Sarsengali Abdygalievich Abdymanapov, Serik Altynbek, Anton Begehr and Heinrich Begehr*

# 1, 2, 3, some inductive real sequences and a beautiful algebraic pattern

https://doi.org/10.1515/anly-2020-0014
Received April 12, 2020; accepted August 11, 2021

**Abstract:** By rewriting the relation $1 + 2 = 3$ as $\sqrt{1}^2 + \sqrt{2}^2 = \sqrt{3}^2$, a right triangle is looked at. Some geometrical observations in connection with plane parqueting lead to an inductive sequence of right triangles with $\sqrt{1}^2 + \sqrt{2}^2 = \sqrt{3}^2$ as initial one converging to the segment $[0, 1]$ of the real line. The sequence of their hypotenuses forms a sequence of real numbers which initiates some beautiful algebraic patterns. They are determined through some recurrence relations which are proper for being evaluated with computer algebra.

**Keywords:** Sequence of real numbers, convergence, plane parqueting

**MSC 2010:** 40A05

## 1 An inductive sequence of positive numbers

With the initial numbers $m_1 = \sqrt{3}$ and $m_2 = \frac{2}{\sqrt{3}}$ a real sequence is defined inductively via

$$m_{k+2} = \frac{\alpha_k m_{k+1} + \beta_k}{\alpha_k + \beta_k m_{k+1}}, \quad \alpha_k = m_k m_{k+1} - 1,\ \beta_k = m_k - m_{k+1},\ k \in \mathbb{N}.$$

By introducing positive numbers $r_k$ by $r_k^2 = m_k^2 - 1$, $k \in \mathbb{N}$, a sequence of circles $|z - m_k| = r_k$ in the complex plane is given which is part of some parqueting of the complex plane [4]. The parqueting-reflection principle is always initiating iteratively given sequences of complex numbers; see [2] for another sample. The sequence presented here served in [1, 3] for treating problems in a hyperbolic domain. The triangles appear by taking one tangent to the unit circle through the center $m_k$ of a circle, the radius of the unit circle perpendicular to the tangent, and the segment $[0, m_k]$ as the hypotenuse; see Figure 1.

**Lemma 1.1.** *The sequence* $\{m_k\}$ *is monotonic decreasing with limit* 1. *In particular,*

$$0 < m_{k+2} - 1 \le q_1^{k+1}(m_1 - 1), \quad k \in \mathbb{N},$$

*where*

$$q_1 = \frac{m_1 + 1}{m_1 - 1}\frac{m_2 - 1}{m_2 + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} < 1.$$

*Proof.* From

$$m_{k+2} \pm 1 = \frac{\alpha_k \pm \beta_k}{\alpha_k + \beta_k m_{k+1}}(m_{k+1} \pm 1)$$

and

$$q_k = \frac{\alpha_k - \beta_k}{\alpha_k + \beta_k} = \frac{m_k + 1}{m_k - 1}\frac{m_{k+1} - 1}{m_{k+1} + 1},$$

***Corresponding author: Heinrich Begehr,** Institut für Mathematik, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany, e-mail: begehrh@zedat.fu-berlin.de. https://orcid.org/0000-0003-0316-0897
**Sarsengali Abdygalievich Abdymanapov, Serik Altynbek,** Kazakh University of Economics, Finance and International Trade, Zhubanov str. 7, 010005 Nur-Sultan, Kazakhstan, e-mail: rector@kuef.kz, serik_-aa@bk.ru
**Anton Begehr,** Fakultät für Informatik und Mathematik, Universität Passau, Innstraße 33, 94032 Passau, Germany, e-mail: a.begehr@fu-berlin.de

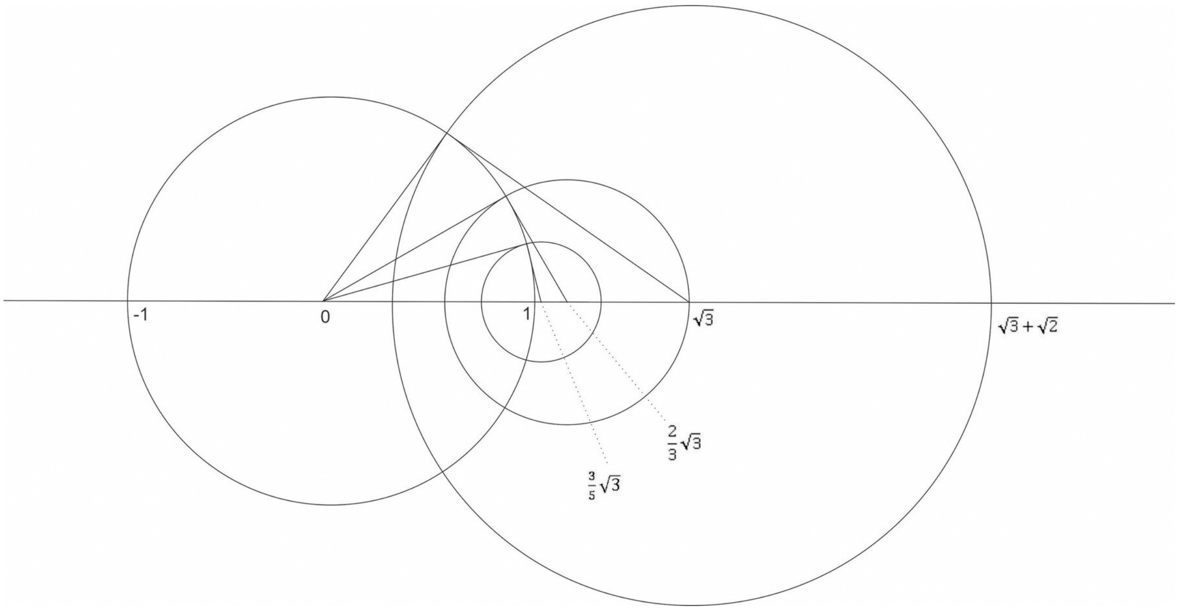


**Figure 1:** First triangles of the sequence.

the estimates

$$0 < \frac{(m_k+1)(m_{k+1}-1)^2}{\alpha_k+\beta_k m_{k+1}} = m_{k+2}-1 \le \frac{\alpha_k-\beta_k}{\alpha_k+\beta_k}(m_{k+1}-1)$$

and in particular

$$\frac{m_{k+2}-1}{m_{k+2}+1} = \frac{\alpha_k-\beta_k}{\alpha_k+\beta_k}\frac{m_{k+1}-1}{m_{k+1}+1} = q_k\frac{m_{k+1}-1}{m_{k+1}+1}$$

follow. The last equation shows $q_k = q_{k+1}$, and hence the $q_k$ do not depend on $k$:

$$q_k = q_1 = \frac{m_1+1}{m_1-1}\frac{m_2-1}{m_2+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} < 1.$$

The inequalities $m_{k+2}-1 \le q_1(m_{k+1}-1)$ imply

$$m_{k+2}-1 \le q_1^{k+1}(m_2-1) \le q_1^{k+1}(m_1-1)$$

for any $k \in \mathbb{N}$.

The monotonicity is seen from

$$m_{k+2}-m_{k+1} = \frac{\beta_k(1-m_{k+1}^2)}{\alpha_k+\beta_k m_{k+1}} < 0 \tag{1.1}$$

together with $m_2 < m_1$. □

The sequence allows a simpler representation.

**Lemma 1.2.** *For any* $k \in \mathbb{N}$,

$$\frac{\alpha_k}{\beta_k} = m_1, \quad m_{k+1} = \frac{1+m_1 m_k}{m_1+m_k}$$

*hold.*

*Proof.* By simple verification,

$$m_{k+1} = \frac{\alpha_k m_k+\beta_k}{\alpha_k+\beta_k m_k}$$

is seen. Combining this with

$$m_{k+2} = \frac{\alpha_k m_{k+1} + \beta_k}{\alpha_k + \beta_k m_{k+1}}$$

in the definitions for $\alpha_{k+1}$ and $\beta_{k+1}$ shows

$$\alpha_{k+1}(\alpha_k + m_k\beta_k)(\alpha_k + m_{k+1}\beta_k) = r_k^2 r_{k+1}^2 \alpha_k, \quad \beta_{k+1}(\alpha_k + m_k\beta_k)(\alpha_k + m_{k+1}\beta_k) = r_k^2 r_{k+1}^2 \beta_k.$$

Thus $\frac{\alpha_k}{\beta_k}$ is independent of $k$, and hence coincides with the value $m_1$ for $k = 1$. □

**Remark 1.3.** Obviously, $\alpha_1 = m_1\beta_1 = 1$. Lemma 1.2 suggests a new definition of the sequence with just one initial value $m_1 = \sqrt{3}$ and

$$m_{k+1} = \frac{1 + m_1 m_k}{m_1 + m_k}, \quad k \in \mathbb{N}.$$

The first members of the sequences $\{m_k\}$ and $\{r_k\}$ are presented in Section A.

## 2 A recurrence relation

With the sequence $\{m_k\}$ and its related coefficients $\alpha_k = m_k m_{k+1} - 1$, $\beta_k = m_k - m_{k+1}$, the system of recurrence relations

$$\alpha_{2k}\delta_{2k-1} + \beta_{2k}\gamma_{2k-1} = \delta_{2k+1}, \quad \alpha_{2k}\gamma_{2k-1} + \beta_{2k}\delta_{2k-1} = \gamma_{2k+1}, \tag{2.1}$$

$$\alpha_{2k+1}\delta_{2k} + \beta_{2k+1}\gamma_{2k} = \delta_{2k+2}, \quad \alpha_{2k+1}\gamma_{2k} + \beta_{2k+1}\delta_{2k} = \gamma_{2k+2} \tag{2.2}$$

defines two new sequences with the initial values $\gamma_0 = 0$, $\delta_0 = 1$, $\gamma_1 = 1$, $\delta_1 = m_1$. The first further members of the sequences $\gamma_k$, $\delta_k$ are listed in Section A.

By using the relation $\alpha_k = m_1\beta_k$ according to Lemma 1.2, these sequences are given in a shorter form as

$$\delta_{k+2} = \beta_{k+1}[m_1\delta_k + \gamma_k], \quad \gamma_{k+2} = \beta_{k+1}[m_1\gamma_k + \delta_k], \qquad k \in \mathbb{N}_0.$$

**Theorem 2.1.** *The sequences* $\{\gamma_k\}, \{\delta_k\}$ *defined by equations* (2.1) *and* (2.2) *with the initial numbers* $\gamma_0 = 0$, $\delta_0 = 1$, $\gamma_1 = 1$, $\delta_1 = m_1$ *are given as*

$$\gamma_{2k} = \sum_{\lambda=1}^{k} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda-1}\le 2k} (-1)^{k+\sum_{\mu=1}^{2\lambda-1}\kappa_\mu} \prod_{\mu=1}^{2\lambda-1} m_{\kappa_\mu},$$

$$\delta_{2k} = \sum_{\lambda=1}^{k} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda}\le 2k} (-1)^{k+\sum_{\mu=1}^{2\lambda}\kappa_\mu} \prod_{\mu=1}^{2\lambda} m_{\kappa_\mu} + (-1)^k,$$

$$\gamma_{2k+1} = \sum_{\lambda=1}^{k} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda}\le 2k+1} (-1)^{k+\sum_{\mu=1}^{2\lambda}\kappa_\mu} \prod_{\mu=1}^{2\lambda} m_{\kappa_\mu} + (-1)^k,$$

$$\delta_{2k+1} = \sum_{\lambda=0}^{k} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda+1}\le 2k+1} (-1)^{k+1+\sum_{\mu=1}^{2\lambda+1}\kappa_\mu} \prod_{\mu=1}^{2\lambda+1} m_{\kappa_\mu}$$

*for* $k \in \mathbb{N}_0$.

*Proof.* In order to show that these formulas present solutions to the recurrence relations, equation (2.1) for $\delta_{2k+1}$ is proved, assuming the expressions for $\delta_{2k-1}$ and $\gamma_{2k-1}$ are verified already. Then

$$\begin{aligned}\delta &= \alpha_{2k}\delta_{2k-1} + \beta_{2k}\gamma_{2k-1} \\ &= (m_{2k}m_{2k+1} - 1)\sum_{\lambda=0}^{k-1} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda+1}\le 2k-1} (-1)^{k+\sum_{\mu=1}^{2\lambda+1}\kappa_\mu} \prod_{\mu=1}^{2\lambda+1} m_{\kappa_\mu} \\ &\quad + (m_{2k} - m_{2k+1})\Big[\sum_{\lambda=1}^{k-1} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda}\le 2k-1} (-1)^{k-1+\sum_{\mu=1}^{2\lambda}\kappa_\mu} \prod_{\mu=1}^{2\lambda} m_{\kappa_\mu} + (-1)^{k-1}\Big].\end{aligned}$$

Splitting the ($\lambda = 0$)-term from the first sum and multiplying shows

$$\begin{aligned}\delta = {} & \sum_{\kappa=1}^{2k-1} (-1)^{k+1+\kappa} m_\kappa + (-1)^{k-1} m_{2k} + (-1)^k m_{2k+1} + \sum_{\kappa=1}^{2k-1} (-1)^{k+\kappa} m_\kappa m_{2k} m_{2k+1} \\ & + \sum_{\lambda=1}^{k-1} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda+1}\le 2k-1} (-1)^{k+\sum_{\mu=1}^{2\lambda+1}\kappa_\mu} \prod_{\mu=1}^{2\lambda+1} m_{\kappa_\mu} m_{2k} m_{2k+1} \\ & + \sum_{\lambda=1}^{k-1} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda+1}\le 2k-1} (-1)^{k+1+\sum_{\mu=1}^{2\lambda+1}\kappa_\mu} \prod_{\mu=1}^{2\lambda+1} m_{\kappa_\mu} \\ & + \sum_{\lambda=1}^{k-1} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda\le 2k-1}} (-1)^{k-1+\sum_{\mu=1}^{2\lambda}\kappa_\mu} \prod_{\mu=1}^{2\lambda} m_{\kappa_\mu} m_{2k} \\ & + \sum_{\lambda=1}^{k-1} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda}\le 2k-1} (-1)^{k+\sum_{\mu=1}^{2\lambda}\kappa_\mu} \prod_{\mu=1}^{2\lambda} m_{\kappa_\mu} m_{2k+1}.\end{aligned}$$

The first three terms on the right-hand side form

$$\sum_{\kappa=1}^{2k+1} (-1)^{k+1+\kappa} m_\kappa.$$

The next two sums are composed to

$$\begin{aligned} & \sum_{\lambda=0}^{k-1} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda+1}\le 2k-1<\kappa_{2\lambda+2}=2k<\kappa_{2\lambda+3}=2k+1} (-1)^{k+1+\sum_{\mu=1}^{2\lambda+3}\kappa_\mu} \prod_{\mu=1}^{2\lambda+3} m_{\kappa_\mu} \\ & \quad = \sum_{\lambda=1}^{k-1} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda-1}\le 2k-1<\kappa_{2\lambda}=2k<\kappa_{2\lambda+1}=2k+1} (-1)^{k+1+\sum_{\mu=1}^{2\lambda+1}\kappa_\mu} \prod_{\mu=1}^{2\lambda+1} m_{\kappa_\mu} + \prod_{\mu=1}^{2k+1} m_\mu.\end{aligned}$$

By leaving the next sum unchanged, the last two become

$$\sum_{\lambda=1}^{k-1} \Bigg[ \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda}\le 2k-1<\kappa_{2\lambda+1}=2k} + \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda}\le 2k-1<\kappa_{2\lambda+1}=2k+1} \Bigg] (-1)^{k+1+\sum_{\mu=1}^{2\lambda+1}\kappa_\mu} \prod_{\mu=1}^{2\lambda+1} m_{\kappa_\mu}.$$

This proves

$$\begin{aligned}\delta & = \sum_{\kappa=1}^{2k+1} (-1)^{k+1+\kappa} m_\kappa + \sum_{\lambda=1}^{k-1} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda+1}\le 2k+1} (-1)^{k+1+\sum_{\mu=1}^{2\lambda+1}\kappa_\mu} \prod_{\mu=1}^{2\lambda+1} m_{\kappa_\mu} + \prod_{\mu=1}^{2k+1} m_\mu \\ & = \sum_{\lambda=0}^{k} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda+1}\le 2k+1} (-1)^{k+1+\sum_{\mu=1}^{2\lambda+1}\kappa_\mu} \prod_{\mu=1}^{2\lambda+1} m_{\kappa_\mu} \\ & = \delta_{2k+1}.\end{aligned}$$

In the same manner, the other three formulas can be verified. □

**Remark 2.2.** The expressions for $\gamma_k$ and $\delta_k$ in Theorem 2.1 show that these quantities are combinations of products of the $m_k$. That they in fact form very regular and beautiful algebraic combinations can be seen by writing these expressions down explicitly. These formulas of arbitrary order can even be produced by computer manipulations on the basis of the recurrence relations (2.1) and (2.2). Moreover, the $\gamma_k$ and $\delta_k$ can in the same way be expressed through the $\alpha_k$ and $\beta_k$. Also, these formulas show a very regular and beautiful algebraic structure; see A.3 and A.4.

As the sequences $\{\alpha_k\}$, $\{\beta_k\}$, also $\{\gamma_k\}$, $\{\delta_k\}$ converge to 0, but only the first two are monotone decreasing.

**Theorem 2.3.** *The sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are monotone. They and the sequences $\{\gamma_k\}$ and $\{\delta_k\}$ converge to* 0.

*Proof.* From the monotonicity of $\{m_k\}$, the estimations

$$m_1(\beta_{k+1} - \beta_k) = \alpha_{k+1} - \alpha_k = m_{k+1}(m_{k+2} - m_k) < 0$$

are obvious. That $\{\alpha_k\}$ and $\{\beta_k\}$ are null-sequences follows from $\lim_{k\to\infty} m_k = 1$. The other two sequences consist of non-negative numbers. This is seen from (2.1) and (2.2) as the initial values are non-negative and the $\alpha_k, \beta_k$ are positive numbers. From the assumption

$$\gamma_k, \delta_k \le \frac{r_k^2(m_1 + 1)}{m_1 + m_k},$$

by using (1.1) reformulated as

$$\beta_{k+1} = \frac{r_{k+1}^2}{m_1 + m_{k+1}},$$

both right-hand sides from

$$\delta_{k+2} = \beta_{k+1}[m_1\delta_k + \gamma_k], \quad \gamma_{k+2} = \beta_{k+1}[m_1\gamma_k + \delta_k],$$

can be estimated from above by

$$\frac{r_{k+1}^2(m_1 + 1)}{m_1 + m_{k+1}} \frac{r_k^2(m_1 + 1)}{m_1 + m_k}.$$

The factor

$$\frac{r_k^2(m_1 + 1)}{m_1 + m_k}$$

is less than 1 for $1 < k$, as can be seen from

$$m_k^2 + m_k(m_k m_1 - 1) < m_2^2 + m_2(m_2 m_1 - 1) = m_2^2 + m_2 < 2m_1 + 1.$$

The assumptions made are readily satisfied for $k = 1, 2$. □

# 3 A second recurrence relation

The recurrence relations (2.1) and (2.2) are providing some other recurrence relations.

For $k \in \mathbb{N}_0$, the terms

$$\Delta_{2k} = m_1\delta_{2k} + \gamma_{2k}, \quad \Gamma_{2k} = m_1\gamma_{2k} + \delta_{2k}, \quad \Delta_{2k+1} = m_1\delta_{2k+1} - \gamma_{2k+1}, \quad \Gamma_{2k+1} = m_1\gamma_{2k+1} - \delta_{2k+1}$$

satisfy the systems

$$\Delta_{2k+2} = \alpha_{2k+1}\Delta_{2k} + \beta_{2k+1}\Gamma_{2k}, \quad \Gamma_{2k+2} = \alpha_{2k+1}\Gamma_{2k} + \beta_{2k+1}\Delta_{2k}, \tag{3.1}$$

$$\Delta_{2k+1} = \alpha_{2k}\Delta_{2k-1} + \beta_{2k}\Gamma_{2k-1}, \quad \Gamma_{2k+1} = \alpha_{2k}\Gamma_{2k-1} + \beta_{2k}\Delta_{2k-1}. \tag{3.2}$$

This is easily deduced from (2.1) and (2.2). The first members of these sequences are also listed in Section A.

A direct consequence from Theorem 2.1 is the next statement.

**Theorem 3.1.** *With the initial data* $\gamma_0 = 0$, $\delta_0 = 1$, $\gamma_1 = 1$, $\delta_1 = m_1$, *i.e.* $\Delta_0 = m_1$, $\Gamma_0 = 1$, $\Delta_1 = 2$, $\Gamma_1 = 0$, *systems* (3.1) *and* (3.2) *have the solutions*

$$\Delta_{2k} = 2\sum_{\lambda=1}^{k} \sum_{2\le\kappa_1<\kappa_2<\cdots<\kappa_{2\lambda-1}\le 2k} (-1)^{k+1+\sum_{\mu=1}^{2\lambda-1}\kappa_\mu} \prod_{\mu=1}^{2\lambda-1} m_{\kappa_\mu},$$

$$\Gamma_{2k} = 2\sum_{\lambda=1}^{k-1} \sum_{2\le\kappa_1<\kappa_2<\cdots<\kappa_{2\lambda}\le 2k} (-1)^{k+1+\sum_{\mu=1}^{2\lambda}\kappa_\mu} \prod_{\mu=1}^{2\lambda} m_{\kappa_\mu} + (-1)^{k+1}2,$$

$$\Delta_{2k+1} = 2\sum_{\lambda=1}^{k} \sum_{2\le\kappa_1<\kappa_2<\cdots<\kappa_{2\lambda}\le 2k+1} (-1)^{k+\sum_{\mu=1}^{2\lambda}\kappa_\mu} \prod_{\mu=1}^{2\lambda} m_{\kappa_\mu} + (-1)^{k}2,$$

$$\Gamma_{2k+1} = 2\sum_{\lambda=1}^{k} \sum_{2\le\kappa_1<\kappa_2<\cdots<\kappa_{2\lambda-1}\le 2k+1} (-1)^{k+1+\sum_{\mu=1}^{2\lambda-1}\kappa_\mu} \prod_{\mu=1}^{2\lambda-1} m_{\kappa_\mu}.$$

*Proof.* Exemplarily, the second formula will be verified from

$$\Gamma_{2k} = m_1\gamma_{2k} + \delta_{2k}.$$

Splitting

$$\begin{aligned}m_1\gamma_{2k} &= m_1 \sum_{\lambda=1}^{k} \sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda-1}\le 2k} (-1)^{k+\sum_{\mu=1}^{2\lambda-1}\kappa_\mu} \prod_{\mu=1}^{2\lambda-1} m_{\kappa_\mu} \\ &= m_1\Big[\sum_{\kappa=1}^{2k}(-1)^{k+\kappa}m_\kappa + \sum_{\lambda=2}^{k}\Big(\sum_{1=\kappa_1<\kappa_2<\cdots<\kappa_{2\lambda-1}\le 2k} + \sum_{2\le\kappa_1<\kappa_2<\cdots<\kappa_{2\lambda-1}\le 2k}\Big)(-1)^{k+\sum_{\mu=1}^{2\lambda-1}\kappa_\mu}\prod_{\mu=1}^{2\lambda-1} m_{\kappa_\mu}\Big],\end{aligned}$$

separating the factor $m_1$ and renaming summation indices lead to

$$m_1\gamma_{2k} = m_1^2\Big[(-1)^{k+1} - \sum_{\lambda=1}^{k-1}\sum_{2\le\kappa_1<\cdots<\kappa_{2\lambda}\le 2k}(-1)^{k+\sum_{\mu=1}^{2\lambda}\kappa_\mu}\prod_{\mu=1}^{2\lambda} m_{\kappa_\mu}\Big] + m_1\Big[\sum_{\lambda=1}^{k}\sum_{2\le\kappa_1<\cdots<\kappa_{2\lambda-1}\le 2k}(-1)^{k+\sum_{\mu=1}^{2\lambda-1}\kappa_\mu}\prod_{\mu=1}^{2\lambda-1} m_{\kappa_\mu}\Big].$$

In a similar way, $\delta_{2k}$ is split:

$$\begin{aligned}\delta_{2k} &= (-1)^k + \sum_{\lambda=1}^{k}\sum_{1\le\kappa_1<\cdots<\kappa_{2\lambda}\le 2k}(-1)^{k+\sum_{\mu=1}^{2\lambda}\kappa_\mu}\prod_{\mu=1}^{2\lambda} m_{\kappa_\mu} \\ &= (-1)^k + \sum_{\lambda=1}^{k-1}\sum_{2\le\kappa_1<\cdots<\kappa_{2\lambda}\le 2k}(-1)^{k+\sum_{\mu=1}^{2\lambda}\kappa_\mu}\prod_{\mu=1}^{2\lambda} m_{\kappa_\mu} - m_1\sum_{\lambda=1}^{k}\sum_{2\le\kappa_1<\cdots<\kappa_{2\lambda-1}\le 2k}(-1)^{k+\sum_{\mu=1}^{2\lambda-1}\kappa_\mu}\prod_{\mu=1}^{2\lambda-1} m_{\kappa_\mu}.\end{aligned}$$

Adding the two formulas gives the expression for $\Gamma_{2k}$. □

As the $\{\gamma_k\}$, $\{\delta_k\}$, also $\{\Gamma_k\}$, $\{\Delta_k\}$ converge to 0.

**Remark 3.2.** Remark 2.2 also applies for the $\Gamma_k$ and $\Delta_k$. They are expressible either through the $m_k$ or the $\alpha_k$ and $\beta_k$; see A.5 and A.6. But neither the formulas in Theorem 2.1 nor those in Theorem 3.1 reveal the beauty and regularity of the algebraic pattern. But writing these formulas out or simpler using some computer algebra to develop the single representations on the basis of the respective recurrence relations (2.1) and (2.2) or (3.1) and (3.2) unveils their symmetry.

**Lemma 3.3.** *For $k \in \mathbb{N}$,*

$$\Gamma_{4k-1} = 2\sum_{\lambda=0}^{k-1}\prod_{\varrho=1}^{2k-2\lambda-2}\alpha_{2\mu_\varrho}\prod_{\tau=1}^{2\lambda+1}\beta_{2\nu_\tau}, \quad \Delta_{4k-1} = 2\sum_{\lambda=0}^{k-1}\prod_{\varrho=1}^{2k-2\lambda-1}\alpha_{2\mu_\varrho}\prod_{\tau=1}^{2\lambda}\beta_{2\nu_\tau}$$

*with*

$$\begin{aligned}&1 \le \mu_1 < \mu_2 < \cdots < \mu_{2k-2\lambda-2} \le 2k-1,\\ &1 \le \nu_1 < \nu_2 < \cdots < \nu_{2\lambda+1} \le 2k-1,\\ &\mu_\varrho \ne \nu_\tau,\end{aligned}$$

*and*

$$\begin{aligned}&1 \le \mu_1 < \mu_2 < \cdots < \mu_{2k-2\lambda-1} \le 2k-1,\\ &1 \le \nu_1 < \nu_2 < \cdots < \nu_{2\lambda} \le 2k-1,\\ &\mu_\varrho \ne \nu_\tau,\end{aligned}$$

*respectively. Also,*

$$\Gamma_{4k+1} = 2\sum_{\lambda=0}^{k}\prod_{\varrho=1}^{2k-2\lambda-1}\alpha_{2\mu_\varrho}\prod_{\tau=1}^{2\lambda+1}\beta_{2\nu_\tau}, \quad \Delta_{4k+1} = 2\sum_{\lambda=0}^{k}\prod_{\varrho=1}^{2k-2\lambda}\alpha_{2\mu_\varrho}\prod_{\tau=1}^{2\lambda}\beta_{2\nu_\tau}, \tag{3.3}$$

*where*

$$1 \le \mu_1 < \mu_2 < \cdots < \mu_{2k-2\lambda-1} \le 2k,$$
$$1 \le \nu_1 < \nu_2 < \cdots < \nu_{2\lambda+1} \le 2k,$$
$$\mu_\varrho \ne \nu_\tau,$$

*and*

$$1 \le \mu_1 < \mu_2 < \cdots < \mu_{2k-2\lambda} \le 2k,$$
$$1 \le \nu_1 < \nu_2 < \cdots < \nu_{2\lambda} \le 2k,$$
$$\mu_\varrho \ne \nu_\tau,$$

*respectively.*

*Proof.* By the relations

$$\Gamma_1 = m_1\gamma_1 - \delta_1 = 0, \quad \Delta_1 = m_1\delta_1 - \gamma_1 = m_1^2 - 1 = 2,$$

$m_1$ is eliminated as parameter. From (3.2), then

$$\Gamma_3 = \alpha_2\Gamma_1 + \beta_2\Delta_1 = 2\beta_2, \quad \Delta_3 = \alpha_2\Delta_1 + \beta_2\Gamma_1 = 2\alpha_2$$

follow. By assuming equations (3.3) to hold, formula (3.2) implies

$$\Delta_{4k+3} = 2\sum_{\lambda=0}^{k}\Big[\prod_{\varrho=1}^{2k-2\lambda+1}\alpha_{2\mu_\varrho}\prod_{\tau=1}^{2\lambda}\beta_{2\nu_\tau} + \prod_{\varrho=1}^{2k-2\lambda-1}\alpha_{2\mu_\varrho}\prod_{\tau=1}^{2\lambda+2}\beta_{2\nu_\tau}\Big]$$

with

$$1 \le \mu_1 < \cdots < \mu_{2k-2\lambda+1} = 2k+1,$$
$$1 \le \nu_1 < \cdots < \nu_{2\lambda} < 2k+1,$$

and

$$1 \le \mu_1 < \cdots < \mu_{2k-2\lambda-1} < 2k+1,$$
$$1 \le \nu_1 < \cdots < \nu_{2\lambda+2} = 2k+1,$$
$$\mu_\varrho \ne \nu_\tau,$$

respectively. Shifting the summation for the second term on the right-hand side replacing $\lambda$ by $\lambda - 1$ gives

$$\Delta_{4k+3} = 2\sum_{\lambda=0}^{k+1}\prod_{\varrho=1}^{2k-2\lambda+1}\alpha_{2\mu_\varrho}\prod_{\tau=1}^{2\lambda}\beta_{2\nu_\tau}$$

with

$$1 \le \mu_1 < \cdots < \mu_{2k-2\lambda+1} \le 2k+1,$$
$$1 \le \nu_1 < \cdots < \nu_{2\lambda} \le 2k+1,$$
$$\mu_\varrho \ne \nu_\tau.$$

In the same way, the part for $\Gamma_{4k+3}$ can be handled. By repeating the procedure, the respective formulas for the index $4k+5$ can be achieved, completing the proof. □

**Lemma 3.4.** *For $k \in \mathbb{N}$,*

$$\Gamma_{4k} = 2\sum_{\lambda=0}^{k-1}\Big[m_2\prod_{\tau=1}^{2\lambda}\alpha_{2\nu_\tau+1}\prod_{\varrho=1}^{2k-2\lambda-1}\beta_{2\mu_\varrho+1} + \prod_{\varrho=1}^{2k-2\lambda-1}\alpha_{2\mu_\varrho+1}\prod_{\tau=1}^{2\lambda}\beta_{2\nu_\tau+1}\Big], \tag{3.4}$$

$$\Delta_{4k} = 2\sum_{\lambda=0}^{k-1}\Big[m_2\prod_{\varrho=1}^{2k-2\lambda-1}\alpha_{2\mu_\varrho+1}\prod_{\tau=1}^{2\lambda}\beta_{2\nu_\tau+1} + \prod_{\tau=1}^{2\lambda}\alpha_{2\nu_\tau+1}\prod_{\varrho=1}^{2k-2\lambda-1}\beta_{2\mu_\varrho+1}\Big]. \tag{3.5}$$

*Here the indices involved are partitions of the set*

$$\{1, 2, \ldots, 2k-1\} = \{\mu_1, \mu_2, \ldots, \mu_{2k-2\lambda-1}\} \cup \{\nu_1, \nu_2, \ldots, \nu_{2\lambda}\}$$

*satisfying*

$$1 \le \mu_1 < \cdots < \mu_{2k-2\lambda-1} \le 2k-1, \quad 1 \le \nu_1 < \cdots < \nu_{2\lambda} \le 2k-1.$$

*Moreover,*

$$\Gamma_{4k+2} = 2\sum_{\lambda=0}^{k-1} m_2 \prod_{\tau=1}^{2\lambda+1} \alpha_{2\nu_\tau+1} \prod_{\varrho=1}^{2k-2\lambda-1} \beta_{2\mu_\varrho+1} + 2\sum_{\lambda=0}^{k} \prod_{\varrho=1}^{2k-2\lambda} \alpha_{2\mu_\varrho+1} \prod_{\tau=1}^{2\lambda} \beta_{2\nu_\tau+1}, \tag{3.6}$$

$$\Delta_{4k+2} = 2\sum_{\lambda=0}^{k} m_2 \prod_{\varrho=1}^{2k-2\lambda} \alpha_{2\mu_\varrho+1} \prod_{\tau=1}^{2\lambda} \beta_{2\nu_\tau+1} + 2\sum_{\lambda=0}^{k-1} \prod_{\tau=1}^{2\lambda+1} \alpha_{2\nu_\tau+1} \prod_{\varrho=1}^{2k-2\lambda-1} \beta_{2\mu_\varrho+1}, \tag{3.7}$$

*where the indices again are decompositions of* $\{1, 2, \ldots, 2k\}$*, each subset ordered according to size.*

*Proof.* Starting from

$$\Gamma_0 = \delta_0 = 1, \quad \Delta_0 = m_1\delta_1 = m_1$$

or from

$$\Gamma_2 = m_1\gamma_2 + \delta_2 = m_1^2 - 1 = 2, \quad \Delta_2 = m_1\delta_2 + \gamma_2 = 2m_2$$

creates similar but different expressions. By neglecting the right-hand sides from the last two equations, $m_1$ can be kept as parameter. By using the latter relations for starting, from formulas (3.1),

$$\Gamma_4 = 2m_2\beta_3 + 2\alpha_3, \quad \Delta_4 = 2m_2\alpha_3 + 2\beta_3$$

follow. By assuming (3.4) and (3.5) to hold, from (3.1) follow

$$\Delta_{4k+2} = \alpha_{4k+1}\Delta_{4k} + \beta_{4k+1}\Gamma_{4k} = 2m_2\Sigma_1 + 2\Sigma_2,$$

where

$$\Sigma_1 = \sum_{\lambda=0}^{k-1}\Big[\prod_{\varrho=1}^{2k-2\lambda} \alpha_{2\mu_\varrho+1} \prod_{\tau=1}^{2\lambda} \beta_{2\nu_\tau+1} + \prod_{\tau=1}^{2\lambda} \alpha_{2\nu_\tau+1} \prod_{\varrho=1}^{2k-2\lambda} \beta_{2\mu_\varrho+1}\Big]$$

with

$$1 \le \mu_1 < \cdots < \mu_{2k-2\lambda} = 2k, \quad 1 \le \nu_1 < \cdots < \nu_{2\lambda} < 2k,$$

and

$$\Sigma_2 = \sum_{\lambda=0}^{k-1}\Big[\prod_{\varrho=1}^{2k-2\lambda-1} \beta_{2\mu_\varrho+1} \prod_{\tau=1}^{2\lambda+1} \alpha_{2\nu_\tau+1} + \prod_{\tau=1}^{2\lambda+1} \beta_{2\nu_\tau+1} \prod_{\varrho=1}^{2k-2\lambda-1} \alpha_{2\mu_\varrho+1}\Big]$$

with

$$1 \le \mu_1 < \cdots < \mu_{2k-2\lambda-1} < 2k, \quad 1 \le \nu_1 < \cdots < \nu_{2\lambda+1} = 2k.$$

Reflecting the summation index, i.e. interchanging $\lambda$ with $k-\lambda$, in the second part of $\Sigma_1$ gives for this part the same expression as in the first part, but the summation is taken between $\lambda = 1$ and $\lambda = k$. The indices now vary according to

$$1 \le \mu_1 < \cdots < \mu_{2\lambda-1} < 2k, \quad 1 \le \nu_1 < \cdots < \nu_{2k-2\lambda+1} = 2k.$$

Thus,

$$\Sigma_1 = \sum_{\lambda=0}^{k} \prod_{\varrho=1}^{2k-2\lambda} \alpha_{2\mu_\varrho+1} \prod_{\tau=1}^{2\lambda} \beta_{2\nu_\tau+1}$$

with

$$1 \le \mu_1 < \cdots < \mu_{2k-2\lambda} \le 2k, \quad 1 \le \nu_1 < \cdots < \nu_{2\lambda} \le 2k.$$

In the same way, $\Sigma_2$ is handled, where in the second part besides reflecting also shifting the summation index is used. Finally, (3.5) is attained. By the symmetry, thus also (3.4) is proved.

To finish the proof, the procedure has to be repeated to get the formulas for $\Gamma_{4(k+1)}$ and $\Delta_{4(k+1)}$ from (3.6) and (3.7). This part is skipped. □

In order to give an impression of the latter pattern, the first elements of the sequences $\{\Gamma_{2k}\}$, $\{\Gamma_{2k+1}\}$, $\{\Delta_{2k}\}$, $\{\Delta_{2k+1}\}$ are also listed in Section A.

# A Visualization of the sequences

**A.1.** $\{m_k\}$:

$$\sqrt{3}, \frac{2}{3}\sqrt{3}, \frac{3}{5}\sqrt{3}, \frac{7}{12}\sqrt{3}, \frac{11}{19}\sqrt{3}, \frac{26}{45}\sqrt{3}, \frac{41}{71}\sqrt{3}, \frac{97}{168}\sqrt{3}, \frac{153}{265}\sqrt{3}, \frac{362}{627}\sqrt{3}, \ldots .$$

**A.2.** $\{r_k\}$:

$$\sqrt{2}, \frac{\sqrt{3}}{3}, \frac{\sqrt{2}}{5}, \frac{\sqrt{3}}{12}, \frac{\sqrt{2}}{19}, \frac{\sqrt{3}}{45}, \frac{\sqrt{2}}{71}, \frac{\sqrt{3}}{168}, \frac{\sqrt{2}}{265}, \frac{\sqrt{3}}{627}, \ldots .$$

**A.3.** $\{\gamma_k\}$, $\{\delta_k\}$ expressed through $\{m_k\}$:

$$\begin{aligned}
\gamma_0 &= 0,\\
\delta_0 &= 1,\\
\gamma_1 &= 1,\\
\delta_1 &= m_1,\\
\gamma_2 &= m_1 - m_2,\\
\delta_2 &= \alpha_1 = m_1 m_2 - 1,\\
\gamma_3 &= m_1 m_2 - m_1 m_3 + m_2 m_3 - 1,\\
\delta_3 &= m_1 m_2 m_3 - m_1 + m_2 - m_3,\\
\gamma_4 &= m_1 m_2 m_3 - m_1 m_2 m_4 + m_1 m_3 m_4 - m_2 m_3 m_4 - m_1 + m_2 - m_3 + m_4,\\
\delta_4 &= m_1 m_2 m_3 m_4 - m_1 m_2 + m_1 m_3 - m_1 m_4 - m_2 m_3 + m_2 m_4 - m_3 m_4 + 1,\\
\gamma_5 &= m_1 m_2 m_3 m_4 - m_1 m_2 m_3 m_5 + m_1 m_2 m_4 m_5 - m_1 m_3 m_4 m_5\\
&\qquad + m_2 m_3 m_4 m_5 - m_1 m_2 + m_1 m_3 - m_1 m_4 + m_1 m_5\\
&\qquad - m_2 m_3 + m_2 m_4 - m_2 m_5 - m_3 m_4 + m_3 m_5 + 1,\\
\delta_5 &= m_1 m_2 m_3 m_4 m_5 - m_1 m_2 m_3 + m_1 m_2 m_4 - m_1 m_2 m_5 - m_1 m_3 m_4\\
&\qquad + m_1 m_3 m_5 - m_1 m_4 m_5 + m_2 m_3 m_4 - m_2 m_3 m_5 + m_2 m_4 m_5\\
&\qquad - m_3 m_4 m_5 + m_1 - m_2 + m_3 - m_4 + m_5,\\
&\vdots
\end{aligned}$$

**A.4.** $\{\gamma_k\}$, $\{\delta_k\}$ expressed through $\{\alpha_k\}$, $\{\beta_k\}$:

$$\begin{aligned}
\gamma_0 &= 0,\\
\delta_0 &= 1,\\
\gamma_2 &= \beta_1,\\
\delta_2 &= \alpha_1,\\
\gamma_4 &= \alpha_3\beta_1 + \beta_3\alpha_1,\\
\delta_4 &= \alpha_3\alpha_1 + \beta_3\beta_1,\\
\gamma_6 &= \alpha_5\alpha_3\beta_1 + \alpha_5\beta_3\alpha_1 + \beta_5\alpha_3\alpha_1 + \beta_5\beta_3\beta_1\\
\delta_6 &= \alpha_5\alpha_3\alpha_1 + \alpha_5\beta_3\beta_1 + \beta_5\alpha_3\beta_1 + \beta_5\beta_3\alpha_1,\\
\gamma_8 &= \alpha_7\alpha_5\alpha_3\beta_1 + \alpha_7\alpha_5\beta_3\alpha_1 + \alpha_7\beta_5\alpha_3\alpha_1 + \beta_7\alpha_5\alpha_3\alpha_1\\
&\qquad + \alpha_7\beta_5\beta_3\beta_1 + \beta_7\alpha_5\beta_3\beta_1 + \beta_7\beta_5\alpha_3\beta_1 + \beta_7\beta_5\beta_3\alpha_1,\\
\delta_8 &= \alpha_7\alpha_5\alpha_3\alpha_1 + \alpha_7\alpha_5\beta_3\beta_1 + \alpha_7\beta_5\alpha_3\beta_1 + \alpha_7\beta_5\beta_3\alpha_1\\
&\qquad + \beta_7\alpha_5\alpha_3\beta_1 + \beta_7\alpha_5\beta_3\alpha_1 + \beta_7\beta_5\alpha_3\alpha_1 + \beta_7\beta_5\beta_3\beta_1,
\end{aligned}$$

$$\begin{aligned}
\gamma_{10} = {} & \alpha_9\alpha_7\alpha_5\alpha_3\beta_1 + \alpha_9\alpha_7\alpha_5\beta_3\alpha_1 + \alpha_9\alpha_7\beta_5\alpha_3\alpha_1 + \alpha_9\beta_7\alpha_5\alpha_3\alpha_1 \\
& + \beta_9\alpha_7\alpha_5\alpha_3\alpha_1 + \alpha_9\alpha_7\beta_5\beta_3\beta_1 + \alpha_9\beta_7\alpha_5\beta_3\beta_1 + \alpha_9\beta_7\beta_5\alpha_3\beta_1 \\
& + \alpha_9\beta_7\beta_5\beta_3\alpha_1 + \beta_9\alpha_7\alpha_5\beta_3\beta_1 + \beta_9\alpha_7\beta_5\alpha_3\beta_1 + \beta_9\alpha_7\beta_5\beta_3\alpha_1 \\
& + \beta_9\beta_7\alpha_5\alpha_3\beta_1 + \beta_9\beta_7\alpha_5\beta_3\alpha_1 + \beta_9\beta_7\beta_5\alpha_3\alpha_1 + \beta_9\beta_7\beta_5\beta_3\beta_1 \\
\delta_{10} = {} & \alpha_9\alpha_7\alpha_5\alpha_3\alpha_1 + \alpha_9\alpha_7\alpha_5\beta_3\beta_1 + \alpha_9\alpha_7\beta_5\alpha_3\beta_1 + \alpha_9\alpha_7\beta_5\beta_3\alpha_1 \\
& + \alpha_9\beta_7\alpha_5\alpha_3\beta_1 + \alpha_9\beta_7\alpha_5\beta_3\alpha_1 + \alpha_9\beta_7\beta_5\alpha_3\alpha_1 + \beta_9\alpha_7\alpha_5\alpha_3\beta_1 \\
& + \beta_9\alpha_7\alpha_5\beta_3\alpha_1 + \beta_9\alpha_7\beta_5\alpha_3\alpha_1 + \beta_9\beta_7\alpha_5\alpha_3\alpha_1 + \alpha_9\beta_7\beta_5\beta_3\beta_1 \\
& + \beta_9\alpha_7\beta_5\beta_3\beta_1 + \beta_9\beta_7\alpha_5\beta_3\beta_1 + \beta_9\beta_7\beta_5\alpha_3\beta_1 + \beta_9\beta_7\beta_5\beta_3\alpha_1
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
\gamma_1 &= 1, \\
\delta_1 &= m_1, \\
\gamma_3 &= \alpha_2 + \beta_2 m_1, \\
\delta_3 &= \alpha_2 m_1 + \beta_2, \\
\gamma_5 &= (\alpha_4\beta_2 + \beta_4\alpha_2)m_1 + \alpha_4\alpha_2 + \beta_4\beta_2, \\
\delta_5 &= (\alpha_4\alpha_2 + \beta_4\beta_2)m_1 + \alpha_4\beta_2 + \beta_4\alpha_2, \\
\gamma_7 &= (\alpha_6\alpha_4\beta_2 + \alpha_6\beta_4\alpha_2 + \beta_6\alpha_4\alpha_2 + \beta_6\beta_4\beta_2)m_1 + \alpha_6\alpha_4\alpha_2 + \alpha_6\beta_4\beta_2 + \beta_6\alpha_4\beta_2 + \beta_6\beta_4\alpha_2, \\
\delta_7 &= (\alpha_6\alpha_4\alpha_2 + \alpha_6\beta_4\beta_2 + \beta_6\alpha_4\beta_2 + \beta_6\beta_4\alpha_2)m_1 + \alpha_6\alpha_4\beta_2 + \alpha_6\beta_4\alpha_2 + \beta_6\alpha_4\alpha_2 + \beta_6\beta_4\beta_2, \\
\gamma_9 &= (\alpha_8\alpha_6\alpha_4\beta_2 + \alpha_8\alpha_6\beta_4\alpha_2 + \alpha_8\beta_6\alpha_4\alpha_2 + \beta_8\alpha_6\alpha_4\alpha_2 \\
&\qquad + \alpha_8\beta_6\beta_4\beta_2 + \beta_8\alpha_6\beta_4\beta_{+}\beta_8\beta_6\alpha_4\beta_2 + \beta_8\beta_6\beta_4\alpha_2)m_1 \\
&\qquad + \alpha_8\alpha_6\alpha_4\alpha_2 + \alpha_8\alpha_6\beta_4\beta_2 + \alpha_8\beta_6\alpha_4\beta_2 + \beta_8\alpha_6\alpha_4\beta_2 \\
&\qquad + \alpha_8\beta_6\beta_4\alpha_2 + \beta_8\alpha_6\beta_4\alpha_2 + \beta_8\beta_6\alpha_4\alpha_2 + \beta_8\beta_6\beta_4\beta_2, \\
\delta_9 &= (\alpha_8\alpha_6\alpha_4\alpha_2 + \alpha_8\alpha_6\beta_4\beta_2 + \alpha_8\beta_6\alpha_4\beta_2 + \beta_8\alpha_6\alpha_4\beta_2 \\
&\qquad + \alpha_8\beta_6\beta_4\alpha_2 + \beta_8\alpha_6\beta_4\alpha_2 + \beta_8\beta_6\alpha_4\alpha_2 + \beta_8\beta_6\beta_4\beta_2)m_1 \\
&\qquad + \alpha_8\alpha_6\alpha_4\beta_2 + \alpha_8\alpha_6\beta_4\alpha_2 + \alpha_8\beta_6\alpha_4\alpha_2 + \beta_8\alpha_6\alpha_4\alpha_2 \\
&\qquad + \alpha_8\beta_6\beta_4\beta_2 + \beta_8\alpha_6\beta_4\beta_{+}\beta_8\beta_6\alpha_4\beta_2 + \beta_8\beta_6\beta_4\alpha_2,
\end{aligned}$$

$$\vdots$$

**A.5.** $\{\Gamma_k\}$, $\{\Delta_k\}$ expressed through $\{m_k\}$:

$$\begin{aligned}
\Gamma_0 &= 1, \\
\Delta_0 &= m_1, \\
\Gamma_2 &= 2, \\
\Delta_2 &= 2m_2, \\
\Gamma_4 &= 2(m_2m_3 - m_2m_4 + m_3m_4 - 1), \\
\Delta_4 &= 2(m_2m_3m_4 - m_2 + m_3 - m_4), \\
\Gamma_6 &= 2(m_2m_3m_4m_5 - m_2m_3m_4m_6 + m_2m_3m_5m_6 - m_2m_4m_5m_6 + m_3m_4m_5m_6 - m_2m_3 + m_2m_4 \\
&\qquad - m_2m_5 + m_2m_6 - m_3m_4 + m_3m_5 - m_3m_6 + m_4m_5 - m_4m_6 + m_5m_6 + 1), \\
\Delta_6 &= 2(m_2m_3m_4m_5m_6 - m_2m_3m_4 + m_2m_3m_5 - m_2m_3m_6 + m_2m_4m_5 - m_2m_5m_6 \\
&\qquad + m_3m_4m_5 - m_3m_4m_6 + m_3m_5m_6 - m_4m_5m_6 + m_2 - m_3 + m_4 - m_5 + m_6),
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
\Gamma_1 &= 0,\\
\Delta_1 &= 2,\\
\Gamma_3 &= 2(m_2 - m_3),\\
\Delta_3 &= 2(m_2 m_3 - 1),\\
\Gamma_5 &= 2(m_2m_3m_4 - m_2m_3m_5 + m_2m_4m_5 - m_3m_4m_5 - m_2 + m_3 - m_4 + m_5),\\
\Delta_5 &= 2(m_2m_3m_5m_5 - m_2m_3 + m_2m_4 - m_2m_5 - m_3m_4 + m_3m_5 - m_4m_5 + 1),\\
\Gamma_7 &= 2(m_2m_3m_4m_5m_6 - m_2m_3m_4m_5m_7 + m_2m_3m_4m_6m_7 - m_2m_3m_5m_6m_7\\
&\qquad + m_2m_4m_5m_6m_7 - m_3m_4m_5m_6m_7 - m_2m_3m_4 + m_2m_3m_5 - m_2m_3m_6\\
&\qquad + m_2m_3m_7 - m_2m_4m_5 + m_2m_4m_6 - m_2m_4m_7 - m_2m_5m_6 + m_2m_5m_7\\
&\qquad - m_2m_6m_7 + m_3m_4m_5 - m_3m_4m_6 + m_3m_4m_7 + m_3m_5m_6 - m_3m_5m_7\\
&\qquad + m_3m_6m_7 - m_4m_5m_6 + m_4m_5m_7 - m_4m_6m_7 + m_5m_6m_7\\
&\qquad + m_2 - m_3 + m_4 - m_5 + m_6 - m_7),\\
\Delta_7 &= 2(m_2m_3m_4m_5m_6m_7 - m_2m_3m_4m_5 + m_2m_3m_4m_6 - m_2m_3m_4m_7 - m_2m_3m_5m_6\\
&\qquad + m_2m_3m_5m_7 - m_2m_3m_6m_7 + m_2m_4m_5m_6 - m_2m_4m_5m_7 + m_2m_4m_6m_7\\
&\qquad - m_2m_5m_6m_7 - m_3m_4m_5m_6 + m_3m_4m_5m_7 - m_3m_4m_6m_7 + m_3m_5m_6m_7\\
&\qquad - m_4m_5m_6m_7 + m_2m_3 - m_2m_4 + m_2m_5 - m_2m_6 + m_2m_7 + m_3m_4 - m_3m_5\\
&\qquad + m_3m_6 - m_3m_7 + m_4m_5 - m_4m_6 + m_4m_7 + m_5m_6 - m_5m_7 + m_6m_7 - 1),\\
&\ \vdots
\end{aligned}$$

**A.6.** $\{\Gamma_k\}, \{\Delta_k\}$ expressed through $\{\alpha_k\}, \{\beta_k\}$:

$$\begin{aligned}
\Gamma_0 &= 1,\\
\Delta_0 &= m_1,\\
\Gamma_2 &= 2,\\
\Delta_2 &= 2m_2,\\
\Gamma_4 &= 2[m_2\beta_3 + \alpha_3],\\
\Delta_4 &= 2[m_2\alpha_3 + \beta_3],\\
\Gamma_6 &= 2[m_2(\alpha_3\beta_5 + \beta_3\alpha_5) + \alpha_3\alpha_5 + \beta_3\beta_5],\\
\Delta_6 &= 2[m_2(\alpha_3\alpha_5 + \beta_3\beta_5) + \alpha_3\beta_5 + \beta_3\alpha_5],\\
\Gamma_8 &= 2[m_2(\alpha_3\alpha_5\beta_7 + \alpha_3\beta_5\alpha_7 + \beta_3\alpha_5\alpha_7 + \beta_3\beta_5\beta_7) + \alpha_3\alpha_5\alpha_7 + \alpha_3\beta_5\beta_7 + \beta_3\alpha_5\beta_7 + \beta_3\beta_5\alpha_7],\\
\Delta_8 &= 2[m_2(\alpha_3\alpha_5\alpha_7 + \alpha_3\beta_5\beta_7 + \beta_3\alpha_5\beta_7 + \beta_3\beta_5\alpha_7) + \alpha_3\alpha_5\beta_7 + \alpha_3\beta_5\alpha_7 + \beta_3\alpha_5\alpha_7 + \beta_3\beta_5\beta_7],\\
&\ \vdots\\
\Gamma_1 &= 0,\\
\Delta_1 &= 2,\\
\Gamma_3 &= 2\beta_2,\\
\Delta_3 &= 2\alpha_2,\\
\Gamma_5 &= 2(\alpha_2\beta_4 + \beta_2\alpha_4), \quad \Delta_5 = (\alpha_2\alpha_4 + \beta_2\beta_4),\\
\Gamma_7 &= 2(\alpha_2\alpha_4\beta_6 + \alpha_2\beta_4\alpha_6 + \beta_2\alpha_4\alpha 6 + \beta 2\beta_4\beta_6),\\
\Delta_7 &= 2(\alpha_2\alpha_4\alpha_6 + \alpha_2\beta_4\beta_6 + \beta_2\alpha_4\beta_6 + \beta_2\beta_4\alpha_6),\\
\Gamma_9 &= 2(\alpha_2\alpha_4\alpha_6\beta_8 + \alpha_2\alpha_4\beta_6\alpha_8 + \alpha_2\beta_4\alpha_6\alpha_8 + \beta_2\alpha_4\alpha_6\alpha_8 + \alpha_2\beta_4\beta_6\beta_8 + \beta_2\alpha_4\beta_6\beta_8 + \beta_2\beta_4\alpha_6\beta_8 + \beta_2\beta_4\beta_6\alpha_8),\\
\Delta_9 &= 2(\alpha_2\alpha_4\alpha_6\alpha_8 + \alpha_2\alpha_4\beta_6\beta_8 + \alpha_2\beta_4\alpha_6\beta_8 + \alpha_2\beta_4\beta_6\alpha_8 + \beta_2\alpha_4\alpha_6\beta_8 + \beta_2\alpha_4\beta_6\alpha_8 + \beta_2\beta_4\alpha_6\alpha_8 + \beta_2\beta_4\beta_6\alpha_8),\\
&\ \vdots
\end{aligned}$$

## References

[1] M. Akel and H. Begehr, Neumann function for a hyperbolic strip and a class of related plane domains, *Math. Nachr.* **290** (2017), 490–506.
[2] S. Altynbek and H. Begehr, A pair of rational double sequences, *Georgian Math. J.*, to appear.
[3] H. Begehr, Green function for a hyperbolic strip and a class of related plane domains, *Appl. Anal.* **93** (2014), 2370–2385.
[4] H. Begehr, H. Lin, H. Liu and B. Shupeyeva, An iterative real sequence based on $\sqrt{2}$ and $\sqrt{3}$ providing a plane parqueting and harmonic Green functions, *Complex Var. Elliptic Equ.* **66** (2021), no. 6–7, 988–1013.