for every sentence ϕ, if $T \vdash \phi$, then $\phi \in T$.

to be few and self-evident[159]. His aim for logic is to ground the entirety of arithmetic, and thus, of mathematics. Mathematical structures would be logical structures.

The standard contemporary foundation of mathematics is Zermelo–Fraenkel set theory with the axiom of choice, which we call *ZFC*. ZFC is formulated in FOL, being a FOL theory – a collection of axioms. But, in contrast with Frege's idea of logic, the axioms of ZFC are neither few nor self-evident. In ZFC, an infinity of axioms is necessary, e.g. in the axiom schema of replacement, which states that the image of a set under a function is also a set. And such axioms as that of infinity, which states that there is an infinite set, are not self-evident and go beyond what FOL is able to say[160]. ZF-style systems were carefully constructed to balance benefits and costs: what you get depends on what you put in. FOL expresses, but does not justify the foundations of mathematics.

11.1.3. Universality

Frege is also a universalist. His ontology justifies his Begriffsschrift[161] as being the correct logic for all fields of knowledge, based on the assumed existence of the True and False, the definition of *function* and *concept*[162], his distinction between sense and reference and so on. Putting logicism and universalism together, logic is, for Frege, the universal foundation. While FOL is defended as the universal or *one true* logic, the claim is weaker in this case too, for at least three reasons.

Firstly, FOL is one of many available formal systems studying valid inference, of disagreeing philosophical premises. For example, constructive logic avoids such rules as double negation elimination and reductio in order to better model constructive proof. And the paradoxical theorems

[159] For both points see *Preface* to the *Grundgesetze der Arithmetic*, *Volume I:* "It cannot be required that everything be proved, because that is impossible; but we can demand that all propositions used without proof be expressly declared as such, so that we can see upon what the whole construction is based. We must then strive to reduce the Number of these primitive laws to a minimum, by proving everything that can be proved." The only axiom he sees as partly non-obvious is Axiom V, which he was to blame later on for Russell's Paradox. Gottlob Frege, "Grundgesetze der Arithmetic" in *The Frege Reader,* ed. Michael Beaney, 194-196.

[160] Note that both are needed. As Montague showed, ZFC cannot be finitely axiomatized. Secondly, a first-order theory of the Peano arithmetic, even with infinite axioms, cannot characterize the natural numbers. Richard Montague, "Fraenkel's addition to the axioms of Zermelo", in *Essays on the foundations of mathematics*, ed. Yehoshua Bar-Hillel, E. I. J. Poznanski, M. O. Rabin, Abraham Robinson (Jerusalem, Magnes Press. 1961).

[161] I use *Begriffsschrift* without quotation marks to denote Frege's envisioned logic.

[162] Frege's mature work takes *function* as a primary notion. The function is defined, on the model of mathematical functions, as a mapping of objects (first-level functions) or functions (for second-level functions) as arguments to objects as values of the function. Functions can be either one-place (monadic) or two-place (dyadic). Monadic functions that map their argument only to the truth values (the True and the False) are called *concepts*.

illustrated above have motivated the development of alternatives to the material reading of *if… then…*, such as in modal and relevant logics[163].

Secondly, FOL is not committed to Frege's ontology, but is still open to metaphysical questioning. Truth in FOL is standardly spoken of through a model-theoretical semantics, which adds to our formal language a metalanguage to define *validity.* Incidentally, the popularity of model-theoretical semantics coincided with the disappearance of second-order logic from mathematics[164]. A distinction between objects and properties is introduced, which seems uncontroversial. But the model is itself standardly built on set theory, which is somewhat disconcerting. Does truth presuppose the notion of set membership[165]? In any case, the adequacy of the model is a second topic of philosophical disagreement.

Thirdly, FOL is neither the most metalogically robust, nor the most expressive logic. Frege's hope was that logic be maximal in terms of power[166]. He does distinguish 'first-level' (later first-order) functions from second-order functions, but he makes nothing of the distinction, his being a second-order calculus[167]. But others studied the differences between logical systems by the level of quantification. Frege's contemporary Charles Peirce distinguished propositional i.e. not quantified, first-order and second-order logic, while raising philosophical objections to the last one.[168] Results in the first-half of the 20th century showed that while expressiveness increases, metalogical robustness decreases from propositional to FOL and then to second-order logic.[169] FOL is more expressive than propositional logic and more robust than second-order logic.

[163] Priest writes "conditionals are about as central to logic as one can get" Graham Priest, *An Introduction to Non-Classical Logic: From If to Is*, xviii.
[164] Matti Eklund, "On How Logic Became First-Order", *Nordic Journal of Philosophical Logic*, 1(2), 1996, 165.
[165] The use of set-theory for speaking about truth in FOL raised the question of what should we use to speak of the truth of set-theory. It is nowadays standard to use English for this task.
[166] E.g. he says that the realm of arithmetic is the enumerable and it comprises everything. Gottlob Frege, "Letter to Marty", in *The Frege Reader,* ed. Michael Beaney, 80.
[167] Many Frege scholars argue that his viewpoint rejected the very concept of metalogic. Jamie Tappenden, "Frege on Axioms, Indirect Proof, and Independence Arguments in Geometry: Did Frege Reject Independence Arguments?" in *Notre Dame Journal of Formal Logic* (vol 41, no 2, 2000), galley proofs, Accessed Oct 14, 2019 https://pdfs.semanticscholar.org/e88e/cbd02e314e8a9a01bd39e6c46ae1ab20eb4d.pdf, 15-17.
[168] Peirce's terms were 'non-relative', 'first-intentional' and 'second-intentional' C. S. Peirce, "On the Algebra of Logic: A Contribution to the Philosophy of Notation". *American Journal of Mathematics*, Vol. 7, No. 2 (Jan., 1885).
[169] Only propositional logic is decidable. Both it and FOL are sound and complete. Lindström's theorem holds that FOL is the strongest logic for which both the compactness theorem and the Löwenheim-Skolem theorem hold.

In short, while criticized by proponents of other ways of founding mathematics, FOL is the most expressive foundation system facilitating mathematical reasoning[170].

11.2. Frege on indeterminateness

Frege's work states a number of times that natural language is fine as it is. He once states that we do not need much precision to lead our daily lives. And that the North Sea is objective despite "the fact that it is a matter of our arbitrary choice which part of all the water on the Earth's surface we mark off and elect to call the 'North Sea'"[171]. But often, when his immediate aim is to make mathematical definition precise, he uses examples of bad common language reasoning and states that such uses show that natural language lacks meaning. Famously, in the "Basic Laws of Arithmetic (2nd volume)" #56, under the heading "Principle of completeness", he writes:

> "A definition of a concept (a possible predicate) must be complete; it has to determine unambiguously for every object whether it falls under the concept or not (whether the predicate can be applied to it truly). Thus, there must be no object for which, after the definition, it remains doubtful whether it falls under the concept, even though it may not always be possible, for us humans, with our deficient knowledge, to decide the question. Figuratively, we can also express it like this: a concept must have sharp boundaries. If one pictures a concept with respect to its extension as a region in a plane, then this is, of course, merely an analogy and must be treated with care, though it can be of service here. A concept without sharp boundaries would correspond to a region that would not have a sharp borderline everywhere but would, in places, be completely blurred, merging with its surroundings. This would not really be a region at all; and, correspondingly, a concept without sharp definition is wrongly called a concept. Logic cannot recognize such concept-like constructions as concepts; it is impossible to formulate exact laws concerning them. The

170 See the debate around Shapiro's 1991 argument for second-order logic. Craig Smorynski writes: "Can anyone imagine developing non-standard analysis using second-order logic with its categorical set of real numbers? Or basing a computer language like PROLOG on a logic in which Herbrand's Theorem (i.e., compactness) fails? What other logic allows the calculation of explicit bounds from proofs in Analytic Number Theory— as announced by Kreisel in the late '50s, and currently demonstrated by Luckhardt and his students? In M[athematics], as well as in M[athematical]L[ogic], the preference for first-order logic is well-founded". Craig Smoryński, "Review of Stewart Shapiro, Foundations without foundationalism; A case of second-order logic", *Mod. Log.* 4 (1994), no. 3, 342. To compare see Stewart Shapiro, *Foundations without Foundationalism. A case for second-order logic* (Oxford University Press, 1991).

171 Gottlob Frege, *The Foundations of Arithmetic*, 34.

> law of excluded middle is in fact just the requirement, in another form, that concepts have sharp boundaries. Any object Δ either falls under the concept Φ or it does not fall under it: tertium non datur. Would, for example, the proposition 'Every square root of 9 is odd' have any graspable sense if square root of 9 were a concept without sharp boundaries? Does the question, 'Are we still Christians', indeed have a sense if it is not determined to whom the predicate Christian can be truly applied and from whom it must be withheld?"[172]

I will call this the *completeness fragment*. Therein, Frege's main purpose is not to discuss natural language reasoning, as it was often cited[173]. It is to press the importance of complete definition in mathematics[174]. The fragment is followed by a detailed critique of *piecemeal* definitions given by Frege's contemporary mathematicians. What Frege understands by *piecemeal* is the habit of introducing and modifying new terms as one likes: He writes it

> "… consists in providing a definition for a special case – for example, for the positive whole numbers – and putting it to use and then, after various theorems, following it up with a second explanation for a different case – for example for the negative whole numbers and for Zero – at which point, all too often, the mistake is committed of once again making determinations for the case already dealt with"[175] .

Applying this to natural language is not straightforward. The reason for which piecemeal definition is unacceptable seems to be that in mathematics, one can define and redefine anything. But natural language may resist unprincipled redefinitions, if one assumes there are such things as linguistic norms. Outside the legislative or scientific realms, you cannot stipulate 'tall' to mean whatever you want. Thus, we need to presuppose that speakers have some, possibly implicit, definitions of common terms, in order for Frege's argument to be relevant. They may acquire them on learning the language, to the same effect as the explicit – even if piecemeal – definitions for mathematical concepts. Then Frege is justified in saying that, if *Christian* neither applies nor does not apply to a Mormon man, it does not fit his definition of a concept. That is, to map any object to the True or the False.

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[172] Gottlob Frege, *Basic Laws of Arithmetic* (Oxford University Press, 2016), 69 (III #56).
[173] Kit Fine, "Vagueness, Truth and Logic", 279.
[174] He introduces the fragment by saying "it will be beneficial to lay down and justify in advance some principles of definition that are disregarded by nearly all authors", meaning mathematicians. Gottlob Frege, *Basic Laws of Arithmetic,* 69 (III #55).
[175] Gottlob Frege, *Basic Laws of Arithmetic,* 70 (III #57).

Some commentators argue that natural-language terms have no meaning for Frege, at least until they are given a rigorous definition in his Begriffsschrift[176]. But this is hard to square with Frege's suggestions, as at the end of the fragment, that such predicates as *Christian* may be acceptable now. And Frege takes many linguistic forms as determinative of logical distinctions. He insists that the definite article marks the difference between objects and concepts, up to the point of hypostatizing enigmatic objects corresponding to expressions of the form "the concept 'man'"[177]. He introduces a special function \ξ to play the role of definite article of turning a concept into an object when appropriate, by way of his Axiom VI of his "Basic Laws of Arithmetic"[178]. He also sees indefinite article as determinative of concepts[179] and the German subjunctive mood as determinative of indirect reference[180]. Frege defends his appeal to linguistic distinctions thus:

> "... my own way of [basing logical rules on linguistic distinctions] is something that nobody can avoid who lays down such rules at all, for we cannot understand one another without language, and so in the end we must always rely on other people's understanding words, inflexions, and sentence-construction in essentially the same way as ourselves."[181]

This may be compatible with a strategy of ignoring the linguistic priors of logical distinctions, once they are apprehended. Thus, there are commentators that see Frege's mature semantics as applying only to formal languages[182]. But such a strategy would still not explain Frege's insistence on linguistic devices and, most of all, his inclination to exemplify the most precise of topics with ordinary predicates.

These issues foretell the contemporary vagueness debate. Take the completeness fragment above. Firstly, it introduces the metaphor of sharp boundaries. It is everywhere now[183]. There are predicates which are comparable to colored regions on the plane and the predicates which are not so comparable have blurred boundaries. Secondly, the fragment exemplifies bad mathematical

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176 Joan Weiner, "Understanding Frege's Project" in *The Cambridge Companion to Frege*, ed. Michael Potter and Tom Ricketts (New York, Cambridge University Press, 2010), 42.
177 Gottlob Frege, "Comments on Sinn and Bedeutung" in *The Frege Reader*, ed. Michael Beaney, 174-177.
178 "here, then, we have a substitute for the definite article of language, which serves to form proper names out of concept-words". Gottlob Frege, *Basic Laws of Arithmetic,* 19 (#11), also 34.
179 "As soon as a word is used with the indefinite article or in the plural without any article, it is a concept word". Gottlob Frege, *The Foundations of Arithmetic*, 64 (#51).
180 Gottlob Frege, "On Sinn and Bedeutung" in *The Frege Reader*, ed. Michael Beaney, 162.
181 Gottlob Frege, "On Concept and Object" in *The Frege Reader*, ed. Michael Beaney, 184.
182 Michael Dummett, "Wang's paradox", 109-110.
183 E.g. Kit Fine, "Vagueness, Truth and Logic", 279, Crispin Wright, "Language-mastery and the sorites paradox", 154, or R.M. Sainsbury, "Is There Higher-Order Vagueness?", The Philosophical Quarterly, Vol. 41, No. 163 (Apr., 1991), 167

definitions with *Christian*, which is a natural-language predicate. Thus, the problem with it might be incomplete definition. Thirdly, it singles out the classical law of excluded middle as requiring that an object is either a negative or a positive case of the predicate. This will be later distinguished as the *principle of bivalence*[184], while the undecided cases will be called *borderline cases*. Blurred borders or imprecision, incomplete definition, failure of bivalence or borderline cases. These are three definitions of vagueness, of which the theories discussed in Chapter 12 will make use.

Therefore, Frege's completeness fragment anticipates philosophical theories of vagueness:

a) **Nihilists and incoherentists** will accept that common vocabulary does not have "exact laws". For them, predicate tolerance shows the inconsistency of common vocabulary, as underlined by the *problem of the many* discussed in 7.6

b) A **contagious trivaluationism** is read[185] in Frege's insistence that an indeterminate term removes the "graspable sense" of any proposition containing it. It implies that there can be no reasoning with the third, indeterminate value.

c) **Epistemicism** will generalize the "[impossibility] … for us humans, with our deficient knowledge, to decide the question". It holds that ordinary predicates have boundaries, but we do not know where they lie.

d) **Fuzzy logic** will generalize the idea of a point belonging to a region in a way that is "…blurred, merging with its surroundings". It holds that an object can be a positive case of a predicate to an intermediate degree and simultaneously a negative case to an intermediate degree.

e) **Supervaluationism** generalizes incomplete definition. There are borderline cases because we have not yet bothered to make a decision upon them[186].

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[184] Rosanna Keefe and Peter Smith, "Introduction: theories of vagueness", 2.
[185] Kit Fine, "Vagueness, Truth and Logic", 298 (note 3).
[186] Rosanna Keefe and Peter Smith, "Introduction: theories of vagueness", 33. For Frege's supervaluationist bend, see Joan Weiner, "Understanding Frege's Project", 48.

11.3. Frege and the sorites

Frege's single discussion of the sorites is in "Begriffsschrift"[187]. The work is built around the notion of *conceptual content* (*German*: "begrifflicher Inhalt") as being that kind of entities between which rigorous proofs can be given by logical means alone[188]. The early Frege only has a formalistic notion of *function* as the invariant part of a unitary sub-expression replaceable by some other symbols in its places. Together with arguments, it forms a conceptual content. A *judgment* is the assertion of truth about such a content. That being said, towards the end of "Begriffsschrift", Frege defines consecutively the notions of a property being hereditary in a sequence[189], the notion of an object following another in a sequence and then arrives at the base proposition in mathematical induction[190]. Frege expresses it in words and adds an aside:

> "We can translate (81) thus:
>
> *If x has a property F that is hereditary in the f-sequence, and if y follows x in the f-sequence, then y has the property F*
>
> For example, let *F* be the property of being a heap of beans; let *f* be the procedure of removing one bean from a heap of beans; so that, *f(a,b)* means the circumstance that *b* contains all beans of a heap *a* except one and does not contain anything else. Then by means of our proposition we would arrive at the result that a single bean, or even none at all, is a heap of beans if the property of being a heap of beans is hereditary in the *f*-sequence. This is not the case in general, however since there are certain z for which $F(z)$ cannot become a judgement on account of the indeterminateness of the notion 'heap'".[191]

Firstly, Frege denies that *removing one bean* is tolerant towards *heap*, by saying that there are cases distanced by one grain where predicate tolerance fails. But secondly, he also diagnoses the source of the problem as the indeterminateness of *heap*. It makes some *heap* predications unable to "become a judgement". The notion of unjudgeable contents is only once more discussed in "Begriffsschrift".

[187] This discussion is not well-known, for example Wright recently writes that Frege "does not cite the sorites paradox" as "threatening basic logic". Crispin Wright, *Wang's Paradox*, 5.
[188] Gottlob Frege, "Begriffsschrift" in *The Frege Reader*, ed. Michael Beaney, 49 (IV) and 53 (#3).
[189] Formula 69 in Gottlob Frege, "Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens", 55 (#24).
[190] Frege says in a footnote "Bernoulli's induction rests upon this" Gottlob Frege, "Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens", 62 (#27).
[191] Gottlob Frege, "Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens", 62 (#27).

Frege stated at *#3* that contents such as 'house' belong to it. But the proposition 'Eleven beans is a heap of beans' is quite different from 'house'. It seems like there is an easy way of saying why the latter cannot become a judgement, namely, it is not grammatically capable (if turned into a judgement) of becoming true or false. But this is precisely Frege's point: that the former is as unable, the grammatical form of a truth-carrying expression being of no import.

Why does Frege choose to deny linguistic appearance in this case? For him, the *heap* notion does not conform to the expectations of the logic. As illustrated above, Frege treats some linguistic distinctions e.g. definite vs. indefinite article as strictly corresponding to logical distinctions. While some other linguistic distinctions are discarded in favor of a single logical form, e.g. active vs passive predications. Frege's guide in this procedure is the search for the purely logical relations among conceptual contents. Begriffsschrift intends to remove all particular content irrelevant to validity of proof[192], such that a consistent kernel of thought would be revealed. Yet, soritical predicates seem to belie this, as they appear to have *prima-facie* logical relations that can be turned into a contradiction by the newly-introduced laws. The pure thought can be shown to be contradictory. Frege was mystified by the logical appearance of predicate tolerance.

Instead, he could have chosen to deny the reality of (some) such prima-facie relations. The hereditary nature of *heap* into the relation of *removing one bean* could have been rejected on account of our bad intuitions. Instead of cases of indeterminateness there would be cases of dispersion. The mature Frege treats soritical predicates akin to fictional entities such as Pegasus. They have some sense, but no reference – which makes the statements containing them lack truth value. But, were *heap* to apply to groups of 12 beans, not apply to those of 13, apply to those of 14, not apply to those of 15, and so on, Frege's project would be in no danger. The *heap* concept would still map all objects to the True or the False in such an alternative approach. Thus, our job would be to *discover* a logic-adequate definition of concepts, if any, not to *stipulate* them in an accessible way[193]. Williamson attributes to Frege a "definitional picture of meaning" which would surely be incompatible with such an attitude: "Where we do not draw a line, no line is drawn"[194]. Frege supports such a view in places, for example in his "Letter to Marty" he writes:

[192] Gottlob Frege, "Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens",7 (VI).

[193] Sider contrasts the *stipulated* grammar of formal languages with the *discovered* English grammar. Theodore Sider, "Logic for philosophy" (Oxford University Press, 2009), 4.

[194] Timothy Williamson, *Vagueness*, 39.

> "Bald people for example cannot be enumerated as long as the concept of baldness is not defined so precisely that for any individual there can be no doubt whether he falls under it"'[195]

But in other places, Frege says that innumerability is no problem for sortal predicates such as *red*[196]. Be that as it may, for Frege, even mathematical partial definitions long refined through use were no good. See his criticism of mathematical piecemeal definition above. It is instructive to note that such definitions, while unprincipled and prone to error, need not always go wrong. Great mathematicians gave piecemeal definitions, while avoiding contradiction. To bring rigor[197], Frege felt the need to treat a heuristic issue as constitutive. There is no wonder that the idea of *discovering* adequate definitions of natural language predicates did not arrive at him.

In conclusion, Frege's disregard for natural language was mostly caused by his focus being on science and mathematics, oftentimes speaking as if precision is right around the corner. His unnecessary deference to the logical appearance of predicate tolerance forces him to treat natural language predicate terms as he treats fictional entities, both lacking references. This can be attributed to a definitional picture of meaning or to his commitment to heuristic rigor. His deference to predicate tolerance has been inherited in the debate on vagueness that started in the 1960s. And we will see philosophers of vagueness repeatedly engaging with Frege's project in order to better accommodate vagueness.

[195] Gottlob Frege, "Begriffsschrift" in *The Frege Reader*, ed. Michael Beaney, 80 (163).
[196] Frege: "We can, for example, divide up something falling under the concept 'red' into parts in a variety of ways, without the parts thereby ceasing to fall under the same concept 'red'. To a concept of this kind no finite number will belong." Gottlob Frege, *The Foundations of Arithmetic,* 66 (#54).
[197] Rigor is one of the main motivation of Frege's project, comprising two conditions: nothing coming into a proof unnoticed and conserving truth – the possible syntactic verification of correct derivation but also a theory of definition. The first is obviously heuristic, not constitutive. Gottlob Frege, *The Foundations of Arithmetic*, XXI.

Chapter 12

Theories and logics of vagueness

12.1. Introduction

FOL is an extension of CPL. The main alternatives to classical logic which have been proposed in the vagueness debate are propositional themselves – their treatment of vagueness, in Zardini's words, "already emerges at the sentential level"[198]. That is why we can call them *vague logics.* But we have seen in Chapter 6 that the sorites is a second-order paradox which can be expressed in FOL within a bounded submodel, i.e. putting a finite upper bound on some predicate encompassing all step predicates. If predicate tolerance is a phenomenon arising between a relation and a monadic predicate, a logic for vagueness should be able to express relations and, hence, be above zero-order. However, most of the logics below locate vagueness at the level of semantics or by redefining logical operators. They are nonetheless relevant for the philosophical issues raised by vagueness.

I will summarize (1) three variants of trivaluationism, namely those of Tye[199] and those of Halldén and Körner (both as cited by Williamson)[200], (2) the fuzzy logics started in the 1960s, in Kenton Machina's variant[201], (3) the supervaluationist school started in the 1970s, mainly in Kit Fine's 1975 variant[202], (4) Goguen's framework for non-numerical degrees of truth which he offered simultaneously with his grounding of fuzzy logic[203], (5) Edgington's probabilistic approach to predicate tolerance[204], (6) Smith's recent fuzzy plurivaluationism[205], (7) Weatherson's recent logic of *truer*[206], (8) Zardini's recent non-transitivism[207], (9) Kit Fine's recent compatibilist semantics for vagueness[208]. They disagree with classical logic in one or more points. Finally, I will add (10)

[198] Elia Zardini, "A Model of Tolerance", 4.
[199] Michael Tye, "Sorites paradoxes and the semantics of vagueness" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997).
[200] Timothy Williamson, *Vagueness*, 103-111.
[201] Kenton Machina, "Truth, belief and vagueness" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997).
[202] Kit Fine, "Vagueness, Truth and Logic".
[203] J. A. Goguen, "The Logic of Inexact Concepts".
[204] Dorothy Edgington, "Vagueness by degrees" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997).
[205] Nicholas J. J. Smith, *Vagueness and Degrees of Truth.*
[206] Brian Weatherson, *True, Truer, Truest* (Penultimate draft), Accessed Oct 15, 2019 http://brian.weatherson.org/ttt.pdf.
[207] Elia Zardini, "A Model of Tolerance".
[208] Kit Fine, "The possibility of vagueness".

Williamson's epistemicism[209] to the comparison. Epistemicism is a theory of vagueness that is usually thought of as synonymous with the defense of classical logic. It is rather classical logic plus what I called *predicate-ordering monotony*. Namely, the popular idea that any man of equal or more cm than another is tall provided the latter is, that any man of equal or less hairs than another is bald provided the latter is, etc. And therefore, that there is a standard of separation[210] (or otherwise put, a *boundary*) between positive and negative cases.

The alternative logics to be discussed are created with philosophical aims, by authors well-acquainted with classical logic. They compete for a logico-philosophical challenge: what is the minimal modification of classical logic to accommodate the inexactitude and graduality which we associate with vagueness, while simultaneously illuminating us in its regard? Non-classical logicians can be compared to someone adapting her grandmother's gown for her daughter's prom night. The gown can be cut in some places, resewn in others, but the enterprise will fail if the teenager does not see the point of wearing it and does not act her part. We will see negation defined in terms of possible interpretations, non-linear truth values with an absolute minimum, valid arguments from partially true premises to false conclusions and other proposals which will seem strange if the motivation is not adapted to. So I will provide the philosophical motivation, but also the most salient objections to each.

As for the method, I will approach these theories in a comparative manner, so as to highlight the relations between them. There are five axes of comparison: (12.2) truth values, (12.3) operators and truth-functionality, (12.4) semantics, (12.5) truth, validity and the sorites. Across each, the theories are ordered differently, following to the best angle of comparison. I will end with (12.6) comparison tables which serve as a handy summary and (12.7) a concluding section. To make the presentation easier to follow, the objections raised to the theories will be in ***bold italic*** typeface, while the mentions of theories and authors will be in **bold** typeface.

[209] Timothy Williamson, *Vagueness.*
[210] See Chapter 4 and 7.2.

12.2. Truth values

FOL is usually said to have two truth values: true and false. They correspond to a long tradition of philosophy and are remarkably theory-independent[211]. A metalogical commitment to this truth-set of two values is called *bivalence*[212]. But the minimal semantics for FOL or CPL does not need this truth-set. In both, only truth can be defined and negation is false just in case the negated sentence is true, true otherwise. This explains why, before the separation of logic and metalogic, what we call bivalence was seen as a normal reading of the common LEM ($A \lor \neg A$)[213].

12.2.1. Adding truth values

The simplest way to accommodate indecision concerning whether a sentence is true or false is to introduce a third value. **Trivaluationists**, such as (arguably) **Frege**, **Halldén**, **Körner**, and **Tye** add a third – intermediate – value, for the indeterminate, the absurd, the unknown, etc. They can claim 'a man of 172 cm is tall' is neither true not false, but indeterminate, absurd, etc. A third value – the indefinite – is also present in the metalanguage of **supervaluationism**, a theory defended by **Fine** (1975), **Kamp** and **Keefe**[214].

Plurivaluationist logics include **trivaluationism**, the **fuzzy logic** defended by **Goguen** and **Machina**, and **Edgington's probability-based logic**. Except trivaluationism, they adopt the continuum – an uncountable infinity of truth values – using the real interval [0,1] as truth set. In it, 0 corresponds to total falsity and 1 to total truth. They speak of *degrees* of truth – using the popular but different intuition of percentages – to model the idea that statements can be truer or less true. Edgington calls degrees *verities.* **Smith** argues for fuzzy degrees of truth combined with the absence of an unique

[211] "Depending on their particular use, truth values have been treated as unanalyzed, as defined, as unstructured, or as structured entities." Yaroslav Shramko and Heinrich Wansing, "Truth Values", *The Stanford Encyclopedia of Philosophy* (Spring 2018), ed. Edward N. Zalta, https://plato.stanford.edu/archives/spr2018/entries/truth-values.

[212] Edgington contests that her infinite verities – corresponding to probabilities between 0 (false) and 1 (true) make her logic non-bivalent. She argues that bivalence is the commitment to LEM, plus a disquotational property of truth. But that is a non-standard usage of *bivalence*, so we can understand Edgington's ideas without classifying her as a bivalent author. LEM will also be perfectly true in supervaluationism and Zardini's tolerant logics. The disquotational property of truth will be discussed at 12.6 below. Dorothy Edgington, "Vagueness by degrees", 311.

[213] See Frege's completeness fragment above for its expression as LEM. The modern view is succinctly expressed by Quine "Bivalence is a basic trait of our classical theories of nature. It has us positing a true-false dichotomy across all the statements that we can express in our theoretical vocabulary, irrespective of our knowing how to decide them." W. V. Quine, "What price bivalence?", *Journal of Philosophy*, 78 (2) (1981), 94.

[214] Classical places for supervaluationism are Kit Fine, "Vagueness, Truth and Logic", Rosanna Keefe, *Theories of Vagueness*, Achille C. Varzi, "Supervaluationism and Its Logics", *Mind* 116 (2007) and J. Kamp, "Two theories about adjectives", in *Formal Semantics of Natural Language*, ed. E. Keenan (Cambridge University Press, 2011) doi:10.1017/CBO9780511897696.011.

intended interpretation. While his semantics is identical to that of fuzzy logic, he argues that strictly speaking there is no single correct interpretation, his being a **fuzzy plurivaluationism**. We note with /*A*/ the truth value of *A*, especially in degree-theoretical treatments.

It was argued by authors such as Delia Graff that while Kripke semantics is a respectable semantic device, the claim that truth comes in degrees, especially in the infinity postulated by fuzzy logic is metaphysically dubious as applied to vagueness[215]. But, as shown by the experimental philosophy studies[216], there are *some* intuitions that people have according to which there are indeed intermediate truths. A better objection to plurivaluationist logics consists in the difficulty to precisely choose the cardinality of the truth set, since this makes a difference to the metalogical robustness of the logic. First, as Williamson argues, the uncountable infinity of the real interval makes fuzzy logic not sound and complete in the normal sense, but only if validity itself is taken to be a matter of degree[217], as we will see in 12.5 below. But it can be so in a fuller sense if instead of the real interval one were to use a finite large truth set. How are we to decide? Williamson argues for the real interval by saying that human discrimination limits are not a matter of logic[218]. This is true, but insufficient for the purpose. The number of fundamental particles in the observable universe is finite[219], which is not a problem of human discrimination and the universe itself may be finite[220]. Therefore, truth values may not need – even for a correspondence theory of fuzzy truth – to be infinite and uncountably so. Plurivaluationist logics are thus affected by a ***battle of intuitions regarding truth values***. Once they abandon the classical truth set, there are no good arguments to decide whether a three-valued, large finite, countably infinite or uncountably infinite truth-set is best[221], especially if weighted against the corresponding metalogical limitations.

[215] "In the absence of some substantial philosophical account of what degrees of truth are, we have no reason to accept that it should be both natural and common to mistake high degrees of truth for the highest degree of truth, or to mistake a small difference in degree of truth for no difference in degree of truth." Delia Graff, *Shifting Sands: An Interest-Relative Theory of Vagueness*, 10. The objection is defended against in Nicholas J. J. Smith, *Vagueness and Degrees of Truth,* 211-213.
[216] See 13.4.1.b) below.
[217] Timothy Williamson, *Vagueness*, 139-140.
[218] Timothy Williamson, *Vagueness*, 140.
[219] See 6.5.1.
[220] For a similar argument from strict finitism, see Feng Ye, *Strict Finitism and the Logic of Mathematical Applications (draft)*, Accessed on Oct 15, 2019. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.603.1574&rep=rep1&type=pdf*,* 31.
[221] Williamson also shows in *Vagueness,* 290 (note 23) that using the countable rationals instead of the uncountable reals makes quantifiers for a first-order fuzzy logic not well-defined, because the set of rational numbers does not have the least-upper-bound property. This holds only if definition by abstraction is allowed along the lines of $A = \{x \in \mathbb{Q}: (2*x)^2<2\}$ in the semantics. If no such abstraction is necessary in modelling natural language reasoning, a rational set of truth values would work.

12.2.2. Truth values in a lattice and designated values

Some philosophers of different persuasions agreed that having more than two truth values offers flexibility in modelling natural language, but wanted to avoid the linearity of truth values. They are non-standard plurivaluationists, using a lattice-based semantics. **Goguen** founded fuzzy logic, but he also wanted to give a general plurivaluationist **framework of non-numerical degrees of truth**, which should not require any two such degrees be directly comparable – either equal, higher or lower than one another[222]. **Weatherson' logic** for *truer* intends all contradictions to be perfectly false and all classical truths to hold, while some sentences be truer than others, which forces a similar setup[223]. **Zardini's non-transitivism** intended to limit the transitivity of the consequence relation and he writes that many different *tolerant logics* can be supplemented with this limitation in his style. He thus chooses a lattice of truth values in order to offer the non-transitivity of consequence to other lattice-based logics, although **Zardini** strictly does not need the freedom afforded by lattices[224]. Technically, all three leave it open to interpretation which values of truth are out there, within some constraints[225]. Firstly, there should be some truth values, so the set is specified as non-empty and for **Weatherson**, as containing at least two: absolute truth and absolute falsity. Secondly, a partial ordering ≤ is defined on the set, such that any two values have a lower upper bound and a greatest lower bound – the set forms a lattice with ≤. In this way, any values are comparable through some other/s, but not necessarily directly.

Some non-classical logicians also speak of *designated values*, a subset of the truth values, which will be used to define validity in a way which is parallel to Tarski's classical definition of validity as truth-preservation. Working backwards, classical logic has only the true as designated. **Trivaluationists** choose either the true (e.g. **Tye**) or the true and the intermediate value (e.g. **Halldén**, **Körner**) as designated. **Zardini's non-transitivism** only requires designated values to be specified i.e. that there should be one non-empty set of designated values, included in the truth-set[226].

[222] J. A. Goguen, "The Logic of Inexact Concepts", 340-341 and 351.
[223] Brian Weatherson, *True, Truer, Truest*, 14.
[224] Indeed, the point of tolerant logics goes through even for a fuzzy-like real interval of truth values.
[225] Zardini claims that his many-valued theory is not about truth, but about a parallel quality of sentences to be *good* for various purposes. In this way, validity is to be read as a relation about levels of goodness being vouched for by other levels of goodness. Two objections can be raised. First, it is a distinction without a difference, since truth does not play a role in Zardini's consequence. Secondly, as I objected in Chapter 3, assuming the same level of goodness for sentences of different soritical predicates blocks a large part of reasoning for Zardini, certainly larger that the sorites. This can only be justified if the levels of goodness correspond to how things are, being truth values. For Zardini's argument, see Elia Zardini, "A Model of Tolerance", 10.
[226] Elia Zardini, "A Model of Tolerance", 7.

12.2.3. Fine's compatibilism

Fine recently abandoned the **supervaluationism** he helped ground in the 1970s and proposes a new **compatibilist** semantics of predicate tolerance. He starts on a classical note: only truth is defined, namely those atomic sentences which hold in the present situation (*use* for **Fine**). But whereas CPL allows understanding a negation as false whenever the negated sentence is true and true otherwise, Fine defines negation model-theoretically. It is true just in case the negated sentence is true at no situation accessible from the current situation[227]. The classical closure clause is missing: if a negation it is not true, it does not mean that it is false for Fine. Since both the conditions for the truth of a sentence and the conditions for the truth of its negation may not be fulfilled, there will be three outcomes: truth of the sentence, falseness (truth of its negation) and neither (outside the rules). In brief, **Fine's compatibilism** properly has one truth state and two truth values as CPL does, but three truth outcomes as trivaluationism[228].

12.3. Operators and truth-functionality

CPL has sixteen possible binary operators corresponding to possible combinations of true and false values for two sentences. Conjunction is true just in case both conjuncts are true, negation is true just in case the negated sentence is false and the conditional is false just in case the antecedent is true and the consequent is false. Thus, truth-functionality is the most straightforward approach to large-scale compositionality of language. All sentences have truth values induced by a main operator from the sentences it connects. Those sentences, if not atomic, have themselves a main operator and so on. Truth-functionality allows the inductive definition of truth for sentences of arbitrary length.

Most alternatives to CPL are truth-functional. They maintain classical behavior if the values are classical – T/F in **trivaluationism** and **supervaluationism**, 1 and 0 in **fuzzy logic**, two values as members of the truth set in **non-transitivism** or **Goguen's framework**. They face, however, the need to decide the behavior of the newly introduced values.

[227] Kit Fine, "The possibility of vagueness", 3707.
[228] Compare Forbes's analysis "compatibilist semantics isn't a three-valued logic in the usual sense." Graeme Forbes, "Fine on Vagueness", forthcoming in *Metaphysics, Meaning, and Modality: Themes from Kit Fine*, ed. Mircea Dumitru (Oxford University Press, 2020), author's manuscript of Jul 21, 2017, 15.

12.3.1. Truth-functionality in plurivaluationism

Trivaluationism splits in two general directions. One may be called *contagious*, such as with **Frege** and **Halldén**, for whom any operator applied to a sentence of the intermediate value will also yield the intermediate, employing Kleene's weak truth tables[229]. Even a disjunction of a true sentence with a sentence of intermediate value will have the third value, corresponding to an intuition that the third value is the *absurd*. The other trivaluationist direction, of **Körner**'s and **Tye**'s may be called *charitable,* aiming to maximize the truth of compound sentences by employing Kleene's strong truth tables. A disjunction with a true disjunct is always true and a conjunction with a false conjunct is always false, corresponding to an intuition that the third value is the *unknown*. Both directions agree that operators connecting solely intermediate sentences yield the intermediate, e.g. the negation of *intermediate* is *intermediate*. Of course, there is a battle of intuitions regarding the third value between the two approaches, since vagueness can be argued to be both absurdity and ignorance. But it is not unsolvable, since each side can express what the other is saying, by using another operator of those available, so as to express it.

Fuzzy logic has even more truth values, and coincidentally many ways of defining truth-functional operators. Most theorists agree with **Machina** in using mostly Łukasiewicz's definitions. A conjunction takes the lowest degree of its conjuncts. A disjunction takes the highest degree of its disjuncts. Negation takes 1 minus the degree of the negated sentence. To accommodate the intuition that each soritical step in the conditional sorites is a little false and the small falsities can accumulate to a final full falsity, the conditional takes 1 whenever the consequent is as true or truer that the antecedent and the difference of their degrees of truth in all other cases. If the antecedent has 0.8 and the consequent has 0.7, the conditional will have 0.9, but if the consequent has 0.82, it will have 1. So you can lose 0.01 truth from 0.99 to 0.98, 0.01 from 0.98 to 0.97 and so on, getting to 0 in small steps of 0.99 degree each, providing a popular explanation of the conditional sorites. The disadvantage is the ***non-equivalence of mpp, mpt, mtp, mtt.*** Namely, this definition of ⊃ breaks the link between $A \supset B$, $\neg(A \wedge \neg B)$, $\neg A \vee B$ and $\neg B \supset \neg A$ on which their equivalence stood. Thus, the intuitive fuzzy treatment of the conditional sorites does not replicate with the other propositional soritical forms[230], leading to the objection that the ***paradox can be restated***.

For this reason, **Smith's fuzzy plurivaluationism** restores the classical link, abandoning the attractive treatment of the conditional sorites and justifying its appeal by the intuitive character of

[229] Stephen Cole Kleene, *Introduction to Metamathematics* (North-Holland, Amsterdam, 1952).
[230] Brian Weatherson, *True, Truer, Truest*, 11.

tolerance, not by the high degree of truth of each soritical step[231]. **Smith**'s and standard **fuzzy logic** can also avoid a battle of intuitions with trivaluationism regarding the definition of operators by defining as many of them as necessary. Williamson cites the fuzzy bivalent monadic operator J_n, which results in 1 if the sentence has degree *n*, and 0 otherwise[232]. Then there can be a – let us say – *absurdity conjunction A * B* which is 0.5 if J_1A or J_1B are 0, replicating **Halldén**'s behavior, 0 iff J_0A and J_0B are 1 and 1 otherwise (when both |A| and |B| are *1*).

It has been objected that truth-functionality for conjunction, disjunction and negation means that – under any **plurivaluationism** – general logical laws such as LEM and NC are not perfectly true. The best trivaluationism can do is have them at least intermediate, so not false. In a parallel way, fuzzy logic has them at least of 0.5 degree. This was argued to be unacceptable, by authors such as **Williamson**[233]**, Fine, Edgington** and **Weatherson**, but defended by fuzzy theorists such as **Machina** and **Smith**[234], a dispute that I will discuss more extensively at 13.4.4 below. My conclusion will be that both classical FOL and fuzzy logic can supply weak and strong variants of LEM and NC, to partially capture what the other side wants to express, so it is finally a matter of intuitive appeal, which tips the scale to the advantage of classical logic. This moderation is justified by the flexibility of fuzzy semantics, which is not the case, however, with the lattice-based semantics of **Goguen** and **Zardini**. They have no way of saying that a contradiction is indeed completely false, since absolute falsity is not defined.

Finally, an influential divergence of opinions was raised by the standard definition of the degree of truth of a conjunction as the least of the degree of its conjuncts: how is having half of two qualities having precisely half of their combination, as most proponents of **fuzzy logic** argue? Structurally, many operations can play the role of generalizing set intersection to the fuzzy sets introduced by Lofti Zadeh[235]. Such sets can be read as specific sets of ordered triples <a, b, c> such that *a* is the belonging element, *b* is the set and *c* is the degree of membership out of [0,1]. So to have a property is to have a degree of membership in a class. The intersection of two such sets should itself be a similarly fuzzy set, the question is of what degree for elements in common. It is classical that if both

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[231] "In fact, we do not need the Łukasiewicz conditional. It is ironic that one of the few things that most philosophers have found attractive about the standard fuzzy account is its resolution of the Sorites—when in fact this resolution fails to solve the problem." Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 265-268.
[232] Timothy Williamson, *Vagueness*, 114.
[233] Timothy Williamson, *Vagueness*, 118 and 136. See echoes in Brian Weatherson, *True, Truer, Truest*, 2.
[234] Kenton Machina, "Truth, belief and vagueness", 185.
[235] L.A. Zadeh, "Fuzzy Sets", *Information and control* 8 (1965), 339. J.A. Goguen, "The Logic of Inexact Concepts", 340.

conjuncts had degree 1 (complete truth), it should be 1 and if one had 0 (complete falsity), it should be 0. Many *t-norms* respect these conditions, among them the standard fuzzy conjunction, the probabilistic product function above, but also Łukasiewicz's strong conjunction (the greatest of 0 and *|A|* + *|B|* - 1) which corresponds with the collective difference from truth[236]. The best reason for preferring the weak conjunction (|A ∧ B| = least of |A|, |B|) is that it is easy to generalize to order theory, the minimum function returning the greatest lower bound. In other words, we do not need any other mathematics to compute it, except direct comparison of the truth values, which can be argued to be logical. The conjunction of tallness of John and baldness of John has one of their values, a property other t-norms do not have.

12.3.2. Truth-functionality in a lattice

Goguen's framework, **Zardini's non-transitivism** and **Weatherson's logic** for *truer* only specify some structural rules which make the operators roughly resemble classical operators. To illustrate, for all of them, the negation should have a truth value which is ≤ than the negated sentence. This is quite little to say. Williamson's criticism against **Goguen**'s structural constraints is that of ***abstractness of framework.*** Those constraints fail at the task of proposing a logic to be evaluated and used. Williamson writes: "It is easy to specify the kind of structure that might be desirable for such a semantics; the difficulty is in specifying a plausible instance of that structure."[237] **Goguen**'s general framework cannot work for negation, since not enough is said about what it is for non-numerical degrees of truth to be the negation of each other. **Zardini** faces Williamson's objection as well, since he uses a lattice of truth values, but without a way to identify one good negation. For example, we have seen that there is a ***battle of intuitions regarding truth values*** inside plurivaluationism, i.e. what is the cardinality of the truth set? The objection is that lattice-based plurivaluationism does not escape these battles simply by omission: they should propose a system to be contrasted with CPL and other alternatives to it, if they find them wrong.

I think that **Weatherson** escapes this objection, since all his operators behave classically (e.g. his negation is not only order-inverting but double negation is also redundant) and his lattice of truth values has a minimum – i.e. the complete falsity he attributes to all contradictions. That is a good reason to have a lattice of truth values – to allow all contradictions be perfectly false without having

[236] Radim Belohlavek, Joseph W. Dauben, and George J. Klir, *Fuzzy Logic and Mathematics - A Historical Perspective* (Oxford University Press, 2017), 48. Also Petr Hájek, *Metamathematics of Fuzzy Logic* (Kluwer Academic Publishers, Dordrecht, Boston, and London, 1998), 27-30.
[237] Timothy Williamson, *Vagueness*, 135.

only two truth values. His negation has a stricter sense: it results in a truth value which is (downward) connected with the value of the negated sentence only through the absolute falsity which is posited to exist.

12.3.3. Weakening truth-functionality

Supervaluationism splits from **trivaluationism** for the reason above of LEM not being perfectly true, choosing a definition of truth as truth across all complete (and acceptable) precisifications, according to which LEM and NC are perfectly true, as we will see at 12.5 below. However, as Varzi writes, “supervaluationism is not truth-functional; in particular, there is a difference between super-falsifying a conjunction and super-falsifying at least one conjunct, just as there is a difference between super-verifying a disjunction and super-verifying at least one disjunct”[238]. This leads to a ***counter-intuitive semantics***, best expressed by what Varzi cites as the “objection from upper-case letters”, as Jamie Tappenden calls it: “You say that <<either φ or ψ>> is true, so EITHER φ OR ψ [stamp the foot, bang the table] must be true”.[239]

Fine’s compatibilism follows Kripke’s semantics for intuitionistic logic by abandoning truth-functionality by half. The value of a compound sentence is still determined by the values of its sub-sentences. But not only by such values at the present situation, but also by their values at situations accessible from the present situation, which themselves might depend on valuations at further inaccessible situations. The implication $A \supset B$ will be true for Fine at a situation just in case (a) $A \wedge B$ is true at that situation or (b) at all situations accessible from it, B is true if A is true[240]. Now take $\neg A \supset B$ and remember that negation is true just in case no accessible situation makes true the negated sentence. Then by (b), the value of $\neg A \supset B$ depends on all situations accessible from the current one, but by the definition of truth of a negation, it depends in its turn on all situations connected to them, even if not connected with the current one. Fine is the single author to take the relationship between situations which he calls *uses* as fundamental, prior to falsity and operator values. As we will see at 12.4 below, his logic is one of agreement: you cannot get a true negation if you accept other situations which take the negated sentence as true. And you partially import their relationships with other

238 Achille C. Varzi, "Supervaluationism and Its Logics", 647.
239 Achille C. Varzi, "Supervaluationism and Its Logics", 647, citing J. Tappenden, “The Liar and Sorites Paradoxes: Toward a Unified Treatment”, Journal of Philosophy, 90 (1993), 564.
Journal of Philosophy, 90,
240 Kit Fine, “The possibility of vagueness”, 3708.

situations as well when accepting them. It is precisely Fine's intent to do without some true negations, since, then, contradictions - i.e. paradox – cannot be obtained.

Finally, Edgington's probabilistic logic of vagueness abandons truth-functionality in favor of probabilistic structure. This is not to say that her verities – degrees of truth – are not determinate. They are, but they need to take into account the relation between the sentences to be connected. This allows **Edgington** to strictly enforce ***predicate-ordering monotony*** ('If a man of n cm is tall, so is a man of n+1 cm'). Even if a man of 170 cm is tall to a verity of 0.5, for **Edgington** the sentence 'if a man of 170 cm is tall, then a man of 171 cm is tall' should be completely true. Were the antecedent to be true, the consequent would always follow. Thus, the verity of a conjunction $A \wedge B$ is defined as the verity of B given A, times the verity of A. If A and B are independent, it is the verity of A times the verity of B. The verity of a conditional $A \supset B$ is only discussed as $\neg(A \wedge \neg B)$[241]. One difference with fuzzy conjunction is that $A \wedge \neg A$ is completely false, the second conjunct being impossible given the first. Another is that, if A, B, C, D, E all sentences of independent probabilities of half-degree (0.5), then a conjunction of two $A \wedge B$ has degree 0.25, a conjunction of three $A \wedge B \wedge C$ has 0.125, decreasing to 0.03125 for $A \wedge B \wedge C \wedge D \wedge E$, in contrast to 0.5 under fuzzy conjunction. Both differences are controversial. We discussed the first at plurivaluationism above.

The second may be called the objection of ***probabilistic conjunction.*** It was argued repeatedly that vagueness-based indecision seems to act fuzzily, not probabilistically[242]. The sentence 'he is tall and he is bald', written $A \wedge B$, where the conjuncts are independent and both have verity 0.5 should also have verity 0.5, not 0.25 as for **Edgington**. The intuition is that vagueness-relation indecision is about reality, that the referent of 'he' is truly half-tall and half-bald, so that it is half-(tall and bald). It is clear that the probabilistic structure would hold only if the possible states are only fully-bald and zero-bald: saying that he is bald one time in two (0.5) and tall one time in two (0.5) is exposing the combinatory frequency, so he is bald and tall in one in four. So the objection is correct against Edgington, as long as the same person is indeed half-bald and half-tall. However, these are not the only possible options. As argued in this work, vagueness allows two kinds of readings of 'humans are tall'. The first is the classical ∀x. Hx ⊃ Tx, while the second is, for example, the statistically quantified $\exists_{>9/10(H,\, m)}$ x . Tx[243], reading 'At least 90% of humans of which there are no more than m are tall'.

[241] Dorothy Edgington, "Vagueness by degrees", 309.
[242] John MacFarlane, "Fuzzy Epistemicism", in *Cuts and Clouds: Vagueness, its Nature, and Its Logic*, ed. Richard Dietz and Sebastiano Moruzzi (Oxford University Press, 2009), 450-453. Also Steven Schiffer, *The Things We Mean* (Oxford University Press, 2003), 210-211.
[243] See 5.3.5 and Part 3 of this work.

Therefore, a statistical conjunction would apply to the second reading: if at least 90% of humans are tall and at least 90% of humans are bald, then at least 80% of humans are tall and bald: $\exists_{>8/10(H,\,m)}\, x\,.\, Tx \wedge Bx$. Statistics is not probability theory, and the objection against Edgington does not apply to the theory of this work, since it is classical, because John himself is either tall or short.

12.4. Semantics

In CPL, interpretations are *simple* in the sense that they contain nothing beyond one truth value for each atomic formula. By truth-functionality, all sentences are true or false. This corresponds to a strong realism about truth. All shading needs to be discarded and all that is both necessary and sufficient for truth is truth, a condition closely related to Tarski's material adequacy[244]. In this sense, **trivaluationism (with the exception of Tye)**, **fuzzy logic**, **Goguen's framework** and **Zardini's non-transitivism** have *simple* Tarski-like semantics as well.

12.4.1. Supervaluationism and its friends

The best known extension to classical model-theoretical semantics is the Kripke semantics for modal logic. A frame is defined such that it contains a set of points called *worlds* (out of which one may be designated as the *actual world*) and an accessibility relation between them, saying which of the worlds are accessible from each world. In **supervaluationism**, the frame is usually called *precisification space* and each world a *precisification* with a root interpretation that can access all others. Then, each precisification is given a trivaluationist interpretation except *complete* precisifications which are classical. If a precisification accesses another, the latter maintains all definite (T, F) values of the former and they all implement what **Fine (1975)** calls *penumbral connections:* both ***predicate-ordering monotony*** ('if a man of *n* cm is tall, so is a man of *n*+1 cm') and all theorems of CPL[245]. In Kripke semantics, necessity of a sentence at some world is defined as that sentence being true in all the worlds accessible from the respective world. The necessity operator $\Box$ is used to define the possibility operator $\Diamond$ as *not necessarily not* – not impossible. Now, **supervaluationism** added a third indeterminate truth value compared to CPL, but wants to keep LEM. Then a precisification space will contain all the acceptable resolutions of the intermediate

[244] Alfred Tarski, "The Concept of Truth in Formalized Languages" in *Logic, Semantics, Metamathematics, second edition*, ed. J. Corcoran, (Indianapolis: Hackett, 1983), 187-188.
[245] Kit Fine, "Vagueness, Truth and Logic", 276, see discussion at 2.4.

sentences from the root interpretation. Thus, for a soritical series for *tall* from 170 cm to 190 cm, there is a complete precisification where 170 cm is the absolute border between *tall* and *short*, another where 171 cm is, another where 171.1 cm is and so on[246]. **Supervaluationists** are thus able to introduce a *Definitely* operator corresponding to the necessity operator ranging over all complete precisifications of the root interpretation. And define supertruth as truth in all complete precisifications. LEM is then supertrue because all complete precisifications are classical. We will discuss the iteration of the *Definitely* operator in 12.5.3 below.

Weatherson's semantics of *truer* is similar to **supervaluationism**, starting from a weak normal modal logic KT, where the single restriction is reflexivity ($\Box A \supset A$)[247]. He modifies however the formation rules so that $\Box$ can only be introduced to conditionals (if '$A \supset B$' is well-formed, so is '$\Box(A \supset B)$'). This allows him to add a substitution rule such that '$\Box(A \supset B)$' can be rewritten '$B \geq_T A$' , read *'B is truer than A'*, including inside sentences. But since any sentence A is classically equivalent with $(A \supset A) \supset A$, '$\Box((A \supset A) \supset A)$' rewritten '$A \geq_T (A \supset A)$' behaves as $\Box A$ did, suggesting that his logic is not less expressive than classical KT[248]. **Weatherson** then proceeds to build maximal consistent sets (i.e. sets of sentences such that they are closed under modus ponens, contain all theorems, and one of any sentence or its negation) for his logic of $\geq_T$, by using this substitution rule on the maximal consistent sets from the canonical model of KT. Finally, he says English is one of these maximal consistent sets, not specifying exact which[249]. If, for **supervaluationism**, the iteration of the necessity operator (read 'Definitely') corresponds to different ways, sets of ways, sets of sets of ways, etc. to precisify a vague language [250], for **Weatherson**, the *truer* operator is fundamental, English being such that is contains sentences of infinitely many iterations – a maximal consistent set of the canonical model.

12.4.2. Counter-intuitive semantics

Fine's recent compatibilism puts Kripke semantics to a radically different use. His frame of situations (he calls them *uses*) is linked by a symmetric, reflexive, but not transitive, relation. As mentioned, he takes truth of sentences in different situations as elementary. But he takes a negation

[246] It was argued that supervaluationism is mistaken because unsharpenable vague predicates (i.e. without a complete precisification) can be found. John Collins and Achille C. Varzi, "Unsharpenable Vagueness", *Philosophical Topics* 28:1 (2000).
[247] Brian Weatherson, *True, Truer, Truest*, 8.
[248] Brian Weatherson, *True, Truer, Truest*, 8.
[249] Brian Weatherson, *True, Truer, Truest*, 10.
[250] Kit Fine, "Vagueness, Truth and Logic", 292-293.

as true just in case the negated sentence is true at no situations accessible from the current one. This means that if a situation where A is not true 'sees' another where A is true, $\neg A$ being true does not obtain. Does this mean that $\neg A$ is then false? Not necessarily, because the present situation can *see* another where $\neg A$ holds, therefore it cannot get $\neg\neg A$. On the other hand, $A \wedge B$ is true at a situation just in case A is true and B is true at that situation. Therefore, $\neg(A \wedge \neg A)$ (NC) will be true at all situations, since at no situation $A \wedge \neg A$ will be true, that in its turn since if a situation has A true, it cannot have – by reflexivity of accessibility – $\neg A$ also true[251]. What **Fine** is really modelling is an agreement environment where everyone can have their positive opinion[252], but no one will deny anything if anyone with which they connect has it as positive opinion. In the real world, this would result in moral hazard. One can get accepted by as many others as possible and adopt as many opinions as possible. Because of a first mover's advantage, the situations with the most own truths influence the values of other situations the most. **Fine's compatibilism** would allow a model where two mutually accessible situations divide propositional letters, each corresponding sentence being true at only one of them. In that case, all negations of atomic sentences will be indeterminate in the entire frame. The relative symmetry of steps in a soritical series, such as they all count equally, is not captured by Fine's construction, justifying the objection of a ***counter-intuitive semantics***.

For **Edgington's probabilistic theory**, there are two ways of providing a semantics. One way is her suggestion to use the interpretations of **supervaluationism**[253] and count the proportion of precisifications having A true in which B is also true in order to determine the probability of B given A. This extends the proposal of **Kamp**[254] to define **supervaluational** degrees of truth and the logic of adjectival comparatives and superlatives. Its upshot is that ZFC allows the introduction of a cardinality comparison between the sets of precisifications in which particular sentences hold. Then, some sentences are said to hold in more interpretations than others. Edgington does not see this variant of providing an interpretation as more than a "heuristic device"[255], with good reason. Using **supervaluationist** structure seems unmotivated if the rest of the **supervaluational** apparatus is discarded: how can precisifications be requested to be classical extensions of a root interpretation if the root interpretation does not have a third, indeterminate, value to eliminate? The other way to understand **Edgington's theory** is to use the set-theoretical developments of probability theory that

[251] This characteristic of his conjunction helps avoid Fine's previous argument that vagueness is impossible, from Kit Fine, "The Impossibility of Vagueness", *Philosophical Perspectives*, 22, Philosophy of Language (2008).
[252] See Kit Fine, "The possibility of vagueness", 3720, for the application of the semantics to disagreement.
[253] Dorothy Edgington, "Vagueness by degrees", 315.
[254] As first elaborated in J. Kamp, "Two theories about adjectives", 137-139.
[255] Dorothy Edgington, "Vagueness by degrees", 315.

deal with the probability of propositions. Therein, not all propositions have probabilities[256], which is incompatible with Edgington's theory that should assume that given any proposition A, there is an independent probability of A. It seems like Edgington's theory cannot have anything but a ***counter-intuitive semantics***. Finally, defining the conditional through probabilities such as |B given A| as suggested by Edgington raises the prospect of a radically different deduction system, as that of conditional probability logic, of much lower reliability than classical modus ponens or tollens[257].

Smith calls his position **fuzzy plurivaluationism**, to distinguish it from the standard fuzzy treatment of vagueness as found for example at **Machina**. Besides rejecting ***non-equivalence of mpp, mpt, mtp, mtt*** and introducing a classical consequence relation, as we will see in 12.5.1 below, he wants to meet the classical objection to fuzzy logic that it introduces an ***artificial precision.*** As formulated by Keefe, it is that "In so far as a degree theory avoids determinacy over whether *a* is F, the objection here is that is does so by enforcing determinacy over the *degree* to which *a* is F"[258]. Now, what the objection is based on is that the essence of being indeterminate is not having any precision, while degrees of truth are a precise device. This can be answered partially by saying that the fuzzy real interval is just a model of the real situation, and the idea of indeterminateness is captured by intermediate truth values diverging from perfect truth (1) and falsity (0). Smith thus says that the objection is really aimed at the *determination* of the precise intermediate degree, not the very fact that it is intermediate or numerical[259]. He further identifies the formal device of the *intended interpretation* as causing the trouble: "there is no fact of the matter concerning which fuzzy interpretation of a given vague discourse is the unique intended one"[260]. His position is, then, that a natural discourse has many possible interpretations, all subject to reasonable constraints paralleling Fine's supervaluationist penumbral connections. However, he refuses to aggregate all acceptable interpretations in a single superinterpretation: "Unlike in the supervaluationist picture, the language is not in a unique (higher-order) semantic state. Semantic states are individuated by interpretations, and there are many of them."[261] So the rejection of artificial precision is in refusing to choose an interpretation, not semantics. Is this tenable? But the constraints on the interpretations *are* semantic, because they

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[256] Brian Weatherson, *Probability in Philosophy*. Lecture Notes, Lectures 2-3, June 2008. Accessed Oct 18, 2019 http://brian.weatherson.org/PL2.pdf, 101.
[257] Niki Pfeifer and Gernot Kleiter, "Inference in conditional probability logic", *Kybernetika*, Vol. 42, No. 4 (2006): 397-398.
[258] Rosanna Keefe, *Theories of Vagueness*, 113, discussed in Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 277-278.
[259] Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 279.
[260] Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 284.
[261] Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 287.

describe which interpretations are acceptable or not. And it would seem like some statements have a minimum or equal degree of truth in all acceptable interpretations, and it seems quite little to say of them only, as Smith writes, that "if a sentence has a certain degree of truth, say 0.3, on every acceptable interpretation, then we can *talk* as if there is just one intended interpretation, on which the sentence is 0.3 true". We not only can talk, but the formal apparatus for describing them as unanimous is readily available. Once more than one interpretation is available, we can count the number of acceptable interpretations on which the statements has that degree of truth and, if computed, the problem of ***artificial precision*** is reintroduced. This justifies an objection to **Smith** as having a ***counter-intuitive semantics***.

12.4.3. Epistemicism and epistemic logic

Williamson's epistemicism is the best-known variant of **epistemicism**, the theory that natural-language predicates have definite borders, but they are unknown. By 'border' we understand a single number of cm separating all short from tall people, a single number of hairs separating all bald from hirsute people and so on[262]. Thus, **epistemicism** defends both classical logic and ***predicate-ordering monotony***. For this reason, it is confronted with the question of why we do not know such borders: the number of cm separating tall and short people, the number of hairs separating bald and hirsute people, etc. **Williamson** answers that the ignorance is justified by *margins of error*, explained through epistemic logic. His first thesis is the failure of the KK principle, that is, the transitivity of knowledge. We can know something without knowing that we know it. In epistemic logic using K as necessity operator: $Ka \nvDash KKa$. The second thesis is that it is mandatory to restrain judgment – refuse predication – in cases where we do not know that we know their respective truth or falsity. In a sorites from 190 cm to 160 cm for *tall*, we affirm that a man of 190 cm is tall because we know that we know that. And similarly for 189 cm. But, say, at 184 cm, we do not know that we know the corresponding statement, so we refuse to either affirm or deny it. That can happen even though a man of 184 cm is tall in any case, if we do not know that we know it. The difference between truth and truth known as known is a *margin of error.* It is reasonable, **Williamson** says, as a kind of epistemic insurance. He gives the example of a stadium full of people[263]. We should not deny that there is a precise number of spectators in it, but saying which number it is at any moment is impossible. We know that the number is not 100. We say that it is not 100 because we have independent justification to know the fact that our opinion that it is not 100 is well-founded. We have

[262] See also *standard of separation* and *class of separation* in Chapters 4 and 7.
[263] Timothy Williamson, *Vagueness*, 217-226.

previously seen similar situations and counted the people in them, such that any opinion according to which the number is not 100 is knowledge. The same cannot be said for 65.012. We may believe it and maybe it is true, i.e. the number of spectators is 65.012, but we do not have reasons to believe that our belief is knowledge[264]. Then, since we do not know that we know that there are 65.012 spectators, we should stop expressing an opinion either way. The epistemic logic in which **Williamson** builds his explanation of ignorance is the normal modal logic KT[265].

However, it was objected to epistemicism that ignorance about such precise borders is unbelievable: it seems like there is no fact of the matter of which to be knowledgeable about. Cian Dorr objects that epistemicism conflates unknowability and ignorance[266]. Therefore, it is important to note that a defense of classical logic *tout court*, abandoning ***predicate-ordering monotony*** has no need to appeal to margins of errors or ignorance, as **Williamson** does. That is because classical logic only requires any particular man be tall or not, not that there should be any single number of cm separating tall men from short men, nor any conceptual border to be knowledgeable or ignorant about. For the same reason, a non-epistemicist defense of classical logic can express the gradual nature of predicate tolerance through the proportion of tall to short men at each number of cm. This is what I will propose in the third part of this work.

12.5. Truth, validity and the sorites

ZFC is standardly used to speak of the truth of FOL through an inductive definition of truth, as in 12.4 above. And, in FOL, the consequence is the relation between the (possibly empty) premises and the conclusion, such that in all interpretations, the former cannot be all true without the latter being true as well. An argument is valid just in case its conclusions are a consequence of its premises. A formula is valid just in case it is a consequence of the empty set of premises. Truth also has the disquotational property which allows the removal of quotation marks. '"Snow is white" is true just in case snow is white'. It connects language and metalanguage, easily visible by replacing the left side with the translation of the phrase in French. There is a second property of quotation marks, that of connecting

[264] This incidentally vies well with some justifications for plurivaluationism where degrees of belief are not distinguished from truth-values: "What actually happens is that we become less and less convinced of men's shortness as their height gradually increases" J. A. Goguen, "The Logic of Inexact Concepts", 330.
[265] Timothy Williamson, *Vagueness*, 272.
[266] Cian Dorr, "Vagueness Without Ignorance". *Philosophical Perspectives*, 17, Language and Philosophical Linguistics (2003): 105..

meaning and assertion in indirect speech. For example, Frege did not use quotation marks, but his early theory of *judgeable content* was based on the affirmation of a conceptual content as a logical act, denoted by the *judgement* vertical stroke[267] as distinct from the horizontal (content) stroke. The content was put forward as true. This sense is the one that allows propositional contents to be embedded in indirect speech records as objects of belief, thought, consideration, etc. Affirming 'snow is white' is affirming that snow is white[268]. Let us call this property that of *truth assertion*.

12.5.1. Plurivaluationism

Edgington's probability-based theory, **trivaluationism** and **fuzzy logic**, both classical and in Smith's variant, are given metalogical constructions in which the disquotational property holds. These constructions are *homophonic*, so called because they duplicate in the metalanguage the phenomenon of the object language. As **Edgington** puts it: "There is no reason to deny the equivalence of 'It is true that A' and 'A', or of 'It is false that A' and '¬A'. If v(A) = 0.5, v(It is true that A) = 0.5 […]. Parallel claims can be made if we prefer to treat 'true' as a metalinguistic predicate."[269] However, plurivaluationism is committed to ***indeterminate truth assertion***. If a sentence is half-true, the removal of quotation marks does not indicate it is half-true. Unlike in the classical picture, affirming that something holds does not commit one to affirming its truth. Suppose your utterance corresponds to a 0.5 degree-sentence, should we take you as affirming that is perfectly true or half-true? To escape this objection, **Smith** introduces degrees of belief that behave in some contexts probabilistically, for uncertainty-based situations and fuzzily, for vagueness-based situations, being able to be combined for situations where both are present[270].

A separate problem for **trivaluationism** is that the third value is meant to give account of those sentences that are not true and not false. In a sorites, some step predications in the middle are to be intermediate, neither tall nor false. Say, all men above 185 cm are tall, all those below 170 cm are short and those between 170 and 185 cm are neither tall nor short. But which value can give account

[267] Gottlob Frege, "Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens", 11.
[268] Frege writes "Mr. Peano has no such sign […] From this it follows that for Mr. Peano it is impossible to write down a sentence which does not occur as part of another sentence without putting it forward as true" Gottlob Frege, *Collected Papers on Mathematics, Logic, and Philosophy*, ed. Brian McGuinness (Basil Blackwell. 1984), 247. The passage is discussed in Nicholas J.J. Smith, *Frege's Judgement Stroke and the Conception of Logic as the Study of Inference not Consequence*. To appear in Philosophy Compass. Accessed Oct 18, 2019 http://www.personal.usyd.edu.au/~njjsmith/papers/smith-freges-js-logic.pdf
[269] Dorothy Edgington, "Vagueness by degrees", 310.
[270] This distinction is discussed at length in Steven Schiffer, *The Things We Mean,* 204-213. Smith's solution is in Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 233-240.

of those sentences that are not true, not intermediate and not false? Because a sorites could be built from 185 to 185.1 cm using a soritical step of 0.001 cm, where it would seem a new – second-order – intermediate value is required. The general problem is called ***higher-order vagueness*** and we will discuss it in Chapter 17, but this leg can be discussed in isolation: it seems like the motivation for the introduction of a third value just as strongly supports the introduction of a fourth, fifth, sixth and so on. **Tye** is the single **trivaluationist** with a coherent reply, in a way which is Fregean in spirit. He affirms that all sentences, no matter how slightly intermediate, are indeterminate and any statement about their indeterminacy is also indeterminate[271]. This means that the sentence 'there is a truth value that is not true, false or intermediate' is itself intermediate. This is however difficult to accept, because it implies the indeterminacy of metalogical and philosophical statements about the intermediate truth value. For example, even set membership is indeterminate for Tye who does not use fuzzy sets, but sets for which it is an indeterminate manner whether an element belongs to them or not[272], a very ***counter-intuitive semantics.***

Of course, **fuzzy logic** responds to the challenge of ***higher-order vagueness*** by positing an infinity of truth values between any distinct two values, which allowed the gradual treatment of the conditional sorites described in 12.4 above. It is then objected that the precision of the assignment of say 0.776 is incompatible with it being second-order indefinite. **Smith**'s reply is to distinguish the problem of ***artificial precision***, which we have discussed in 12.4.2 above from the question of indeterminateness, noting that there are no borders: "while we may distinguish borderline cases of 'bald' (say) from objects assigned 0 or 1 by the fuzzy set of bald things, we do not get boundaries between the clear cases and the borderline cases, and the clear countercases and the borderline cases"[273].

However, **fuzzy theorists** disagree on whether the sorites is valid or not, just as **trivaluationists** do. As Williamson argued[274], **plurivaluationism** has three ways of dealing with validity. Firstly, as the preservation of designated values. For example, **trivaluationists** such as **Halldén** and **Körner** take also the intermediate value as designated, such that all classical formulas[275] will be valid i.e. the conclusion will not be false provided the premises are not false. The disadvantage is that modus ponens would be invalid. Williamson's example is 'p ⊃ q' and 'p' where p is intermediate and q is

[271] Michael Tye, "Sorites paradoxes and the semantics of vagueness", 285, 292.
[272] Michael Tye, "Sorites paradoxes and the semantics of vagueness", 283-284.
[273] Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 310-311.
[274] Timothy Williamson, *Vagueness*, 102.
[275] Timothy Williamson, *Vagueness*, 102-110.

false: designated-value premises have a false i.e. undesignated conclusion. **Tye** being coherent with his interdiction of speaking about the third value, he defines validity without taking it into account. He takes only the true as designated, as Łukasiewicz did in his introduction of three-valued logics[276], resulting in a logic in which most classical formulas are invalid, as for example LEM ($A \vee \neg A$ is intermediate when A is intermediate). Secondly, in the case of **fuzzy logic**, a more natural option is to make validity a matter of degree itself, as **Machina** proposes: "rather than a notion of tautology, I propose we use a notion of a *minimally n-valued formula*: a formula is minimally n-valued iff it can never have a value less than n. [...] this notion can be generalized to the notion of the *degree of truth-preservation* possessed by an argument form"[277]. For example, define the degree of validity as the difference between the truth value of the conclusion and that of the lowest of the premises. In this case, the soritical steps can be 0.99 valid, corresponding to the 0.99 degree of the conditional defined as above. As Machina observes, this keeps the classical parallelism between the truth-functional definition of ⊃ (in Łukasiewicz fuzzy definition) and the consequence relation. Finally, validity *tout court* is defined by **Machina** as the conclusion having a truth value equal or higher than the lowest of the premises[278]. In this case even an argument from premises of 0.1 degree to a 0.1 conclusion would be valid. The obvious objection is that it ***allows too much bad reasoning,*** as in the example counting as purely valid. The second is that this validity is not classical, for example if |A| is 0.5, |B| is 0.1, |A ⊃ B| will be 0.4. So a modus ponens of $A \supset B$; $A \therefore B$ will not be valid, since the conclusion has degree 0.1 but the lowest of the premises has 0.4.

Smith avoids all disadvantages above by combining the variants to give a definition of validity for his **fuzzy plurivaluationism** which is classical. For him, an argument is valid just in case the conclusion is at least 0.5 true whenever all premises are strictly above 0.5 true[279]. That threshold is justified as separating values which are *good for reasoning* as premises from those *good enough* for conclusions and yields a classical consequence relation, for example in the modus ponens example above. As in CPL, the standard sorites from 200 cm to 100 cm for *tall* would be valid but unsound, since some of the premises are less than 0.5 true. For example, ***predicate-ordering monotony*** motivates all versions of plurivaluationism to take people at middle heights such as 165-175 cm as having

[276] J. Łukasiewicz, "On three-valued logic", in *Selected works by Jan Łukasiewicz*, ed. L. Borkowski (Amsterdam, North–Holland, 1970), 87-88.
[277] Kenton Machina, "Truth, belief and vagueness", 197.
[278] Kenton Machina, "Truth, belief and vagueness", 197.
[279] Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 222-224.

intermediate degrees of tallness[280]. This means that some men will be tall to a degree less than 0.5, so any soritical step in which they feature will also have a degree of less than 0.5 by Smith's definitions.

Edgington's theory allows valid arguments with false conclusions, provided that the sum of the unverities (an unverity is 1 minus the verity) is higher than 1[281]. For her, a valid argument is one in which the premise does not go further from truth than the sum of the differences from truth of the premises. But this is far from the classical picture, and it can be objected that **Edgington *allows too much bad reasoning***. She gets a definition of the sorites as valid, at the price of saying that an argument from two premises of 0.5 degree to a completely false conclusion is valid. Validity is about rigor in handling truth, and an argument with partly true premises and a false conclusion is not rigorous.

Moreover, just like the standard **fuzzy theory** of the conditional sorites, **Edgington**'s approach to the sorites is vulnerable to **Weatherson**'s ***discrete terms paradox***[282]. Indeed, their assumption is that a credible sorites needs a large number of steps, so that small, barely noticeable deviations from full truth accumulate to falsity[283]. But an intuitive sorites on *few children* can be played on one, two, three and four children. It is intuitive that one child is few and four are not, and the four steps are too few for small untruths to accumulate. I argue that such discrete terms tell in favor of vagueness as dispersion: some groups of two children are of few children and some are not of few children, depending on their relationship, socio-economic milieu, convergence with parents' wishes, and so on. Were Edgington to abandon ***predicate-ordering monotony***, her probabilistic degrees could be defined in classical FOL as the proportion of few to non-few groups of children at each rank i.e. number of children. Then statistical reasoning would apply to such proportions, while having LEM and NC true, as she wants. As we will see in the third part of this work, while any man is either tall or short, but there is a sense that they may be among those 80% likely to be tall, namely if 80% of the people of the same number of cm are tall.

[280] Keefe objects that this parallelism between heights and fuzzy truth values show the latter to be unnecessary and unmotivated devices. Rosanna Keefe, *Theories of Vagueness*, 134. Smith convincingly replies that degrees of truth can be taken as the range of a function that is distinct, yet monotonous with height. Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 216-217.
[281] Dorothy Edgington, "Vagueness by degrees", 303.
[282] Brian Weatherson, *Vagueness as Indeterminacy,* 4.
[283] For example Goguen: "this phenomenon corresponds to our feeling that the deductive process […] is 'fairly valid' for *x* and *y* of similar height, but becomes less and less valid as the number of applications of *modus ponens* increases." J. A. Goguen, "The Logic of Inexact Concepts", 335.

12.5.2. Zardini's non-transitivism

Smith's distinction between *inference-grade* and *conclusion-grade* sentences[284] is also the essence of **Zardini's non-transitivism**, allowing him to redefine the consequence relation to be non-transitive. For him, there is a function which will have one designated value (becoming semantic entailment) just in case the truth value of the consequent is tolerated by the truth value of the antecedent. By constraint, each truth value tolerates itself and its tolerated values form an upset under ≤[285]. In this way, in a sorites from 190 cm to 160 cm, the value of the statement that 'a man of 190 cm is tall' may tolerate the value associated to 'a man of 180 cm is tall', but not that of 'a man of 179 cm is tall'. The statement that 'a man of 189 cm is tall' may then tolerate the value of 'a man of 179 cm is tall' but not that of 'a man of 178 cm is tall'. Consequence is not transitive, although it is transitive on narrow spans. Semantic entailment will be used to define the conditional, making a conditional between the first sentence and any sentence of tolerated values have a designated value. Therefore, **Smith** and **Zardini** have a functionally intermediate status of sentences. They behave classically as conclusions, but cannot be used as premises. For Zardini, this distinction can be attacked as being a limitation which is avoidable, because the ***paradox can be restated***. As we saw Williamson arguing with regard to all substructuralist restrictions of chaining in Chapter 3, as long as something is accepted as conclusion, it is reasonable to start a new argument with it, independently of the previous one.

12.5.3. Supervaluationism and Weatherson's logic of *truer*

Supervaluationism is 'classical at a remove'[286]. For it, a sentence is true just if supertrue – true in all acceptable complete precisifications. Thus, intermediate heights generate indeterminate propositions, say A: 'a man of 178 cm is tall'. But in all complete precisifications, either A holds or not, so $A \vee \neg A$ is true, without A or $\neg A$ being true i.e. supertrue. Therefore, supervaluationism validates all classical theorems, since all complete precisifications are classical. But, as Williamson objected, it is ***lacking the disquotational property***: from the fact that A ∨ ¬A is true one can infer A or ¬A. But from that we cannot infer that 'A' is supertrue or '¬A' is superfalse[287]. As for validity, there are two main ways of defining validity in **supervaluationism**. The first is global validity, called by **Varzi** variant A ("An argument is valid iff, necessarily, if every premise is: T on all precisifications, then some conclusion is: T on all precisifications"), which is preservation of supertruth. The second is local

[284] Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 223.
[285] Elia Zardini, "A Model of Tolerance", 6-7.
[286] Kit Fine, "Vagueness, Truth and Logic", 278.
[287] Timothy Williamson, *Vagueness*, 162.

validity, called by **Varzi** α ("An argument is valid iff, necessarily, on all precisifications: if every premise is T, then some conclusion is T"), which is preservation of truth[288]. But since **supervaluationism** is built around supertruth, Williamson argued it should use *global* validity i.e. preservation of supertruth. However, using global validity leads **supervaluationism** to blocking inference rules such as conditional proof, reduction at absurdum where premises contain the *Definitely* operator[289]. **Keefe** maintains global validity but proposes some replacements including inside sentences[290] whose status as inference versus heuristic rules are unclear[291]. Therefore, supervaluationism in its most popular definition of validity (i.e. global) partially ***does not keep CPL theorems and rules***. It also ***blocks higher-order vagueness***[292], because the iteration of the *Definitely* operator can be replaced with a transitive *Definitely** operator which seems to draw a hard border between definite* statements and any other[293]. In contrast, Varzi's reply to Williamson is that it is fair when dealing with a vague language to check whether indeterminate premises lead to true conclusions (hereby supertruth playing no role), as it is for a set of perfectly precise interpretations where truth is supertruth[294]. Therefore, **Varzi** argues for adopting local validity (while accepting the ***lacking of the disquotational property***) for supervaluationism.

To compare, **Weatherson's logic of *truer*** takes some argument as valid just if the premises taken collectively cannot be truer than the conclusion. What this means is that he isolates all models for $\geq_T$ such that the conclusion is $\geq_T$ than the greatest lower bound of the premises (premises taken as a conjunction). Note that the glb of the premises exists axiomatically, since there exists an absolute minimum in the truth set: the false value. He affirms that the constraints of his logic define a "Boolean lattice over equivalence classes of sentences with respect to $=_T$"[295], Boolean algebras being models for classical logic, including classical inference rules which supervaluationism abandons. However, one should note that this classicality only holds for language fragments not containing **Weatherson**'s $\geq_T$, his logic containing more valid sequents than CPL otherwise (stemming from his definition of $\geq_T$

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[288] Achille C. Varzi, "Supervaluationism and Its Logics", 643-646.
[289] Timothy Williamson, *Vagueness*, 149-151.
[290] Rosanna Keefe, *Theories of Vagueness*, 179-180.
[291] Achille C. Varzi, "Supervaluationism and Its Logics", 656-657.
[292] "Can one save A-validity by making sense of the idea that all vagueness is in fact first-order? To most people, this would just be biting the bullet". Achille C. Varzi, "Supervaluationism and Its Logics", 659.
[293] Williamson "Define 'Definitely* A' to mean the infinite conjunction: A and definitely A and definitely definitely A and . . . . The definition guarantees that if definitely* A then definitely definitely* A and indeed definitely* definitely* A" ,Timothy Williamson, *Vagueness*, 160. Varzi agrees with the argument, Achille C. Varzi, "Supervaluationism and Its Logics", 658-660.
[294] Achille C. Varzi, "Supervaluationism and Its Logics", 646. Smith goes with Williamson "It should be obvious that the latter definition in fact has no plausibility at all" Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 82.
[295] Brian Weatherson, *True, Truer, Truest*, 7.

from implication plus necessity)[296]. This seems a partial improvement over **supervaluationism**, but it can be argued that at least partially, his logic ***does not keep CPL theorems and rules***. What tells in favor of **Weatherson**'s logic is his treatment of the sorites. For **supervaluationism**, the sorites paradox is a manifestation of our indecision concerning intermediate cases, each soritical step which is not true being false in at least one complete acceptable interpretation. I take **Weatherson** to say that the sorites is the accidental listing together of many strict affirming-the-consequent fallacies: sentences which are truer that others (i.e. because they are strictly implied by them in any interpretation where the latter are true) are said to imply them. Thus, (a) 'a man of 180 cm is tall' is for **Weatherson** truer than (b) 'a man of 179 cm is tall', because it seems that the latter cannot hold without the former in all interpretations – what we called ***predicate-ordering monotony***. But the sorites says that the latter holds if the former does in a typical soritical step ('If a man of 180 cm is tall, so is a man of 179 cm'), which reverses the implication. Thus truer sentences imply less true sentences in a chain. This explanation is somewhat similar to that given by the fuzzy treatment of conditional sorites – small differences accumulating – but it is obtained with the weapons of supervaluationism[297]. In contrast to the **fuzzy** treatment, Weatherson cannot analyze 'almost true'[298], so we rely on a partial ordering to understand the graduality of the soritical steps. Under **supervaluationism** 'there is a single number of cm separating all tall men from all short men' is supertrue[299], since all complete precisifications are classical and they all respect penumbral truths, i.e. ***predicate-ordering monotony***. Under **Weatherson**'s theory, I take it that many soritical steps may be not true.

12.5.4. Fine's compatibilism

Fine's recent compatibilism defines validity classically, as truth-preservation in all interpretations[300]. As mentioned, the truth valuations are also classical, for each situation a sentence is true if it is assigned truth by the valuation function for the situation. But since negation and the conditional depend on all other accessible situations (under the reflexive symmetric relationship connecting Fine's situations called *uses*), the logic will be very different, since most classical inferences involve the negation or conditional. Interpretations for Fine are assigned globally, the

[296] Brian Weatherson, *True, Truer, Truest*, 7.
[297] Brian Weatherson, *True, Truer, Truest*, 1-2.
[298] Brian Weatherson, *True, Truer, Truest*, 11.
[299] Brian Weatherson, *Review of Rosanna Keefe, Theories of Vagueness*, Cambridge University Press, 2000. Accessed Oct 15, 2019 http://brian.weatherson.org/keefe.pdf, 4.
[300] Kit Fine, "The possibility of vagueness", 3722.

relations between the situations being of equal importance with the truth valuations. The removal or addition of a situation inside the frame can modify the truth of all negations and implications in its accessible situations and then in all others through the frame. Fine's motivation is to have modus ponendo tollens, also called *conjunctive syllogism* fail. If it fails, he can describe a soritical series from 170 to 190 cm through the truth of the denial of all combinations such that two number of cm diverge (one is true or false). Let us cite his illustration[301] (situations in italic, sentences in bold):

```
              A
              •u*
       /             \
B •w                  •u ¬B
       \             /
              •v
              ¬A
```

Situations: $\{u^*, u, v, w\}$

Accessibility: $\{<u^*, u^*>, <u, u>, <v, v>, <w, w>, <u^*, u>, <u, u^*>, <u, v>, <v, u>, <v, w>, <w, v>, <w, u^*>, <u^*, w>\}$

Valuation: V(u^*):{A}, V(u): ∅, V(v): ∅, V(w):{B}

Then:

$u^* \vDash$ q; $u \vDash \neg$B (because $u^* \nvDash$ B and $v \nvDash$ B);

$w \vDash$ B; $v \vDash \neg$A (because $u \nvDash$ A and w $\nvDash$ A);

NC holds because ¬(A ∧ ¬A) and ¬(B ∧ ¬B) are true at all four situations. But ¬((A ∨ ¬A) ∧ (B ∨ ¬B)) also holds, because both (A ∨ ¬A) and (B ∨ ¬B) do not hold together at any of the four situations. This is the key of Fine's description of the sorites. He calls it a *global approach* because it only acts when describing a series, not an individual situation (wherein LEM and NC hold). And a *logical* approach because he affirms that there is nothing in vagueness beyond this logical behavior[302].

Both points are attractive, but the logic can be objected to. Firstly, defining negation model-theoretically can lead to a model with no true negations of atomic sentences, as indicated in 12.4.2, which is a ***counter-intuitive semantics***. Secondly, together with Zardini, Fine's logic ***blocks too large a part of reasoning***, by not discriminating between arguments for *tall* and *bald*. Once somebody has opinions about tallness and baldness, he cannot reason classically from one to the other, even though vagueness is paradoxical only in arguments using exclusively one of them. Thirdly, Fine's logic displays ***non-equivalence of mpp, mpt, mtp, mtt***. He is the single one in which mpt (conjunctive

[301] Kit Fine, "The possibility of vagueness", 3709.
[302] Kit Fine, "The possibility of vagueness", 3705-3706.

syllogism) fails, although it has been constantly cited as the strongest, most reliable inference rule, from Chrysippus[303] to Edgington[304]. He keeps the validity of mtp (disjunctive syllogism) and the mpp (modus ponens), appealing to philosophical arguments to block their corresponding soritical forms, which I discussed in Chapter 2. A classical logical treatment for both *globalism* and *logicalism* is provided by my notational extension to FOL, as developed in the third part of this work. Vagueness is global because it is about tolerance across ranks of a total preorder, all individual objects at any rank being either tall or not tall themselves and it is logical because it can be given a purely extensional definition through the interplay of a monadic predicate and orderings corresponding to measurable properties.

Finally, **epistemicism** is **supervaluationism** without possible-world semantics[305]. There is a single number of cm separating all tall from short people, but we do not know it, and, if the explanation in 12.4.3 above is correct, we cannot know it because of natural margins of error. The sorites has a single false soritical step, because of the acceptance of ***predicate-ordering monotony***.

[303] Timothy Williamson, *Vagueness*, 25-26.
[304] Dorothy Edgington, "Vagueness by degrees", 310.
[305] For a critique of the distinction between epistemicism and supervaluationism, based viz. on the similarity between the sharp cutoffs of epistemicism and the supertruth remarked above of the existence of a single border see Andrew Bacon, *Vagueness and Thought*, 20-23.

12.6. Comparison

12.6.1. Main comparison table

	Truth values (**bald:** truth monism, brackets: outcomes)	Truth-functionality	Negation (¬A)	Conjunction (A ∧ B)	Conditional (A ⊃ B)	All contradictions are F		mpp, mpt, mtp, mtt are valid	No failure of transitivity
Trivaluationism (Halldén)	*T*,*F*,*I* (3)	✓	*I* if *A* is *I*, else classical	*I* if any is *I*, else classical					✓
Trivaluationism (Körner)	*T*,*F*,*I* (3)	✓		*F* if any is *F*, *I* if any is *I*, else *T*	*T* if A is *F* or B is *T*, then *I* if any is *I*, else classical				✓
Trivaluationism (Tye)	*T*,*F*,*I* (3)	✓							✓
Supervaluationism	*T*,*F*,*I* (3)	½	*T*/*F* if *T*/*F* in all complete precisifications, *I* otherwise			✓		✓	✓
Fuzzy logic	[*0*,*1*]	✓	1-|A|	least of |A|, |B|	*1* if |B| ≥ |A|, else |A|-|B|	½			✓
Smith's fuzzy plurivaluationism	[*0*,*1*]	✓	1-|A|	least of |A|, |B|	as ¬(A ∧ ¬B) or ¬A ∨ B	½		✓	½
Goguen's non-numerical framework	Lattice	✓	≤ |A|	glb of |A|, |B|	Least truth-value |C| such that |A ∧ C| ≤ |B|				✓
Zardini's non-transitivism	Lattice	✓	≤ |A|	glb of |A|, |B|	Has a designated value iff |B| is tolerated by |A|			✓	
Weatherson's logic of *truer*	Lattice plus *T*,*F*	✓	≤ |A| with |¬¬A| = |A|	glb of |A|, |B|	as ¬(A ∧ ¬B) and ¬A ∨ B	✓		✓	✓
Edgington's probability theory	[*0*,*1*]		*1*-|A|	|B given A| * |A|	*(suggested)* |B given A|	✓		✓	✓
Fine's compatibilist semantics	***T***,*F* (3)	½	*T* iff A *T* in no compatible use	*T* iff both *T*	*T* iff (1) A ∧ B or (2) all compatible uses have B *T* if A *T*	✓			
Williamson's epistemicism	***T***,*F* (2)	✓	*T*/*F* iff A is *F*/*T*	*T* iff both *T*	as ¬(A ∧ ¬B) and ¬A ∨ B	✓		✓	✓

CPL and Fine’s compatibilism have a single truth state: truth. Both can be said to have two truth values, adding the false. However, Fine’s theory allows neither the truth of a sentence, nor that of its negation obtaining, thus getting a possible third truth outcome, just like trivaluationism.

I write ‘½’ at ‘Truth-functionality‘ for supervaluationism because a disjunction can be true without any disjunct being true and at Fine because, as discussed, he defines negation and implication model-theoretically.

I write ‘½’ at ‘All contradictions are F‘ for fuzzy logic and Smith, because, as discussed, fuzzy semantics can define a function mimicking classical contradiction.

I write ‘½’ at ‘No failure of transitivity’ for Smith, because he has conclusions of 0.5 degree which are unusable as premises for further argument. For Zardini, consequence fails of transitivity. For Fine, the accessibility relation between uses does.

12.6.2. Comparison of validity and soritical treatment

	Kripke semantics may be used in the approach to... (*italic* indicates philosophical, not logical approaches)	Validity is ...	A sorites is...
Trivaluationism (Halldén, Körner)		Preservation of *T*, *I*	Valid but unsound. One middle step is intermediate.
Trivaluationism (Tye)		Preservation of *T*	
Supervaluationism	Truth, validity	Preservation of supertruth / truth on all precisifications *(debated)*	Valid but unsound. 'Exactly one step is false' is supertrue.
Fuzzy logic		When the least premise ≤ conclusion	Valid and partially sound *(debated)*. Many steps are intermediate, the consequents getting falser.
Smith's fuzzy plurivaluationism	*Rejecting artificial precision*	When premises are all > 0.5 and conclusion ≥ 0.5	Valid but unsound. Many soritical steps are ≤ 0.5. As fuzzy logic otherwise.
Goguen's framework		Preservation of designated values	Valid and partially sound *(debated)*.
Zardini's non-transitivism		If all premises take designated values, one conclusion takes a tolerated value	Invalid. Only short spans of steps are linked through consequence.
Weatherson's logic of *truer*	Operators, validity	When the least premise $\leq_T$ conclusion	Valid but unsound. One or more steps are not true.
Edgington's probability theory	Truth values (suggested)	When the sum of the unverities of the premises ≥ the unverity of the conclusion	Valid and sound. Many soritical steps are intermediate, but not enough to make the argument unsound.
Fine's compatibilism	Operators, truth, validity	Preservation of *T*	Mtp and mpp sorites are valid but misguided, so unsound. Mpt is invalid.
Williamson's epistemicism	*Rejecting the forced march*	Preservation of *T*	Valid but unsound. One (unknown) step is false.

12.6.3. Comparison of objections

	Battle of intuitions regarding truth values	Non-equivalence of mpp, mpt, mtp, mtt	Abstractness of framework	Probabilistic conjunction	Counter-intuitive semantics	Lacking the disquotational property	Discrete terms paradox	Paradox can be restated in a new argument	Blocks higher-order vagueness	Artificial precision	Indeterminate truth assertion	Does not keep CPL theorems and rules	Blocks too large a part of reasoning	Allows too much bad reasoning	Predicate-ordering monotony
Trivaluationism (Halldén)	✓		✓						✓		✓	½			½
Trivaluationism (Körner)	✓	✓	✓						✓		✓	½			½
Trivaluationism (Tye)		✓	✓		✓						✓	½			½
Supervaluationism						✓			½		✓	✓			✓
Fuzzy logic	✓	✓					✓	✓		✓	✓	✓		✓	½
Smith's fuzzy plurivaluationism	✓				✓										½
Goguen's non-numerical framework	✓	✓	✓			✓					✓	✓			½
Zardini's non-transitivism			✓			✓		✓			½	½	✓		½
Weatherson's logic of *truer*												½			✓
Edgington's probability theory	✓	✓		✓	✓		✓				✓	✓		✓	✓
Fine's compatibilist semantics		✓			✓							✓	✓		✓
Williamson's epistemicism															✓

I write '½' at 'Blocks higher-order vagueness' for supervaluationism, because of the debate discussed above about its definition of validity. I write '½' at 'Indeterminate truth-assertion' for Zardini's non-transitivism because of his distinction between truth and levels of goodness.

I write '½' at 'Does not keep CPL theorems and rules' for trivaluationism, Zardini's non-transitivism and Weatherson's logic because they display, respectively, either failures of some classical modes of inference, limited transitivity of classical consequence or have extra valid sequents.

I write '½' at 'Predicate-ordering monotony' for trivaluationism, the three fuzzy theories, Zardini's non-transitivism because, while motivated by the intuition that a gradation in the ordering (i.e. number of cm) corresponds to a truth value, this intuition does not have a direct semantic effect. In contrast, epistemicism and supervaluationist argue for single borders, Edgington gives verity 1 to expressions of *predicate-ordering monotony*, Fine's compatibilism rejects disjunctive syllogism on its account[306] and I understand Weatherson as reading it in a series of strict conditionals generating his $\geq_T$ relation.

12.7. Conclusion

To conclude, out of the variants of trivaluationism, the most consistent is Tye's. It has the serious disadvantage of forbidding metalogical statements about the intermediate value and has a counter-intuitive semantics, by making set membership indeterminate. Smith's fuzzy plurivaluationism is the best continuum-valued logics have to offer to the philosophy of vagueness. I found Weatherson's logic of *truer* preferable to standard supervaluationism, because, in my reading, it is not committed to the truth of 'a single soritical step is false' and avoids more elegantly the objections related to higher-order vagueness. I believe the intuitions behind Edgington's probabilistic logic of vagueness and Fine's globalist and logicalist position are better captured by expressing in FOL that a person is tall but they may be so at a height where most others are not. This logic would correspond to an epistemicism without predicate-ordering monotony, and thus without ignorance.

[306] See Chapter 2.

Chapter 13

General objections to vague logics

13.1. Logic and metalogic

Since the middle of the 20th century, ZFC has been the standard of mathematical correctness. Just as metamathematics uses ZFC to study mathematics itself, metalogic uses ZFC theories to study logic itself. Thus, metalogic gives a secondary, rigorous sense of *logic*. Let us exemplify with Elia Zardini's "A Model of Tolerance"*.* Zardini first defines a formal *language*, composed of an alphabet of symbols and a recursive definition of well-formed formulas (*wffs*). Then, using ZFC, he builds the notion of a *sequence* of wffs, that is, any n-tuple formed of wffs indexed to the ordinals. Finally, we can construct the notion of a *logic* as any set of the ordered pairs composed of two such sequences.[307]

Each ordered pair may be called a *sequent* or *ZFC argument*. The two elements, each a sequence of wffs, can be thought of as *premises* and *conclusions*[308]. Thus, a logic is a set of paired premises and conclusions out of all wffs available. FOL can be similarly built on the familiar language of FOL, by gathering all and only those pairs of premises and conclusions that correspond to the application of classical inference rules. To work with a familiar single-conclusion variant of FOL, we can take the *conclusions* to be the singleton of the single conclusion. We require it to be non-empty. Note that the set of premises may be empty, when the conclusion is a theorem. One adds constraints on the logic by specifying which pairs of premises and conclusions are included, in a structural manner, paralleling the NK introduction and elimination rules. Each constraint is a constraint on *validity,* i.e. the relation between some FOL premises and the conclusion that holds just in case, under all interpretations, the conclusion is true when all the premises are true.

Let us call *Sequent logic* or *ZFC logic* any logic understood in this metalogical sense, i.e. a set of all pairs of premises and conclusions in some language, which are allowable according to some constraints expressible in ZFC. Then there are infinitely many ZFC logics that use the language of FOL, not all having the intuitive qualities of *logic*. Most non-classical alternatives maintain the classical language but alter the constraints.

[307] For all, Elia Zardini, "A Model of Tolerance", 5-6.
[308] There are logics with one conclusion and with multiple conclusions, the second case being more general.

There is a tension between this metalogical sense of *logic*, and our initial sense. Logic is supposed to describe truth-preservation in a fundamental way. Philosophical disagreements about the correct logic should then be fundamental disagreements. Yet, metalogic shows that at least some *logics* can be constructed from classical ZFC as any other mathematical theory, which suggests that those disagreements with classical logic are not fundamental.

There are three more specific worries therein. The first is that some such ZFC logics may be philosophically *misleading* if they pretended to achieve something belied by their metalogical definition. For example the set corresponding to CPL, but with all wffs containing ∧ eliminated, if it was put forward as a logic incapable to express contradictions. Most would agree that since it is otherwise classical, $\Theta \vee \neg\Theta$[309], that is, LEM, together with metalogical statements, would still capture the sense of NC. The second worry is that some such ZFC logics may be *too strange or weak* to be *logics*. For a simple example, the set of all and only pairs of the form $<\{\Theta\}, \{\neg\Theta\}>$[310]. The arguments of this logic would have a single sentence as premise and obtain its negation as conclusion, a not very intuitively logical setup. And the third worry is that some may be *unnecessary*, the fundamental justification for them being accommodated more or less easily by classical logic. For example, suppose someone proposing a traditional syllogistic logic (*TSL*) built within ZFC, as the true logic of quantification[311]. It is not that TSL is wrong, but FOL can do as good a job and FOL is already used, as we have seen, in expressing ZFC itself.

I follow the literature for the first two worries. The third one prepares the ground for the third part of this work – a classical logic for vagueness.

13.2. The classicality of non-classical logics

Let us examine the first worry, that vague logics may be philosophically misleading as compared with their metalogical construction in ZFC, since ZFC is a theory formulated in classical FOL. Does this mean that any ZFC logic is itself classical? There are three arguments to that effect, in ascending order of soundness.

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[309] In this paragraphs, Θ is a metavariable ranging over well -formed formulas.

[310] Discussed in Theodore Sider, *Logic for philosophy*, 8, he uses 'genuine' where I use 'fundamental'.

[311] TSL usually is taken as including syllogistic modes such as *Celarent: 'MeP; SaM ∴ SeP', read 'No people are extraterrestrial; All Martians are extraterrestrial; No people are Martian'.*

The first argument is that any logic that can be given a semantics in ZFC is through this very fact classical. For example, Heyting algebras generalize Boolean algebras and serve as the sound and complete semantics for intuitionistic propositional logic. Just as Boolean algebras serve as the semantics for CPL. But Heyting algebras (or Kripke semantics for intuitionistic logic for that matter) do not prove that intuitionistic logic is classical. First, Heyting algebras can be characterized by first-order intuitionistic logic, so that the latter can give itself a semantics, in the propositional case, just as FOL can achieve it through ZFC. Secondly, the characterizability of intuitionistic logic in ZFC is a consequence of classical logic being stronger than intuitionistic logic, just as Boolean algebras are a subset of Heyting algebras, not a case of them being in any sense equivalent. So this argument can be rejected.

The second argument is that, because in ZFC, the law of non-contradiction (NC) is true, then, any ZFC logic displays NC in its metalogic. This can be rejected, as for example the logic L_{LNC} identified with the set of all and only arguments of either the forms $<\emptyset, \{\Theta\}>$, $<\emptyset, \{\neg\Theta\}>$. In L_{LNC}, for any Θ, it can be that both $\vdash_{LNC} \Theta$ and $\vdash_{LNC} \neg\Theta$, since $\vdash_{LNC}$ ranges over the relation between premises and conclusions, i.e. the first and the second members in each pair. The NC of ZFC applies only to the level of metametalogic where we can say that it is not the case that the set corresponding to the logic both contains and does not contain any pair. With L_{NC} the ZFC logic set and $<\{B\}, \{B\}>$ a ZFC argument, that would be: $\vdash_{ZFC}$[312] $\neg\, (<\emptyset, \{\Theta\}> \in L_{NC} \land <\emptyset, \{\Theta\}> \notin L_{NC})$. ZFC logics are combinations of alowable strings of characters from some vocabulary. So it is understandable that mathematics can study any such logic. As we have seen above, most vague logics can be given *homophonic* semantics, so called because they duplicate in the metalanguage the phenomenon of the object language. For example, if the negation in the object language is not standard, they can define a function *neg()* in the metalanguage with which to correspond, instead of defining the non-standard behavior of '¬' by cases.[313] Thus, the second argument can be rejected as well.

Williamson's 1994 "Vagueness" proposes a better argument, aimed against fuzzy logic, but which can apply to other vague logics. As Andrew Bacon puts it, it is that "the very phenomena responsible for non-classicality occur in the field of semantics as much as they do elsewhere"[314]. The fuzzy Łukasiewicz conditional takes 1 whenever the consequent is as true or truer that the antecedent and the difference of their degrees of truth in all other cases. But this metalogical definition depends on

[312] I take $\vdash_{ZFC}$ as indicating the formula is a theorem of ZFC.
[313] An example is Zardini's *impl()* function, used to define the conditional. Elia Zardini, "A Model of Tolerance", 9.
[314] Andrew Bacon, *Non-classical metatheory for non-classical logics*. Version of February 13, 2012. Accessed Oct 15, 2019 https://andrew-bacon.github.io/papers/Non-classical%20metatheories.pdf, 1.

LEM itself, because of the closure condition ending with 'in all other cases'[315]. Although fuzzy philosophers argue with Lofti Zadeh that "much of human reasoning is approximate rather than precise in nature"[316], they seem to rely on classicality precisely in the study of reasoning, where they cannot exemplify a single instance of LEM failing. Williamson cites in an endnote Goguen's defense of this behavior:

> "Our models are typical purely exact constructions, and we use ordinary exact logic and set theory freely in their development. This amounts to assuming we can have at least certain kinds of exact knowledge of inexact concepts. […] It is hard to see how we can study at all rigorously without such assumptions"[317].

But this defense is adequate against the first (rejected) argument above, not against the present one. Constructions being exact does not necessarily mean that LEM should be relied on, as any intuitionist would argue. A respectable vague logic should not rely, in formulating its laws, on the very phenomenon it wants to avoid[318]. Williamson attempts to formulate this condition and he does so, with a qualification that it is "not perfectly precise":

> "…what is an appropriate logic for a vague language? It should have at least this feature: when combined in the metalanguage with an appropriate degree-theoretic semantics for the object language, it should permit one to prove its validity as a logic for the object language […] If one combines a classical logic in the meta language with a continuum-valued semantics for the object-language, one can prove that classical logic is not valid for the object-language"[319].

That is because, through T-sentences, the classicality of the metalanguage can be passed down to the object language. Williamson's example is LEM holding to the effect that "Either it is at least as wet as

[315] Timothy Williamson, *Vagueness*, 130 and 292.
[316] L.A. Zadeh, "Fuzzy Logic and Approximate Reasoning (In Memory of Grigore Moisil)", *Synthese* Vol. 30, No. 3/4: (1975), 407.
[317] J. A. Goguen, "The Logic of Inexact Concepts", 327. Cited in Timothy Williamson, *Vagueness*, 292.
[318] Incidentally, this affects intuitionism as well since the proofs of completeness for intuitionistic logic usually use contraposition which is only partially valid in intuitionism - Arthur Zito Guerriero, *Classical Inferences in the Meta-Theory of Non-Classical Logics*. Lecture Notes. 02.05.2017. Accessed Oct 15, 2019 https://www.ruhr-uni-bochum.de/phdsinlogicix/Slides/Contributed/Arthur_Guerriero.pdf, 4. Also, Bacon cites McCarty for showing that "intuitionistic predicate logic is provably incomplete, within intuitionistic metamathematics, with respect to models of broadly the same kind considered here". Andrew Bacon, *Non-classical metatheory for non-classical logics,* 3.
[319] Timothy Williamson, *Vagueness*, 130, cited in Andrew Bacon, *Non-classical metatheory for non-classical logics*, 8.

it is cold or it is not at least as wet as it is cold”[320] is perfectly true for fuzzy logic and cannot be interpreted differently.

What can be replied to Williamson? Firstly, Andrew Bacon proves that non-classical logics strong enough to express modus ponens and the paradox of material implication ($\varphi \vdash \psi \supset \varphi$) have second-order variants that can express a model theory for their own propositional forms. They build a standard bivalent semantics, but Bacon relaxes membership conditions for being a set in the relevant set-theory, i.e. developing a non-standard set theory[321] . Since being a member of a set no longer satisfies bivalence, the model theory is *faithful*, i.e. corresponds to a possible way of interpreting the non-classical logic[322]. Bacon manages to prove soundness and weak completeness for all such logics that he axiomatizes. Weak completeness is the statement that given the theory and that φ is true in all models, φ is a theorem[323]. However, strong completeness (all tautologies are theorems) cannot be obtained. Bacon suggests that since the syntax of a logic is exact (one could say that its string transformation rules act like a ZFC theory), provability is a precise matter while validity is not, at least intuitively. Bacon writes

> „we have reason to think that validity is a vague or indeterminate notion, while provability is not. In these cases we should not expect both strong soundness and completeness to hold, since no vague or indeterminate notion can be determinately equivalent to a determinate one.”[324]

Bacon’s result is carefully qualified: he primarily uses Tarski’s original sense of metalogic – i.e. higher-order language, not set theory, his result only covers propositional logics, the semantics is bivalent. But he shows that there is a sense of fulfilling Williamson’s condition above on its terms: fuzzy logic and other vague logics can prove soundness for at least their own propositional variants.

Secondly, a correspondentist fuzzy theory can claim that degrees of truth are real properties of reality. And bivalence is a real characteristic of philosophical and mathematical theories, since they strive for perfect truth. As we have seen[325], the function J_1 can express bivalence in fuzzy logics for either having perfect truth (1) or any other degree (0). Fuzzy metatheory is perfectly true, so it can be read classically. This line of thought is used by Smith against Williamson with a distinction of two

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[320] Timothy Williamson, *Vagueness*, 129.
[321] Andrew Bacon, *Non-classical metatheory for non-classical logics*, 7.
[322] Andrew Bacon, *Non-classical metatheory for non-classical logics*, 2.
[323] Andrew Bacon, *Non-classical metatheory for non-classical logics*, 15.
[324] Andrew Bacon, *Non-classical metatheory for non-classical logics*, 17.
[325] See 12.3.1.

kinds of truth. Disquotational truth is the only one to correspond to reality. Perfect truth (a truth predicate mimicking the behavior of J_1) is the bivalent one. Saying "Either it is at least as wet as it is cold or it is not at least as wet as it is cold" is justified for the latter if understood theoretically i.e. as the truth of the science determining the exact degree of reality[326]. Otherwise, it is not disquotational[327].

I have a third, but very sketchy answer to Williamson. He makes the observation that classical logic coupled with a continuum-valued semantics can be proved unsound for the fuzzy object logic. But it may still be the case, as it is argued[328], that fuzzy logic may be proved to be probability theory in disguise. Then Williamson's result should fail, for some classical elaboration of probability theory in the metalanguage. How can it fail? I think it can do so if the terms of the object language no longer correspond to objects of the metatheory, but to some mathematical structures such as ranks of an ordering or groups of objects. A fuzzy logic of individual propositions (or objects) would correspond to a metalogic of statistical constructs. This would tie well with the spirit of the third part of this work, identifying vagueness with dispersion, but it will not be further elaborated in this work.

To summarize, a general objection was raised that vague logics belie their philosophical justification, since they are generally given a semantics using ZFC. But there are good reasons to reject the three forms of this objection.

13.3. Universality and topic-specificity

The second worry was that some vague logics may be *too strange or weak* to be *logics*. The claim that they are strange was mostly spent in the previous chapter where they were examined, and I listed the relevant objections. But another argument against non-classical logics starts from their *weakness* as logics. In Chapter 11, we've seen that the classicality of FOL is based on having a proof system, a truth theory, an expression of the foundations of mathematics and some metalogical robustness. It is

326 As Williamson cites Machina's defense of fuzzy logic, he says his perspective is only epistemological. Timothy Williamson, *Vagueness*, 292.

327 For this paragraph, Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 274-275.

328 For a probability view see Peter Cheeseman, *In defense of probability,* Accessed Oct 15, 2019 https://www.ijcai.org/Proceedings/85-2/Papers/064.pdf. For contrast, see Lotfi A. Zadeh, "Discussion: Probability Theory and Fuzzy Logic Are Complementary Rather Than Competitive", *Technometrics* Vol. 37, No. 3:271-276 (Aug., 1995).

reasonable that any replacement of FOL should display these same characteristics to have a competing claim of being the universal logic.

The mathematical half, that is, the last two points, make FOL difficult to replace. Even constructive logic, a non-classical logic of good mathematical motivation, struggles to ground as much mathematics as practicing mathematicians like. For example, it cannot prove the intermediate value theorem, only approximations of it which are classically equivalent but intuitionistically distinct.[329]. In contrast, intuitionists claim that such a weakness is in fact a strength. For example, the recent constructivist program of Bishop and Bridges as applied by Feng Ye gives up ZFC and aims to reduce all mathematics to strict finitism *SF*[330]. The classical picture is that the intermediate value theorem should be proven for 0, because that is the ideal object for which it holds. Finitists object that it is rather the infinitists i.e. the classical mathematicians, who provide an unrealistic model for our finite world. Ye writes that, as long as the world above the Planck scale is finite and discrete according to scientific consensus, "infinity and continuity in mathematics are idealizations to gloss over microscopic details or generalize beyond an unknown finite limit, in order to get simplified mathematical models of finite and discrete natural phenomena."[331]. Classical logic is better for capturing the ideal nature of objects, constructive logic is better for what is actually happening in the world. The point is that unlike the trivial logic of the form <∅, {¬Θ}> above, some non-classical logics may have the internal philosophical motivation to be credible candidates for universality, at the price of some classical mathematical concepts, such as the continuum.

In any case, intuitionistic logic is not a candidate logic for vagueness[332]. Of the logics in Chapter 12, fuzzy logic is the only one whose proponents think it suitable to replace FOL as universal logic. These views are grouped under the name of *mathematical fuzzy logic*[333]. However, it has considerably lower

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[329] E.g. Feng Ye, *Strict Finitism and the Logic of Mathematical Applications (draft)*, 108.
[330] SF is "a fragment of quantifier-free primitive recursive arithmetic (PRA) with the accepted functions restricted to elementary recursive functions. Elementary recursive functions are the functions constructed from some base arithmetic functions by composition and bounded primitive recursion" Feng Ye, *Strict Finitism and the Logic of Mathematical Applications (draft)*, vi. Therefore SF Is "so that it is 'a realistic theory about concrete computational devices.'" Feng Ye, *Strict Finitism and the Logic of Mathematical Applications (draft)*, 59. As Nigel Vinckier puts it: "The bounded PR guarantees that we will be able to make sense of a finite model." Nigel Vinckier and Jean Van Bendegem, "A Case Study In Strict Finitism Feng Ye's Strict Finitism and the Logic of Mathematical Applications", *Postgraduate Studies in Logic, History and Philosophy of Science*, 16.
[331] Feng Ye, *Strict Finitism and the Logic of Mathematical Applications (draft)*, v.
[332] Intuitionistic logic was briefly discussed in the 1980s as a logic for vagueness, but abandoned because the sorites goes through. Hilary Putnam, "Vagueness and Alternative Logic.", *Erkenntnis* (1975-), vol. 19, no. 1/3, 1983, Stephen Read and Crispin Wright, "Hairier than Putnam Thought.", *Analysis*, vol. 45, no. 1, 1985.
[333] Radim Belohlavek, Joseph W. Dauben, and George J. Klir, *Fuzzy Logic and Mathematics - A Historical Perspective*, 152, 435.

robustness than FOL. Depending on which t-norm[334] and truth set is chosen, many of its variants have been proven incomplete[335] and most of them have been proven not recursively axiomatizable[336], a result which according to Pelletier may call for "the death knell for fuzzy logic"[337]. It is accepted that the only recursively axiomatizable fuzzy logics are the Gödel logics, a sub-variant which uses the common t-norm of conjunction as least degree of the conjuncts as we did, but gives up Łukasiewicz's other operators. Also, they are intermediate logics i.e. intermediate in strength between intuitionistic and classical logic[338]. A second reason of skepticism is the debate around the issue of whether fuzzy logic is a disguised probability theory, pertaining to classical mathematics[339]. A modest conclusion is that the prospects of fuzzy logic as universal logic are not bright.

Of course, most proponents of vague logics do not claim universality, thus avoiding these hurdles. They limit the discussion to issues which are specific to some area of philosophical concern, such as soriticality. We thus have *topic-specific* non-classical logics. That is, logical calculus or model-theoretical semantics serving to illustrate philosophical theories of the respective topic. As for example, epistemic logics are logics that deal with the epistemic states such as *know*, *knowledge, evidence, etc.* vague logics would be systems trying to model only the phenomena associated with vagueness. Their adequacy may be topic-limited.

However, as Williamson argues in "Alternative Logics and Applied Mathematics", applied mathematics would conflict with any deviant vague logic, even if the latter purported to apply only to some restricted class of phenomena. Such a logic would illustrate philosophical argument, but fail at truth-preservation, which was the point of logic. One argument of Williamson's against fuzzy logic is

> "Pick any positive real number δ, however small. Then on some valuation which assigns an atomic formula P a value within δ of a classical value, 0 or 1, some classical tautology built up

[334] See 12.3.3 above.
[335] Timothy Williamson, *Vagueness,* 139.
[336] Łukasiewicz (strong conjunction) and Goguen (product). Radim Belohlavek, Joseph W. Dauben, and George J. Klir, *Fuzzy Logic and Mathematics - A Historical Perspective*, 152. Also Petr Hájek, *Metamathematics of Fuzzy Logic*, 127. In contrast, Gödel variant of the logic, which is axiomatizable.
[337] Francis Jeffry Pelletier, "Review: Petr Hájek, Metamathematics of Fuzzy Logic", *Bull. Symbolic Logic 6* (2000), no. 3, https://projecteuclid.org/euclid.bsl/1182353709, 346.
[338] Norbert Preining, "Complete Recursive Axiomatizability of Gödel Logics." (2003). Accessed Oct 15, 2019 https://pdfs.semanticscholar.org/a663/c6ab16b9766227682e3dae788df93c3079f2.pdf?_ga=2.226829341.1204890606.1571762590-665552017.1571573530, 58.
[339] For a pro-probability theory view see Peter Cheeseman, "In defense of probability". For a fuzzy view see Lotfi A. Zadeh, "Discussion: Probability Theory and Fuzzy Logic Are Complementary Rather Than Competitive".

> only from P, ↔, and ¬ comes out perfectly false. On such an approach, the slightest degree of vagueness can falsify some classical theorems."[340]

This means that if applied mathematics predicts something of a situation, interpreting the situation as in any way a little fuzzy may drive the prediction completely from truth to falsity. Since the predictions of applied mathematics are justified by their popularity and general use, it is the non-classical logic that should find substantial applications where it is equally or more popular than classical logic. Once again, fuzzy logic is the only vague logic that is claimed to have some applications, especially in engineering and control systems. But therein, the fuzzy part can be replaced losslessly by classical mathematics, mostly through probability theory. Pelletier asks ironically "Does it really require fuzzy logic to model a dial that maps minutes-on-barbecue into degree-of-doneness?"[341]

Williamson constructs another argument[342], which applies mainly to supervaluationism but also to other vague logics that weaken the structural rules of classical logic. Williamson starts from supposing an argument which is valid in classical mathematics, which the non-classical logicians reject for a given atomic component E in it, on account of vagueness (say E is a predication of *tall*). But E can be replaced throughout by mathematical V, in which case they should say that it is once again acceptable, since mathematics is otherwise classical. But, Williamson writes, it is the same argument form. Denying that a non-vague V can substitute a vague E is either – if nonspecific – denying the applicability of mathematics to vague terms such as E, which is irrational, or – if specific – aiming to reconstruct mathematics i.e. a non-classical logic that is not anymore topic-specific, but must clear the universality hurdles above.

Note that such an objection does not apply to epistemic logic. Epistemic logic does not seek to describe processes already well-described by classical applied mathematics. Vague logics purport to model such processes i.e. common life situations that feature in natural language reasoning and soritical arguments. Logics of vagueness are, therefore, not any philosophical logics. They concern the application of logic, and thus, mathematics to natural language reasoning, thus, to the description of the world. The concurrent successful applicability of classical mathematics to the world makes topic-specificity untenable.

Zardini's answer to this line of argument is revealing. He argues that only non-transitivism conserves the concept of knowing a good way to cook a risotto guaranteeing that a teaspoon of butter would

[340] For this paragraph, Timothy Williamson, *Alternative Logics and Applied Mathematics*, 10.
[341] Francis Jeffry Pelletier, "Review: Petr Hájek, Metamathematics of Fuzzy Logic", 342.
[342] For this paragraph, Timothy Williamson, *Vagueness*, 13-14.

not result in a bad way to cook a risotto[343]. For him, this shows that alternatives to non-transitivism cripple our reasoning as much as it does, since they do not keep such inferences. He is right that this pattern of reasoning is common, yet what it expresses is the rejection of a definite border in teaspoons of butter between good and bad ways of cooking a risotto. Non-classical solutions and epistemicism support predicate-ordering monotony, resulting in a *border* between all good and all bad ways to cook the risotto. But FOL does not require predicate-tolerance monotony. FOL can express that if the way to cook the risotto was strictly good (i.e. all ways of that much butter are good), then the way to cook it with one more teaspoon of butter will be broadly good (i.e. most ways with this new quantity of butter are good as well), as I show in Chapter 16. Non-transitivism does not have an advantage over vanilla FOL, yet it blocks mathematical reasoning.

Zardini writes "no one is proposing to do mathematics in a non-transitive logic"[344] and this is the sentiment of most vague logicians. But people mix mathematical and non-mathematical reasoning in natural language reasoning and one intermediate conclusion can be reused with a different standard of laxity. A theory that can only approximate (some part of) lax reasoning would then imprecisely describe more mathematical reasoning. The promise of fuzzy logicians in 12.5.2 above was for validity itself to be a matter of degree. This may apply to common reasoning understood fuzzily, but Williamson's example above of theorems becoming false on very slight introduction of fuzziness makes it unsuitable for the mathematical part of the mix. It is better to search for a way to apply precise reasoning to imprecise situations, on the model of mathematicians replacing = with ≈[345], with a stated standard of approximation.

In short, vague logics are not adequate to classical, pure and applied, mathematical reasoning, although fuzzy logic comes closest. Their supporters are thus committed to topic-specificity. With them, the realm of mathematics, including mathematics applied to everyday situations and the realm of natural language reasoning cannot be one. But speakers do mix mathematical reasoning in everyday situations and applications of classical mathematics are highly successful. Both points should count against vague logics and in favor of the universality of classical logic[346].

[343] Elia Zardini, *Non-Transitivism*, 5,

[344] Elia Zardini, *Non-Transitivism*, 5.

[345] "Admittedly, some standard notation of ordinary working mathematics is vague by intention: for instance, $x \approx y$ is read 'x is approximately equal to y', and $x \ll y$ is read 'x is much smaller than y'. However, for the sake of argument we may concede that in principle such vague notation can always be eliminated in favour of something more precise". Timothy Williamson, *Alternative Logics and Applied Mathematics*, 2.

[346] As Williamson writes "Classical logic and semantics are vastly superior to the alternatives in simplicity, power,

13.4. Notational extensions of first-order logic

The third general worry regarding vague logics is that they may be unnecessary, because FOL is adequate to the task which they claim to do. This section aims only to illustrate how FOL, with a purely notational extension, can model a small part of the debate on vagueness: that around the law of non-contradiction. Let us see the philosophical reasons for which NC may fail in natural language. I will then build a minimal fuzzy logic that displays failure of NC, a FOL theory that attempts to do the same and compare them. Under the comparison, the classical treatment will turn out to be preferable.

13.4.1. Reasons to doubt non-contradiction

The reasons of doubting NC in natural language have a common source: non-stipulationism. Meaning by stipulation spans both formal and natural language. In formal semantics, if predicate letter P is assigned the set {α, β}, it stands for the respective set only. If John says 'I will use "tall" as meaning only Michael Jordan' and Mark follows that rule, they share a meaning-by-stipulation of the predicate. Related to stipulation is the acquisition of meaning in scientific discourse, carried out in a rarefied variant of natural language, where explicit definitions are respected. In these cases, verification of correct use consists in most part in conformity with explicit definitions or criteria. In contrast, there are no such criteria in natural language. Three more specific worries can be cited.

a) Language-change

Firstly, it may be claimed that since language changes, any static picture of it is wrong. And any interpretation is static. Suppose white pieces in chess are in the process of being renamed to 'yellow'. Some speakers still use 'white', some are using both 'white' and 'yellow', while some exclusively use 'yellow' for the same pieces. Are there *white* and *yellow* predicates in the classical sense? Are not white pieces also not white?

b) Popular heresies

Secondly, in natural language, speakers agree to statements that are heresy to classical logic. As studies show, in some experiments over three people in ten agree that people at some heights are

past success, and integration with theories in other domains" Timothy Williamson, "Vagueness and ignorance" in *Vagueness: A Reader*, ed. Rosanna Keefe and Peter Smith (MIT Press, 1997), 279.

both tall and not tall, a direct denial or NC. Over four in ten agree that people at some heights are neither tall nor not tall, a denial of LEM or directly of bivalence[347]. Of course, it is safe to assume that speakers also support the classical ‘Everything that is yellow is not white’. Moreover, the previous point raises two seemingly-contradictory intuitions: ‘White pieces of chess are white’ and ‘White pieces of chess are not white’.

c) Observability and family resemblances

Thirdly, Ludwig Wittgenstein famously claimed that the meaningful uses of common predicates such as *game* are linked together not by some set of necessary and sufficient conditions, but by what he called “family resemblances”[348]. Relatedly, the meaning of common terms may not be expressible in language. That is because it can be formed in many ways: for example for *white* some speakers may take it to require solely “rough and ready” - Wright’s expression[349] – impressions to apply. Graphic-design specialists may use a color book. Blind or remote people may apply it by relying on the opinions of aleatory people. Of these three cases, the first and the last cannot be expressed discursively: reliance on faulty senses on a cloudy morning or reliance on the opinions of the people one meets cannot be turned into explicit rules. Something may be called *white* by Wright’s speaker today and *not white* by them tomorrow.

13.4.2. Failure of non-contradiction in fuzzy logic

Let us see how the tradition of fuzzy logic accommodates the philosophical doubts about NC. The elementary notion is that of *fuzzy set*. A fuzzy set is usually defined as an ordered pair $\langle S, f\rangle$ where *S* is a classical set and f: $S \rightarrow [0,1]$, a membership function associating each member $e \in S$ with a value on the real interval. However, one should ensure that the same set is not paired with different

[347] Both from the review of experimental literature made by P. Égré and J. Zehr, "Are Gaps Preferred to Gluts? A Closer Look at Borderline Contradictions" in *The Semantics of Gradability, Vagueness, and Scale Structure. Experimental Perspectives* (Berlin, Springer, 2016), 27-31. In my reading, it is fair to say that rejection is LEM is more popular than rejection of NC and that the latter never has a majority of speakers, which will count against the fuzzy way of weakening NC, in 13.4.4 below. Same for Pablo Cobreros, Paul Egré, David Ripley and Robert Van Rooij “Tolerant, Classical, Strict”, 383-384.

[348] “We are inclined to think that there must be something in common to all games, say, and that this common property is the justification for applying the general term ‘game’ to the various games”. Ludwig Wittgenstein, *The Blue and Brown Books* (Blackwell Publishers. 1958), 17.

[349] Crispin Wright, "Language-mastery and the sorites paradox", 157. His vague predicate is *heap,* not *white.*

membership functions – that an element does not belong to the same set in two degrees. There are various ways to do this, so we assume there are such fuzzy sets[350].

To weaken NC, we want to define a possibly intermediate degree of the negation in terms of the intermediate degree of the non-negated formula. For this, we build a fuzzy object language and a minimal logic L_{fuzzy}:

a) The language contains finite predicate letters {P, Q}, constant letters {a, b}, operators {¬, ∧}

b) Atomic formulas are those of the form $\Pi\xi$ where Π ranges over predicate letters and ξ over constant letters. They are well-formed. If φ is a wff, so is $\neg\varphi$. If φ and ψ are wff, so is $\varphi \wedge \psi$. Nothing else is a wff.

c) An interpretation for L_{fuzzy} contains a domain of elements, a reference function *R* and a valuation function *V*. The domain contains some elements, on the model of (10.1.1) above. *R* associates each constant letter with a member of the domain. *V* associates each predicate letter with a single fuzzy set. That is, the valuation function ensures that each predicate letter is associated a fuzzy set of some elements of the domain, each belonging to it to a single degree of the real interval [0,1].

d) The *degree of truth* of a well-formed formula φ, noted $|\varphi|$ is defined as:

i. for an atomic formula $\Pi\xi$, the degree of membership of the object identified by ξ to the fuzzy set identified by Π. That is $|\Pi\xi| = m$ iff there is a fuzzy set $\langle S, f\rangle$ such that $R(\xi) \in S$, $V(\Pi) = S$ and $\langle R(\xi), m\rangle \in f$.

ii. $|\neg\varphi| = 1-|\varphi|$.

iii. $|\varphi \wedge \psi| = \min(|\varphi|, |\psi|)$ i.e. the lowest degree of the conjuncts.

Suppose we have the interpretation:

a) *Domain:* $\{e_1, e_2\}$.

b) *R:* $\{\langle a, e_1\rangle, \langle b, e_2\rangle\}$

[350] One way to do it is to start from an infinite domain. We form the class of all pairs of sets from its power set and membership functions, then the concept of a fuzzy world, meaning any subset of the class, respecting the condition that there be exactly one pair per ZFC set. Then, to be a fuzzy set is to be a member of such a world. In brief: (1) $D \overset{\text{def}}{=} \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \dots\}$. (2) $C_{fuzzy} \overset{\text{def}}{=} \{\langle x,y\rangle \mid x \in \mathcal{P}(D); y: x \rightarrow [0,1]\}$. (3) $W_{fuzzy} \subset C_{fuzzy} \wedge \forall xyz\, (\langle x, y\rangle \in W_{fuzzy} \wedge \langle x, z\rangle \in W_{fuzzy} \supset y = z) \wedge \forall xy\, (\langle x, y\rangle \in C_{fuzzy} \supset \exists t\, \langle x,t\rangle \in W_{fuzzy})$.

c) *V:* $\{<P, <\{e_1\}, \{< e_1, 0.3>\} >>, <Q, <\{ e_1, e_2\}, \{< e_1, 0.9>, < e_2, 1>\} >>\}$

Now, in L_{fuzzy}, the following is a well-formed formula:

$Pa \land \neg Pa$

Which has degree of truth 0.3 under our fuzzy interpretation, while its negation has 0.7. We can write:

(Deg) $|Pa \land \neg Pa| = 0.3$

(DegNeg) $|\neg(Pa \land \neg Pa)| = 0.7$

Under this fuzzy interpretation, some objects belong and do not belong to a predicate, both in intermediate degrees. But this is equivalent with saying in the standard set-theoretical language that there exists such fuzzy sets that some object in a set is assigned 0.3 by the respective membership function. Thus, a metatheoretical definition of truth can be found in ZFC for a non-classical logic. We defined some entities such that they resemble the building blocks of classical logic, with partially altered functionality. Instead of two truth values or membership of a set being a matter of *yes* or *no*, we now have an infinity of truth values and fuzzy set membership. That is why NC can also have an intermediate degree of truth.

In conclusion, fuzzy logic can express a doppelganger of NC which is much weaker than the NC of classical logic, allowing a contradiction to have an intermediate degree of truth.

13.4.3. Weakening NC in a FOL theory

Let us now turn to FOL and ask whether it can be notationally extended to accommodate the philosophical doubts about NC in 13.4.1. Then, we will compare the fuzzy and the classical approach.

a) Building blocks

Adding axioms in FOL can force an infinite domain. One way to do so is through a successor-like relation (i.e. transitive, asymmetric and extended):

(SL) $\exists xy\, Rxy \land \forall xyz\, (Rxy \land Ryz \supset Rxz) \land \forall xy\, (Rxy \supset \neg Ryx) \land \forall xy \exists z\, (Rxy \supset Ryz)$[351]

[351] Similar to that given by Stephen Cole Kleene, *Mathematical Logic* (New York. 1967 - Dover ed. 2002), 293.

On the side of finiteness, one can state that there is a minimum or maximum number of elements belonging to some predicate, say P. This can be shortened in special quantifiers:

(Eq.Q) $\exists_{=n}x\ Px \overset{\text{def}}{=} \exists v_1 \ldots v_n\ .\ Pv_1 \land \ldots \land Pv_n$

$\land\ v_1 \neq v_2 \land v_1 \neq v_3 \ldots v_1 \neq v_n \land \ldots \land v_{n-1} \neq v_1 \land v_{n-1} \neq v_2 \land \ldots \land v_{n-1} \neq v_n$[352]

$\land\ \forall x\ (Px \supset x = v_1 \lor \ldots \lor x = v_n)$

(Min.Q) $\exists_{>n}x\ Px \overset{\text{def}}{=} \exists v_1 \ldots v_n\ .\ Pv_1 \land \ldots \land Pv_n$

$\land\ v_1 \neq v_2 \land v_1 \neq v_3 \ldots v_1 \neq v_n \land \ldots \land v_{n-1} \neq v_1 \land v_{n-1} \neq v_2 \land \ldots \land v_{n-1} \neq v_n$

(Max.Q) $\exists_{<n}x\ Px \overset{\text{def}}{=} \forall v_1 \ldots v_{n+1}\ .\ Pv_1 \land \ldots \land Pv_{n+1} \supset (v_{n+1} = v_1 \lor \ldots \lor v_{n+1} = v_n)$

With these tools we can create a minimal FOL theory for illustrative purposes. We aim to model the same intuition as L_{fuzzy} above, namely that sometimes non-contradiction (NC) is not perfectly true. Obviously, NC *is* true, but this applies also in the metametatheory of L_{fuzzy} above. The fair comparison is whether FOL can express something – by notational means – that is as intuitive as L_{fuzzy}.

Start from this intuition: there is a sense in which there is an infinity of things belonging to any predicate. There are white things. If we count *possible* things as well, those usually modeled by modal logics, there may surely be an infinity of them. Modal realists such as David Lewis are committed to the existence of an infinity of worlds[353], therefore to the existence of an overall infinity of elements to populate them. This is compatible with each world having only a finite number of things[354]. Suppose we gather all possible white things in a predicate *possibly white* and all finite white things on Earth in a predicate *earthly white.* For this, we create a FOL theory, call it $T_{WEAK\text{-}NC}$.

b) $\mathbf{T_{WEAK\text{-}NC}}$

The FOL we are working with is standard. The predicate symbols are $P_1^0, P_1^1, P_1^2, \ldots, P_2^0, P_2^1, P_2^2, \ldots$, where superscripting indicates the arity and subscripting indicates the position of the predicate in the dictionary for that arity, both infinite.

---

[352] The suspension points should be read as to exclude any affirmation of non-identity with itself for any variable. So theres no $v_m \neq v_m$ sub-clause for any $1 \leq m \leq n$. This applies to (Min.Q) below as well.
[353] David Lewis, *On The Plurality of Worlds* (Oxford, Blackwell, 1986), 86.
[354] For example, in Chapter 6 we have cited that the number of baryons in the universe is below 10^{81}.

Our $T_{WEAK\text{-}NC}$ theory has the following axioms.

a) For each monadic predicate letter P_n^1 with *n* divisible by two (e.g. 2, 4, ...), we say that it has at least two elements and all its elements are linked by a successor-like relation, denoted by P_n^2, i.e. that the predicate has an infinity of objects, similar to (SL) above.

(ML.1) $\exists_{>2}x\ P_n^1x$

(ML.2) $\forall xy\ .\ P_n^1x \land P_n^1y \leftrightarrow P_n^2xy \lor P_n^2yx$

(ML.3) $\forall xyz\ (P_n^2xy \land P_n^2yz \supset P_n^2xz) \land \forall xy\ (P_n^2xy \supset\neg P_n^2yx) \land \forall x\forall y\exists z\ (P_n^2xy \supset P_n^2yz)$

b) For each monadic predicate letter P_n^1 with *n* not divisible by two (e.g. 1, 3 ...), we say that it has at most 10^{81} elements. Let us note 10^{81} by *m.*

(ML.4) $\exists_{<m}x\ P_n^1x$

c) Each finite predicate (of *n* not divisible by two) is contained by one infinite predicate, namely that denoted by the next predicate letter in the vocabulary:

(ML.5) $\forall x\ .\ P_n^1x \supset P_{n+1}^1y$

We now add notational flexibility to FOL. We introduce easier-to-read letters with a convention to indicate monadic predicates of *n* not divisible by 2 (e.g. 1, 3…) by an uppercase letter, their corresponding relation of same *n* by the same uppercase letter subscripted with 'REL' and monadic predicates of *n* divisible by 2 (e.g. 2, 4…) by the same uppercase letter subscripted with 'inf'. We then re-express (ML.1-5) for a letter such as *W*, with *m* still 10^{81}:

(ML.1.2) $\exists_{>2}x\ W_{inf}x$

(ML.2.2) $\forall xy\ .\ W_{inf}x \land W_{inf}y \leftrightarrow W_{REL}xy \lor W_{REL}yx$

(ML.3.2) $\forall xyz\ (W_{REL}xy \land W_{REL}yz \supset W_{REL}xz) \land \forall xy\ (W_{REL}xy \supset\neg W_{REL}yx) \land \forall x\forall y\exists z\ (W_{REL}xy \supset W_{REL}yz)$

(ML.4.2) $\exists_{<m}x\ Wx$

(ML.5.2) $\forall x\ .\ Wx \supset W_{inf}y$

Of course, we will have five such axioms for each uppercase letter we use. This theory allows us to prove a weak denial of non-contradiction for each predicate letter:

(NK.W) $\vdash_{FOL} \exists x \,.\, W_{inf}x \wedge \neg Wx$[355]

Read: there are possibly white things which are not white.

To recapitulate, by notational means, we have separated all monadic predicate letters in two. One half were stated to correspond to predicates of infinite cardinality, by means of a successor-like relation, denoted by its corresponding dyadic predicate letter. The other half were bounded in number by 10^{81} and each included in one predicate of the first half.

13.4.4. Comparison between the fuzzy and the classical approach to weakening NC

We have two minimal ways to weaken NC: fuzzy logic $\mathbf{L_{fuzzy}}$ and classical FOL theory $\mathbf{T_{WEAK\text{-}NC}}$. In the former, each object is associated to a predicate to a degree. In the latter, there is a purely notational sense that some white things are not white: some W_{inf} things are not W. Is this more or less adequate than what L_{fuzzy} does? As seen above in Chapter 12, most philosophical opinion argues with Aristotle that a contradiction is always false, Williamson writing[356]:

> "More disturbing is that the law of non-contradiction fails …. $\neg(p \wedge \neg p)$ always has the same degree of truth as $p \vee \neg p$, and thus is perfectly true only when p is either perfectly true or perfectly false. When p is half-true, so are both $p \wedge \neg p$ and $\neg (p \wedge \neg p)$."

and

> "At some point [of waking up] 'He is awake' is supposed to be half-true, so 'He is not awake' will be half-true too. Then 'He is awake and he is not awake' will count as half-true. How can an explicit contradiction be true to any degree other than 0?"

This intuition is plainly denied by fuzzy theorists, for example Kenton Machina writes:

> "It should by now be clear that in so characterizing Jones' belief, or its negation, I am not thereby trying to say part of the proposition he believes is true and part of it is false. Nor that in some respect, or in some sense, or at some time it is true, and in some other respect, sense, or time it is false. i.e., I am carefully avoiding those common misunderstandings of the law of

[355] Proof is immediate from the infinity forced by (ML.3.2) versus the finite bound of (ML.4.2).

[356] For Aristotle, see 5.1.1 above. The following two excerpts are from Timothy Williamson, *Vagueness*, 118 and 136, also cited in Brian Weatherson, *True, Truer, Truest*, 2.

non-contradiction which lead people to suppose the law false for bad reasons. I am trying to indicate, instead, that Jones' belief and its negation are both neither completely true nor completely false in the same respect, in the same sense, at the same time"[357].

However, as we have seen in 13.2, when faced with the objection according to which fuzzy logic cannot use a classical meta-language, fuzzy theorists such as Smith can appeal to their introduction of a function J_1 which expresses bivalence in fuzzy logics when needed. Relying on classical LEM, J_1 takes either the value 1 for perfect truth (1) or 0 for any other degree of truth. Therefore, something like $J_1(Pa) \wedge \neg J_1(Pa)$ will be perfectly true. So, in that sense, fuzzy logic can differentiate itself between respects, senses and times in which a contradiction is completely or incompletely true. The remaining difference is, then, which sense of non-contradiction to take as primary. For Williamson, the primary sense is the absolute one, in which NC is perfectly true, while supposedly allowing that in some respects, variants of NC may fail, while Machina's position is the reverse, taking the failure of NC as primary.

Since in empirical studies, only a minority of speakers relativize NC, as we have seen at 13.4.1 b) above and the weight of philosophical tradition, mathematics and science is on the side of avoiding non-contradiction, I side with Williamson. NC is always true, as in classical logic, so we need to only express the ways in which respects, senses and times correspond to weakened versions of it. And this is precisely what $T_{WEAK\text{-}NC}$ does. It allows us to speak of any predicate as having negative cases which are also positive cases, not at the same time, sense and respect, but as a *possibility*, i.e. in some own correlated predicate. This kind of theory shows why FOL is a better logic for studying vagueness than CPL: it can, at least partially, express through notational means intuitions which CPL cannot.

Now, how can the three philosophical objections in 13.4.1 be classically answered in a FOL theory such as $T_{WEAK\text{-}NC}$? In classical set-theoretical semantics, there is a multitude of properties and relations between objects. This is so even for those theories that say that vague sets correspond to vague properties, such as Michael Tye's[358]. As long as sets are vague *on the inside*, in determining whether an object belongs or not, there still are multiple sets corresponding to multiple properties. Thus, we have many other ways of interpreting each utterance of 'white'. For a domain of ten objects, there are 2^{10} properties[359]. Taking extensionality seriously means searching for one or more of the available

[357] Kenton Machina, "Truth, belief and vagueness", 185. See also Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 256-260 for a defense of the intuitive nature of degrees of truth.
'[358] See Chapter 12 above.
[359] Assuming that the power set can be built, as it is standard.

sets that match what each speaker is saying[360]. And choosing them such as that what they say come out as reasonable. It is only if we cannot do this that classical logic fails.

For example, when a non-philosopher utters 'some things are both white and not white', the best strategy is to understand it as 'there are things which are both white in some sense and not white in some other sense', where the two predicates significantly overlap, such as the W_{inf} and W of (ML.1-5.2) above. That is, going against Machina. There are white pieces of chess that are not milky white. Both are however white things in a general sense. We can take *white in a general sense* to have the maximal sense of corresponding to the union of all the predicates interpreted as corresponding to any utterance of 'white', in our case W_{inf}, *possibly white.* Let us now apply this strategy to the three objections of 13.4.1 above.

13.4.5. An interpretation and answering the challenges to NC

We will extend our $T_{WEAK\text{-}NC}$. In order to improve readability, I will use some strings of letters for our predicates. We have an Infinite domain, ten elements listen explicitly {α, β, γ, δ, ε, ζ, φ, χ, ψ, ω, …}, and the following predicate letters:

Letter	*Replacement name*	*Reading*	*Corresponding set*[361]
A	B_{CHESS}	Black chess piece	{α, β, …}
B	P_{CHESS}	Chess piece	{α, β, γ, δ, …}
C	W_{CHESS}	White chess piece	{γ, δ, …}
W_{inf}	W_{inf} (same)	White in a general sense	{γ, δ, φ, χ, ψ, ω, …}
E	Y_{CHESS}	Yellow chess piece	{γ, δ, …}
Y_{inf}	Y_{inf} (same)	Yellow in a general sense	{γ, δ, ε, ζ, …}
G	W_{STRICT}	Strictly white	{ψ, ω, …}
H	W_{TODAY}	Today-seemingly white	{δ, χ, ω, …}
I	$W_{YESTERDAY}$	Yesterday-seemingly white	{φ, χ, ω, …}
J	$W_{PASSERBY}$	White for some passerby	{γ, φ, χ, ψ, ω, …}
W	W (same)	White	{φ, χ, ψ, ω, …}
Y	Y	Yellow	{ε, ζ, …}

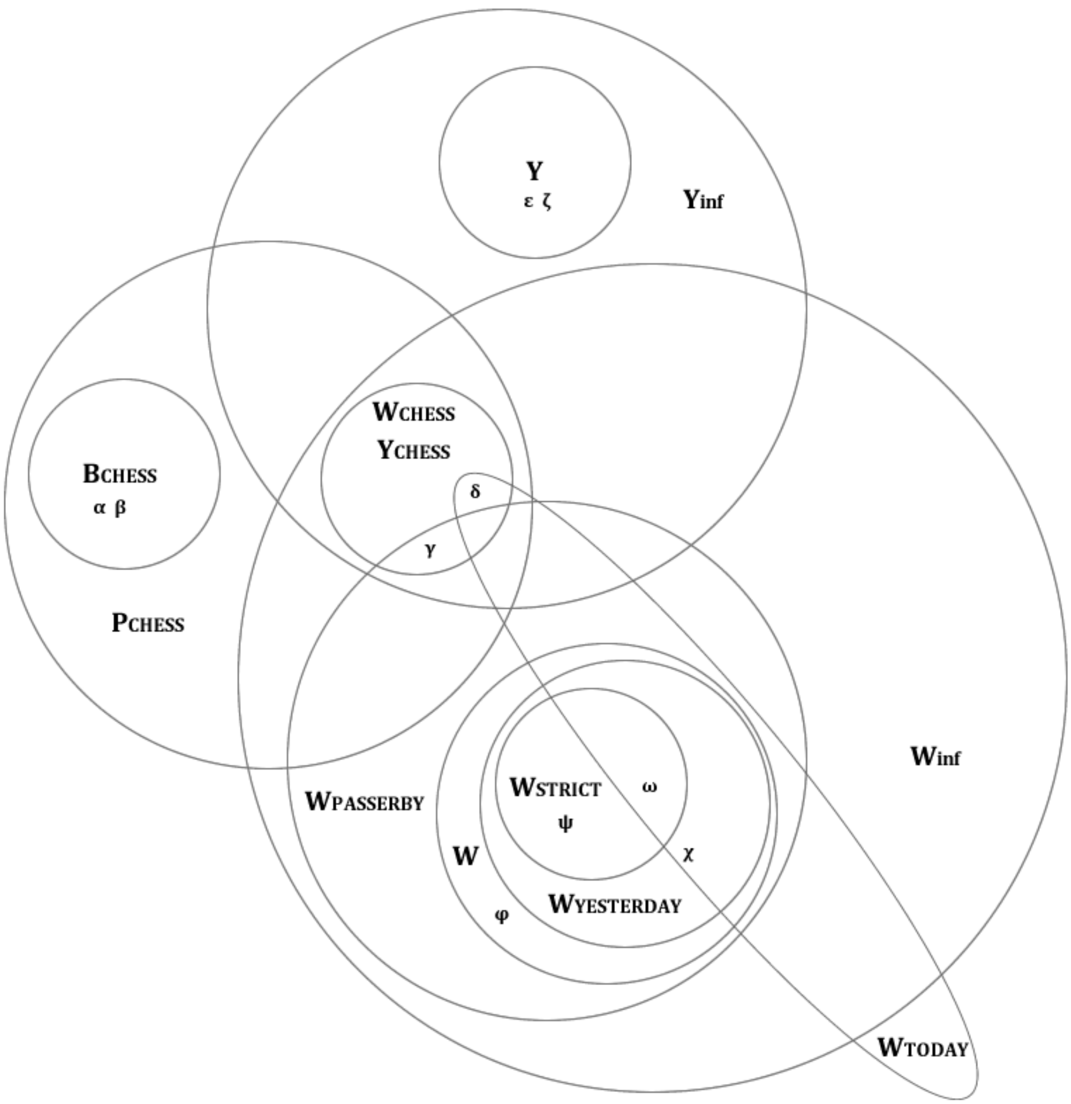
[360] See the same objection brought to Frege, above at 11.3.
[361] The rule for the enumeration below is that if one of α, β, γ, δ, ε, ζ, ϕ, χ, ψ, ω, belongs to a set corresponding to predicates in the table, it is explicitly listed thusly. So, for example, the fact that β is not listed at J ($W_{PASSERBY}$) signifies that β does not belong to the respective set.

Axioms:

Utterance	*Formalization and reading*
(W.1) Non-black chess pieces are white.	$\underline{\text{(W.1.1) } \forall x\ .\ \neg B_{CHESS}x \land P_{CHESS}x \supset W_{CHESS}x}$ *Chess pieces which are not black are white chess pieces.*
(W.2) Non-black chess pieces are white and yellow.	$\underline{\text{(W.2.1) } \forall x\ .\ \neg B_{CHESS}x \land P_{CHESS}x \supset W_{CHESS}x \land Y_{CHESS}x}$ *Chess pieces which are not black are white chess pieces and yellow chess pieces.* (W.2.2) $\forall x\ .\ W_{CHESS}x \leftrightarrow Y_{CHESS}x$ *A thing is a white chess piece just in case it is a yellow chess piece.*
(W.3) Non-black chess pieces are yellow.	$\underline{\text{(W.3.1) } \forall x\ .\ \neg B_{CHESS}x \land P_{CHESS}x \supset Y_{CHESS}x}$ *Chess pieces which are not black are yellow chess pieces.*
(W.4) White chess pieces are white.	$\underline{\text{(W.4.1) } \forall x\ .\ W_{CHESS}x \supset W_{inf}x}$ *White chess pieces are white in a general sense.*
(W.5) White chess pieces are not white.	$\underline{\text{(W.5.1) } \forall x\ .\ W_{CHESS}x \supset \neg Wx}$ *White chess pieces are not white.*
(W.6) Some things are both white and not white.	$\underline{\text{(W.6.1) } \exists x\ .\ W_{inf}x \land \neg Wx}$[362] *Some things are both white in a general sense and not white.*
(W.7) Some things are neither white nor not white.	$\underline{\text{(W.7.1) } \exists x\ .\ \neg W_{STRICT}x \land \neg\neg Wx}$ *Some things are neither strictly white nor not white.* (W.7.2) $\forall x\ .\ W_{STRICT}x \supset Wx$ *Everything that is strictly white is white.*
(W.8) Yellow things are not white and white things are not yellow.	$\underline{\text{(W.8.1) } \forall x\ .\ Yx \supset \neg Wx}$ *Yellow things are not white and white things are not yellow*[363].
(W.9) Some things which were white according to quick impressions yesterday are not white according to them today.	$\underline{\text{(W.9.1) } \exists x\ .\ W_{YESTERDAY}x \land \neg W_{TODAY}x}$ *Some things which were white according to quick impressions yesterday are not white according to them today.*
(W.10) Some things are white according to some passerby but not white by the color book.	$\underline{\text{(W.10.1) } \exists x\ .\ W_{PASSERBY}x \land \neg W_{STRICT}x}$ *Some things are white according to some passerby but not strictly white.*

[362] Corresponds to (NK.W) above.
[363] The second half of the reading can be derived by modus tollens.

Main formalizations are underlined. Non-underlined formalizations are ancillary to the main ones, clarifying the relation between the newly introduced predicates and those already introduced. We can get the following diagram:



The first objection ('language change') above raised the issue of language change, e.g. renaming white chess pieces to yellow. But we can take names of being no importance. Since the objects to which 'white chess pieces' and 'yellow chess' pieces refer are the same, the three speakers of the objection only use different names for the same set of objects. This can be accommodated in the vocabulary of the interpretation, which uses an infinite number of predicate letters.

The second objection ('popular heresies') raised the issue of denials of non-contradiction, excluded middle, in surveys of speakers. It can be answered by taking each utterance of 'white' and 'yellow' to refer to different – yet significantly overlapping – predicates. This is, of course, not taking the speakers of speaking of the same thing, even in the same sentence. But a speaker who says that things can be both white and not white and that things cannot be both yellow and white should not be taken as speaking of the same thing.

The third objection ('observability and family resemblances') raised the issue that no definition of predicates may be possible, because of family resemblances. Some games are linked by history, some by common athletic characteristics, and some by resemblance with other activities widely seen as games. Against this formulation we have the technique of set union. It corresponds to a disjunction of conditions which are at least partially extensional. It would be verbose, along the lines of 'Something is a game just if a) it shares a common source in the 19th century with the games of association football or b) it requires physical movement of two or more persons for at least ten minutes or c) it is chess'. It may be the case that such a definition be too unwieldy for anyone to understand it, making Wittgenstein's point correct. But in FOL its extensional treatment is available, as for other examples inspired by Wright. The speaker relying on momentary sense impressions picked up a different predicate yesterday than the one he picks up today by 'white'. The speaker relying on a color book may single out the predicate of *strictly white*, that whose affirmation and negation in the same sentence cannot be understood other than as a contradiction[364]. Finally, the speaker relying on some passerby may pick up a distinct predicate corresponding to 'white' each time he imports someone else's understanding as his.

Therefore, extensionality gives a plausible reading of the examples in the three objections, although we sacrificed some of the speakers' linguistic competence to do so. They may not accept at first that they use the same word differently today as compared to yesterday, or even in the same sentence. Philosophers such as Wright might doubt that applying *white* by the color book is compatible with having the mastery of this observational concept. Might taking extensionality seriously mean taking natural language as unserious? I note that an extensional approach maximizes the truth that can be had by speakers' utterances. The best common strategy if something does not make sense to you in

[364] For the moment, take *strictly white* things to be those of whom no reinterpretation is possible. We saved (W.6) by interpreting it as (W.6.1). But there is no way to understand a similar-looking statement for *strictly white*, such as: "Some things are both strictly white and not strictly white" other than a contradiction. In 16.3., I will define *strictly tall* through the interplay of the predicate *tall* and the total preorder *to have less or equal number of cm than*, making it relevant for vagueness.

casual talk is to ask ‘do you have an example?’ or ‘what are you referring to?’ It seems like interpretative charity requires that a recourse to specific cases be read in any incomprehensible statement[365].

In conclusion, it is unlikely for vague logics to replace FOL as universal logic, especially because of the role played by the latter in mathematics. Yet, topic-specific logics that ignore mathematical reasoning conflict with the large domain of applied mathematics, as Williamson shows. They also cannot model the mix of mathematical and non-mathematical reasoning that occurs in natural language reasoning. Finally, we can compare what fuzzy logic achieves in weakening non-contradiction to what can be achieved by notationally extending classical FOL. The latter technique is preferable and will be developed in the next chapters, with the aim of defining vagueness in FOL.

[365] For both points Donald Davidson “We make maximum sense of the words and thoughts of others when we interpret in a way that optimises agreement” and “In giving up the dualism of scheme and world, we do not give up the world, but reestablish unmediated touch with the familiar objects whose antics make our sentences and opinions true or false.” Donald Davidson, "On the Very Idea of a Conceptual Scheme", *Proceedings and Addresses of the American Philosophical Association* (Vol. 47, 1973 -1974), 19-20. Also see for a model of interpretative charity without linguistic knowledge (i.e. intensions) his Donald Davidson, “A Nice Derangement of Epitaphs” in *Truth and Interpretation: Perspectives on the Philosophy of Donald Davidson*, ed. Ernest Lepore (Blackwell, 1986).

Part III

The logic of vagueness

The conditional sorites (CS) of Chapter 1 started with 'A man of 200 cm is tall'. It raises the issue of the connection between the predicate *having 200 cm* and *tall,* but moreover, between the genus of the former, namely height in cm and *tall.* The monadic predicate *tall* is moreover linked in natural language with relations *taller, less tall,* and *as tall as*, and is often qualified: *strictly tall, broadly tall*, *very tall*, *little tall, tallish*.

No philosophical theory says specifically what semantic characterization should be given to men of 180 cm. Plurivaluationism does not say that they have the property of tallness to 0.87 degree, but only that somewhere around there the degree is intermediate between 1 and 0. Supervaluationism does not say that they are definitely tall, but only that somewhere around there, there will be men who will not be definitely tall. Thus, the challenge to classical logic should not be that it does not say whether people of 180 cm are tall or not. It is, firstly, that under the epistemicist-driven acceptance of predicate-ordering monotony, there is a threshold somewhere, say between 179 and 178 cm that separates all tall people from all short people, which seems strange, and secondly, that it does not have the resources of non-classical logic for expressing the graduality, measurability and comparability of vagueness. I think we should accept the first objection and give up predicate-ordering monotony. In this part I present how classical first-order logic (*FOL*) can deal with the second objection and thus serve as logic for vagueness.

FOL can define vagueness and higher-order vagueness, with some notational extensions and with a limitation. The notational extensions are pain-free. The limitation is that we need some (high) finite upper bound on the number of objects we study, in order to express statistics in FOL. The analysis of the sorites paradox in Chapter 6 showed that its general formulation can only be given in second-order logic, it being dependent on the notion of transitive closure of a relation R. But by placing an upper bound of, say, 10^{12} on $\bigcup$R, we are able express it in FOL. This is the technique we will use, to construct FOL statements about the ordering of a predicate by a total preorder. I will define vagueness and higher-order vagueness using some statistical statements, in order to express our intuitions about vague language.

Chapter 14

Natural dimensions and total preorders

14.1. Measurements

Precise measurements determine the application of vague predicates, the latter supervening on the first. Williamson indicates Halldén as the originator of the "idea that if a vague property can be attributed at all, it can be attributed on the basis of precise properties"[366]. Therefore, I aim to give a show how vague properties can be applied on the basis of precise binary relations corresponding to measurements[367], and how vagueness emerges from the interplay of the predicate and its preferred one such ordering. In Chapter 6, when I derived the general form of the Sorites, the numeric predicates were then replaced with a binary relation I called '*small difference'.* Let us take *to have 200 cm* as soritical starting point. First, the specific numeric value is of little import. Using imperial measurements, the information would be conveyed as *to have 78.74',* using mm it would be *to have 2000 mm* and so on. Second, all the numeric predicates in the series are connected using the unit of measurement: all objects of 200 cm are objects having one cm more than objects of 199 cm[368] . And all objects of 3 cm are objects having one cm more than objects having one cm more than objects having one cm more than objects of 0 cm. Therefore, we can write that having 200 cm is *to be 200 steps away from having 0 cm*. With *T* the predicate of *having 200 cm, N* the predicate of *having 0 cm* and *S* the relation of *having 1 less cm than*:

(NumElim) $Ta \leftrightarrow \forall v_1 ... v_{200} (Nv_1 \wedge Sv_1v_2 \wedge Sv_2v_3 \wedge .... \wedge Sv_{199}v_{200} \supset Sv_{200}a)$

Now, we can always choose a smaller comparison interval instead of 1 cm, just like the soritical step gets smaller and smaller in the literature of vagueness, as discussed in 6.6. But, however small, the ordering of the objects classified under the numeric predicates concerned will stay the same. Because real, rational and natural numbers each form a total order under *to be less or equal to*. But two people

366 Timothy Williamson, *Vagueness*, 107.

367 As argued below at 14.3, I assume that precise natural dimensions come first. Therefore, it is possible to describe the world using only precise properties, at least for paradigmatically vague predicates. This corresponds to the argument against metaphysical vagueness of Chapter 8 above, but it is a contested matter for some non-paradigmatically vague properties, such as psychological states. See Williamson's interpretation of Waismann: "Beliefs, desires, intentions and other psychological states also have vague contents; how can they be fully described in a precise language?" Timothy Williamson, *Vagueness*, 93.

368 See 6.2.

can have the same number of cm, so an ordering of people by height should allow people having the same place in the ordering.

We will use 'R' or the '≤' symbol to denote any such *total preorder*, also called *preference relation*. Thus, for two distinct elements *a* and *b* we can have (1) $a \leq b \land \neg(b \leq a)$ (2) $b \leq a \land \neg(a \leq b)$ or (3) $a \leq b \land b \leq a$, which comes to saying that their number of cm can be larger, smaller or the same. For example that denoted in English by '*to have less or equal cm of height than*'[369].

(Ranks.1) Example of a total order for a finite subset of $\mathbb{Q}$**:**

1		*2*		*3*		*4*		*5*
170	≤	171	≤	172.5	≤	173	≤	175

Example of a relative total preorder for a finite set of people of those same heights[370]

1		*2*		*3*		*4*		*5*
Mary	≤	Ann	≤	Vick	≤	Vince	≤	Ally
John		Nick						
		Tim						

14.2. Total orders and total preorders

Total orders are defined as a set together with a binary relation on it that is transitive, reflexive, antisymmetric, and has the connex property.

(14.1) Q is transitive: $\forall xyz\ .\ Qxy \land Qyz \supset Qxz$

(14.2) Q is reflexive: $\forall x\ Qxx$

[369] Since this binary predicate can be expressed in inches or mm, a strictly appropriate name would be *to have lesser height than*.

[370] Mary and John have 170 cm, Ann, Nick and Tim have 171 cm, Vick has 172.5 cm, Vince has 173 cm and Ally has 175 cm.

(14.3) Q has the connex property: $\forall xy . Qxy \lor Qyx$

(14.4) Q is antisymmetric: $\forall xy . Qxy \land Qyx \supset x = y$

To get a total *preorder* from the definition of a total order, we give up antisymmetry (14.4) above, but intend to keep the others: the relation is transitive, is reflexive (any element has it with itself) and has the connex property (all elements are connected one way or the other). Nonetheless, we can prove a doppelganger of antisymmetry for total preorders, namely *indistinguishability under R*. But because we will adopt what we called earlier the *bounded submodel* variant[371], we will also explicitly state that R only holds between humans. So we define *total preorder for a bounded submodel* thus:

(14.5) R is transitive: $\forall xyz . Rxy \land Ryz \supset Rxz$

(14.6) R is reflexive for H elements: $\forall x . Hx \supset Rxx$

(14.7) R has the connex property for H elements: $\forall xy . Hx \land Hy \supset Rxy \lor Ryx$

(14.8) R only applies to H elements $\forall xy . Rxy \supset Hx \land Hy$

(14.9) Symmetry implies indistinguishability under R: $\forall xy . Rxy \land Ryx \supset \forall z ((Rxz \leftrightarrow Ryz) \land (Rzx \leftrightarrow Rzy))$[372]

Total preorders are generalizations of total orders. Two different people can have the same height in cm while two numbers are identical if equal. However, total orders and total preorders share the concept of *rank*[373]: i.e. the 1-5 numbers written above the examples above. That is precisely what indistinguishability under R above (14.9) means: that two elements at the same rank of a total preorder have the same R relations, meaning they are at the same place in the ordering. That place can be indicated by a unique natural number, as for total orders.

[371] See 6.5.1.

[372] *Proof*: Suppose any elements a, b such that Rab and Rba. First, suppose there is some c such that Rac. By transitivity, since Rba and Rac, Rbc. Secondly, suppose there is some c such that ¬Rac. By the connex property, Rbc or ¬Rbc and if the first, transitivity would lead to the absurd Rac. So it is Rbc. Similarly for the last two cases ■

[373] The rank is generated by a dense ranking function that can be written along the lines of (15.9) below.

14.3. Discussion and possible objections

Any measured quantity generates a number on a scale, which, being a number, is part of a total order, thus a fortiori a total preorder, by ≤. Measurement of height yields centimeters. Measurement of kyphosis yields degrees of spine curvature. Measurement of coiffure height yields centimeters. Even color is reducible to a set of total preorders by measurement. A hair has a certain red length number from 0 to 255, and similarly for green and blue, thus defining the hair's rank in three total orders. Putting together those ranks on a computer gives a standard RGB color[374].

This is an essential point, because the natural dimensions themselves are numerical and expressible as total preorders. Only after establishing the place of an object in these orderings according to its real natural properties (e.g. height, degrees of kyphosis, etc.), we will discuss whether it has vague properties. This is quite different from the approach taken by Rosanna Keefe in her "Theories of Vagueness"[375]. She skips the exact natural dimensions and analyzes directly the property of being *tall, hot*, *heavy*, in terms of the relations *to be at least as tall as*, *to be at least as hot as*, *to be at least as heavy as*[376]. But it is obvious that tallness is not height: height is paradigmatically precise while tallness is paradigmatically imprecise. So we should first discuss the logical properties of measurable dimensions and then see how they can create a place for vagueness to emerge. Secondly, *to be as tall as* is a relation, while *tall* is a property, they should be carefully distinguished.

Moreover, it needs to be remarked that we only need measurability, not physicality or predicate-ordering monotony. For example Williamson writes "… since thinness supervenes on exact physical measurements, the generalization 'Everyone with physical measurements *m* is thin' expresses a necessary truth"[377]. Firstly, the measurements involved need not be physical in a strict sense, since people may be thin partially based on their domicile, just as women of Seoul are tall at a different height than men of New York. But geographical position and statistical distribution among other close inhabitants are measurable dimensions, for which reason I call them all *natural*, not *physical*. Secondly, saying that measurements determine thinness is not saying that only height and weight determine it, just like not only height determines tallness. And neither that anyone of both more

[374] Of course, the very idea of measurement can be attacked by skeptics. The same height may not be measured in a constant way by any device, not all devices may output the same result or there may not be normal conditions of the kind necessary to assume that measurements are objective. But this kind of skepticism is incompatible with the idea that there is such a fact as a man having 180 cm, which is the assumed starting point of the debate.
[375] Rosanna Keefe, *Theories of Vagueness*, 125-138.
[376] Rosanna Keefe, *Theories of Vagueness*, 126.
[377] Timothy Williamson, *Vagueness*, 230.

height and less weight than a thin person must also be thin[378]. Measurability and supervenience of vague on precise properties do not imply predicate-ordering monotony.

Let us now analyze some possible objections against the use of total preorders for capturing the series of numeric predicates which characterize natural measurements, e.g. *having 200 cm*, *having 199 cm*, etc.

14.3.1. Too many total preorders

On any finite domain, the number of total preorders (up to isomorphism[379]) can be computed. It is large. The Online Encyclopedia of Integer Sequences gives a number of 13 total preorders for 3 elements, 75 for 4, rising to 2677687796244384203115 for 20 elements[380]. Let us examine the 13 for 3 elements *a, b, c*:

(Ranks.2) Table of total preorders for 3 elements. Vertical alignment signifies common ranking.

(1) a b c	(3) a b c	(5) a b c	(7) c a b	(9) a c b	(11) c a b	(13) b c a
(2) a c b	(4) b a c	(6) b a c	(8) a b c	(10) b a c	(12) c b a	

Total preorder (1) would correspond to three people having the same height in cm, a not very interesting situation for what vagueness of tallness may be about. On the opposite size, scenarios (8) to (13) have as many ranks as there are elements, which is similarly counter-intuitive for our purpose. However, with a domain of three elements, we are left with six intermediate ways of grouping them into ranks. For 10^{12} humans as there may have ever lived[381], the number of such intermediate distinct preorders is higher than the number of baryons in the universe.

[378] Provided, of course that vagueness depends also on other measurable dimensions.
[379] 'Up to isomorphism' indicates that the order inside ranks is of no import. For example if in scenario (1) below, the vertical order was *b a c* or *c a b* instead of the actual *a b c*, it would be the same order. Indeed, total preorders have an analogue of the antisymmetry of total orders, which we may call *indistinguishability under R*, see above.
[380] Up to isomorphism, https://oeis.org/A000670 .
[381] Discussed in 5.3.5.

As I affirmed in Chapter 13 with regard to understanding predicates extensionally i.e. a predicate corresponds to some member of the power set of the domain, any ranking of all people will be found among the multitude of total preorders available combinatorically. The fact that we do not *know* all the total preorders on account of their high number does not mean they are not available to us as determinate mathematical constructions. Whatever heights in cm humans have, since the number of humans is finite, there is a single total preorder of them by the binary relation of *having less or equal cm of height than.*

14.3.2. Abstracting from accuracy and lack of accuracy

The number of ranks is a factor of the accuracy of the measurement. If the device measuring height only returns whole numbers of cm, there will be no people between 180 cm and 181 cm. In contrast, if the scale is more accurate, it may be the case that there are as many ranks as people i.e. no two people share the same number measuring height. Using total preorders allows us to abstract from the accuracy of measurement and ignore the issue whether *man of 180 cm* applies to men of 180.5 cm or not.

For example, suppose a more discriminating device returns the measurements:

(Ranks.3) People: Mary 170.002 cm, John 170.01 cm, Ann 169.96 cm, Nick 171.02 cm, Tim 171.025 cm, Vick 172.5 cm, Vince 172.98 cm and Ally 175.07 cm

Then the total preorder would look like:

1		2		3		4		5		6		7		8
Mary	≤	John	≤	Ann	≤	Nick	≤	Tim	≤	Vick	≤	Vince	≤	Ally

Secondly, since ranks are formed by a finite number of people populating them through their numeric measurements, there is such a thing as a maximum rank for humans – that of the man with the greatest number of cm – and a minimum rank – that of the man with the lowest number. For example, 8 and 1 above. Thus, ranks correspond with the intuition that there is a limited space of the relevant dimension – in this case, height – that can be partitioned in more than one way. There is a sense that we can split the ranks in two (say high ranks of height in cm and low ranks), in three (high, intermediate and low), in five (high, high-to-intermediate, intermediate, intermediate-to-low, low) and so on. For example, one way of splitting *(Ranks.3)* in three is taking Mary, John and Ann for low, Nick and Tim for intermediate and Vick, Vince and Ally for high. That is easy because ranking converts absolute values (180, 181, 182, ...) to relative values (1st, 2nd, 3rd, ...). However, this raises another objection.

14.3.3. Relative versus absolute total preorders

Our purpose is for *tall* to be determined jointly by height and by kyphosis[382]. We can take the total preorder of all people by number of cm and the total preorder of all people by degree of spine curvature. But in a relative ranking of degrees of kyphosis, individuals of few degrees of spine curvature would cluster around the low end, while those of high spine curvature would populate the high end of rankings. Therefore, it may be objected that tallness does not depend even partially on being, say, in the last third of people ordered ascendingly by spine curvature, but on the real degree of kyphosis. To wit, having 3*0 degrees of curvature* is the same property as having *0.5236 radians* and it may be included in a formula determining tallness, a formula that can be restated for radians, but is in essence the same. Since information is lost when converting absolute values to relative values, we would not have the means to apply the formula if we used the *ordering* of people by degree of spine curvature, not that degree itself.

You cannot recover the fact that John has 228 cm from the fact that he is at the last rank among people by height in cm. What we can do is to work with two total preorders for each measurement: a *relative* total preorder which has ranks only where there is at least a person in the population and an *absolute* total preorder which has one rank for each smallest interval discernible between the measurements of two persons. Let us illustrate with the same measurements as in (Ranks.1).

[382] Strangely enough, tall has often been cited as a case of a unidimensional predicate, although it seems obvious that kyphotic people are short at heights where other are tall and that, without specifying a comparison class, speakers accept that men are tall people at different heights than women. That is, there is a global meaning of tall that is determined by height, sex, kyphosis etc. Contrast "Vague terms are typically associated with some dimension or dimensions of comparison. The predicate 'tall' is one-dimensional (with respect to some comparison class) as it merely governs the dimension of heights" Patrick Greenough, "Vagueness: A Minimal Theory", Mind, Vol. 112, No. 446 (Apr., 2003), 240.

(Ranks.4) Absolute ranking

1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
Mary	.	Ann	Vick	.	.	Vince	.	.	.	Ally
John		Nick								
		Tim								

We could easily start *(Ranks.4)* from 0 cm, to keep all the information contained in *having 170 cm*. Then any formula expressed in cm could be restated for our absolute ranks by multiplication of the rank number with 2 to get the number of cm, since the rank distance in (Ranks.4) is 0.5. Absolute rankings also force us to extend the domain with one object per minimal unit of measurement, symbolized above with dots, because each rank needs to be populated by at least one object. I will not use such absolute orders in the definitions for vagueness in the following chapters, but such objects can be found. One way is to use for this purpose the possible predicates of infinite elements of $T_{\text{WEAK-NC}}$ in 13.4.3 above (such as *possible people)* or to introduce the numbers in the domain, resulting a two-sorted logic.

In short, we can eliminate the series of numeric predicates for natural dimensions, they being reducible to relative or absolute total preorders. One example is the comparative measurement relation *to have less or equal cm of height than*. The height in cm of a person is recoverable from their position among other people by height (from a relative point of view) and the position of their number of cm of height among other numbers (from an absolute point of view). I will use relative total preorders to define vagueness in FOL.

Chapter 15

First-order expressions

15.1. The interplay of a total preorder and a predicate

Let us see how we can say in FOL that something is at the first rank of a total preorder, the last rank, the *n^th^* rank, or the *n*th rank from the last rank. We can also express a new relation that only holds between adjacent decreasing[383] ranks (that is, that corresponding to *predicate tolerance*) or adjacent increasing ranks (that is, corresponding to *predicate-ordering monotony*).

We will then split the rank space of a total preorder into three: an initial chain of ranks which is *negative* (all elements at those ranks are negative cases of the predicate), a dispersion zone of ranks (which each contains both negative and positive cases of the predicate) and a final chain which is positive (all elements at those ranks are positive cases). We start with basics.

(15.1) An universal quantifier limited to predicate H, to shorten expressions:

$\forall_H x\ \varphi \stackrel{\text{def}}{=} \forall x\ .\ Hx \supset \varphi$

(15.2) An existential quantifier limited to predicate H, to shorten expressions:

$\exists_H x\ \varphi \stackrel{\text{def}}{=} \exists x\ .\ Hx \wedge \varphi$

(15.3) Affirming that there are exactly *n* elements such that...

$\exists_{=n} x\ \phi x \stackrel{\text{def}}{=} \exists v_1 \ldots v_n\ .\ \phi v_1 \wedge \ldots \wedge \phi v_n$

$\wedge\ v_1 \neq v_2 \wedge v_1 \neq v_3 \ldots v_1 \neq v_n \wedge \ldots \wedge v_{n-1} \neq v_1 \wedge v_{n-1} \neq v_2 \wedge \ldots \wedge v_{n-1} \neq v_n$[384]

[383] That is because I will use the total preorder of *having less or equal cm of height than.* Had I used the mirror *having more or equal cm of height than*, predicate tolerance for *tall* would concern adjacent increasing ranks and predicate-ordering monotony would concern decreasing ranks.

[384] This should be read as to exclude any affirmation of non-identity with itself for any variable. So theres no $v_m \neq v_m$ sub-clause for any $1 \leq m \leq n$.

$\wedge \forall x\, (\phi x \supset x = v_1 \vee \ldots \vee x = v_n)$

(15.4) Affirming that there are at least *n* elements such that...

$\exists_{>n} x\, .\, \phi x \stackrel{\text{def}}{=} \exists v_1 \ldots v_n\, .\, \phi v_1 \wedge \ldots \wedge \phi v_n$

$\wedge\, v_1 \neq v_2 \wedge v_1 \neq v_{3\ \ldots}\, v_1 \neq v_n \wedge \ldots \wedge v_{n-1} \neq v_1 \wedge v_{n-1} \neq v_2 \wedge \ldots \wedge v_{n-1} \neq v_n$[385]

(15.5) Affirming that there are at most *n* elements such that...

$\exists_{<n} x\, .\, \phi x \stackrel{\text{def}}{=} \forall v_1 \ldots v_{n+1}\, .\, \phi v_1 \wedge \ldots \wedge \phi v_{n+1} \supset (v_{n+1} = v_1 \vee \ldots \vee v_{n+1} = v_n)$

(15.6) Affirming that an element *a* is at rank 1 in the total preorder

$\forall_H x\, .\, \neg Rxa \vee Rax$

(15.7) Affirming that an element *a* is at the final rank

$\forall_H x\, .\, \neg Rax \vee Rxa$

(15.8) Affirming that an element *b* is at next or previous rank from *a*. Shorten for reuse

$\mathrm{Next}(b,a,R) \stackrel{\text{def}}{=} Rab \wedge \neg Rba \wedge \nexists_H x\, (Rax \wedge Rxb \wedge \neg Rxa \wedge \neg Rbx)$ [386]

$\mathrm{Prev}(b,a,R) \stackrel{\text{def}}{=} Rba \wedge \neg Rab \wedge \nexists_H x\, (Rax \wedge Rxb \wedge \neg Rxa \wedge \neg Rbx)$

(15.9) Affirming that an element *a* is at rank 4, or *n* generally

$\exists_H xyz\, .\, \forall_H t\, (\neg Rtx \vee Rxt) \wedge \mathrm{Next}(y,x,R) \wedge \mathrm{Next}(z,y,R) \wedge \mathrm{Next}(a,z,R)$

385 Idem.
386 At total orders that would be simply *Rab* $\wedge \nexists x$ *(Rax* $\wedge$ *Rxb)* but here we need to take into consideration that more elements can be at the same rank.

$\text{Rank}^{<}(a,n,R) \overset{\text{def}}{=} \exists_H v_1 \ldots v_{n-1} \,.\, \forall_H t\ (\neg Rtv_1 \vee Rv_1t) \wedge \text{Next}(v_2,v_1,R) \wedge \ldots \wedge \text{Next}(v_{n-1},v_{n-2},R) \wedge \text{Next}(a, v_{n-1},R)$ [387]

(15.10) Affirming that an element *a* is at negative rank 4, where negative rank 1 is the final one, negative rank 2 the penultimate, etc. Shorten for *n* generally

$\exists_H xyz \,.\, \forall_H t\ (\neg Rxt \vee Rtx) \wedge \text{Prev}(y,x,R) \wedge \text{Prev}(z,y,R) \wedge \text{Prev}(a,z,R)$

$\text{Rank}^{>}(a,n,R) \overset{\text{def}}{=} \exists_H v_1 \ldots v_n \,.\, \forall_H t\ (\neg Rv_1t \vee Rtv_1) \wedge \text{Prev}(v_2,v_1,R) \wedge \ldots \wedge \text{Prev}(v_{n-1},v_{n-2},R) \wedge \text{Prev}(a, v_{n-1},R)$

(15.11) Affirming that all elements of rank 3 are T

$\forall_H x \,.\, \text{Rank}^{<}(x,3,R) \supset Tx$

(15.12) Affirming that all elements of rank 3 and lower are not T. Shorten for *n* instead of 3

$\forall_H x \,.\, (\text{Rank}^{<}(x,1,R) \supset \neg Tx) \wedge (\text{Rank}^{<}(x,2,R) \supset \neg Tx) \wedge (\text{Rank}^{<}(x,3,R) \supset \neg Tx)$

$\text{Start}(n,\neg T,R) \overset{\text{def}}{=} \forall_H x \,.\, (\text{Rank}^{<}(x,1,R) \supset \neg Tx) \wedge \ldots \wedge (\text{Rank}^{<}(x,n,R) \supset \neg Tx)$

(15.13) Affirming that all elements in the last 3 ranks are T. Shorten for *n* instead of 3

$\forall_H x \,.\, (\text{Rank}^{>}(x,1,R) \supset Tx) \wedge (\text{Rank}^{>}(x,2,R) \supset Tx) \wedge (\text{Rank}^{>}(x,3,R) \supset Tx)$

$\text{End}(n,T,R) \overset{\text{def}}{=} \forall_H x \,.\, (\text{Rank}^{>}(x,1,R) \supset Tx) \wedge \ldots \wedge (\text{Rank}^{>}(x,n,R) \supset Tx)$

(15.14) Affirming that the maximum rank is 3. Shorten for *n* instead of 3

$\exists_H x\ \text{Rank}^{<}(x,3,R) \wedge \nexists_H x\ \text{Rank}^{<}(x,4,R)$

$\text{MaxRank}(n,R) \overset{\text{def}}{=} \exists_H x\ \text{Rank}^{<}(x,n,R) \wedge \nexists_H x\ \text{Rank}^{<}(x,n+1,R)$

[387] This definition should be read as to allow Rank<(x,1,R) to be equivalent to the statement of (15.6).

(15.15) Affirming that the maximum rank for elements which are not T and whose rank is lower than that of any element that is T is 3, i.e. there is an *initial negative chain* of 3 ranks. Shorten for *n* instead of 3

$\exists_H x\ (\neg Tx \wedge Rank^{<}(x,3,R) \wedge Start(3,\neg T,R)) \wedge \nexists_H x\ (\neg Tx \wedge Rank^{<}(x,4,R) \wedge Start(4,\neg T,R))$

$StartLen(n,\neg T,R) \stackrel{def}{=} \exists_H x\ (\neg Tx \wedge Rank^{<}(x,n,R) \wedge Start(n,\neg T,R)) \wedge \nexists_H x\ (\neg Tx \wedge Rank^{<}(x,n{+}1,R) \wedge Start(n{+}1,\neg T,R))$

(15.16) Affirming that the maximum negative rank[388] for elements that are T and whose rank is higher than that of any element that is not T is 3, i.e. there is a *final positive chain* of 3 ranks. Shorten for *n* instead of 3

$\exists_H x\ (Tx \wedge Rank^{>}(x,3,R) \wedge End(3, T,R)) \wedge \nexists_H x\ (Tx \wedge Rank^{>}(x,4,R) \wedge End(4,T,R))$

$EndLen(n,T,R) \stackrel{def}{=} \exists_H x\ (Tx \wedge Rank^{>}(x,n,R) \wedge End(n,T,R)) \wedge \nexists_H x\ (Tx \wedge Rank^{>}(x,n{+}1,R) \wedge End(n{+}1,T,R))$

(15.17) Affirming that the minimum rank for elements that are T and whose rank is higher than that of any element that is not-T is 3. Shorten for *n* instead of 3

$\exists_H x\ (Tx \wedge Rank^{<}(x,3,R) \wedge \forall_H y\ (Ty \vee \neg Rxy)\ ) \wedge \nexists_H x\ (Tx \wedge Rank^{<}(x,2,R) \wedge \forall_H y\ (Ty \vee \neg Rxy)\ )$

$EndStart(n,T,R) \stackrel{def}{=} \exists_H x\ (Tx \wedge Rank^{<}(x,n,R) \wedge \forall_H y\ (Ty \vee \neg Rxy)\ ) \wedge \nexists_H x\ (Tx \wedge Rank^{<}(x,n{-}1,R) \wedge \forall_H y\ (Ty \vee \neg Rxy)\ )$

(15.18) Affirming that all elements from rank 3 to the end are T. Shorten for *n* instead of 3

$\forall_H x\ .\ (Rank^{<}(x,3,R) \supset Tx) \wedge \forall_H y\ (Rxy \supset Ty)$

$EndPos(n,T,R) \stackrel{def}{=} \forall_H x\ .\ (Rank^{<}(x,n,R) \supset Tx) \wedge \forall_H y\ (Rxy \supset Ty)$

(15.19) Affirming that rank *n* is intermediary i.e. after the end of the initial negative chain and before the start of the final positive chain

[388] See (15.10) above.

$\text{InterRank}(n,\neg T,T,R) \stackrel{\text{def}}{=} \neg\text{Start}(n,\neg T,R) \land \neg\text{EndPos}(n,T,R) \land \exists_H x\ \text{Rank}^{<}(x,n,R)$

(15.20) Affirming that an element *a* is at an intermediate rank

$\text{ElemInter}(a,\neg T,T,R) \stackrel{\text{def}}{=} \exists_H x\ (Rax \land \neg Tx \land \nexists_H y(\neg Ty \land Rxy \land \neg Ryx)\ ) \land \exists_H x\ (Rxa \land Tx \land \nexists_H y(Ty \land Ryx \land \neg Rxy))$

(15.21) Affirming that an element *a* is inside the *safe zone*, i.e. the initial negative chain or final positive chain excluding the first rank, the end of the initial negative chain, the start of the final positive chain and the last rank

$\text{ElemSafe}(a,\neg T,T,R) \stackrel{\text{def}}{=} \neg\text{ElemInter}(a,\neg T,T,R) \land \neg\text{Rank}^{<}(x,1,R) \land \neg\nexists_H xy\ (\ \text{Next}(x,a,R) \land \text{Next}(y,x,R) \land Ty) \land \neg\nexists_H xy\ (\ \text{Prev}(x,a,R) \land \text{Prev}(y,x,R) \land \neg Ty)\ \land \neg\text{Rank}^{>}(x,1,R)$

15.2. Statistical expressions

With the shortened expressions of the previous subchapter, we can define in FOL some statistical properties that will help define vagueness. This is only possible with the specification of a finite upper bound, such as 10^{12} for humans. Let us call this bound *m*.

We want first, to split each of the initial negative chain, the dispersion zone and the final positive chain of the total preorder into halves. Secondly, we want to be able to characterize a rank as being composed of a proportion of negative to positive cases. Finally, we want to express the proportion of elements at intermediate ranks (i.e. in the dispersion zone) versus non-intermediate ranks (i.e. initial and final chain).

I will use $Ceil_{n/2}$ for *n* divided by 2 and rounded-up to the next integer and $\text{Floor}_{n/2}$ for *n* divided by 2 and rounded-down to the previous integer. It is our task to compute them i.e. they are extra-logical, given here in variable form for shortening purposes only.

(15.22) A majority quantifier "for most objects of H of which there are no more than *m*":

$\exists_{>\frac{1}{2}(H,\,n)}x\ \phi \overset{\text{def}}{=} (\exists_{=n}x\ Hx \supset \exists_{>Floor n/2+1}x\ \phi) \lor (\exists_{=n-1}x\ Hx \supset \exists_{>Floor(n-1)/2+1}x\ \phi) \lor \ldots \lor (\exists_{=1}x\ Hx \supset \exists_{>Floor 1/2+1}x\ \phi)$

(15.23) Affirming that at least ½ of elements are inside the *safe zone* (i.e. non-intermediary, not at the first or last rank, not at the end of the initial negative chain or the start of the final positive chain), using upper bound *m*

$\text{Safe50}(m,\neg T,T,R) \overset{\text{def}}{=} (\exists_{>Ceil\,m/2}x\ \text{ElemSafe}(x,\neg T,T,R) \land \exists_{<Ceil\,m/2}x\neg\text{ElemSafe}(x,\neg T,T,R)) \lor (\exists_{>Ceil(m-1)/2}x\ \text{ElemSafe}(x,\neg T,T,R) \land \exists_{<Ceil(m-1)/2}x\neg\text{ElemSafe}(x,\neg T,T,R)) \lor \ldots \lor (\exists_{>Ceil\,1/2}x\ \text{ElemSafe}(x,\neg T,T,R) \land \exists_{<Ceil\,1/2}x\neg\text{ElemSafe}(x,\neg T,T,R))$

(15.24) Affirming that an element *a* is in the first half of the ranks in the initial negative chain

We first affirm that element *a* is in the first half between rank 1 and some rank *n:*

$\text{FirstHalfBefore}(a, n, R) \overset{\text{def}}{=} \text{Rank}^{<}(a,Ceil_{n/2},R) \lor \text{Rank}^{<}(a,Ceil_{(n-1)/2},R) \lor \ldots \lor \text{Rank}^{<}(a,Ceil_{1/2},R))$

We then get what we wanted, using the upper bound *m*:

$\text{HalfStart}(a,m,\neg T,R) \overset{\text{def}}{=} (\text{StartLen}(m,\neg T,R) \land \text{FirstHalfBefore}(a, m, R)) \lor (\text{StartLen}(m-1,\neg T,R) \land \text{FirstHalfBefore}(a, m-1, R)) \lor \ldots \lor (\text{StartLen}(1,\neg T,R) \land \text{FirstHalfBefore}(a, 1, R))$

(15.25) Affirming that an element *a* is in the last half of the ranks in the final positive chain

We first affirm that element *a* is in the last half of the ranks found between some negative rank[389] *n* and the final rank*:*

$\text{LastHalfAfter}(a, n, R) \overset{\text{def}}{=} \text{Rank}^{>}(a,Ceil_{n/2},R) \lor \text{Rank}^{>}(a,Ceil_{(n-1)/2},R) \lor \ldots \lor \text{Rank}^{>}(a,Ceil_{1/2},R))$

We then get what we wanted, using the upper bound *m*:

$\text{HalfEnd}(a,m,T,R) \overset{\text{def}}{=} (\text{EndLen}(m,T,R) \land \text{LastHalfAfter}(a, m, R)) \lor (\text{EndLen}(m-1,T,R) \land \text{LastHalfAfter}(a, m-1, R)) \lor \ldots \lor (\text{EndLen}(1,T,R) \land \text{LastHalfAfter}(a, 1, R))$

[389] See (15.10) above

(15.26) Affirming that an element is in the first half of the intermediary ranks (i.e. between the end of the initial negative chain and the start of the final positive chain)

We first shorten the notion of an element *a* being in the first half of ranks between *q,* end of initial negative chain and *n*, the start of final positive chain:

HalfIntervalTwo(a,*q*,*n*,¬T,T,R) $\stackrel{\text{def}}{=}$ StartLen(*q*,¬T,R) ∧ EndStart(*n*,T,R) ∧ ($\text{Rank}^<(\text{a},\mathit{Ceil}_{(q+n)/2},\text{R})$ ∨ $\text{Rank}^<(\text{a},\mathit{Ceil}_{(q+n-1)/2},\text{R})$ ∨ ... ∨ $\text{Rank}^<(\text{a},\mathit{Ceil}_{(q+1)/2},\text{R})$)

We then shorten the notion of an element *a* being in such a first half, for any interval generated by some *n*, start of the final positive chain:

HalfIntervalOne(a,*n*,¬T,T,R) $\stackrel{\text{def}}{=}$ HalfIntervalTwo(a,*n*,*n*,¬T,T,R) ∨ HalfIntervalTwo (a,*n*-1,*n*,¬T,T,R) ∨ ... ∨ HalfIntervalTwo(a,1,*n*,¬T,T,R)

We then get what we wanted, using the upper bound *m*:

HalfInterval(a,*m*,¬T,T,R) $\stackrel{\text{def}}{=}$ HalfIntervalOne(a,*m*,¬T,T,R) ∨ HalfIntervalOne(a,*m*-1,¬T,T,R) ∨ ... ∨ HalfIntervalOne(a,1,¬T,T,R)

(15.27) Affirming that in a rank *r* at least ½ of elements are T, using the upper bound *m*

RankProp50(*r*,*m*,T,R) $\stackrel{\text{def}}{=}$ ($\exists_{=m}$x $\text{Rank}^<$(x,*r*,R) ∧ $\exists_{>\text{Ceil}m/2}$x (Tx ∧ $\text{Rank}^<$(x,*r*,R))) ∨ ($\exists_{=m-1}$x $\text{Rank}^<$(x,*r*,R) ∧ $\exists_{>\text{Ceil}(m-1)/2}$x (Tx ∧ $\text{Rank}^<$(x,*r*,R))) ∨ ... ∨ ($\exists_{=1}$x $\text{Rank}^<$(x,*r*,R) ∧ $\exists_{>\text{Ceil}1/2}$x (Tx ∧ $\text{Rank}^<$(x,*r*,R)))

(15.28) Affirming that an element *a* is in a rank in which at least ½ of elements are T, using the upper bound *m*

$\text{RankProp50}^<$(a,*m*,T,R) $\stackrel{\text{def}}{=}$ ($\text{Rank}^<$(a,*m*,R) ∧ RankProp50(*m*,*m*,T,R)) ∨ ($\text{Rank}^<$(a,*m-1*,R) ∧ RankProp50(*m-1*,*m*,T,R)) ∨ ... ∨ ($\text{Rank}^<$(a,*1*,R) ∧ RankProp50(1,*m*,T,R))

(15.29) Affirming that an element *b* is in a rank in which the proportion of elements which are T is at least as high as that of the rank of element *a*

We first affirm that an element *a* is in a rank with *n* elements:

RankNum $(a,n,\mathrm{T},\mathrm{R}) \stackrel{\mathrm{def}}{=} \exists_{=n}x\ (\mathrm{R}xa \wedge \mathrm{R}ax)$

We then affirm that an element *a* is in a rank with at most *n* elements which are T:

RankMaxP$(a,n,\mathrm{T},\mathrm{R}) \stackrel{\mathrm{def}}{=} \exists_{<n}x\ (\mathrm{R}xa \wedge \mathrm{R}ax \wedge \mathrm{T}x)$

We then affirm that an element *b* is in a rank of *n* elements and it has a larger proportion of elements which are T than the rank of element *a*, provided the latter rank has at most *o* elements:

ElemHigherPropOne$(b,a,n,o,\mathrm{T},\mathrm{R}) \stackrel{\mathrm{def}}{=}$ RankNum$(b,n,\mathrm{T},\mathrm{R})$

$\wedge$ (

(RankNumP$(b,n,\mathrm{T},\mathrm{R})$

$\supset$ ((RankNum$(a,o,\mathrm{T},\mathrm{R}) \supset$ RankMaxP$(a,\mathrm{Floor}_{(n/n)*o},\mathrm{T},\mathrm{R})$)

$\wedge$ (RankNum$(a,o\text{-}1,\mathrm{T},\mathrm{R}) \supset$ RankMaxP$(a,\mathrm{Floor}_{(n/n)*(o-1)},\mathrm{T},\mathrm{R})$)

$\wedge$ …

$\wedge$ (RankNum$(o,1,\mathrm{T},\mathrm{R}) \supset$ RankMaxP$(a,\mathrm{Floor}_{(n/n)*1},\mathrm{T},\mathrm{R})$)

)

)

$\wedge$ (RankNumP$(b,n\text{-}1,\mathrm{T},\mathrm{R})$

$\supset$ ((RankNum$(a,o,\mathrm{T},\mathrm{R}) \supset$ RankMaxP$(a,\mathrm{Floor}_{((n-1)/n)*o},\mathrm{T},\mathrm{R})$)

$\wedge$ (RankNum$(a,o\text{-}1,\mathrm{T},\mathrm{R}) \supset$ RankMaxP$(a,\mathrm{Floor}_{((n-1)/n)*(o-1)},\mathrm{T},\mathrm{R})$)

$\wedge$ …

$\wedge$ (RankNum$(a,1,\mathrm{T},\mathrm{R}) \supset$ RankMaxP$(a,\mathrm{Floor}_{((n-1)/n)*1},\mathrm{T},\mathrm{R})$)

)

)

$\wedge$ …

$\wedge$ (RankNumP$(b,1,\mathrm{T},\mathrm{R})$

$\supset$ ((RankNum$(a,o,\mathrm{T},\mathrm{R}) \supset$ RankMaxP$(a,\mathrm{Floor}_{(1/n)*o},\mathrm{T},\mathrm{R})$)

∧ (RankNum(a,$o-1$,T,R) ⊃ RankMaxP(a,Floor$_{(1/n)*(o-1)}$,T,R))

∧ …

∧ (RankNum(a,1,T,R) ⊃ RankMaxP(a,Floor$_{(1/n)*1}$,T,R))

)

)

)

We finally get what we want, using the upper bound m:

ElemHigherProp(b,a,m,T,R) $\stackrel{\text{def}}{=}$ ElemHigherPropOne(b,a,m,m,T,R) ∨ ElemHigherPropOne(b,a,$m-1$,m,T,R) ∨ … ∨ ElemHigherPropOne(b,a,1,m,T,R)

Chapter 16

Defining vagueness

We can now define vagueness. The definition will only cover being vague for a total preorder, so we should only be able to call *tall* vague when providing the ordering relation, for example that of height in cm.

16.1. Minimal definition

Predicate T is *vague for a relation* R in a bounded submodel iff:

(Vag.1) Bounded submodel

There is a finite upper bound *m* on the number of elements of a predicate (here *humans*):

$\exists_{<m}x$ Hx

(Vag.2) Total preorder

R is a total preorder on the bounded predicate H (here *humans*). See (14.5-8) above for transitivity, reflexivity, the connex property and limitation to H[390].

(Vag.3) Negative and positive chain

There are initial two ranks of R to only contain elements which are not T (i.e. there is an initial negative chain) and two final ranks only of elements which are T (i.e. there is a final positive chain).

$\mathrm{Start}(2,\neg T,R) \land \mathrm{End}(2, T,R) \land \exists_H x\ \mathrm{Rank}^{<}(x,2,R) \land \exists_H x\ \mathrm{Rank}^{>}(x,2,R)$

(Vag.4) Dispersion

There are at least two switching points i.e. intermediate ranks between ¬T and T:

$\exists_H xyzt\ .\ \mathrm{Next}(y,x,R) \land \mathrm{Next}(t,z,R) \land \neg Tx \land Ty \land \neg Tz \land Tt \land \neg(Rxz \land Rzx) \land \neg(Rxt \land Rtx) \land \neg(Rzy \land Ryz)$

[390] This makes *m* an upper bound of $\bigcup R$, as we wanted.

(Vag.5) At least ½ of the elements are in the safe zone

The minimum number of elements at intermediary ranks, the first or last rank, the end of the initial negative chain or the start of the final positive chain is ½ of *m*. That is, dispersion does not happen over more than half of the elements:

Safe50(m,¬T,T,R)

This is intended as a minimal definition.

(Vag.1) and (Vag.2) say that vagueness is a property of the ordering of the variable of belonging to a predicate into ranks by the respective total preorder, in a bounded submodel i.e. for a finite numbers of persons such as 10^{12}.

(Vag.3) says that there is a chain of negative cases at the beginning and a chain of positive cases at the end, such as at least two adjacent short persons at small heights in cm and two adjacent tall persons at large heights in cm. We here have only two ranks defining a chain, because it is a *minimal* definition, which is easy to model.

(Vag.4) says that there is at least some dispersion i.e. failure of predicate-ordering monotony, such that there is an intermediate zone of heights in cm where tall and short people are intermingled. This is minimal in the sense that it excludes a *border* i.e. having a single switching point from negative to positive cases. Further intuitive conditions can be added, most likely one that as the rank increases after the end of the initial negative chain, the proportion of positive cases increases too (see (17.1) below).

(Vag.5) says that there is no majority of persons outside the safe zone. This means that predicate tolerance will hold for at least half of persons by height. Moreover, it implies that more than half of persons are either among those adjacently short because of small height in cm or adjacently tall because of large height in cm.

16.2. Advantages of the minimal definition

Firstly, this definition meets the requirement indicated by Andrew Bacon[391] that people have similar expectations of vague terms whether in English or Russian. The structural logical structure of vagueness is on display. It explains why *tall* is vague in function of *to have less or equal cm of height*, *bald* in function of *to have more or equal hairs*, *rich* in function of *to have less or equal money*, and, respectively, why *tall* is not vague in function of *to have more or equal hairs,* and so on. Vagueness is a logical phenomenon between precision (understood as the presence of a threshold), and chaotic dispersion.

Secondly, (Vag.5) minimally captures the intuition that predicate tolerance is likely although false, since at most ranks it holds, as discussed at 5.3.5 above. The statistical reading of the major premise was:

(M. Statistical) $\exists_{>1/2(H,\, m)}\, x \forall_H y\ .\ C_n x \land Tx \supset (C_{n-1} y \supset Ty)$

Read 'for the majority of humans of which there are no more than *m,* if one of a number of cm of height is tall, any other man having one less cm is also tall'. We can now express it in a short form, taking adjacent decreasing ranks to correspond to the difference of 1 cm (a fortiori, if the formulation holds for any ranking step, it holds for 1 cm):

(M. Statistical.2) $\exists_{>1/2(H,\, m)}\, x \forall_H y\ .\ \mathrm{Prev}(y, x, R) \supset \neg(Tx \land \neg Ty)$

Since by definition, a majority of elements is inside the safely non-intermediary ranks, for them tolerance will hold. This is very intuitive, but minimal. Stronger defining conditions can be added to guarantee (M.Statistical) holds for a majority of ranks, or even for all ranks, or for its analogue using 9/10 instead of ½. To wit, it may be required that all ranks contain at least two elements and the proportion of tall cases increases by no more than 10% with each rank.

Finally, this minimal definition captures the essence of vagueness, i.e. the failure and likeliness of predicate-ordering monotony: some element is a negative case of a predicate, even when ranked equal or higher than a positive case. That is why we can now express in FOL both the bivalent and the comparable nature of vagueness, as intended:

[391] See 5.3.2 above.

(VagRank) An element *a* is not T while being in a rank that is at least ½ T, using upper bound *m*

$\neg Ta \wedge \text{RankProp50}^{<}(a,m,T,R)$

(VagRank.2) Elements *a* and *b* are not T while *b* is in a rank of a higher proportion of T elements, using upper bound *m* (i.e. *b* is more vaguely short than *a*)

$\neg Ta \wedge \neg Tb \wedge \text{ElemHigherProp}(b,a,m,T,R)$

16.3. The relatives of *tall*

In order to meet the aim of Chapter 14, we now want to define some new predicates i.e. specify both when they apply and when they do not. The superscripted predicate names that we introduce, such as 'T^{R1}' should be treated as textual variants of predicate letters such as 'G', 'H', 'I', etc., to illustrate what we want from our notational extension of FOL.

All the following apply only with (Vag.1-5) in the mix. So R is a total preorder on humans, *m* is the explicit upper bound, etc.

16.3.1. *Strictly tall*

Define T^{R0}, read 'Strictly T', here *Strictly tall*, as that predicate holding only inside the final positive chain:

$\forall_H x \,.\, Tx \wedge \nexists_H y(\neg Ty \wedge Rxy) \leftrightarrow T^{R0}x$

Define $!T^{R0}$, read 'Strictly not T', here *Strictly short*, as holding only inside the initial negative chain:

$\forall_H x \,.\, \neg Tx \wedge \nexists_H y(Ty \wedge Ryx) \leftrightarrow !T^{R0}x$

16.3.2. *Broadly tall*

Define T^{R1}, read 'Broadly T', Tolerant T', here *Broadly tall* or *Tallish*, as that predicate holding outside the initial negative chain:

$\forall_H x \,.\, \exists_H y(Ty \wedge \nexists_H z(Tz \wedge Rzy \wedge \neg Ryz) \wedge Ryx) \leftrightarrow T^{R1}x$

Define $!T^{R1}$, read 'Broadly not T', 'Tolerant not T', here *Broadly short* or *Shortish*, as that predicate holding outside the final positive chain:

$\forall_H x \,.\, \exists_H y(\neg Ty \wedge \nexists_H z(\neg Tz \wedge Ryz \wedge \neg Rzy) \wedge Rxy) \leftrightarrow {!T^{R1}}x$

It can be checked that *Broadly tall* is the complement of *Strictly short* and *Broadly short* the complement of *Strictly tall*:

$\forall_H x \,.\, {!T^{R0}}x \leftrightarrow \neg T^{R1}x$

$\forall_H x \,.\, T^{R0}x \leftrightarrow \neg\, {!T^{R1}}x$

16.3.3. *Arguably tall*

Define T^{Ra}, read 'Arguably T or not T', here *Arguably tall* or *Arguably short*, as the intersection of T^{R1} and $!T^{R1}$, here *Broadly tall* and *Broadly short* i.e. all elements in the dispersion zone:

$\forall_H x \,.\, T^{R1}x \wedge {!T^{R1}}x \leftrightarrow T^{Ra}x$

Or $\forall_H x \,.\, \mathrm{ElemInter}(x,\neg T,T,R) \leftrightarrow T^{Ra}x$

Define $!T^{Ra}$, read 'Unarguably T or not T', here *Unarguably tall or not tall* or *Unarguably short or not short,* as the complement of T^{Ra}:

$\forall_H x \,.\, \neg T^{Ra}x \leftrightarrow {!T^{Ra}}x$

We see *Broadly tall* is the union of *Arguably tall* with *Strictly tall* and *Broadly short* is the union of *Arguably short* with *Strictly short*:

$\forall_H x \,.\, T^{R1}x \leftrightarrow T^{Ra}x \vee T^{R0}x$

$\forall_H x \,.\, {!T^{R1}}x \leftrightarrow T^{Ra}x \vee {!T^{R0}}x$

Therefore *Unarguably tall or short* is the union of *Strictly tall* with *Strictly short*:

$\forall_H x \,.\, {!T^{Ra}}x \leftrightarrow T^{R0}x \vee {!T^{R0}}x$

16.3.4. *Probably tall*

Define *T^{Rp}*, read 'Probably T', here *Probably tall,* as all elements found at ranks that are at least ½ T, using upper bound *m*:

$\forall_H x$. RankProp50$^<$(x,*m,*T,R) $\leftrightarrow T^{Rp}x$

Define *!T^{Rp}*, read 'Probably not T', here *Probably short,* as the complement of T^{Rp}:

$\forall_H x . \neg T^{Rp}x \leftrightarrow !T^{Rp}x$

16.3.5. *Ideally tall*

Define *T^{Ri}*, read 'Ideally T', here *Ideally tall,* as the union of *Strictly tall* (T^{R0}) with the second half of *Arguably tall* (the dispersion zone), using upper bound *m*:

$\forall_H x . T^{R0}x \lor (T^{Ra}x \land \neg\mathrm{HalfInterval}(x,m,\neg T,T,R)) \leftrightarrow T^{Ri}x$

Define *!T^{Ri}*, read 'Ideally not T', here *Ideally short*, as the complement of T^{Ri}:

$\forall_H x . \neg T^{Ri}x \leftrightarrow !T^{Ri}x$

16.4. A graphical interpretation and discussion

A graphical visualization of an interpretation that satisfies this vague structure of predicates would look like this:

<table>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>2</td><td>1</td><td>3</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>2</td><td>4</td><td>3</td><td>2</td><td>4</td><td>2</td><td>2</td><td>5</td><td>4</td><td>5</td><td>3</td><td>3</td></tr>
<tr><td>a</td><td>b</td><td>c</td><td>d</td><td>e</td><td>f</td><td>g</td><td>h</td><td>i</td><td>j</td><td>k</td><td>l</td><td>m</td><td>n</td><td>o</td><td>p</td><td>q</td><td>r</td><td>s</td><td>t</td><td>u</td><td>v</td><td>w</td><td>x</td><td>y</td><td>z</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>*</td><td>&</td><td>/</td><td>\</td><td>^</td></tr>
<tr><td>3</td><td>3</td><td>4</td><td>2</td><td>3</td><td>2</td><td>3</td><td>1</td><td>1</td><td>1</td><td>1</td><td>3</td><td>3</td><td>3</td><td>2</td><td>3</td><td>3</td><td>2</td><td>2</td><td>1</td><td>0</td><td>2</td><td>0</td><td>0</td><td>0</td><td>0</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td colspan="11">$!T^{Ra}$ – unarguably short/tall</td><td colspan="23">T^{Ra} – arguably tall / arguably short</td><td colspan="6">$!T^{Ra}$ – unarguably short/ tall</td></tr>
<tr><td colspan="22">$!T^{Ri}$ – ideally short</td><td colspan="18">T^{Ri} – ideally tall</td></tr>
<tr><td colspan="34">$!T^{R1}$ – broadly short</td><td colspan="6">T^{R0} – strictly tall</td></tr>
<tr><td colspan="11">$!T^{R0}$ – strictly short</td><td colspan="29">T^{R1} – broadly tall</td></tr>
<tr><td colspan="19"></td><td colspan="7">T^{Rp} – probably tall</td><td colspan="2"></td><td colspan="12">T^{Rp} – probably tall</td></tr>
<tr><td colspan="19">$!T^{Rp}$ – probably short</td><td colspan="7"></td><td colspan="2">$!T^{Rp}$</td><td colspan="12"></td></tr>
</table>

Legend: The ranks of R are represented ascendingly from left to right by all letters, numbers or symbols between *a* an ^. Ranks in **bold** typeface are inside the safe zone of (15.21). Above them we list the number of elements that are T at that rank and below them the number of not-T elements at that rank. We underline when RankProp50(*rank,*40,T,R) does not hold for that rank and overline when it does.

These secondary predicates reflect the structural characteristics of a total preordering with regard to the monadic predicate T. It is because of the way R orders T's and not T's that we can discern them.

As we have seen in the previous chapters, vagueness has been approached in different manners. I submit that this structure of predicates captures its possible senses.

Firstly, the classical sense is plainly captured by the distinction between T and not T, the underlying logic being classical: a man is either tall or short. However, we are inclined to extract from this the further thesis of epistemicism: that there exists a single threshold in cm between *short* and *tall* people. It has no experimental basis but an intuitive appeal: were all predicates established stipulationally, we would introduce a threshold in cm between tall and short people, mostly for heuristic reasons[392]. This is an *ideal* situation, captured by my *Ideally tall* above*. Ideally tall* is a predicate which is precise under the R ordering, i.e. is has a single switching point, so it corresponds to the common idealization of vague predicates: 'but surely there must be a threshold for our ideal conception about *tall* be true'.

Secondly, the laxer sense of vagueness, that there are elements which are not, depending on philosophical preference, - *definitely, clearly, determinately, knowably, certainly* - *tall,* is captured by our definition of *Strictly T* and *Broadly T.* Their definition helps us answer any sorites, by translating the major premise(s) therein. As shown below, we say without paradox that if a man of 170 cm is tall, a man of 171 cm will be broadly tall and that if a man of 170 cm is strictly tall, a man of 169 cm will be broadly tall. Relatedly, *Arguably tall* i.e. the dispersion zone is defined as the intersection of *Broadly tall* and *Broadly short* or as the difference of humans with *strictly tall* and *strictly short*. This means that it dispersion zone can in turn be taken as primary to define the others.

Thirdly, the probabilistic sense of vagueness, as expounded by Edgington, is captured by my *Probably tall*. It is defined as the union of all ranks where there are more elements that are at least ½ T. What is interesting is that in some models all T^{Ra} (Arguably T) elements will be T^{Rp} e.g. when all intermediary ranks have two T elements out of three, while in some other vague models there will not be any T^{Rp} elements outside T^{R0} e.g. when all intermediary ranks have one T element out of three. This opens the door to a variant of higher-order vagueness, as we will see.

Finally, the comparable sense of vagueness is captured by expressing in FOL the proportion of T elements at each rank or succession of ranks. This eliminates the advantage of fuzzy logic over other logics for vagueness: the expressions of the previous chapter illustrate how we can compare vague predicates without fuzzy semantics.

Let us now see how the different perspectives on vagueness are expressible in a coherent manner with the use of the newly defined predicates.

[392] See 12.3 for the effect of confounding this distinction and the importance of heuristics in Frege's analysis of the sorites and indeterminacy.

16.5. Philosophical and logical statements

Assume only (Vag.1-5). We can formulate some results, which validate our FOL definition for vagueness as appropriate for common intuitions of vagueness[393].

16.5.1. Some *NC-looking* clauses fail

(NC.1) 'Some men are both broadly tall and broadly short'

$\exists_H x \,.\, T^{R1}x \wedge\ !T^{R1}x$

(NC.2) 'Some men are both broadly tall and short'

$\exists_H x \,.\, T^{R1}x \wedge \neg Tx$

(NC.3) 'Some men are both arguably tall and short'

$\exists_H x \,.\, T^{Ra}x \wedge \neg Tx$

(NC.4) 'Some men are both ideally tall and short'

$\exists_H x \,.\, T^{Ri}x \wedge \neg Tx$

(NC.5) 'If there are tall and short people of the same height, some men are both probably tall and short or some men are both probably short and tall'

$\exists_H xy\ (\neg Tx \wedge Ty \wedge Rxy \wedge Ryx) \supset \exists_H x\ (\ (T^{Rp}x \wedge \neg Tx) \vee (!T^{Rp}x \wedge\ Tx))$

16.5.2. Some *LEM-looking* clauses fail

(LEM.1) 'Some men are neither strictly tall nor strictly short'

$\exists_H x \,.\, \neg(T^{R0}x \vee\ !T^{R0}x)$

(LEM.2) 'Some men are neither strictly tall nor short'

$\exists_H x \,.\, \neg(T^{R0}x \vee \neg Tx)$

[393] Sketches of proofs are in Appendix 1.

16.5.3. Weak versions of *predicate tolerance* and *predicate-ordering monotony* hold

(WPT) Predicate tolerance: 'If a man is strictly tall, anyone one rank lower by height is broadly tall':

$\forall_H xy \,.\, Prev(y,x,R) \supset \neg(T^{R0}x \wedge \neg T^{R1}y)$

(WPOM) Predicate-ordering monotony: 'If a man is tall, anyone one rank higher by height is broadly tall':

$\forall_H xy \,.\, Next(y,x,R) \supset \neg(Tx \wedge \neg T^{R1}x)$

16.5.4. Unproblematic intuitions hold

(UI.1) 'If someone is strictly tall, they are tall'

$\forall_H x \,.\, T^{R0}x \supset Tx$

(UI.2) 'Is someone is strictly tall, they are probably tall'

$\forall_H x \,.\, T^{R0}x \supset T^{Rp}x$

(UI.3) 'If someone is strictly tall, they are ideally tall'

$\forall_H x \,.\, T^{R0}x \supset T^{Ri}x$

(UI.4) 'If someone is tall, they are broadly tall'

$\forall_H x \,.\, Tx \supset T^{R1}x$

16.5.5. Classical (non-tolerant) positions hold

(LEM.T) 'A man is either tall or not tall'

$\forall_H x \,.\, Tx \vee \neg Tx$

(NC.T) 'A man is not both tall or not tall'

$\forall_H x \, \neg(Tx \wedge \neg Tx)$

(NT.1) 'But surely NC and LEM hold even for tolerant predicates, you cannot be both broadly tall and not broadly tall or not be either strictly tall or not strictly tall'

$\forall_{H}x \,.\, \neg(T^{R1}x \land \neg T^{R1}x)$

$\forall_{H}x \,.\, T^{R0}x \lor \neg T^{R0}x)$

(NT.2) 'There are two unique heights i.e. ranks that separate ideally short and ideally tall people, i.e. a threshold'

$\exists_{H}xy \,.\, \neg T^{Ri}x \land T^{Ri}y \land Next(y,x,R) \land \forall_{H}zt(\neg T^{Ri}z \land T^{Ri}t \land Next(t,z,R) \supset Rxz \land Rzx \land Ryt \land Rty\,)$

Thus, I applied to vagueness the strategy of subchapter 13.4, trying to express in FOL different senses of *tall* and its relatives, so that our common discourse about vagueness, both tolerant and non-tolerant can be expressed in a coherent manner. Let us now approach the remaining aspects of vagueness.

Chapter 17

Graduality, higher-order vagueness and multidimensionality

17.1. Gradual vagueness

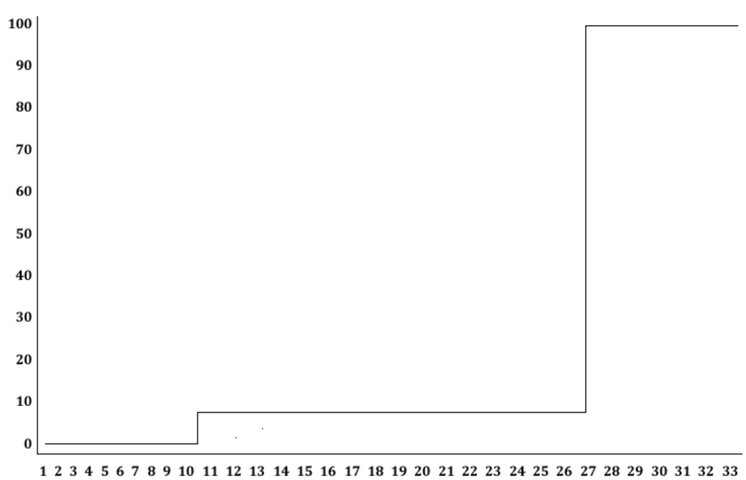
(VagRank) and (VagRank.2) above displayed the bivalent and comparable nature of vagueness. The former stated that a man is short while being at a height where most men are tall. The latter stated that the height of some man has a higher proportion of tall men than that of another. This illustrates the fact that we can extract such numerical proportions, approximating the continuous degrees of fuzzy logic, without committing ourselves to there being more than two truth-values. Then, we can define the relation of *being as vague or vaguer than*, in this case, *being as vaguely or more vaguely tall than*, applying only to broadly tall men.

(17.1) Define 'T^{RV}', read 'to be as vaguely or more vaguely T than' as

$\forall_H xy.\ T^{R1}x \wedge T^{R1}y \wedge \text{ElemHigherProp}(x,y,m,T,R) \leftrightarrow T^{RV}xy$

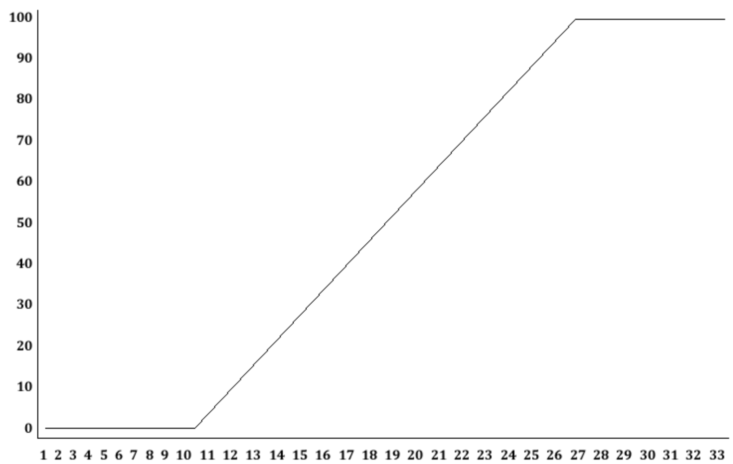
A gradient of such proportions from the initial negative chain to the final positive chain is what we may call *graduality*. It presupposes comparability, to which it adds the highly intuitive condition that the proportion should increase across ranks. People of more cm of height are more often tall that those of less cm of height.

Our minimal definition of vagueness is compatible with the following **(Graph.1)** of proportions, in which all intermediate ranks have the same (low) proportion of tall people:



Legend: Ranks of R are on the x axis, percentage of tall men from total at each rank are on the y axis.

While our intuition better aligns with the following **(Graph.2)**, in which the proportion of tall men increases with the intermediate rank number:



Legend: Ranks of R are on the x axis, percentage of tall men from total at each rank are on the y axis.

Therefore, we can minimally define *gradual vagueness*. Predicate T is *gradually vague* for a relation R iff:

(GradVag.1) T is vague for R

(GradVag.2) The proportion of elements which are T increases across ranks. This can be split in:

a) The proportion of elements which are T does not decrease between consecutive ranks

b) For any group of $\text{Ceil}_{n/100}$ consecutive ranks, with *n* the number of intermediate ranks (i.e. 1 for 100 intermediate ranks, 10 for 999, etc.), the proportion of elements which are T is higher at the last rank of the group than at the first.

We will not formalize a) and b) here, but it should be clear from the expressions in Chapter 15 how this can be achieved in FOL.

This definition of *gradually vague* is also intended as minimal i.e. capture the intuition that vague predicates are gradual, but without opening the door to higher-order vagueness, to which we now turn.

17.2. Higher-order vagueness

The definitions of *vague predicate* in Chapter 16 and of *gradually vague predicate* in 17.1 are minimal definitions, approximating our first-hand intuitions of vagueness and allowing us to express both tolerant and non-tolerant positions concerning vagueness, as displayed in 16.5 above. But it has been argued that vagueness is characterized by a special kind of deep lack of borders, called *higher-order vagueness*. Let us introduce it succinctly, the objections to our minimal definitions, and propose an approach to it in the present framework.

17.2.1. Is higher-order vagueness relevant?

In Chapter 12, solving the indecision characteristic of vagueness by inventing a third category was rejected as insufficient, as the indecision will show up between the negative cases and the third category or between the latter and positive cases[394]. Second-order vagueness concerns plainly the indecision whether a case is vaguely tall or not. In modalist language, the claim of higher-order vagueness is than any number *n* of Definitely operators cannot justify the application of *n*+1 such operators[395]. This, taken maximally, seems to require a recursive mechanism without which any definition of vagueness would be inappropriate.

Still, higher-order vagueness is extremely controversial. Not only are the terms used to introduce it 'indecision', 'category' metaphorical, but philosophical opinion is divided. First, as alluded, there have been proposals defining vagueness as higher-order vagueness, which supposedly implies an infinite hierarchy of Definitely operators[396]. There is also a majority opinion that there are intuitions for 2nd

[394] E.g. "It soon appears that the idea that there is a sharp division between the positive cases and the borderline ones, and between the borderline cases and the negative ones, can no more be sustained than can the idea that there is a sharp division between positive and negative cases" R.M. Sainsbury, "Is There Higher-Order Vagueness?" *The Philosophical Quarterly*, (Vol. 41, No. 163, 1991), 168.

[395] i.e. The modal system is weaker than S4. For an argument to that effect, see Crispin Wright, "Is Higher Order Vagueness Coherent?" Analysis, Vol. 52, No. 3 (Jul., 1992), 137. The S4 axiom has been defended as distinct from the system S4 and conserving higher-order vagueness by Susanne Bobzien, "If it's clear, then it's clear that it's clear, or is it? - Higher-order vagueness and the S4 Axiom", in *Episteme, etc.: Essays in Honour of Jonathan Barnes*, ed. Ben Morison and Katerina Ierodiakonou (Oxford University Press. 2012). doi:10.1093/acprof:oso/9780199696482.003.0010,

[396] "For those unwilling to accept epistemicism, it might seem that vagueness just is higher-order vagueness." Delia Graff and Timothy Williamson, "Introduction", in *Vagueness*, ed. Delia Graff and Timothy Williamson. Routledge. (Ashgate, Aldershot, 2002), xxii. Such a logical construction neutral between epistemicism and supervaluationism can be found in Timothy Williamson, "On the Structure of Higher-Order Vagueness", *Mind*, Vol. 108, No. 429 (Jan., 1999), 136-137.

or 3rd order vagueness, but no further[397]. Finally, there is another minority of opinion according to which higher-order vagueness does not exist: all vagueness is first-order[398]. Moreover, even against the classic argument against trivaluationism, it has been objected that there is not much empirical evidence that anyone would reject trivalence in favor of 5-valence and 5-valence in favor of 7-valence and so on. Secondly, that preferring 5-valence to tri-valence would be a philosophical remain of stipulationism[399]. On this latter line of thought, it has also been argued that any ZFC-based treatment of vagueness as higher-order vagueness will not be able to avoid boundaries, so both the trivaluationist solution and the infinite iteration of Definitely operators is misguided in describing *boundarilessness*[400]. Finally, Zardini argued that a sorites can be built even against unlimited higher-order vagueness, so the latter cannot be justified as avoiding the paradox[401].

Not only is the existence and explanatory status of higher-order vagueness controversial, but it seems to me that the very basis of introducing *clearly, definitely, determinately, knowably, certainly* into a logic for vagueness can be objected to, by parity of reasoning. The fundamental logical intuition seems to be that *Definitely tall* implies *tall* and iterating the operator of higher-order vagueness does not transgress the negation of the predicate, i.e. it will not apply truthfully to short people at any order. Let us call this view of the iterable *Definitely* operator *internal*: if someone is definitely tall, then he cannot be not tall and presumably, there are tall men that are not definitely tall. Let us describe an alternative position as *external*: there may be men that are tolerantly tall without being tall. According to the internal view, as we add *Definitely* operators, we retreat towards the kernel of truth,

[397] For a first such argument, J. A. Burgess, "The Sorites Paradox and Higher-Order Vagueness", *Synthese*, Vol. 85, No. 3 (Dec., 1990).

[398] For an argument from supervaluationism, see Achille C. Varzi, "Supervaluationism and Its Logics", 662-663. For an argument from plurivaluationism: "What is going on here is that a poor characterization of vagueness is accepted (i.e. vagueness as possession of borderline cases), and then our intuitive reservations about the characterization are given outlet in the positing of an additional phenomenon, over and above mere vagueness—i.e.higher-order vagueness" Nicholas J. J. Smith, *Vagueness and Degrees of Truth*, 182.

[399] Wright: "The claim that there are borderline cases of a certain concept is, after all, partly an empirical sociological claim: to make it is to predict that possessors of the concept will not react with verdicts about its application that collectively converge on a sharp distinction between positive and negative cases. How do Russell and Dummett know this in advance, sitting in their armchairs? […] The answer, presumably, is that we think we know already what the outcome of an experiment would be. But why do we think that?—It is not, after all, as if we have often made stipulations of the Dummett–Russell sort and experience has taught that they do not work. […] In going along with the prediction of uneliminated vagueness, we are reporting something about our own sense of limitation in response to the kind of stipulation hypothetically envisaged". Crispin Wright, "The Illusion of Higher-Order Vagueness", in *Cuts and Clouds: Vagueness, its Nature, and Its Logic*, ed. Richard Dietz and Sebastiano Moruzzi (Oxford University Press, 2009), 544-545.

[400] R.M. Sainsbury, "Is There Higher-Order Vagueness?" *The Philosophical Quarterly*, Vol. 41, No. 163 (Apr., 1991), 182.

[401] Elia Zardini, *Higher-Order Sorites Paradox*, July 27, 2011 version. Accessed Feb 15, 2019 http://www.eliazardini.eu/papers/2013/Higher-Order-Sorites-Paradox.pdf

leaving a larger and larger protection zone. An intuition that can correspond to it is the recognition of truth. Say, those truths that are recognized by a group of people are definite for them, those that are recognized by their superiors are definitely definitely true, those recognized at the top of their organizational hierarchy are definitely definitely definitely true for them and so on. Kit Fine's metaphor of the tree of language may seem apt here[402]. Another example is Timothy Williamson's solution to the forced march variant of the sorites paradox: we need to stop before we get close to unclear cases, even if we are proximate only to clear cases[403].

On the other hand, there is a case to be made for the external view. It starts from generality in the propagation of truth. If something is true, it is remote. Once expressed, it becomes broadly true according to a theory. Then it becomes broadly broadly true according to a theory of the theory and so on. It would correspond to wider and wider approximation, so that truth is approximated by scientific truth, scientific truth is approximated by truth according to science popularization and science popularization is approximated in its turn by common knowledge. No matter how much we try to keep truth across this progression, we may still fail, even in science. We can recall that out of the things that are now or have been cited at one time as definitely or clearly true, some proved false. But the repetition of the *Broadly* operator would correspond to propagation of truth and increasing generality, describing a halo of truth inside falsity, not a protection zone inside truth. So, as we have seen broadly tall men which are short, we presumably should have even more short men under *Broadly broadly tall*.

Now, the problem is that accounts of genuine higher-order vagueness that I know of do not discuss the external view[404]. Yet in common speech, it is as intuitive to iterate 'broadly' or 'in an even broader sense' as 'strictly' or 'in an even stricter sense'. That is why I have not yet proposed a definition of *Definitely tall* or *Clearly tall*. I do not want to argue here for the external view, just that a case could be made for it. And, since common use language does not advantage one view over another, we should restrain from basing a theory of vagueness only upon its treatment of higher-order vagueness.

Thus, the existence and explanatory status of higher-order vagueness is extremely controversial, and it is not clear whether it has at its basis a usable logical intuition. Since higher-order vagueness seems a theory-laden notion, I think it is better to base the claim that FOL is the logic of vagueness solely on the vague structure of predicates of 16.3 above, which showed their utility in the statements of 16.5.

[402] Kit Fine, "Vagueness, Truth and Logic", 275.
[403] See Chapter 12
[404] Except perhaps Williamson's 'Definitely* A' can be put to that use if negated. See 12.5.3 for a discussion of it against super-valuationism. Timothy Williamson, *Vagueness*, 160.

I will now only illustrate iterability within the present construction, leaving the choice to philosophical preference.

17.2.2. Objections

There are two obvious objections to our minimal definitions of *vague predicate* and *gradually vague predicate*.

Firstly, let us call it the *Border objection*, namely that our definition of *Broadly tall* and *Strictly short* creates a unique border between them, i.e. two unique ranks[405], which separate the predicates. Note that under our definitions there is no such unique border between *Broadly tall* and *tall*, neither between *tall* and *short*. So an option is to reject this objection on the lines of the argument in the previous section, namely that it is an issue of solely philosophical interest, without common linguistic intuitions to decide[406]. However, the present construction can handle iterability for both the internal and the external views, as we will see, if the philosopher so chooses.

Secondly, let us call it the *Deep dispersion objection,* namely that our definition of *gradually vague* is too strict: it forces each 100^{th} of consecutive ranks to display an increasing proportion of positive cases of the predicate. Suppose kyphotic men are concentrated at some close heights in cm, then our definition will not be met and *tall* will not be a gradually vague predicate by height in cm, which seems wrong.

I propose to sketch a single answer to both objections. But let us first eliminate some ways of defining higher-order vagueness that would not do:

a) Taking 'Definitely tall' as 'Strictly tall' of 14.2 above i.e. the final positive chain and give up iterating. This would work only if the philosophical position was that higher-order vagueness does not exist.

b) Defining *n^{th} order vague* as there being outside a margin of error of, say *n*% of the ranks, around the threshold rank between *strictly tall* and *broadly short*. This would give up the intuitive criterion used to define *Strictly* and *Broadly* above, namely that they apply when all elements at those ranks are similarly classified under the predicate, i.e. when the proportion is 100%.

[405] See (NT.2) above for an analogue expressing this in FOL.

[406] Indeed, it seems to me intuitive that there be a border between *Broadly tall* and *strictly short*, since some sense is strict if there are well-defined limits to it and a dichotomy of *broadly-strictly* is common.

c) Defining n^{th} *order vague* as there being at least n+1 switching points, see (Vag.4).

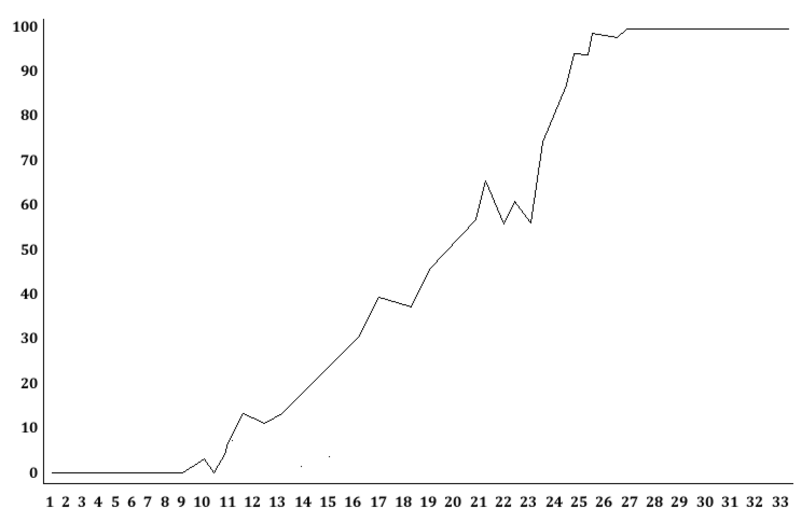
d) Defining n^{th} *order vague* as there being at least n*2 intermediate ranks.

e) Defining n^{th} *order vague* as there being at least n-1 ranks in the initial negative chain and n-1 ranks in the final positive chain.

The common reason for which c-e) would not work is that we may be working with a very large number of ranks. It would be quite counterintuitive when facing millions of intermediary ranks to treat a single or few ranks as going up in the higher-order vagueness hierarchy. Since our position is that vagueness is dispersion, it may be expected that any definition of higher-order vagueness make use of statistical approximations to iterate the structure of predicates introduced in 16.3.

17.2.3. A sketch of higher-order vagueness

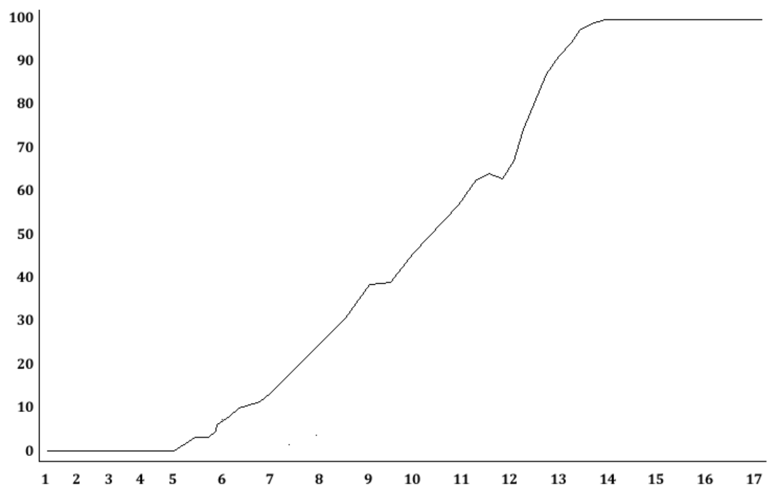
Let us first provide a more intuitive alternative to the linearity of (Graph.2), as indicated by the *Deep dispersion* objection. This is **(Graph.3):**



Legend: Ranks of R are on the x axis, percentage of tall men from total at each rank are on the y axis.

While generally the proportion of tall men increases across intermediary ranks, as in (Graph.2), here it does not increase in a linear manner.

Moreover, let us compare (Graph.3) with the following **(Graph.4)**, obtained by dividing the rank number by two and applying a smoothing function to them:



Legend: Ranks of R are on the x axis, percentage of tall men from total at each rank are on the y axis.

We observe in (Graph.3) and (Graph.4):

a) There can be graduality as in (Graph.3), without the linearity of (Graph.2). In (Graph.3), there are intermediate ranks of a lower proportion of *tall* men than their preceding intermediate rank, yet a general graduality is maintained. Indeed, the definition of *gradually vague* above will need to be weakened so as to include the case of (Graph.3).

b) We observe that the *Strictly tall* of (Graph.4) will contain two more ranks than the *Strictly tall* of (Graph.3), because the smoothing function removes the deviation from 100% tall men of the last intermediate rank in (Graph.3).

c) Based on b), suppose we treat *Strictly* as once iterable and we want to model once-iterability by the ordered pair containing the set of elements corresponding to *Strictly tall* in (Graph.3) and the set of elements corresponding to *Strictly tall* in (Graph.4). Then, there are two variants:

i) If we treat it as <Strictly tall of (Graph.4), Strictly tall of (Graph.3)>, then ‘Strictly tall’ has a larger part of the rank space (and of elements[407]) than that of ‘Strictly strictly tall’. It thus corresponds to the internal view. But the consequence would be that plain language use would correspond to having already applied a smoothing function i.e. an approximation of the real situation.

[407] This should be sensitive to the definition of the smoothing function.

ii) If we treat it as <Strictly tall of (Graph.3), Strictly tall of (Graph.4)>, then ‘Strictly tall’ has a smaller part of the rank space (and of elements[408]) than that of ‘Strictly strictly tall’. It thus corresponds to the external view. By the smoothing function, ranks of (Graph.4) include men which are not *tall*, but their rank proportion has been smoothed-out, so *Strictly strictly tall* will include short men, whereas *Strictly tall* did not. This is intuitive for *Broadly*, not for *Strictly*.

The last suggestion of i), corresponding to the internal view, can be given an epistemological justification. Since speakers do not have access to all other men and their natural measurements, they estimate the rank space of height by reducing the possibly million ranks to a smaller rank number. When talking of height, we speak of intervals of only tens to hundreds of cm, and we have an intuition that short men are classed lower than tall men. When we speak in a *strict sense* of tall men, we mean those whose height is the single determining dimension i.e. there is no other man of such height, even kyphotic, not to be tall. If we try to iterate ‘strictly’, as ‘in an even stricter sense’ we need to get more information and examine the situation ‘in more detail’. This corresponds to applying a weaker smoothing function to the real situation, and so on. Iterability of higher-order vagueness under this justification would mean examining more measurements and deciding questions about their interaction, getting closer to the real situation.

On the other hand, replacing in ii) above ‘strictly’ with ‘broadly’, we can give an equally intuitive justification of the iteration of ‘in a broad sense’, corresponding to the external view. You know someone who has a large number of cm, and they have some degree of kyphosis. They are not tall, but they are close to being tall, were the number larger or the degree smaller. You also know that such small differences often disappear when generalizing. So you know that the person is short, while they are broadly tall. Then, in an even broader sense of the term, even some people which are short in a strict sense have some characteristics which would make them to be considered tall, since when generalizing further, they will be lumped together by proportion with many people who are already broadly tall. Iterability under this justification would correspond to providing more general rules and classifications, getting further from the real situation.

I will not formulate them here, but appropriate definition of the smoothing function to achieve these effects can be found. The *Deep dispersion objection* is answered by applying the definition of gradually vague not on the base interpretation, but on the result of applying the smoothing function so as to render, for example, only ten ranks of the relation. The *Border objection* is answered by a hierarchy

[408] This should be sensitive to the definition of the smoothing function.

of applying *n* times the smoothing function. Higher-order vagueness implies, under this view, that when speaking strictly we walk from approximations to reality, and when speaking broadly, we go from reality to approximations. It matters, for us humans, whether we are examining a rank space of one million or one hundred ranks of height, in order to decide who is tall or not. No matter how many times we iterate, since the smoothing function works on the proportions, not the elements themselves, the criteria of Chapter 16 will apply in both situations: *Strictly* and *Broadly* will apply for all ranks where the proportion is 100%. Paradox does not arise, since the soritical steps can be reformulated as indicated in 16.5.3 above.

17.3. Multidimensionality and vagueness

We now need investigate two final issues. First, to discuss what a *multidimensional predicate* is and see how multidimensionality invalidates predicate-ordering monotony. This failure is the key to the misalignment between the preferred ordering *to have equal or lower cm of height* and *tall*. Secondly, to investigate the philosophical import of this misalignment.

17.3.1. Combining total preorders

I argued that all vague predicates are multidimensional, their application depending one more than one dimension. Since *dimensions* are expressed as numeric predicates and we have eliminated the latter in favor of total preorders, it is natural to take multidimensionality as total preorders functioning in some way as criteria for the predicate. For example, since *being tall* depends somewhat on height and somewhat on lacking kyphosis, it depends somewhat on number of cm and somewhat on degrees of spine curvature. Which, according to what has been said until now, means that it depends on the rank among people ordered by number of cm and the rank among people ordered by degrees of spine curvature.

From any number of total preorders defined on the same elements a new total preorder can be derived with some basic properties corresponding to the intuitive notion of *combining* them. For example, if work performance is determined 70% by hours worked and 30% by clients attended to, managers can rest assured that they can obtain a single ranking of their employees (with possibly more than one employee per rank), even if the implementation details may vary. There is at least another relation $R_{combined}$ which is such a combination of total preorder R to factor v (e.g. 0.7) with total preorder S to factor w (e.g. 0.3), respecting the following:

(Ranks.5) $R_{combined}$ respects the orderings of R and S.

$\forall xy \,.\, Rxy \wedge Sxy \supset R_{combined}\, xy$

(Ranks.6) $R_{combined}$ respects the coverings of R and S.

$\forall xy \,.\, Rxy \wedge Next(y,x,R) \wedge Sxy \wedge Next(y,x,S) \supset R_{combined}\, xy \wedge Next(y,x,R_{combined})$

(Ranks.7) $R_{combined}$ has at least as many ranks as the least of R and S.

To express this, we first define the concept of rank n being the least maximum rank of two total preorders:

$MaxRankTwo(n,R,S) \stackrel{def}{=} (MaxRank(n,R) \wedge \exists x\, Rank^{<}(x,n,S)) \vee (MaxRank(n,S) \wedge \exists x\, Rank^{<}(x,n,R))$

Then we define the concept of the maximum rank n of relation U being at least as high as the least maximum rank of two total preorders R and S:

$MaxRankComparedTwo(n,U,R,S) \stackrel{def}{=} MaxRank(n,U) \wedge (MaxRankTwo(n,R,S) \vee MaxRankTwo(n\text{-}1,R,S) \vee \ldots \vee MaxRankTwo(1,R,S))$

Then we get what we want (one total preorder having at least as many ranks as the least of two others), using the finite upper bound *m:*

$MinMaxRankThree(m,L,R,S) \stackrel{def}{=} MaxRankComparedTwo(m,L,R,S) \vee MaxRankComparedTwo(m\text{-}1, L,R,S) \vee \ldots \vee MaxRankComparedTwo(1, L,R,S)$

These are reasonable. A combination function must allow one input relation to weigh more than others (e.g. 70%-30% above), should conserve the orderings (two elements ordered in the same way by all input relations cannot be ordered inversely by the output relation), should conserve the coverings (two elements ordered by all input relations such that there is no third between them cannot have a third between them in the output) and should not have fewer ranks than the input relations. One example of such a function is given in Appendix 1.

17.3.2. Determinative versus preferred ordering

Total preorders, being relations, are ordered pairs of elements that have structural extensional properties, their relevance is a philosophical matter, not logical. As discussed in 14.3.1, there is a huge number of such extensional relations, much higher than the number of elements themselves.

The philosophical import is that people can be ordered relatively and absolutely by their height, degree of kyphosis or height of hair arrangement and that these relative and absolute rankings can be converted into one ranking which weights them together. This shows that a *formula* for height that takes into account kyphosis to make a person of more cm than a tall person short can be found among such combinations of total preorders. Thus, a way of converting numeric measurements into precise determination criteria for a predicate will correspond to a total preorder where all negative cases of the predicate are ranked lower than all positive cases, i.e. is precise and displays predicate-ordering monotony.

Let us call such total preorder corresponding to the multidimensional determining formula, and under which the predicate is precise *a determinative total preorder.* This is an ideal concept. Since bivalence holds, such a total preorder exists[409]. Under it, *tall* is precise: an analogue of predicate-ordering tolerance holds and an analogue of predicate tolerance fails:

(PT.Det) If a ≤ b in the determinative total preorder for *tall*, *b* cannot be tall without *a* being tall as well.

(PM.Det) If a ≤ b in the determinative total preorder for *tall*, *a* cannot be tall without *a* being tall as well.

For an example, suppose our Ally of 175 cm has a high-degree kyphosis (say 30°) to the effect that her head is standing at an even lower level from the ground than Mary who has 170 cm and no kyphosis, that Ally is not thought of as athletic while Mary is and has few other characteristics usually associated with tallness. Whatever the precise formula for the determinative total preorder for *tall*, it seems like Ally will be ordered lower than Mary. Predicate-ordering monotony just failed, since we take the predicate to be *tall* and the ordering *to have less or equal cm than*, which we use for height measured as it is standardly along the spine.

[409] It is trivial to find a total preorder making *tall* precise, if bivalence holds: take the total preorder of two ranks, the first containing all short men, the second containing all tall men. Of course, the real determinative total preorder for *tall* will have much more ranks, viz. the minimum of the ranks of ordering by height, kyphosis and the other measurable dimensions involved.

On the other hand, the sorites paradox is based on choosing a single ordering relation as decisive: number of hairs for *bald*, number of cm of height for *tall*, number of grains for *heap* and so on. I called such a relation the *preferred ordering* of the predicate. While *tall* is precise under its ideal determinative total preorder, it is vague under *to have less or equal cm of height than.*

So, in order to be able to express vagueness in FOL, we need to introduce in the semantics, for each predicate that is to be vague, at least a corresponding relation, namely its preferred ordering, and give an interpretation such as in 16.4 above, under which (Vag.1-5) are satisfied for the predicate and the respective ordering[410]. So the assumption of a classical semantics for vagueness is that each vague predicate comes with a preferred ordering relation with which it is misaligned. We now turn to the philosophical issues this connection raises.

17.3.3. *Taller*, *very tall* and the source of predicate-ordering monotony

The current philosophical debate on vagueness stands on the assumption of predicate-ordering monotony. For example, Patrick Greenough proposed a *minimal definition of vagueness* as epistemic tolerance and offered a proof that "vagueness *qua* epistemic tolerance entails vagueness *qua* borderline cases"[411], making use of the assumption that it is known that anyone of more cm than some tall person is tall. This assumption was criticized for not clearly being extendable to other vague predicates, because we may not know the parameters of application[412]. I think that would only work for discussing non-soritical vague predicates, which are outside the scope of this work. That is because the sorites is based, as discussed, on iterating a *small difference*, so such a *small difference* relation must be associated with the predicate. Thus, I accept that vague predicates have a single presumed dimension of comparison. But it is obvious that they do not apply simply upon its consideration. As we have cited, there is also distribution of hair partially determining baldness, kyphosis determining tallness, the collation of the sand particles determining heapness, and so on. So the way to reply to Greenough's argument is to plainly reject the assumption: we do not know that a kyphotic man of more cm than a non-kyphotic man is tall, provided the latter is. But why do philosophers find such an assumption irresistible?

[410] See Appendix 1 for a sketch of the notational extension of FOL.
[411] Patrick Greenough, "Vagueness: A Minimal Theory", 267.
[412] Brian Weatherson, *Vagueness as Indeterminacy,* 13

The reason is that the relatives of *tall* include *to be as tall as, to be taller than* and *to be less tall than*, which exert a strong linguistic pull[413]. Of course, they should not be bundled together in a single notion of "the concept 'tall'"[414], since *tall* is a predicate and *taller* is a relation. But their common source forces us to treat as analytic the proposition that anyone taller than a tall men is tall. Thus, predicate-ordering monotony has been called 'a fortiori'[415] or 'a priori'[416] in the literature.

There are two options to reject it. The first is to reject analyticity. One reason is that there are not explicit definitions such as for 'bachelors' and 'unmarried men'. The second is that analyticity for common language predicates is too much. We may have a language without analyticity but with normativity of meaning. For example, imagine three speakers. A takes 'optimal number' to be 2 with the sense of 'around 2', B takes 'optimal number' to be 3 with the sense of 'around 3' and C takes it to be 4 with the sense 'around 4'. If what they say does not allow agreement, they all have a duty to vary their understanding of 'optimal number' to an average of the options available, rounded to the nearest odd number if not an integer, as to reach agreement. Then, whenever communicating, they will agree on a common referent, which will be 3. Hence, in this model, communication works, there is a single extension, meaning is normative and each of the speakers means something by the term, i.e. takes some statement as true, but there is no analytic truth.

The second option is to retain analyticity for the connection between *bald*, *tall,* and *heap* and *hairier*, *taller,* and *larger*, but deny the identity between *to be taller than* with *to have less or equal cm of height than*. It is not clear to me that having the meaning of *taller* means having a common method of measuring height, such as along the spine, to output cm. Or that *hairier* presupposes counting the individual hairs. Thus, the skepticism about measurement in 14.3 above is reintroduced, with the effect that *taller* needs a separate semantic definition than *to have less or equal cm of height than*. The effect is that, of course, there will be two men A and B, A taller than B although B has more cm than A. I believe this option is more plausible than the first. Measurement of height in cm along the spine is motivated by scientific reasons, while *taller* is motivated in a larger proportion by appearance. So if B's head generally stands above the ground lower than A's, he may be less tall, although when standardly measured, they have more cm of height.

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[413] For *heap*, the pull is generated by *to be a larger heap,* etc.
[414] Already cited Patrick Greenough, "Vagueness: A Minimal Theory", 240.
[415] Timothy Williamson, *Vagueness*, 13
[416] Forbes, "Fine on Vagueness", 8. A priori – a posteriori is an epistemological distinction famously paralleling the analytic-synthetic semantic distinction

Then we would have:

a) *Tall* is a monadic predicate.

b) *To have equal or more cm of height than* is the preferred ordering, a total preorder under which *tall* is vague, because of dispersion, respecting (Vag.1-5).

c) *To be as tall or taller than* is a different total preorder, namely the determinative total preorder for *tall*, under which *tall* is precise.

Under this construction, *Very tall* and *Little tall* could be defined as a proper subset of *Strictly tall*, and *Strictly short,* respectively, respecting the condition that there is no dispersion for them under the determinative total preorder *to be as tall or taller than.* That is, there is no very tall man less tall than a not very tall men. But this allows *very tall* to have dispersion i.e. be vague under the preferred ordering under which *tall* itself is vague: *to have equal or more cm of height than*. Then, both *tall* and *very tall* are vague under the same comparison relation, while *very tall* implies *strictly tall*, which is an advantage. It avoids the classical objection to fuzzy logic that it results in men who are very tall not being fully tall or that all fully tall people are also very tall under fuzzy logic[417]. In the present construction, being very tall implies being strictly tall.

Is my rejection of predicate-ordering monotony too unintuitive since speakers clearly take it as true? An incoherentist analysis of vagueness was provided by Matti Eklund in his “What Vagueness Consists in”, whose conclusion is:

> “… vagueness of an expression consists in it being part of competence with the expression to be disposed to accept a tolerance principle for it (where, as the later discussion illustrated, we must be careful both about how to construe tolerance and about how to construe the claim about the disposition). Now, this thesis, even if right as far as it goes, does not immediately imply anything about the semantic values of vague expressions. I went on to argue that vague expressions are second-level indeterminate (no assignment of semantic values to vague expressions is uniquely the best) but they are not first-level indeterminate (all acceptable assignments are classical).”[418]

[417] Timothy Williamson, *Vagueness*, 125-126.
[418] Matti Eklund, "What Vagueness Consists in". Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition, Vol. 125, No. 1 (Jul., 2005), 55.

The similarity is the classicality of logical interpretations of vague contexts and the attribution of tolerance principles to semantic (linguistic) competence. But tolerant principles can be weakened on the line of 16.5.3 above, e.g. 'If a man is strictly tall, anyone one rank lower by height is broadly tall'. This weakening meets Eklund's criterion that speakers' disposition to accept the predicate as tolerant "can be overridden, for example when it is learned that tolerance principles can never be satisfied"[419]. Therefore, if we take such weakened principles as true, we can give up the second-level indeterminateness Eklund discusses and settle for FOL as logic of vagueness.

To conclude, FOL can serve as logic for vagueness as long as we accept three theses. Firstly, vagueness applies only to the natural world, with a finite upper bound on the number of elements. Secondly, all vague predicates are multidimensional. Thirdly, a vague predicate comes, because of linguistic reasons, with a preferred ordering under which it is imprecise, yet their interplay create a vague structure of predicates which allows us to express both tolerant and non-tolerant discourse and generates a sorites.

[419] Matti Eklund, "What Vagueness Consists in", 41.

Conclusion

The debate over which logic is better for vagueness has been held mostly on the field of propositional logic. Therein, both plurivaluationism and supervaluationism have had some advantages over classical propositional logic, while still running into philosophical trouble.

I argued that vagueness cannot be approached propositionally. It may be seen as a statistical phenomenon concerning the dispersion of a predicate variable along the rankings of an ordering relation. Bivalence holds and any man is tall or not tall. Yet, there is a clear graduality of assertibility between saying of people of 200 cm that they are tall and saying of people of 170 cm that they are tall. Classical first-order logic allows expressing two things at the same time, namely that a man is tall while he is, by height, among a majority of short people. Vagueness is where positive and negative cases are intermingled. We defined *broadly tall* as the union of *tall* with the dispersion zone i.e. those intermingled tall and short people. And *strictly tall* as their difference. We expressed a non-paradoxical sorites argument, by claiming that for any two people proximal in height, if the first is strictly tall, the second is broadly tall. We saw also a NC-looking clause fail: 'There are people that are broadly tall and broadly short', as well as a LEM-looking one: 'There are people that are neither strictly tall nor strictly short'.

A controversial consequence is that there are tall people of equal or less height than short people. I believe that this is what reality tells us: when becoming ill, chained healthy days are not followed by chained ill days and when one lets their hair grow, days of having short hair are intermingled with days of having long hair, hair arrangement becoming more important than length as measured with a ruler. If we accept that vague predicates such as *tall* are multidimensional, we must accept that their application may depend on something else than measurement of height in cm. For example, kyphosis can make persons of 185 cm short. However, this thesis and its connection with theories of meaning deserve a separate research.

Appendix 1: Formal aspects

1. Sketch of proofs for *Philosophical and logical results (16.5)*

1.1. (NC.1) $\exists_H x . T^{R1}x \wedge\ !T^{R1}x$ 'Some men are both broadly tall and broadly short'

By (Vag.3) there is an initial negative chain of at least 2 ranks and a final positive chain of at least 2 ranks. By (Vag.4), there are at least 2 pairs of elements $<x, y>$ and $<z, t>$ such that the first one (which is ¬T) precedes the last (which is T) in total preorder R. There are two options: *y* or *t* are at the first rank after the initial negative chain or they are not. If they are not, call any T element at the first rank after the initial negative chain *u*. Similarly, there are two options: *x* or *z* are at the last rank before the final positive chain or they are not. If they are not, call any ¬T element at the last rank before the initial negative chain *v*. (Vag.4) guarantees that $x \neq y$, $z \neq t$ (no element succeeds itself in the relation ranks), $x \neq z$ and $u \neq v$ since *u* is T and *v* is ¬T. The two ranks of $<x, y>$ are either before those of $<z, t>$ or after. In the first case, by the definition of `$T^{R1}$` in 16.3.2, y will be T^{R1} since for it exists a T element, namely *y* or *u*, such that for the latter there is no other T element at a preceding rank of R and by the definition of `$!T^{R1}$` in 16.3.2, *y* will be $!T^{R1}$ since for it exists a ¬T element, namely *z*, or *v*, such that for the latter there is no other ¬T element at a succeeding rank of R. So in this case *y* is both T^{R1} and $!T^{R1}$. And similarly for *t* if the ranks of $<x, y>$ are after those of $<z, t>$. So either *y* or *t* are both T^{R1} and $!T^{R1}$ ■

1.2. (NC.2) $\exists_H x . T^{R1}x \wedge \neg Tx$ 'Some men are both broadly tall and short'

In 1.1. above, note that if the ranks of $<x, y>$ are before those of $<z, t>$, *z* is also T^{R1} since for it exists a T element, namely *y* or *u*, such that for the latter there is no other T element at a preceding rank of R, so *z* is both T^{R1} and ¬T. And similarly for *x* if the ranks of $<x, y>$ are after those of $<z, t>$. So either *z* or *x* are both T^{R1} and ¬T ■

1.3. (NC.3) $\exists_H x . T^{Ra}x \wedge \neg Tx$ 'Some men are both arguably tall and short'

Immediate from 1.1, 1.2 and the definition of 'T^{Ra}' in 16.3.3 ■

1.4. (NC.4) $\exists_H$x . T^{Ri}x ∧ ¬Tx ‘Some men are both ideally tall and short’

By (Vag.3) there is an initial negative chain of at least 2 ranks. By (Vag.4), there are at least 2 pairs of elements <*x*, *y*> and <*z*, *t*> such that the first one (which is ¬T) precedes the last (which is T) in total preorder R. By the definition of ‘T^{Ri}’ in 16.3.5, an element *a* is T^{Ri} either by being in the initial negative chain or by both T^{Ra}a and ¬HalfInterval(a,*m*,¬T,T,R) holding. For the first case, the proof is immediate. For the second case, take the proof 1.1. above. Note that if the ranks of <*x*, *y*> are before those of <*z*, *t*>, there exists a T element, namely *y* or *u* (at the first intermediate rank), such that for it there is no other T element at a preceding rank of R. And there exists a ¬T element (at the final intermediate rank), namely *z* or *v*, such that for it there is no other ¬T element at a succeeding rank of R. Also from 1.1, *y*, *u*, *z* and *v* are all distinct and both T^{R1} and !T^{R1}, so T^{Ra} by the definition of ‘T^{Ra}’ in 16.3.3. Since either *z* and *v* are at the last intermediate rank and, by (Vag.4), there are at least two intermediate ranks, either *z* and *v* are not in the first half of intermediate ranks, so either ¬HalfInterval(z,*m*,¬T,T,R) or ¬HalfInterval(v,*m*,¬T,T,R), by the definition of ‘HalfInterval’ in (15.26). So either *z* or *v* fulfil the conditions of (NC.4). And similarly for either *x* or *v* if the ranks of <*x*, *y*> are after those of <*z*, *t*>. So one of *z*, *v*, or *x* are both T^{Ri} and ¬T ■

1.5. (NC.5) $\exists_H$xy (¬Tx ∧ Ty ∧ Rxy ∧ Ryx) ⊃ $\exists_H$x ((T^{Rp}x ∧ ¬Tx) ∨ (!T^{Rp}x ∧ Tx)) ‘If there are tall and short people of the same height, some men are both probably tall and short or some men are both probably short and tall’

From the antecedent, let us call the two people, one short, one tall, at the same rank in the relation *u* (¬T) and *v* (T) and call their rank *r. r* either contains at least ½ T elements or not. If it does, $\mathrm{RankProp50}_{<}$(z,*m*,T,R) as defined in (15.28) holds, because since there are maximum *m* elements, there cannot be more than *m* ranks. No rank can contain more than *m* elements, so we can check in each such rank that if there are *m*, *m*-1, …, 1 elements, at least $\mathrm{Ceil}_{m/2}$,$\mathrm{Ceil}_{(m-1)/2}$, …, $\mathrm{Ceil}_{1/2}$ are T, as defined in (15.27). Since *u* is ¬T, the first disjunct of the consequent follows, under the definition of ‘T^{Rp}’ in 16.3.4. Secondly, if *r* does not contain at least ½ T elements, since *v* is T, the second disjunct of the consequent follows, under the definition of ‘!T^{Rp}’ in 16.3.4. The assumption of the antecedent guarantees the consequent. ■

1.6. (LEM.1) $\exists_H x \,.\, \neg(T^{R0}x \vee\ !T^{R0}x)$ 'Some men are neither strictly tall nor strictly short'

By (Vag.3) there is an initial negative chain of at least 2 ranks and a final positive chain of at least 2 ranks. By (Vag.4), there are two pairs of elements $<x, y>$ and $<z, t>$ such that the first one (which is ¬T) precedes the last (which is T) in total preorder R. The two ranks of $<x, y>$ are either before those of $<z, t>$ or after. In the first case, by the definition of `$T^{R0}$` in 16.3.1, *z* will not be T^{R0} since it is ¬T. And, by the definition of `$!T^{R0}$` in 16.3.1 *z* will not be $!T^{R0}$ since for it, there is an element, namely *y*, such that *y* is T and Ryz. And similarly for *x* if the two ranks of $<x, y>$ are after those of $<z, t>$. So either z or x are neither T^{R0} nor $!T^{R0}$ ■

1.7. (LEM.2) $\exists_H x \,.\, \neg(T^{R0}x \vee \neg Tx)$ 'Some men are neither strictly tall nor short'

In 1.6. above, note that if the ranks of $<x, y>$ are before those of $<z, t>$, *y* is also not T^{R0} by the definition of `$T^{R0}$` in 16.3.1, since for it exists a ¬T element, namely *z*, such that Ryz, so *y* is both $\neg T^{R0}$ and T. And similarly for *t* if the ranks of $<x, y>$ are after those of $<z, t>$. So either *y* or *t* are neither T^{R0} nor ¬T ■

1.8. (WPT) $\forall_H xy \,.\, Prev(y,x,R) \supset \neg(T^{R0}x \wedge \neg T^{R1}y)$ 'If a man is strictly tall, anyone one rank lower by height is broadly tall'

Assume the antecedent. Call the two adjacent ranks of *y* and *x* *r* and *p*. By (Vag.3) there is an initial negative chain of at least 2 ranks and a final positive chain of at least 2 ranks. By (Vag.4), there are at least two intermediate ranks. So *r* and *p* can be either in the initial negative chain, the final positive chain or intermediary. If *r* is in the initial negative chain, *y* cannot be T^{R0}, since all T^{R0} elements need be T, by the definition in 16.3.1, so (WPT) follows. If *r* is intermediate, *y* cannot be T^{R0}, by the definition in 16.3.1, so (WPT) follows. If both *r* and *p* are in the final positive chain, (WPT) follows, both *y* and *x* will be both T^{R0} and T^{R1}. Since there are intermediary ranks and by the definition of 'Prev' in 15.8 *r* and *p* are adjacent, it is impossible for *r* to be in the final positive chain and *p* in the initial negative chain. So the only case left is when *r* is in the final positive chain and *p* is intermediary. Since an element is at an intermediary rank by (15.20) just in case there is a preceding or same-ranked T element such that for it there is no other preceding T element, by the definition of 'T^{R1}' in 16.3.2, *y* is T^{R1}. So in all cases, (WPT) holds ■

1.9. (WPOM) $\forall_H xy$. Next(y,x,R) $\supset \neg(\text{T}x \wedge \neg\text{T}^{R1}x)$ 'If a man is tall, anyone one rank higher by height is broadly tall'

Assume the antecedent. Call the two adjacent ranks of *y* and *x* *r* and *p*. By (Vag.3) there is an initial negative chain of at least 2 ranks and a final positive chain of at least 2 ranks. By (Vag.4), there are at least two intermediate ranks. So *r* and *p* can be either in the initial negative chain, the final positive chain or intermediary. If *p* is in the initial negative chain, *y* cannot be T, so (WPOM) follows. If *p* is in the final positive chain, since *r* is higher than *p*, *r* is also in it, so both *y* and *x* will be both T and T^{R1}, so (WPOM) follows. Let us examine the case when *p* is intermediary. Then since *r* is higher than *p*, *r* cannot be in the initial negative chain. So *r* is in the final positive chain or intermediary. The former case guarantees *y* is T^{R1}, as above. Finally, if both *p* and *r* are intermediary, since an element is at an intermediary rank by (15.20) just in case there is a preceding or same-ranked T element such that for it there is no other preceding T element, by the definition of 'T^{R1}' in 16.3.2, *y* is T^{R1}. So in all cases, (WPOM) holds■

1.10. (UI.1) $\forall_H x$. $\text{T}^{R0}x \supset \text{T}x$ 'If someone is strictly tall, they are tall'

Immediate from the definition of 'T^{R0}' in 16.3.1■

1.11. (UI.2) $\forall_H x$. $\text{T}^{R0}x \supset \text{T}^{Rp}x$ 'Is someone is strictly tall, they are probably tall'

By the definition of 'T^{R0}' in 16.3.1, there is no ¬T element at all ranks of T^{R0} elements. So the proportion of T elements at them is 1. So for all such elements, RankProp50$_<$(z,*m*,T,R) as defined in (15.28) holds, also used in 1.5 above. Under the definition of 'T^{Rp}' in 16.3.4, they are all T^{Rp}■

1.12. (UI.3) $\forall_H x$. $\text{T}^{R0}x \supset \text{T}^{Ri}x$ 'If someone is strictly tall, they are ideally tall'

Immediate from the definition of 'T^{Ri}' in 16.3.5■

1.13. (UI.4) $\forall_H x$. $\text{T}x \supset \text{T}^{R1}x$ 'If someone is tall, they are broadly tall'

Assume the antecedent and call *a* an arbitrary T element. *a* is either in the initial negative chain, intermediary or in the final positive chain. If it is the initial negative chain, (UI.4) holds, by the paradox of material implication. If it is in the final positive chain, a will be both T and T^{R1}, so (UI.4) follows. If it is intermediary, since an element is at an intermediary rank by (15.20) just in case there is a preceding or same-ranked T element such that for it there is no other preceding T element, by the definition of 'T^{R1}' in 16.3.2, *a* is T^{R1}. So in all cases, (UI.4) holds■

1.14. (LEM.T) $\forall_H x \,.\, Tx \vee \neg Tx$ 'A man is either tall or not tall'

Immediate from non-contradiction and DeMorgan■

1.15. (NC.T) $\forall_H x\, \neg(Tx \wedge \neg Tx)$ 'A man is not both tall or not tall'

Immediate from non-contradiction ■

1.16. (NT.1) $\forall_H x \,.\, \neg(T^{R1}x \wedge \neg T^{R1}x) \wedge \forall_H x \,.\, T^{R0}x \vee \neg T^{R0}x)$ 'But surely NC and LEM hold even for tolerant predicates, you cannot be both broadly tall and not broadly tall or not be either strictly tall or not strictly tall'

Immediate from non-contradiction and DeMorgan■

1.17. (NT.2) $\exists_H xy \,.\, \neg T^{Ri}x \wedge T^{Ri}y \wedge Next(y,x,R) \wedge \forall_H zt(\neg T^{Ri}z \wedge T^{Ri}t \wedge Next(t,z,R) \supset Rxz \wedge Rzx \wedge Ryt \wedge Rty\,)$ 'There are two unique heights i.e. ranks that separate ideally short and ideally tall people, i.e. a threshold'

By (Vag.4), there are at least two intermediary ranks. By the definition of 'HalfInterval' in (15.26), the elements *x* at the first half of intermediary satisfy HalfInterval(x,*m*,¬T,T,R) and all other intermediary elements do not. Call *a* an element at the last intermediary rank whose elements *x* satisfy HalfInterval(x,*m*,¬T,T,R) and *b* and element at the first intermediary rank whose elements *y* do not satisfy HalfInterval(y,*m*,¬T,T,R). Then *a* is not T^{Ri} and *b is* T^{Ri}, *b* succeeds *a* under R and any other elements fulfilling both condition will be at the same respective ranks as *a* and *b.* So there are two unique ranks that separate ideally tall and ideally short people ■

2. Example of a function that combines total preorders by weights

We start with a finite domain of objects and the standard definition for predicates and their complements.

$D = \{e_1, e_{2, \dots,} e_n\}$

$Pow_D \stackrel{\text{def}}{=}$ the power set of D.

(Def.1) P is a predicate iff $P \in Pow_D$

We assume the standard definition of wffs such as Pa, Rab, as well as operators and quantifiers ∼, ∧, ∨, ⊃, ↔, ∀, ∃ etc. Especially, ∼Pa iff a ∉ P.

We define a *total preorder* on D:

(Def.2) A binary relation R (or ≤) on D is a *total pre-order* iff:

2.1 R is transitive: ∀xyz . Rxy ∧ Ryz ⊃ Rxz

2.2 R is reflexive: ∀x . Rxx

2.3 R has the connex property: ∀xy . Rxy ∨ Ryx

For any total preorder R on D, there is[420] a *unique dense ranking function r for R,* that we'll call $r_{R,}$ such that:

2.4. $r : D \to \mathbb{N}^*$

2.5. r(x) respects the ordering of R: ∀xyz . Rxy ⊃ $r(y) \geq r(x)$

2.6. r(x) respects the coverings of R: ∀xy. Rxy ∧ ∄z (Rxz ∧ Rzy) ⊃ $r(x) = r(y)-1$

2.7. the minimal elements of D under R are assigned 1

We define a *coefficient function for a set of relations on a domain* thus:

[420] See (15.9) above

(Def.3) f is a coefficient function for $\{G_1, ..., G_n\}$ on D iff:

2.8 $f : \{G_1...G_n\} \rightarrow \mathbb{Q}$

2.9 $\forall x \,.\, f(x) \geq 0$

2.10 $G_1, ..., G_n$ total preorders on D

We define a *determination function* thus:

(Def.4) g is a determination function iff:

2.11 $g: \{f_1, ..., f_n\} \rightarrow D \times D$

2.12 Every $f \in$ Domain(g) is a coefficient function, i.e. its domain is a set of total preorders on D and its range consists of positive rational numbers

2.13. g selects its result for x from D × D according to the following rules:

2.13.1 If there is a single G ∈ Domain(x), G = g(x)*;*

2.13.2 If not, determine the future maximal rank:

Let m be 0;

For all $G_1, ..., G_n \in$ Domain(x), called individually G, determine the maximal rank of D under G, times f(G).

That is, let q_G = maximal rank of D * f(G);

m becomes the average of these q.

That is, m = $\sum q_G$ for G ∈ Domain(x) / |Domain(x)|.

At the end, change m to the smallest natural number n such as $n \geq m$;

2.13.3 Determines the individual unweighted rank and the maximum such rank

For all $e_1, e_{2, \ldots ,} e_n \in D$, called individually e here, define a single function that links them with the sum of the products of their rank under each $G_1, ..., G_n$ called individually G, times f(G)

That is, for all $G_1, ..., G_n$ called individually G:

Let $p_{eG} = r_G(e) * f(G)$

And the single function is h*:* D → $\mathbb{Q}$, $h(y) = \sum p_{yG}$ for each G ∈ Domain(x).

Let t be the greatest such h(y);

2.13.4. Determines the final individual rank

Define another function i*:* D → $\mathbb{N}$, i(y) = n ∈ $\mathbb{N}$, n ≥ 1, the smallest number such that n ≥ h(y) * (m / t)

2.13.5. Define a relation R such that for all n ∈ $\mathbb{N}$, m ≥ n, n ≥ 0:

∀zy. z, y ∈ D ∧ i(z) ≥ i(y) ⊃ <*y, z*> ∈ R

∀zy. z, y ∈ D ∧ i(y) ≥ i(z) ⊃ <*z, y*> ∈ R

This R = g(x).

We define a *determination relation for a function* (what we were looking for to) as:

(Def.5) R is a determination relation for function f iff f is a coefficient function and there is a determination function g such that $g(f)$ = R.

3. The notational extension of FOL

To the standard construction of FOL, we add the rules:

3.1. A *vagueness dictionary*, a set Ξ of quadruples of the form <*C*, *n*, *P*, *S*> where C and P are monadic predicate letters, *n* is a positive natural number and S is a dyadic predicate letter. It encodes which finitely bounded relations vaguify which predicates. Any member of the vague dictionary is called a *vague quadruple.* All monadic predicates in the third position are called *vague predicates*.

3.2. If <*C*, *n*, *P, S*> a vague quadruple, the following are monadic predicate sequences: \`$P^{S0}$\`, \`$P^{S1}$\`, \`$P^{Sa}$\`, \`$P^{Sp}$\`, \`$P^{Si}$\`, \`!$P^{S0}$\`, \`!$P^{S1}$\`, \`!$P^{Sa}$\`, \`!$P^{Sp}$\`, \`!$P^{Si}$\`

3.3. If <*C*, *n*, *P, S*> a vague quadruple, the following are dyadic predicate sequences: \`$P^{SV}$\`

3.4. Monadic predicate sequences act as monadic predicate letters under the well-formation rules. Dyadic predicate sequences act as dyadic predicate letters under the well-formation rules.

3.5. For each vague quadruple <*C*, *n*, *P, S*>, add as axioms:

a) (Vag.1-5) with 'H' replaced by 'C', 'm' replaced by 'n', 'T' replaced by 'P', and 'R' replaced by 'S'. That is, C has *n* as finite upper bound, S is a total preorder on C, there is an initial negative and a final positive chain, there is dispersion and at least ½ of the elements of P are in the safe zone of S and P.

b) The statements of: 16.3.1, 16.3.2, 16.3.3, 16.3.4, 16.3.5, and (17.1) with 'H' replaced by 'C', 'm' replaced by 'n', 'T' replaced by 'P', and 'R' replaced by 'S'. That is, the monadic and dyadic predicate sequences can be used and introduced by equivalence according to their definitions.

Appendix 2: Logical notation

This work uses a standard logical notation for FOL with identity, modified as to optimize readability.

1. The syntax[421]

1.1. Symbols

a) Six operators (negation, conjunction, disjunction, material implication, material equivalence, identity): ¬, ∧, ∨, ⊃, ↔, =;

b) Two quantifiers (universal and existential): ∀, ∃;

c) Two brackets (opening and closing): (,);

d) Infinite propositional letters: p_0, p_1, ... , usually represented by A, B, C, ... including subscripted C_1, C_2, etc.;

e) Infinite predicate letters of infinite arity: P_1^0, P_1^1 , P_1^2, ..., P_2^0, P_2^1 , P_2^2, ..., where superscripting indicates the arity and subscripting indicates the position of the predicate in the dictionary for that arity. They are usually represented by P, T, etc for predicates of arity 1 and by R, S, Z, etc for predicates of arity 2. The latter are often called *relations*;

f) Variables: v_1, v_2, ..., usually represented by x, y, z, t, v, u ..., including subscripted v_1, v_2, v_3, etc.;

g) Constants: k_1, k_2, ..., usually represented by a, b, c, ... ;

h) Variables and constants are collectively called *terms*;

1.2. Formation rules

a) If A a propositional letter, *A* is an atomic formula;

b) If τ_1, ..., τ_n are terms and Π is a predicate of *n* arity, $\Pi\tau_1 \ldots \tau_n$ is an atomic formula;

c) If τ_1 and τ_2 are terms, $\tau_1 = \tau_n$ is an atomic formula;

[421] I follow generally Graham Priest, *An Introduction to Non-Classical Logic: From If to Is*, 4-5 and 263-264.

d) Atomic formulas are well-formed formulas;

e) If φ and ψ are well-formed formulas, so are $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \supset \psi)$, $(\varphi \leftrightarrow \psi)$;

f) If φ is a well-formed formula, $\nu_1, \ldots, \nu_n$, with $n \geq 1$ are variables present in φ and there is no other variable in φ besides $\nu_1, \ldots, \nu_n$, then $\Xi\nu_1 \ldots \Theta\nu_n\varphi$ is a well-formed formula, with Ξ and Θ indicating any of the quantifiers in any combination[422];

g) Nothing is a well-formed formula except if recursively so by a-f) above.

2. Readability modifications

2.1. Elimination of brackets for ↔ and ⊃

a) If a formula (quantified or non-quantified) contains two or more distinct operators except ¬ and = and its main operator is ↔, then each of the two subformulas connected by the main operator omits outermost brackets if its main operator is not ↔:

$(\varphi_1 * \varphi_2 \leftrightarrow \psi) \stackrel{\text{def}}{=} ((\varphi_1 * \varphi_2) \leftrightarrow \psi)$, with * one of $\wedge$, $\vee$, and $\supset$

$(\psi \leftrightarrow \varphi_1 * \varphi_2) \stackrel{\text{def}}{=} (\psi \leftrightarrow (\varphi_1 * \varphi_2))$, with * one of $\wedge$, $\vee$, and $\supset$

b) If a formula (quantified or non-quantified) contains two or more distinct operators except ¬ and = and its main operator is ⊃, then each of the two subformulas connected by the main operator omits outermost brackets if its main operator is not ↔ or ⊃:

$(\varphi_1 * \varphi_2 \supset \psi) \stackrel{\text{def}}{=} ((\varphi_1 * \varphi_2) \supset \psi)$, with * one of $\wedge$, or $\vee$

$(\psi \supset \varphi_1 * \varphi_2) \stackrel{\text{def}}{=} (\psi \supset (\varphi_1 * \varphi_2))$, with * one of $\wedge$, or $\vee$

2.2. Elimination of inner brackets for ∧, ∨, and ↔

$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3) \stackrel{\text{def}}{=} (\varphi_1 \wedge (\varphi_2 \wedge \varphi_3))$

$(\varphi_1 \vee \varphi_2 \vee \varphi_3) \stackrel{\text{def}}{=} (\varphi_1 \vee (\varphi_2 \vee \varphi_3))$

$(\varphi_1 \leftrightarrow \varphi_2 \leftrightarrow \varphi_3) \stackrel{\text{def}}{=} (\varphi_1 \leftrightarrow (\varphi_2 \leftrightarrow \varphi_3))$

[422] The rule aims to only accept what are usually called *closed* formulas as well-formed formulas, while accepting all standard combinations of quantifiers, such as $\forall x\, \phi$, $\forall x \exists y\, \phi$, $\exists x \forall y\, \phi$, $\forall x \forall y\, \phi$ etc.

2.3. Outermost brackets are invisible

$\varphi \stackrel{\text{def}}{=} (\varphi)$, only if *(φ)* is not a sub-formula

2.4. Special symbols for the negation of identity and of the existential quantifier

$\nu \neq \xi \stackrel{\text{def}}{=} \neg\nu = \xi$

$\nexists\nu\ \varphi \stackrel{\text{def}}{=} \neg\exists\nu\ \varphi$

2.5. Uniformly quantified formulas keep only the first quantifier

$\forall\nu_1 \ldots \nu_n\, \varphi \stackrel{\text{def}}{=} \forall\nu_1 \ldots \forall\nu_n\ \varphi$; with $n \geq 2$

$\exists\nu_1 \ldots \nu_n\ \varphi \stackrel{\text{def}}{=} \exists\nu_1 \ldots \exists\nu_n\ \varphi$; with $n \geq 2$

$\nexists\nu_1 \ldots \nu_n\ \varphi \stackrel{\text{def}}{=} \nexists\nu_1 \ldots \nexists\nu_n\ \varphi$; with $n \geq 2$

2.6. Main quantifiers replace their bracket, if any, with a dot

$\forall\nu_1 \ldots \nu_n .\ \varphi \stackrel{\text{def}}{=} \forall\nu_1 \ldots \nu_n(\varphi)$; with $n \geq 1$, only when there is a main operator of φ, it is not ¬ or = and ∀ is the main quantifier

$\exists\nu_1 \ldots \nu_n .\ \varphi \stackrel{\text{def}}{=} \exists\nu_1 \ldots \nu_n(\varphi)$; with $n \geq 1$, only when there is a main operator of φ, it is not ¬ or = and ∃ is the main quantifier

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