Fine on the Possibility of Vagueness*

Andreas Ditter

Penultimate Draft

Forthcoming in *Outstanding Contributions to Logic: Kit Fine*

Federico L. G. Faroldi and Frederik Van de Putte (eds), Springer

June 17, 2020

1 Introduction

In his paper 'The possibility of vagueness' (Fine, 2017), Kit Fine proposes a new logic of vagueness, **CL**, that promises to provide both a solution to the sorites paradox and a way to avoid the impossibility result from Fine (2008). The present chapter presents a challenge to his new theory of vagueness. I argue that the possibility theorem stated in Fine (2017), as well as his solution to the sorites paradox, fail in certain reasonable extensions of the language of **CL**. More specifically, I show that if we extend the language with any negation operator that obeys *reductio ad absurdum*, we can prove a new impossibility result that makes the kind of indeterminacy that Fine takes to be a hallmark of vagueness impossible. I show that such negation operators can be conservatively added to **CL** and examine some of the philosophical consequences of this result. Moreover, I demonstrate that we can define a particular negation operator that behaves exactly like intuitionistic negation in a natural propositionally quantified extension of **CL**. Since intuitionistic negation obeys *reductio*, the new impossibility result holds in this propositionally quantified extension of **CL**. In addition, the sorites paradox resurfaces for the new negation.

The chapter will be structured as follows. Section 2 introduces some of the main features of Fine's new theory of vagueness and states the new impossibility result for certain extensions of the language. Section 3 demonstrates a key conservative extension result, considers a propositionally quantified extension of **CL** in which we can define a negation operator that behaves exactly like intuitionistic negation in this extension, and explores the ramifications

*Thanks to Hartry Field, Harvey Lederman, Marko Malink, Stephen Schiffer, as well as two anonymous referees and the editors of this volume for their helpful suggestions and comments on an earlier draft. Special thanks to Kit Fine and Cian Dorr for many helpful discussions of this material and comments on earlier versions of this chapter.

of these results for Fine's theory of vagueness. Two appendices provide proofs of the central technical results appealed to in the main text.

2 Fine's new logic of vagueness

The key feature of Fine's novel approach to vagueness is its emphasis on what he calls the *global* character of vagueness. Here is the basic idea. Suppose we are given a sorites series $b_1, ..., b_{100}$ for the predicate 'bald' that begins with a completely bald man b_1 and ends with a very hairy man b_{100}. According to Fine, we may then correctly say that the predicate 'bald' is indeterminate in its application to the members of the series. The indeterminacy is said to be *global* rather than *local* since it is characterized with respect to a range of cases as opposed to a single case. A paradigm example of local indeterminacy is the notion of a borderline case, which is characterized in terms of the application of a predicate to a *single* member of the series. Fine takes one of the central morals of the impossibility result from Fine (2008) to be that we cannot understand global indeterminacy in terms of local indeterminacy. But he leaves open whether it might be possible to understand local in terms of global indeterminacy. Fine's new theory of vagueness is *globalist* in the sense that it asserts the primacy of global over local indeterminacy.

Given the primacy of global indeterminacy, we are faced with the question of how it is to be understood. Another key feature of Fine's new theory is that global indeterminacy can be understood by logical means alone. That is to say that we can express the global indeterminacy of a predicate with respect to a range of cases by means of our logical vocabulary alone—there is no need for primitive determinacy operators or the like. The basic idea is as follows. Consider again our sorites series from above and let p_k be the proposition that b_k is bald, for each $1 \leq k \leq 100$. A *state description in* the propositions $p_1, ..., p_{100}$ is a conjunction $q_1 \wedge ... \wedge q_{100}$, where each of the propositions q_k is either the proposition p_k or its negation $\neg p_k$. Let $s_1, ..., s_m$ be the state descriptions in the propositions $p_1, ..., p_{100}$. We can now express the global indeterminacy of 'bald' with respect to the series by denying *each* of the state descriptions in the propositions $p_1, ..., p_{100}$, i.e. $\neg s_1 \wedge \neg s_2 \wedge ... \wedge \neg s_m$. According to Fine (2017), this is the "hallmark of indeterminacy". In order to be able to consistently express such global indeterminacy claims, Fine develops a new logic—a logic that radically departs from the logic of vagueness he originally championed in Fine (1975).

I will now introduce some of the main characteristics of Fine's 'compatibilist' logic (**CL**). I begin with the semantics. A compatibilist model $\mathcal{M}_{\mathbf{CL}}$ is a triple $\langle U, \circ, V \rangle$ where

C(i) U is a non-empty set;

C(ii) $\circ$ is a reflexive and symmetric relation on U;

C(iii) V is a function from the set of propositional variables PL into $\mathcal{P}(U)$.[1]

Given a **CL**-model $\mathcal{M}_{\mathbf{CL}} = \langle U, \circ, V\rangle$ and a point $u \in U$, the notion of a formula $\phi \in \mathcal{L}_{\mathbf{CL}}$ being *true in* $\mathcal{M}_{\mathbf{CL}}$ *at point* $u \in U$ is recursively defined as follows:

T(i) $\mathcal{M}_{\mathbf{CL}}, u \vDash p$ iff $u \in V(p)$

T(ii) $\mathcal{M}_{\mathbf{CL}}, u \vDash \phi \wedge \psi$ iff $\mathcal{M}_{\mathbf{CL}}, u \vDash \phi$ and $\mathcal{M}_{\mathbf{CL}}, u \vDash \psi$

T(iii) $\mathcal{M}_{\mathbf{CL}}, u \vDash \phi \vee \psi$ iff $\mathcal{M}_{\mathbf{CL}}, u \vDash \phi$ or $\mathcal{M}_{\mathbf{CL}}, u \vDash \psi$

T(iv) $\mathcal{M}_{\mathbf{CL}}, u \vDash \neg\phi$ iff for all v for which $u \circ v$, $\mathcal{M}_{\mathbf{CL}}, v \nvDash \phi$

T(v) $\mathcal{M}_{\mathbf{CL}}, u \vDash \phi \supset \psi$ iff either (a) $\mathcal{M}_{\mathbf{CL}}, u \vDash \phi$ and $\mathcal{M}_{\mathbf{CL}}, u \vDash \psi$ or (b) for all v, if $u \circ v$ and $\mathcal{M}_{\mathbf{CL}}, v \vDash \phi$, then $\mathcal{M}_{\mathbf{CL}}, v \vDash \psi$

T(vi) never $\mathcal{M}_{\mathbf{CL}}, u \vDash \bot$.

Informally, Fine interprets the points $u \in U$ in a model as representing uses of the language. The accessibility relation $\circ$ between points can then be informally interpreted as a relation of *compatibility* between uses. The clause for negation then says that $\neg\phi$ is true under a use u if there is no use compatible with u under which ϕ is true. Thus, both a sentence and its negation may fail to be true under a use, which corresponds to the idea that a given use of the language may leave open whether a (vague) sentence or its negation is true. The clause for negation entails that two uses will be incompatible if a given sentence is true under one while its negation is true under the other. So, for example, a use under which the sentence 'b is bald' is true is incompatible with a use under which the negation of 'b is bald' is true, but may be compatible with a use under which neither the sentence nor its negation is true. There is of course more to be said about this interpretation of the semantics, but the informal interpretation of the semantics won't matter for our discussion. For present purposes, a purely formal understanding of the semantics will do.[2]

The notion of a formula ϕ being a ***CL****-consequence of a set of formulas* Γ—in symbols, $\Gamma \vDash_{\mathbf{CL}} \phi$—is defined in a standard way: $\Gamma \vDash_{\mathbf{CL}} \phi$ iff for all **CL**-models $\mathcal{M}_{\mathbf{CL}}$ and for all points u in $\mathcal{M}_{\mathbf{CL}}$, if $\mathcal{M}_{\mathbf{CL}}, u \vDash \psi$ for all $\psi \in \Gamma$, then $\mathcal{M}_{\mathbf{CL}}, u \vDash \phi$. A formula ϕ is *valid*—in symbols, $\vDash_{\mathbf{CL}} \phi$—iff ϕ is a consequence of the empty set of formulas. The validity of ϕ is thus tantamount to ϕ being true at every point in every **CL**-model. A set of formulas Γ is *satisfiable* iff for some **CL**-model $\mathcal{M}_{\mathbf{CL}}$ and point u in $\mathcal{M}_{\mathbf{CL}}$, $\mathcal{M}_{\mathbf{CL}}, u \vDash \psi$ for all $\psi \in \Gamma$; Γ is *consistent with* the formula ϕ iff $\Gamma \cup \{\phi\}$ is satisfiable.[3]

[1] Compatibilist models are thus identical to models for the modal logic **B**. This fact will be exploited in the next section.

[2] The reader should consult Fine (2017) for more on the informal interpretation of the semantics.

[3] An axiomatic presentation of the logic is given in Appendix 2. Fine (2017) notes without proof that this axiom system is weakly complete for the compatibilist semantics. In unpublished work, Fine proves strong soundness and completeness. The model-theoretic notion of consistency defined above thus coincides with its syntactic counterpart.

CL has some interesting properties, some of which are immediately relevant to its application as a logic of vagueness. The first feature worth highlighting is the failure of *conjunctive syllogism* in **CL**. Conjunctive syllogism is the rule that from premises ϕ and $\neg(\phi \wedge \psi)$ one may infer $\neg\psi$. Figure 1 depicts a simple **CL**-model in which it fails:

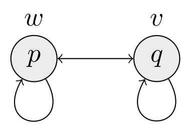

Figure 1

Here is why: p is true at w and $\neg(p \wedge q)$ is true at w, because there is no u accessible from w at which both p and q are true. However, it is not the case that $\neg q$ is true at w, since there is a point accessible from w at which q holds.

The invalidity of conjunctive syllogism is of great significance in the context of a logic of vagueness because it allows one to avoid the sorites paradox under its propositional (i.e. non-quantificational) formulation where the second premise of each transitional inference of the form $\neg p_k, \neg(p_{k-1} \wedge \neg p_k) \vdash \neg p_{k-1}$ is the negation of a conjunction, *viz.* $\neg(p_{k-1} \wedge \neg p_k)$. Call this the *conjunctive version of the sorites.* On this way of formulating the second premise of each transitional inference in a piece of soritical reasoning, the transitional inference from $\neg p_k$ and $\neg(p_{k-1} \wedge \neg p_k)$ to $\neg p_{k-1}$ can be resisted in **CL**. Fine (2017) offers an extended discussion of why this formulation of the second premises in a sorites argument results in a more plausible version of this premise than formulations on which the premise is a conditional $\neg p_k \supset \neg p_{k-1}$ or a disjunction $\neg p_{k-1} \vee p_k$. For the purpose of the present discussion, I will grant that Fine is right about this. The transitional steps in the latter two formulations involve, respectively, applications of modus ponens and disjunctive syllogism; both of these rules are valid in **CL**. Fine's solution to the sorites paradox then consists in showing that the only version of the sorites on which its premises are plausible is the conjunctive version, and that on this version the argument is invalid, due to the failure of conjunctive syllogism.

The invalidity of conjunctive syllogism is intimately connected with the failure of *reductio ad absurdum* in **CL**, the rule that allows us to infer $\Gamma \vdash \neg\phi$ whenever $\Gamma, \phi \vdash \bot$. For if reductio were validity preserving, we could infer the validity of conjunctive syllogism from it. Reductio does in fact hold without side-premises, in which case it is the rule that from $\phi \vdash \bot$ one may infer $\vdash \neg\phi$. Moreover, it is easy to check that the *law of excluded middle* fails. Take the model depicted in Figure 1. In this model, q is not true at w; but neither is $\neg q$, since q is true at v, which is accessible from w. Hence, the disjunction $q \vee \neg q$ is not true at w. Similarly, the rule of *double negation elimination* fails in **CL**. For consider again the model in Figure 1. In this model, $\neg\neg q$ is true at w, because there is no point accessible from w at which $\neg q$ is true; but q is still not true at w.

Most distinctively, perhaps, sentences of the form $\neg((p \vee \neg p) \wedge (q \vee \neg q))$ are satisfiable in **CL**. This is true for negations of longer conjunctions of this form as well, though not for the case of the negation of a single instance of excluded middle.[4] A model in which the above negation holds is depicted in Figure 2. The satisfiability of this type of sentence is crucial for the proof of the possibility theorem in Fine (2017).

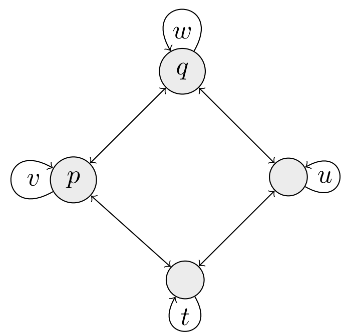

Figure 2

To see this, consider a sorites series $b_1, ..., b_n$ for some predicate and let p_k be the proposition that the predicate applies to b_k. Recall that, according to Fine (2017), the "hallmark of indeterminacy" consists in the denial of *each* of the state descriptions in the propositions $p_1, ..., p_n$. In **CL**, the conjunction of the negations of each state description, i.e. $\neg S_1 \wedge \neg S_2 \wedge ... \wedge \neg S_m$, where S_k represents a state description, is equivalent to $\neg((p_1 \vee \neg p_1) \wedge (p_2 \vee \neg p_2) \wedge ... \wedge (p_n \vee \neg p_n))$. Following Fine (2017) we can thus understand an *indeterminacy claim* to be a proposition of the following form:

$$I(p_1, p_2, \ldots, p_n) := \neg \bigwedge_{1 \leq i \leq n} (p_i \vee \neg p_i)$$

A slightly modified version of I turns out to be crucial to Fine's Possibility Theorem stated below:

$$I^{'}(p_1, p_2, \ldots, p_n) := \neg \bigwedge_{1 \leq i \leq n} (\neg p_i \vee \neg\neg p_i)$$

The intuitive idea behind the Possibility Theorem is as follows. Suppose we are faced with a series $b_1, ..., b_n$ for which a given predicate is indeterminate. A global claim of indeterminacy should then satisfy two conditions: (i) it should be consistent with the *extremal response* $p_1 \wedge \neg p_n$ to the series; (ii) it should be inconsistent with every *sharp response* to the series where, intuitively speaking, a response is sharp if it involves a 'sharp cut-off'. An example of

[4] To see why $\neg(p \vee \neg p)$ is not satisfiable, suppose for a contradiction that $\neg(p \vee \neg p)$ is true at some w. Now either p is true at some v accessible from w, in which case $p \vee \neg p$ is true at v and hence $\neg(p \vee \neg p)$ is not true at w; or p is not true at any such v, in which case $\neg p$ is true at w, and so $\neg(p \vee \neg p)$ is not true at w, since $p \vee \neg p$ is then true at w and w is accessible from itself.

a sharp response would be $p_1, ..., p_k, \neg p_{k+1}, ..., \neg p_n$. According to Fine, vagueness is possible only if there is a proposition that satisfies both of these requirements. His Possibility Theorem shows that there is in fact a proposition expressible in the language of **CL** that satisfies these requirements, namely $I^{'}(p_1, p_2, \ldots, p_n)$.

In order to state the Possibility Theorem, we introduce some terminology from Fine (2017).[5] A *collective response* is a sequence $\phi_1(p), \phi_2(p), ..., \phi_n(p)$ of formulas constructed from the sentence letter p; and $\phi_1, \phi_2, ..., \phi_n$ is said to be a *collective response to* $\psi_1, \psi_2, ..., \psi_n$ if $\phi_1, \phi_2, ..., \phi_n$ are respectively of the form $\phi_1(\psi_1), \phi_2(\psi_2), ..., \phi_n(\psi_n)$, where $\phi_1(p), \phi_2(p), ..., \phi_n(p)$ is a collective response. We say that the collective response $\phi_1, \phi_2, ..., \phi_n$ is *sharp* if:

(i) $\phi_i \neq \phi_j$ for some $i, j \leq n$;

(ii) ϕ_i is inconsistent with ϕ_j or $\phi_i = \phi_j$ whenever $1 \leq i < j \leq n$.

We may similarly talk of a *sharp* response *to* $\psi_1, \psi_2, ..., \psi_n$.[6] As Fine puts it, 'In a sharp response, we give any two questions the same answer or inconsistent answers, with at least two of the answers not being the same' (and similarly in regard to a sharp response to $\psi_1, \psi_2, ..., \psi_n$) (Fine, 2008, p. 129). We call $\{\psi_1, \neg\psi_n\}$ the *extremal response* to $\psi_1, \psi_2, ..., \psi_n$. The Possibility Theorem can now be stated as follows:

> **Possibility Theorem** (Fine, 2017)
>
> For $n \geq 2$, $I^{'}(p_1, p_2, \ldots, p_{n+1})$ is consistent with $\{p_1, \neg p_{n+1}\}$ and inconsistent with every sharp response to $p_1, p_2, \ldots, p_{n+1}$.

Notice that the Possibility Theorem is about sentences in the language of **CL**. Hence, the consistency and inconsistency referred to in the theorem concern sentences of $\mathcal{L}_{\mathbf{CL}}$. The significance of the result crucially depends on its robustness under plausible extensions of the language. It would therefore be important to know whether there are any reasonable extensions of the language in which the Possibility Theorem fails. It turns out that the Possibility Theorem fails in any extension of **CL** that contains a negation operator that obeys at least *reductio ad absurdum.* More precisely, let $\mathcal{L}_{\mathbf{CL}+}$ be the language of **CL** supplemented with a negation operator $\sim$ and let **CL+** be the logic that results from **CL** by adding the rule of reductio ad absurdum for $\sim$:[7]

(RAA) If $\Gamma, \phi \vdash_{\mathbf{CL}+} \psi$ and $\Gamma, \phi \vdash_{\mathbf{CL}+} \sim\psi$ for some ψ, then $\Gamma \vdash_{\mathbf{CL}+} \sim\phi$

[5] The terminology as well as the statement of the Possibility Theorem are adopted almost *verbatim* from Fine (2017); I have only replaced 'incompatible' by 'inconsistent'. The change is purely terminological and is made in the interest of terminological uniformity.

[6] Note that a collective response to $\psi_1, \psi_2, ..., \psi_n$ of the form $\phi_1(\psi_1), \phi_2(\psi_2), ...\phi_n(\psi_n)$ is sharp if the corresponding collective response $\phi_1(p), \phi_2(p), ..., \phi_n(p)$ is sharp.

[7] A precise definition of the logic **CL+** is given in Appendix A1.

We can then show the following impossibility result, the proof of which is given in Appendix 1:

> **Impossibility Theorem**
>
> Let $\phi_0, \phi_1, ..., \phi_{n+1}$, $n \geq 0$, be any formulas of $\mathcal{L}_{\mathbf{CL}+}$. Then there is no set of formulas Δ_0 which is **CL**+-consistent with the extremal response $\{\phi_0, \neg\phi_{n+1}\}$ and yet **CL**+-inconsistent with every sharp response to $\phi_0, \phi_1, ..., \phi_{n+1}$.

In view of this result, it becomes a pressing question for Fine whether any extension of the language of **CL** in which the Impossibility Theorem holds is legitimate. In the next section, I will argue that at least some such extensions are in fact legitimate and that this poses a problem for Fine's theory.

3 A challenge

I will first show that we can conservatively extend **CL** by any negation operator that satisfies a subset of the axioms and rules of classical negation. I will then argue that this conservative extension result provides us with a reason for thinking that at least some such extensions are legitimate. Since (RAA) is among the rules of classical negation, the Impossibility Theorem holds in some conservative extensions of **CL**. Moreover, since conjunctive syllogism is derivable from (RAA), the sorites paradox would appear to resurface for the new negation.

We first show that **CL** is embeddable into the modal logic **B**. We will assume the set of propositional variables of $\mathcal{L}_{\mathbf{B}}$ to coincide with the set of propositional variables of $\mathcal{L}_{\mathbf{CL}}$. The modal logic **B** is the normal modal logic that results from the modal logic **K** by the addition of the T-axiom $\Box\phi \to \phi$ and the B-axiom $\phi \to \Box\Diamond\phi$, and which is sound and complete with respect to the class of reflexive and symmetric Kripke frames. The notion of a **B**-model is defined in the exact same way as that of a **CL**-model, so every **B**-model is a **CL**-model and *vice versa*. The relation of a formula ϕ being true at a point u in a **B**-model $\mathcal{M}_{\mathbf{B}}$ is defined in the standard way.

The translation below will pave the way for our conservative extension result. Furthermore, it will provide a convenient way of understanding **CL** in terms of a logical framework that is more familiar to most philosophers and logicians.

We will show that $\Gamma \vDash_{\mathbf{CL}} \phi$ just in case $\Gamma^* \vDash_{\mathbf{B}} \phi^*$, with $\Gamma^* = \{\phi^* : \phi \in \Gamma\}$.[8] The translation is defined as follows:[9]

(i) $\bot^* = \bot$

[8] The notion of a formula ϕ being a (local) ***B**-consequence of a set of formulas* Γ - in symbols, $\Gamma \vDash_{\mathbf{B}} \phi$ - is defined in the standard way: $\Gamma \vDash_{\mathbf{B}} \phi$ iff for all **B**-models $\mathcal{M}_{\mathbf{B}}$ and for all points u in $\mathcal{M}_{\mathbf{B}}$, if $\mathcal{M}_{\mathbf{B}}, u \vDash \psi$ for all $\psi \in \Gamma$, then $\mathcal{M}_{\mathbf{B}}, u \vDash \phi$.

[9] Here, $\sim$ and $\to$ denote classical negation and the material conditional, respectively.

(ii) $p^* = p$, if p is a propositional variable

(iii) $(\phi \wedge \psi)^* = \phi^* \wedge \psi^*$

(iv) $(\phi \vee \psi)^* = \phi^* \vee \psi^*$

(v) $(\neg\phi)^* = \Box{\sim}\phi^*$

(vi) $(\phi \supset \psi)^* = (\phi^* \wedge \psi^*) \vee \Box(\phi^* \rightarrow \psi^*)$

We first prove the following

Lemma 1. *Suppose that $\mathcal{M} = \langle U, \circ, V\rangle$ is a **CL**- (or, equivalently, **B**-) model. Then for all $\phi \in \mathcal{L}_{\mathbf{CL}}$, for all $w \in U : \mathcal{M}, w \vDash_{\mathbf{CL}} \phi$ iff $\mathcal{M}, w \vDash_{\boldsymbol{B}} \phi^*$.*

Proof. By a straightforward induction on the complexity of ϕ. □

Using the preceding lemma, we can now state and prove the central

Theorem 1. *For any $\Gamma \subseteq \mathcal{L}_{\mathbf{CL}}, \phi \in \mathcal{L}_{\mathbf{CL}} : \Gamma \vDash_{\mathbf{CL}} \phi$ iff $\Gamma^* \vDash_{\boldsymbol{B}} \phi^*$.*

Proof. $\Leftarrow$: By contraposition. Suppose $\Gamma \nvDash_{\mathbf{CL}} \phi$. Then there is a **CL**-model $\mathcal{M}$ and $w \in U_{\mathcal{M}}$ such that $\mathcal{M}, w \vDash_{\mathbf{CL}} \Gamma$ and $\mathcal{M}, w \nvDash_{\mathbf{CL}} \phi$. Since every **CL**-model is a **B**-model, there is a **B**-model, namely $\mathcal{M}$, such that by Lemma 1, $\mathcal{M}, w \vDash_{\mathbf{B}} \Gamma^*$ and $\mathcal{M}, w \nvDash_{\mathbf{B}} \phi^*$. Thus, $\Gamma^* \nvDash_{\mathbf{B}} \phi^*$. The $\Rightarrow$- direction is proved analogously. □

As an immediate consequence of Theorem 1 we get[10]

Corollary 1. *Any unary operator that obeys a subset of the classically valid axioms and rules for negation can be conservatively added to* **CL**.

Note that if we add classical negation to **CL**, the logic that results is a notational variant of **B** in which $\Box$ becomes definable in terms of $\neg$ and $\sim$: $\Box\phi =_{df} \neg{\sim}\phi$. It is worth pointing out that Corollary 1 implies that compatibilist negation is not *uniquely characterized*, because we could add another negation operator $\neg'$ obeying exactly the same principles as $\neg$ to **CL**,

[10]One is naturally reminded here of modal translations of intuitionistic logic into the modal logic **S4**. There is a result similar to Theorem 1 for these logics (see, for example, Chagrov and Zakharyaschev (1997, p.97)). It is worth noting, however, that the result for intuitionistic logic and **S4** does not imply that classical negation can be conservatively added to intuitionistic logic. Indeed, adding classical negation to intuitionistic logic has the effect that intuitionistic negation becomes intersubstitutable with classical negation, thereby making rules like double negation elimination valid for intuitionistic negation. Let $\neg^c$ be a classical negation operator governed by the usual classical rules of inference. Let $\mathcal{L}_{\neg^c}$ be the language of intuitionistic logic supplemented with $\neg^c$. We can then prove in $\mathcal{L}_{\neg^c}$ that for every formula ϕ, $\neg^c\phi \dashv\vdash \neg^i\phi$. *Proof*: By *ex falso quodlibet* for $\neg^c$, we have $\neg^c\phi, \phi \vdash \phi$ and $\neg^c\phi, \phi \vdash \neg^i\phi$. So by *reductio ad absurdum* for $\neg^i$ we get $\neg^c\phi \vdash \neg^i\phi$. The other direction is proved analogously. The fact that the logic licences the substitution of logical equivalents then yields the stronger conclusion that the two copies of negation are intersubstitutable in every sentential context. See Humberstone (1979), Humberstone (2011, p. 591ff.) and Schechter (2011) for discussion of the question under what conditions we can have logics containing both classical and intuitionistic negation.

but $\neg\phi$ and $\neg'\phi$ wouldn't be interderivable.[11] Unique characterization is sometimes viewed as a necessary condition for a connective to be regarded as logical.[12] I will not dwell on this point here, however, since the logicality of compatibilist negation is not my main concern.

In view of the Impossibility Theorem and the potential resurgence of the sorites paradox in the presence of a negation operator obeying (RAA), Fine seems to have two options: (i) Deny that the addition of any negation operator that obeys (RAA) is legitimate, or (ii) Deny that the Impossibility Theorem does in fact pose a problem for his theory of vagueness and that the sorites paradox reappears for the new negation.

I will argue that both options are problematic. Let me begin with the first. There is an influential tradition in the philosophy of logic going back to Belnap (1962), according to which the question of whether a connective can be added conservatively to a given logic serves as a criterion of admissibility for this connective in the context of the logic to which the connective is added.[13] On this line of thought, the conservative extension result above provides a reason for thinking that we can legitimately extend the language of **CL** with some unary operator that obeys a subset of the classically valid rules for negation. One might of course argue that introducing a connective by means of axioms and rules may fail to bestow meaning on the connective in question, and that only a meaningful connective obeying (RAA) poses a potential problem for Fine's theory. However, it is not essential to the present argument that the connective be literally 'introduced' by means of axioms and rules, in the sense that its meaning is to be defined by its inferential behavior.[14] We may instead envisage someone who claims to have an antecedent grasp of the connective to be added; the axioms and rules governing this connective may then be simply regarded as correct principles for the connective in question, without any presumption that they should also be meaning conferring. Given that the new connective is meaningful and doesn't interfere with any of the inferences and truths of the original language, it would appear to be a legitimate addition to the language.

The conservative extension criterion need not be without exception in order to carry weight for the present dialectic. Even if we grant that the conservative extension criterion does not *generally* serve as a sufficient condition for the legitimacy of a connective, it is a plausible default criterion (at least if the meaningfulness of the connective is not in question).[15] The burden of proof is thus on Fine to provide reasons for denying the legitimacy of every connective that obeys (RAA).

[11] Incidentally, the same is true for the compatibilist conditional $\supset$.

[12] See, for instance, Humberstone (2011) for discussion.

[13] Another criterion Belnap proposes is unique characterization. This criterion, although satisfied by many candidate negation operators, is not as pertinent as conservativity here, since, as mentioned above, compatibilist negation is itself not uniquely characterized. By contrast, classical and intuitionistic negation, for example, are uniquely characterized by their usual introduction- and elimination-rules.

[14] Belnap's own concern did involve the question of whether a connective can be understood in terms of its inferential behavior.

[15] See Humberstone (2011, pp. 566 ff.) for a discussion of some cases in which a connective might be deemed illegitimate in a logic despite its conservatively extending the logic.

So far I've spoken about the legitimacy of extending the language of **CL**. But one might think that this raises some irrelevant issues, since, after all, we already speak a language that is expressively much richer than the language of **CL**. We can therefore instead ask whether *our* language contains an operator obeying (RAA). There is at least *some* evidence that (sentence) negation in English is such an operator. And given the conservative extension result, someone who thought that negation in English obeys (RAA)—for example, because they think that negation in English is classical or intuitionistic negation—need not dispute the existence of compatibilist negation. They could, for example, think that it just means 'definitely not'. But even if we were convinced that negation in English means $\neg$, there is a real question as to whether there are any other operators in English that obey (RAA). An argument to the effect that negation in English doesn't obey (RAA) therefore does not suffice for ruling out the existence of all such operators.

I will now argue that there is no way of ruling out the existence or legitimacy of all such operators. The only remaining option for meeting the challenge would thus seem to be to deny that the Impossibility Theorem from the previous section is in fact problematic and that the sorites paradox resurfaces for the new negation.

The reason why it is unpromising to argue that no connective that obeys (RAA) is legitimate from the perspective of **CL** is that we can define such a connective in a natural and unobjectionable propositionally quantified extension of **CL**. So even if there are reasons for thinking that adding such connectives directly to the language of **CL** is problematic, there is another way of adding one of them that is a lot harder to resist.

Consider the language of **CL** supplemented with the propositional quantifier $\exists$. The logic **CL**π is obtained by adding the following axioms and rules for $\exists$ to **CL**:

$\exists 1$	$\phi(\xi/p) \supset \exists p \phi(p)$	
$\exists 2$	If $\vdash \psi(p) \supset \phi$, then $\vdash \exists p \psi(p) \supset \phi$	(p not free in ϕ)
$\exists\wedge$	$(\phi \wedge \exists p \psi(p)) \supset \exists p(\phi \wedge \psi)$	(p not free in ϕ)

These principles seem to be completely unobjectionable from the point of view of **CL**. They are clearly sound for a semantics in which the set of propositions in a model is closed under formulas, such that every formula expresses a proposition in every model. (Notice that axiom $\exists\wedge$ would be derivable if the background logic were at least intuitionistic; in the present context, we have to add it due to the weakness of the underlying proof system. This does not, however, make it any less plausible.) There are in fact other axioms one might want to add to a reasonable propositionally quantified extension of **CL**, but the list above is sufficient to prove our main result here.

In **CL**π we can define a conditional operator $\to$ and a negation operator $\neg^i$ as follows:

(DEF$\to$) $(\phi \to \psi) =^{df} \exists p(p \wedge ((\phi \wedge p) \supset \psi))$, where p is the first variable not occurring free in either ϕ or ψ

(DEF$\neg^i$) $\neg^i\phi =^{df} \exists p(p \wedge \neg(p \wedge \phi))$, where p is the first variable not occurring free in ϕ

It turns out that in $\mathbf{CL}\pi$, $\rightarrow$ and $\neg^i$ behave exactly like the intuitionistic conditional and intuitionistic negation, respectively.[16] In fact, we have the following result, a proof of which is given in Appendix 2:

Embedding Theorem

$\mathbf{CL}\pi$ exactly embeds propositional intuitionistic logic under translation.[17]

Since intuitionistic negation obeys (RAA), the Impossibility Theorem holds in $\mathbf{CL}\pi$. Moreover, the sorites paradox would resurface in $\mathbf{CL}\pi$, since conjunctive syllogism is valid in intuitionistic logic.

The only option for Fine is thus to argue that the Impossibility Theorem as well as the sorites paradox in $\mathbf{CL}\pi$ do not in fact constitute any problem for his theory of vagueness. One way to argue against the significance of the Impossibility Theorem in $\mathbf{CL}\pi$ would be as follows. The Impossibility Theorem rests on an understanding of a sharp response in the context of the new logic $\mathbf{CL}\pi$. This requires adjusting the notion of consistency in the second condition for being a sharp response to the context of the new logic. But one might think that the only relevant sense of consistency for the notion of a sharp response is consistency in **CL**, where a set of sentences of $\mathbf{CL}\pi$ is **CL**-*inconsistent* just in case its members are the result of uniformly substituting formulas of the new language for the variables of a **CL**-inconsistent set in the old language. For example, the set $\{p, \neg^i p\}$ would not be **CL**-inconsistent and so the sequence $p, p, p, \neg^i p, \neg^i p$ would not count as a sharp response. But it is unclear why we should privilege **CL**-inconsistency in this way. For although $\neg$ is logically stronger than $\neg^i$ (in the sense that $\vdash_{\mathbf{CL}\pi} \neg p \supset \neg^i p$ but not $\vdash_{\mathbf{CL}\pi} \neg^i p \supset \neg p$), a contradiction of the form $p \wedge \neg^i p$ is

[16] Anderson et al. (1992, p. 50ff.) give a similar definition of a conditional (with the symbol for relevant entailment in place of $\supset$) in the propositionally quantified relevant logic $\mathbf{E}^{\forall\exists p}$. They take such conditionals to express enthymematic arguments, i.e. arguments with a suppressed true premise, and claim that the defined conditional captures the meaning of the intuitionistic conditional. Interestingly, their defined conditional, just like ours, behaves exactly like the intuitionistic conditional and they show that intuitionistic logic can be exactly embedded in $\mathbf{E}^{\forall\exists p}$. We have an exactly analogous result for $\mathbf{CL}\pi$. However, I do not claim that the conditional defined by (DEF$\rightarrow$) captures the meaning of the intuitionistic conditional, but only that they obey the same logical principles.

[17] It is worth noting that (DEF$\neg^i$) defines classical negation in the logic $\mathbf{CL}\pi+$ in which the propositional quantifier receives its *primary interpretation*. A $\mathbf{CL}\pi+$ model is an ordered quadruple $\mathcal{M} = \langle U, \circ, V, \Pi\rangle$, where the first three components are as in C(i)-C(iii), and Π, the set of propositions, is the powerset of U. Given a $\mathbf{CL}\pi+$-model $\mathcal{M}$, a proposition $P \in \Pi$ and a propositional variable p, $\mathcal{M}[P/p]$ is the model just like $\mathcal{M}$ except that it assigns the proposition P to the propositional variable p. The extension $[\![\cdot]\!]$ of the valuation function V from the propositional variables to Π is determined by T(i)-T(vi) together with the following clause for the propositional quantifier:

T'(vii) $[\![\exists p\phi]\!]^{\mathcal{M}} = \bigcup\{[\![\phi]\!]^{\mathcal{M}[P/p]} : P \in \Pi\}$

We may call the interpretation of the propositional quantifier given by a $\mathbf{CL}\pi+$-model its primary interpretation. We say that $\mathcal{M}$ *validates* ϕ ($\mathcal{M} \vDash \phi$) just in case $[\![\phi]\!]^{\mathcal{M}} = U$. ϕ is $\mathbf{CL}\pi+$-*valid* ($\vDash_{\mathbf{CL}\pi+} \phi$) just in case it is validated by every $\mathbf{CL}\pi+$-model. The logic $\mathbf{CL}\pi+$ is the set of $\mathbf{CL}\pi+$-valid formulas ($\{\phi : \vDash_{\mathbf{CL}\pi+} \phi\}$). The logic $\mathbf{CL}\pi+$ is thus the logic corresponding to the primary interpretation of the propositional quantifier in **CL**.

just as bad as a contradiction of the form $p \wedge \neg p$, because we have $\vdash_{\mathbf{CL}\pi} (p \wedge \neg^i p) \supset (p \wedge \neg p)$. Now if we think of $p_1, ..., p_5$ as representing some sorites series, for instance, a response in which we assert p_1, p_2, p_3 and $\neg^i p_4, \neg^i p_5$ would correspondingly seem to be just as bad as one where we replace $\neg^i$ with $\neg$. The step from p_3 to $\neg^i p_4$ still marks a cut-off of the kind that is supposed to be ruled out by the general idea behind Fine's characterization of indeterminacy and the motivation for the Possibility Theorem. In general, it is implausible to tie the notion of a sharp response to **CL**, and it was in fact not introduced in this way in Fine (2008).

But even if there is a way to argue that the Impossibility Theorem in $\mathbf{CL}\pi$ is unproblematic, the sorites paradox would still resurface for $\neg^i$, so we would have to find some new solution to the sorites for $\neg^i$. Indeed, once we accept each of the minor premises $\neg p_n$ (which entails $\neg^i p_n$) and p_0 together with all 'no cut-off premises' $\neg^i(p_{k-1} \wedge \neg^i p_k), k \leq n$, we can straightforwardly infer not only $\neg^i p_0$ but also $\neg p_0$,[18] thereby recovering the original paradoxical conclusion we would like to avoid. The premises of the sorites do not seem to lose any of their pull when we use $\neg^i$ in place of $\neg$ to formulate them. The only candidate premises to be given up would be the 'no cut-off premises': $\neg^i(p_{k-1} \wedge \neg^i p_k)$. But there seems to be no good reason to believe that these premises should lose their plausibility when formulated with $\neg^i$ in place of $\neg$.[19] In particular, it would seem to be unpromising to argue that the premise is implausible due to the more complex quantificational structure of $\neg^i$. Since given that $\neg^i$ is a legitimate and intelligible connective in $\mathbf{CL}\pi$ we can simply add another primitive negation operator $\sim$ to $\mathbf{CL}\pi$ by adding the axiom schema $(\sim\phi \supset \neg^i\phi) \wedge (\neg^i\phi \supset \sim\phi)$ to the logic, which guarantees that $\sim$ is intersubstitutable with $\neg^i$ in all contexts.[20] The new negation $\sim$ seems to be clearly acceptable from the standpoint of $\mathbf{CL}\pi$ and we can formulate a sorites argument with it that does not have any quantificational structure in the premises.[21]

4 Conclusion

I have argued that Fine's new theory of vagueness faces a challenge. Fine's Possibility Theorem is not robust under extensions of the language with a negation operator that obeys (RAA). But there are good reasons to believe that at least some such extensions are in fact legitimate. I have provided one specific example of a legitimate extension, $\mathbf{CL}\pi$, in which we can define a negation operator that behaves exactly like intuitionistic negation. Moreover, I

[18] This last step follows from the fact that $p_0 \wedge \neg^i p_0$ entails $\neg p_0$.

[19] An anonymous referee has suggested that the existence of an operator like $\neg^i$ is only problematic if it adequately regiments negation in English. I disagree, and the arguments given above are deliberately neutral on the question of which (if any) of the operators in question adequately regiments negation in English. Even if $\neg$ is in fact a better candidate than $\neg^i$ for regimenting negation in English—which is not obvious—it is unclear why this alone would make the premises in a sorites with $\neg^i$ instead of $\neg$ any less plausible.

[20] Alternatively, we could add some set of axioms and rules that uniquely characterize intuitionistic negation. The point has even more force if we already have an independent understanding of intuitionistic negation.

[21] An analogous point applies if one wanted to argue against the significance of the Impossibility Theorem in $\mathbf{CL}\pi$ by appeal to the quantificational structure of $\neg^i$.

have argued that in the presence of this operator, the kind of indeterminacy that Fine takes to be a hallmark of vagueness is impossible, and that the sorites paradox, which resurfaces for the defined negation, remains without a solution.

Appendix 1

This appendix establishes the Impossibility Theorem from section 2. Let $\mathcal{L}_{\mathbf{CL+}}$ be the language of **CL** supplemented with a unary negation operator $\sim$. A *consequence relation*, $\vdash$, for a language $\mathcal{L}$, is a relation between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas that is substitution invariant and closed under the usual structural rules of reflexivity, monotonicity and cut. When $\vdash$ is a consequence relation, we write $\Gamma \vdash \phi$ for $\langle \Gamma, \phi \rangle \in \vdash$, and $\phi_1, \ldots, \phi_n \vdash \psi$ for $\{\phi_1, \ldots, \phi_n\} \vdash \psi$. Let $\vdash_{\mathbf{CL}} := \{\langle \Gamma, \phi \rangle : \Gamma \vDash_{\mathbf{CL}} \phi, \Gamma \subseteq \mathcal{L}_{\mathbf{CL}}, \phi \in \mathcal{L}_{\mathbf{CL}}\}$. We define $\vdash_{\mathbf{CL+}}$ to be the smallest consequence relation for $\mathcal{L}_{\mathbf{CL+}}$ that contains $\vdash_{\mathbf{CL}}$ and satisfies the following condition:

(RAA) If $\Gamma, \phi \vdash_{\mathbf{CL+}} \psi$ and $\Gamma, \phi \vdash_{\mathbf{CL+}} \sim\psi$ for some ψ, then $\Gamma \vdash_{\mathbf{CL+}} \sim\phi$;

Thus, $\vdash_{\mathbf{CL+}}$ is closed under *reductio ad absurdum* for $\sim$. We define the logic **CL+** to be the consequence relation $\vdash_{\mathbf{CL+}}$. Note that the fact that **CL+** contains **CL** and is substitution invariant guarantees that every $\mathcal{L}_{\mathbf{CL+}}$-substitution instance of a valid argument of **CL** is in **CL+**. This entails, for example, that $p, \neg p \vdash_{\mathbf{CL+}} \sim p$, this being a $\mathcal{L}_{\mathbf{CL+}}$-substitution instance of *ex falso quodlibet*, which is a valid rule of **CL**.[22] I will henceforth write $\vdash$ instead of $\vdash_{\mathbf{CL+}}$ whenever no confusion is likely to arise.

A set of formulas Γ of $\mathcal{L}_{\mathbf{CL+}}$ is *inconsistent* just in case $\Gamma \vdash \phi$ and $\Gamma \vdash \sim\phi$, for some formula ϕ. Γ is *consistent* just in case it is not inconsistent.

Lemma 2. *Let Γ be a consistent set of formulas. Then if $\Gamma \cup \{\phi\}$ is inconsistent, $\Gamma \cup \{\sim\phi\}$ is consistent.*

Proof. Suppose that $\Gamma \cup \{\phi\}$ is inconsistent. Then $\Gamma, \phi \vdash \psi \wedge \sim\psi$, for some ψ. By (RAA), $\Gamma \vdash \sim\phi$. Now suppose for reductio that $\Gamma \cup \{\sim\phi\}$ is inconsistent. Then $\Gamma, \sim\phi \vdash \psi \wedge \sim\psi$, for some ψ. Hence, by (RAA), $\Gamma \vdash \sim\sim\phi$. Hence, Γ is inconsistent, contrary to our hypothesis. Thus, $\Gamma \cup \{\sim\phi\}$ is consistent. $\square$

Lemma 3. *For all formulas of $\mathcal{L}_{\mathbf{CL+}}$, $\neg\phi \vdash \sim\phi$.*

Proof. By reflexivity and monotonicity, $\phi, \neg\phi \vdash \phi$. By *ex falso quodlibet* for $\neg$, $\phi, \neg\phi \vdash \sim\phi$. Now by (RAA), $\neg\phi \vdash \sim\phi$. $\square$

Note that, as a consequence of Theorem 1, the converse entailment does not hold. The definitions of the notions of an *extremal response* and a *sharp response* are as in the main

[22] This is easily verified by using the semantics given in the main text.

text, although they should now be understood with respect to the language $\mathcal{L}_{\mathbf{CL}+}$, and the notion of inconsistency appealed to in condition (ii) of the definition of a *sharp response* is now to be understood as **CL**+-inconsistency.

We can now state and prove the Impossibility Theorem from the main text.

Theorem 2 (Impossibility)**.** *Let $\phi_0, \phi_1, ..., \phi_{n+1}$, $n \geq 0$, be any formulas of $\mathcal{L}_{\mathbf{CL}+}$. Then there is no set of formulas Δ_0 which is consistent with the extremal response $\{\phi_0, \neg\phi_{n+1}\}$ and yet inconsistent with every sharp response to $\phi_0, \phi_1, ..., \phi_{n+1}$.*

Proof. The proof is similar to the proof of the Impossibility Theorem in Fine (2008). Let $\phi_0, \phi_1, ..., \phi_{n+1}$ be any set of formulas and let Δ_0 be a set of formulas consistent with the extremal response $\{\phi_0, \neg\phi_{n+1}\}$. We show that Δ_0 is consistent with a sharp response to $\phi_0, \phi_1, ..., \phi_{n+1}$ by extending Δ_0. Let $\Delta_1 = \Delta_0 \cup \{\phi_0, \sim\phi_{n+1}\}$. For $k = 1, 2, ..., n$, we define:

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{\phi_k\}, & \text{if consistent} \\ \Delta_k \cup \{\sim\phi_k\}, & \text{otherwise} \end{cases}$$

Note that by Lemma 2, Δ_k is consistent for $k = 1, ..., n+1$. The consistency of Δ_1 follows from Lemma 3. For suppose Δ_1 is inconsistent. Then (i) $\Delta_0, \phi_0, \sim\phi_{n+1} \vdash \psi \wedge \sim\psi$, for some ψ. From this, by (RAA) and monotonicity, we obtain: (ii) $\Delta_0, \phi_0, \neg\phi_{n+1} \vdash \sim\sim\phi_{n+1}$. By Lemma 3 and monotonicity: (iii) $\Delta_0, \phi_0, \neg\phi_{n+1} \vdash \sim\phi_{n+1}$, in contradiction to the consistency of $\Delta_0 \cup \{\phi_0, \neg\phi_{n+1}\}$. Note further that for every $k \leq n+1$, either $\phi_k \in \Delta_{n+1}$ or $\sim\phi_k \in \Delta_{n+1}$.

We can now define a collective response $\psi_0(p), \psi_1(p), ..., \psi_{n+1}(p)$ and a corresponding collective response $\psi_0(\phi_0), \psi_1(\phi_1), ..., \psi_{n+1}(\phi_{n+1})$ to $\phi_0, \phi_1, ..., \phi_{n+1}$. Where $k = 0, 1, ..., n+1$:

$$\psi_k(p) = \begin{cases} p, & \text{if } \phi_k \in \Delta_{n+1} \\ \sim p, & \text{if } \sim\phi_k \in \Delta_{n+1} \end{cases}$$

We now get:

(1) $\Delta_{n+1} \vdash \psi_k(\phi_k)$, for $k = 0, 1, ..., n+1$.

The claim follows trivially from the reflexivity and monotonicity of $\vdash$. Since Δ_{n+1} is consistent it follows from (1) that Δ_{n+1} is consistent with the response $\psi_0(\phi_0), \psi_1(\phi_1), ..., \psi_{n+1}(\phi_{n+1})$ to $\phi_0, \phi_1, ..., \phi_{n+1}$. Hence, Δ_0 is also consistent with this response, since it is a subset of Δ_{n+1}.

It remains to show that the response $\psi_0(p), \psi_1(p), ..., \psi_{n+1}(p)$ is sharp. The first condition for being a sharp response is satisfied, since $\psi_0(p) = p \neq \sim p = \psi_{n+1}(p)$. The second condition says that any two individual responses are either the same or inconsistent. But every response is either of the form p or of the form $\sim p$. So the second condition is also satisfied. This concludes the proof. □

Appendix 2

This appendix establishes the Embedding Theorem from section 3. The first step is to show that with the conditional $\to$ defined as $(\phi \to \psi) =^{df} \exists p(p \wedge ((\phi \wedge p) \supset \psi))$, where p is the first variable not occurring free in either ϕ or ψ and the negation $\neg^i$ defined as $\neg^i \phi =^{df} \phi \to \bot$, we can derive all the axioms and rules of **IL** in **CL**π. In what follows, we will often use the convention that $\wedge$ and $\vee$ are given precedence over $\supset$ and $\to$ so that, for example, $\phi \wedge \psi \supset \xi$ abbreviates $(\phi \wedge \psi) \supset \xi$ and $\phi \to \psi \vee \xi$ abbreviates $\phi \to (\psi \vee \xi)$. Outer brackets of conjunctions and disjunctions are thus often omitted.

We will use the axiomatization of **CL** from Fine (2017, p. 3723f.). Note that compatibilist negation $\neg$ is a defined connective in this axiomatization: $\neg\phi =^{df} \phi \supset \bot$.

Axioms

A1. $\phi \wedge (\phi \supset \psi) \supset \psi$

A2. $\phi \supset ((\phi \supset \psi) \supset \psi)$

A3. $\phi \supset (\psi \supset \psi)$

A4. $\phi \wedge \psi \supset (\phi \supset \psi)$

A5. $((\phi \supset \psi) \wedge (\psi \supset \xi)) \supset \psi \vee (\phi \supset \xi)$

A6. $\phi \wedge \psi \supset \phi$

A7. $\phi \wedge \psi \supset \psi$

A8. $((\phi \supset \psi) \wedge (\psi \supset \xi)) \supset (\phi \supset (\psi \wedge \xi))$

A9. $\phi \supset \phi \vee \psi$

A10. $\psi \supset \phi \vee \psi$

A11. $((\phi \supset \xi) \wedge (\psi \supset \xi)) \supset ((\phi \vee \psi) \supset \xi)$

A12. $\phi \wedge (\psi \vee \xi) \supset (\phi \wedge \psi) \vee (\phi \wedge \xi)$

A13. $\bot \supset \phi$.

Rules

R1. If $\vdash \phi$ and $\vdash \phi \supset \psi$, then $\vdash \psi$

R2. If $\vdash \phi$ and $\vdash \psi$, then $\vdash \phi \wedge \psi$

R3. If $\vdash \phi \supset \psi$ and $\vdash \psi \supset \xi$, then $\vdash \phi \supset \xi$

In what follows, $\vdash$ denotes derivability in **CL**π, which is obtained by adding the following axioms and rules for the propositional quantifier to **CL**:

$\exists 1$	$\phi(\xi/p) \supset \exists p \phi(p)$	
$\exists 2$	If $\vdash \psi(p) \supset \phi$, then $\vdash \exists p \psi(p) \supset \phi$	(p not free in ϕ)
$\exists\wedge$	$(\phi \wedge \exists p \psi(p)) \supset \exists p(\phi \wedge \psi(p))$	(p not free in ϕ)

We need to show that all of the following are provable in **CL**π:

Axioms	Rules
IL1. $\phi \to (\psi \to \phi)$	MP$\to$. If $\vdash \phi$ and $\vdash \phi \to \psi$, then $\vdash \psi$.
IL2. $(\phi \to \psi) \to ((\phi \to (\psi \to \xi)) \to (\phi \to \xi))$	
IL3. $\phi \to (\psi \to \phi \wedge \psi)$	
IL4. $\phi \wedge \psi \to \phi$	
IL5. $\phi \wedge \psi \to \psi$	
IL6. $\phi \to \phi \vee \psi$	
IL7. $\psi \to \phi \vee \psi$	
IL8. $(\phi \to \tau) \to ((\psi \to \tau) \to (\phi \vee \psi \to \tau))$	
IL9. $(\phi \to \psi) \to ((\phi \to \neg^i \psi) \to \neg^i \phi)$	
IL10. $\neg^i \phi \to (\phi \to \psi)$	

We state without proof the following two elementary lemmas:

Lemma 4. $\vdash \phi \supset \phi$

Lemma 5. *If* $\vdash \phi$, *then* $\vdash \psi \supset \phi$

The use of associativity and commutativity of conjunction and disjunction will be tacit in the proofs to follow.[23] The proofs of these principles themselves are routine. The following central lemma will be useful:

Lemma 6. $\vdash (\phi \supset \psi) \supset (\phi \to \psi)$

Proof.

1. $(\phi \supset \psi) \supset (\phi \supset \psi)$ [Lemma 4]
2. $\phi \wedge (\phi \supset \psi) \supset \psi$ [A1]
3. $(\phi \supset \psi) \supset (\phi \wedge (\phi \supset \psi) \supset \psi)$ [2., Lemma 5]
4. $((\phi \supset \psi) \supset (\phi \supset \psi)) \wedge ((\phi \supset \psi) \supset (\phi \wedge (\phi \supset \psi) \supset \psi))$ [1., 3., R2]
5. $(\phi \supset \psi) \supset (\phi \supset \psi) \wedge (\phi \wedge (\phi \supset \psi) \supset \psi)$ [4., A8, R1]
6. $(\phi \supset \psi) \wedge (\phi \wedge (\phi \supset \psi) \supset \psi) \supset \exists p(p \wedge (\phi \wedge p \supset \psi))$ [∃1]
7. $(\phi \supset \psi) \supset \exists p(p \wedge (\phi \wedge p \supset \psi))$ [5., 6., R3]
8. $(\phi \supset \psi) \supset (\phi \to \psi)$ [7., Def.]

□

From this and the corresponding axioms of **CL** we immediately get IL4, IL5, IL6 and IL7. To prove MP$\to$, we first show:

Lemma 7. $\vdash \phi \wedge (\phi \to \psi) \supset \psi$

Proof. Let p be a variable not free in ϕ or ψ.

[23] The proof strategy below is somewhat similar to the strategy of a similar proof given in Anderson et al. (1992, pp. 57ff.). They show that the propositionally quantified relevance logic $\mathbf{E}_{+}^{\forall\exists p}$ exactly embeds intuitionistic logic under virtually the same translation as the one given here (with $\supset$ replaced by the symbol for relevant entailment).

1. $(\phi \wedge p) \wedge (\phi \wedge p \supset \psi) \supset \psi$ [A1]
2. $\exists p((\phi \wedge p) \wedge (\phi \wedge p \supset \psi)) \supset \psi$ [1. ∃2]
3. $\phi \wedge \exists p(p \wedge (\phi \wedge p \supset \psi)) \supset \exists p((\phi \wedge p) \wedge (\phi \wedge p \supset \psi))$ [∃∧]
4. $\phi \wedge (\phi \to \psi) \supset \psi$ [3., 2., R3, Def.]

□

From Lemma 7 we get MP→ from R2 and R1:

Lemma 8 (MP→). *If* $\vdash \phi$ *and* $\vdash \phi \to \psi$*, then* $\vdash \psi$.

Next, we prove IL1 and IL3. We first state without proof the following auxiliary lemma:

Lemma 9. *If* $\vdash \phi$ *and* $\vdash \psi \wedge \phi \supset \xi$*, then* $\vdash \psi \supset \xi$.

Lemma 10 (IL1). $\vdash \phi \to (\psi \to \phi)$

Proof.
1. $\phi \wedge (\phi \wedge \psi \supset \phi) \supset (\psi \to \phi)$ [∃1, Def]
2. $\phi \supset (\psi \to \phi)$ [1., A6, Lemma 9]
3. $(\phi \supset (\psi \to \phi)) \supset (\phi \to (\psi \to \phi))$ [Lemma 6]
4. $\phi \to (\psi \to \phi)$ [2., 3., R1]

□

Lemma 11 (IL3). $\vdash \phi \to (\psi \to \phi \wedge \psi)$

Proof.
1. $\phi \wedge (\phi \wedge \psi \supset \phi \wedge \psi) \supset (\psi \to \phi \wedge \psi)$ [∃1, Def.]
2. $\phi \supset (\psi \to \phi \wedge \psi)$ [1., Lemma 9]
3. $\phi \to (\psi \to \phi \wedge \psi)$ [2., Lemma 6, R1]

□

For IL2 and IL8 we need the following

Lemma 12 (Exp.). $\vdash ((\phi \wedge \psi) \to \xi) \supset (\phi \to (\psi \to \xi))$

Proof. Let p be a variable not free in ϕ, ψ or ξ.
1. $\exists p(p \wedge ((\phi \wedge \psi) \wedge p \supset \xi)) \wedge \phi \supset \exists p((p \wedge \phi) \wedge (p \wedge (\phi \wedge \psi) \supset \xi))$ [∃∧]
2. $((p \wedge \phi) \wedge ((p \wedge \phi) \wedge \psi \supset \xi)) \supset \exists q(q \wedge (q \wedge \psi \supset \xi))$ [∃1]
3. $\exists p((p \wedge \phi) \wedge ((p \wedge \phi) \wedge \psi \supset \xi)) \supset \exists q(q \wedge (q \wedge \psi \supset \xi))$ [2., ∃2]
4. $(((\phi \wedge \psi) \to \xi) \wedge \phi) \supset (\psi \to \xi)$ [1., 3., R3]
5. $(((\phi \wedge \psi) \to \xi) \wedge ((((\phi \wedge \psi) \to \xi) \wedge \phi) \supset (\psi \to \xi))) \supset (\phi \to (\psi \to \xi))$ [∃1, Def.]
6. $((\phi \wedge \psi) \to \xi) \supset (\phi \to (\psi \to \xi))$ [4., 5., Lemma 9]

□

Lemma 13 (IL2). $\vdash (\phi \to \psi) \to ((\phi \to (\psi \to \xi)) \to (\phi \to \xi))$

Proof.
1. $\phi \wedge (\phi \to \psi) \supset \psi$ [Lemma 7]

2. $\phi \wedge (\phi \to (\psi \to \xi)) \supset (\psi \to \xi)$ [Lemma 7]

3. $\psi \wedge (\psi \to \xi) \supset \xi$ [Lemma 7]

4. $(\phi \wedge (\phi \to \psi)) \wedge (\phi \to (\psi \to \xi)) \supset \xi$ [A6-8, 1.-3., R3]

5. $(\phi \to \psi) \to ((\phi \to (\psi \to \xi)) \to (\phi \to \xi))$ [4., Lemma 6, R1, Exp. (twice)]

□

For the proof of IL8 we require another auxiliary lemma.

Lemma 14. $\vdash (\xi \wedge \chi) \wedge (\phi \vee \psi) \supset (\phi \wedge \chi) \vee (\psi \wedge \xi)$

Proof. 1. $\chi \wedge (\phi \vee \psi) \supset (\chi \wedge \phi) \vee (\chi \wedge \psi)$ [A12]

2. $\xi \wedge (\chi \wedge (\phi \vee \psi)) \supset \xi \wedge ((\chi \wedge \phi) \vee (\chi \wedge \psi))$ [1., A6-8, R3]

3. $\xi \wedge ((\chi \wedge \phi) \vee (\chi \wedge \psi)) \supset (\xi \wedge (\chi \wedge \phi)) \vee (\xi \wedge (\chi \wedge \psi))$ [A12]

4. $\xi \wedge (\chi \wedge \phi) \supset (\phi \wedge \chi) \vee (\psi \wedge \xi)$ [A7, A9, R3]

5. $\xi \wedge (\chi \wedge \psi) \supset (\phi \wedge \chi) \vee (\psi \wedge \xi)$ [A7, A8, R3]

6. $(\xi \wedge (\chi \wedge \phi)) \vee (\xi \wedge (\chi \wedge \psi)) \supset (\phi \wedge \chi) \vee (\psi \wedge \xi)$ [4., 5., A11, R2, R1]

7. $\xi \wedge (\chi \wedge (\phi \vee \psi)) \supset (\phi \wedge \chi) \vee (\psi \wedge \xi)$ [2., 3., 6., R3]

□

Lemma 15 (IL8)**.** $\vdash (\phi \to \tau) \to ((\psi \to \tau) \to (\phi \vee \psi \to \tau))$

Proof. 1. $(\phi \wedge (\phi \to \tau) \supset \tau) \wedge (\psi \wedge (\psi \to \tau) \supset \tau) \supset$
$((\phi \wedge (\phi \to \tau)) \vee (\psi \wedge (\psi \to \tau)) \supset \tau)$ [A11]

2. $(\phi \wedge (\phi \to \tau)) \vee (\psi \wedge (\psi \to \tau)) \supset \tau$ [Lemma 7, 1., R1, R2]

3. $((\phi \to \tau) \wedge (\psi \to \tau)) \wedge (\phi \vee \psi) \supset$
$(\phi \wedge (\phi \to \tau)) \vee (\psi \wedge (\psi \to \tau))$ [Lemma 14]

4. $((\phi \to \tau) \wedge (\psi \to \tau)) \wedge (\phi \vee \psi) \supset \tau$ [2., 3., R3]

5. $((\phi \to \tau) \wedge (\psi \to \tau)) \wedge (\phi \vee \psi) \to \tau$ [4., Lemma 6, R1]

6. $(\phi \to \tau) \to ((\psi \to \tau) \to (\phi \vee \psi \to \tau))$ [5., Exp. (twice)]

□

It only remains to show the two negation axioms IL9 and IL10. But IL9 is just an instance of IL2, with $\bot$ for ξ. IL10 may be proved as follows:

Lemma 16 (IL10)**.** $\vdash (\neg^i \phi) \to (\phi \to \psi)$

Proof. 1. $(\phi \to \bot) \wedge \phi \supset \bot$ [Lemma 7]

2. $\bot \supset \psi$ [A13]

3. $(\phi \to \bot) \wedge \phi \supset \psi$ [1., 2., R3]

4. $(\phi \to \bot) \wedge \phi \to \psi$ [3., Lemma 6, R1]

5. $(\neg^i \phi) \to (\phi \to \psi)$ [4., Exp., R1, Def.]

□

To state the first part of our central result, we define a translation $'$ from $\mathcal{L}_{\mathbf{IL}}$ into $\mathcal{L}_{\mathbf{CL}\pi}$ by taking $(\phi \to^i \psi)' = \exists p(p \wedge ((\phi' \wedge p) \supset \psi'))$ and $(\neg^i \phi)' = \exists p(p \wedge ((\phi' \wedge p) \supset \bot))$, and translating conjunctions, disjunctions and atomic formulas homophonically.[24]

Corollary 2. *If $\vdash_{\mathbf{IL}} \phi$, then $\vdash_{\mathbf{CL}\pi} \phi'$.*

In fact, what we have proven is significantly stronger than Corollary 2. For we have shown that the intuitionistic axioms and rules hold unrestrictedly for all formulas of $\mathcal{L}_{\mathbf{CL}\pi}$ (with $\to$ and $\neg^i$ understood as defined connectives), and not only for translations of formulas of $\mathcal{L}_{\mathbf{IL}}$.

Next we show that the converse of Corollary 2 holds as well, so that the translation of a formula of $\mathcal{L}_{\mathbf{IL}}$ is a theorem of $\mathbf{CL}\pi$ only if its inverse image is an intuitionistic theorem. This is what justifies saying that the embedding of **IL** into $\mathbf{CL}\pi$ is exact.

Let $\mathbf{IL}\pi$ be the propositionally quantified intuitionistic logic whose language is the same as $\mathcal{L}_{\mathbf{IL}}$ with the addition of the propositional quantifier $\exists$, and which results from **IL** by adding the following axioms and rules for the quantifiers:

IL∃1 $\quad \phi(\xi) \to^i \exists p \phi(p)$

IL∃2 $\quad$ If $\vdash \psi(p) \to^i \phi$, then $\vdash \exists p \psi(p) \to^i \phi \quad$ (p not free in ϕ)

We will make use of the following auxiliary translation from $\mathcal{L}_{\mathbf{CL}\pi}$ to $\mathcal{L}_{\mathbf{IL}\pi}$. Define h to be the function from $\mathcal{L}_{\mathbf{CL}\pi}$ to $\mathcal{L}_{\mathbf{IL}\pi}$ that replaces every $\supset$ by $\to^i$, and leaving everything else the same (i.e. translating everything else homophonically). For example, the formula $p \supset q$ gets translated into $p \to^i q$.

We prove the following three claims, for ϕ in $\mathcal{L}_{\mathbf{IL}}$:

1. If $\vdash_{\mathbf{CL}\pi} \phi'$, then $\vdash_{\mathbf{IL}\pi} h(\phi')$
2. If $\vdash_{\mathbf{IL}\pi} h(\phi')$, then $\vdash_{\mathbf{IL}\pi} \phi$
3. If $\vdash_{\mathbf{IL}\pi} \phi$, then $\vdash_{\mathbf{IL}} \phi$

Lemma 17. *If $\vdash_{\mathbf{CL}\pi} \phi$, then $\vdash_{\mathbf{IL}\pi} h(\phi)$*

Proof. The result follows immediately from the observation that every theorem of **CL** is a theorem of **IL** (Theorem 2 in Fine (2017)), the fact that ∃1 and ∃2 are common to both logics (with $\to^i$ in place of $\supset$) and the fact that the intuitionistic analog of $\exists\wedge$ is provable in $\mathbf{IL}\pi$. □

Lemma 18. *For ϕ in $\mathcal{L}_{\mathbf{IL}}$, if $\vdash_{\mathbf{IL}\pi} h(\phi')$, then $\vdash_{\mathbf{IL}\pi} \phi$.*

Proof. We observe that the following equivalences are easily provable in $\mathbf{IL}\pi$:

(i) $\vdash_{\mathbf{IL}\pi} (\phi \to^i \psi) \leftrightarrow^i \exists p(p \wedge ((p \wedge \phi) \to^i \psi))$

[24]The translations above are to be performed in such a way as to avoid problems with free variables in subformulas becoming bound after translation. In other words, the quantified variable p in the above translations should always be such that it doesn't occur free in either ϕ' or ψ'.

(ii) $\vdash_{\mathbf{IL}\pi} \neg^i \phi \leftrightarrow^i \exists p(p \wedge ((p \wedge \phi) \rightarrow^i \bot))$

Now suppose that ϕ is in $\mathcal{L}_{\mathbf{IL}}$ and $\vdash_{\mathbf{IL}\pi} h(\phi')$. We obtain a proof of ϕ from a proof of $h(\phi')$ by noting that ϕ is logically equivalent to its translation $h(\phi')$ by virtue of (i) and (ii) and the fact that substitution of logical equivalents preserves theoremhood in $\mathbf{IL}\pi$. □

Finally, we show that $\mathbf{IL}\pi$ is a conservative extension of $\mathbf{IL}$.

Lemma 19. *For $\phi \in \mathcal{L}_{\mathbf{IL}}$, if $\vdash_{\mathbf{IL}\pi} \phi$, then $\vdash_{\mathbf{IL}} \phi$.*

Proof. We prove the contrapositive. Suppose $\nvdash_{\mathbf{IL}} \phi$. Then by the completeness of the Kripke semantics for $\mathbf{IL}$ there is a Kripke model $\mathcal{M} = \langle W, \leq, V\rangle$ for $\mathbf{IL}$ in which ϕ fails to hold: $\mathcal{M} \nvDash \phi$. But $\mathcal{M}$ can also be regarded as an $\mathbf{IL}\pi$-model in the familiar way; we only need to extend the truth definition by a clause for the existential quantifier (see, for instance, Kremer (1997)). So we still have $\mathcal{M} \nvDash \phi$. Hence by the soundness of this semantics for $\mathbf{IL}\pi$, we obtain $\nvdash_{\mathbf{IL}\pi} \phi$. □

Corollary 3. *For $\phi \in \mathcal{L}_{\mathbf{IL}}$, if $\vdash_{\mathbf{CL}\pi} \phi'$, then $\vdash_{\mathbf{IL}} \phi$.*

Corollaries 2 and 3 imply

Theorem 3 (Embedding)**.** $\mathbf{CL}\pi$ *exactly embeds* $\mathbf{IL}$ *under translation.*

References

Anderson, A. R., Belnap, N. D., and Dunn, J. M. (1992). *Entailment: The Logic of Relevance and Necessity, Vol. II.* Princeton University Press.

Belnap, N. (1962). Tonk, plonk and plink. *Analysis*, 22(6):130–134.

Chagrov, A. and Zakharyaschev, M. (1997). *Modal Logic, volume 35 of Oxford Logic Guides.* Clarendon Press, Oxford.

Fine, K. (1975). Vagueness, truth and logic. *Synthese*, 30(3-4):265–300.

Fine, K. (2008). The Impossibility of Vagueness. *Philosophical Perspectives*, 22(1):111–136.

Fine, K. (2017). The possibility of vagueness. *Synthese*, 194(10):3699–3725.

Humberstone, I. L. (1979). Interval semantics for tense logic: Some remarks. *Journal of Philosophical Logic*, 8(1):171–196.

Humberstone, L. (2011). *The Connectives.* MIT Press.

Kremer, P. (1997). On the complexity of propositional quantification in intuitionistic logic. *Journal of Symbolic Logic*, 62(2):529–544.

Schechter, J. (2011). Juxtaposition: A new way to combine logics. *Review of Symbolic Logic*, 4(4):560–606.