

Non-finitely axiomatisable two-dimensional modal logics

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Abstract

We show the first examples of recursively enumerable (even decidable) two-dimensional products of finitely axiomatisable modal logics that are not finitely axiomatisable. In particular, we show that any axiomatisation of some bimodal logics that are determined by classes of product frames with linearly ordered first components must be infinite in two senses: It should contain infinitely many propositional variables, and formulas of arbitrarily large modal nesting-depth.

1 Introduction

Products of Kripke frames are natural relational structures allowing us to model interaction between different modal operators, representing time, space, knowledge, actions, etc. The product construction shows up in various disguises, and is related to many other logical formalisms, such as algebras of relations in algebraic logic, finite variable fragments of classical, intuitionistic and modal predicate logics, temporal-epistemic logics, dynamic topological logics, modal and temporal description logic, see e.g. [1, 2, 3, 6, 7, 8, 9, 18, 19]. Ever since their introduction [21, 22, 10], products of modal logics —propositional multimodal logics determined by classes of product frames— have been extensively studied, see [9, 16] for comprehensive expositions and further references.

In this paper we consider the problem of finding finite axiomatisations for products of two finitely axiomatisable modal logics. Let us first summarise the known results related to this problem:

(1) If both unimodal logics L_0 and L_1 are such that their classes of Kripke frames are definable by recursive sets of first-order sentences, then their product $L_0 \times L_1$ is a recursively enumerable bimodal logic [10].

(2) If both L_0 and L_1 are finitely axiomatisable by modal formulas having universal Horn first-order correspondents, then $L_0 \times L_1$ is finitely axiomatisable [10]. In fact, these product logics are *product-matching*: One only needs to take the formulas axiomatising L_0 and L_1 , and add the following two bimodal (Sahlqvist) formulas:

$$\Box_1 \Box_0 p \leftrightarrow \Box_0 \Box_1 p, \quad \Diamond_0 \Box_1 p \rightarrow \Box_1 \Diamond_0 p.$$

(These formulas are valid in all product frames, and express that the modal operators of L_0 and L_1 commute and have the Church–Rosser property (confluence).) For example, if each L_i is either **K** (the logic of all frames), or **K4** (the logic of all transitive frames), or **S4** (the logic of all reflexive and transitive frames), or **S5** (the logic of all equivalence frames), then $L_0 \times L_1$ is product-matching.

(3) The result in (2) cannot be generalised to products of logics axiomatised by formulas having universal (but not necessarily Horn) first-order components. Such an example is the finitely axiomatisable modal logic **K4.3**, determined by frames (W, R) , where R is transitive and *weakly connected*:

$$\forall x, y, z \in W (xRy \wedge xRz \rightarrow (y = z \vee yRz \vee zRy)).$$

(A rooted transitive and weakly connected relation is a *linearly ordered sequence of clusters*.) As is shown in [9, Thm.5.15], no product logic of the form **K4.3** \times L is product-matching, whenever L is any Kripke complete modal logic containing **K4** and having the two-element reflexive chain among its frames. So, say, **K4.3** \times **K4** is an example of a recursively enumerable but not product-matching product of two finitely axiomatisable logics. However, it was left open whether any of these product logics were finitely axiomatisable.

(4) Finally, note that the product construction may result in quite complex bimodal logics. There are several examples of non-recursively enumerable, even Π_1^1 -complete, products of finitely axiomatisable logics [11, 20, 23].

In this paper we show the first examples of recursively enumerable (even decidable) two-dimensional products of finitely axiomatisable modal logics that are not finitely axiomatisable. In particular, we show that any axiomatisation of some bimodal logics that are determined by classes of product frames with linearly ordered first components (such as, e.g., **K4.3** \times **K**) must be infinite in two senses: It should contain infinitely many propositional variables, and formulas of arbitrarily large modal nesting-depth. Precise formulations are given in Section 3. These results give negative answers to questions in [10], and to Questions 5.18 and 5.19 in [9].

The structure of the paper is as follows. Section 2 provides the relevant definitions and notation. The main results are listed in Section 3, and proved in Sections 4 and 5. Finally, in Section 6 we discuss the obtained results and formulate some open problems.

2 Products of modal logics

In what follows we assume that the reader is familiar with the basics of possible world semantics for modal logics (see, e.g., [4, 5]). Let us here begin with summarising the necessary notions and notation for the bimodal case. Similarly to (propositional) unimodal formulas, by a *bimodal* formula we mean any formula built up from propositional variables using the Booleans and the unary modal operators \Box_0, \Box_1 , and \Diamond_0, \Diamond_1 . Bimodal formulas are evaluated in *2-frames*: relational structures of the form $\mathfrak{F} = (W, R_0, R_1)$, having two binary relations R_0 and R_1 on a non-empty set W . A *Kripke model based on* \mathfrak{F} is a pair $\mathfrak{M} = (\mathfrak{F}, \vartheta)$, where ϑ is a function mapping propositional variables to subsets of W . The *truth relation* ‘ $\mathfrak{M}, w \models \varphi$ ’, connecting points in models and formulas, is defined as usual by induction on φ . Given a set Σ of bimodal formulas, we write $\mathfrak{M} \models \Sigma$ if we have $\mathfrak{M}, w \models \varphi$, for every $\varphi \in \Sigma$ and every $w \in W$. (We write just $\mathfrak{M} \models \varphi$ for $\mathfrak{M} \models \{\varphi\}$.) We say that φ is *valid in* \mathfrak{F} , if $\mathfrak{M} \models \varphi$ for every model \mathfrak{M} based on \mathfrak{F} . If every formula in a set Σ is valid in \mathfrak{F} , then we say that \mathfrak{F} is a *frame for* Σ .

A set L of bimodal formulas is called a (normal) *bimodal logic* (or *logic*, for short) if it contains all propositional tautologies and the formulas $\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$, for $i < 2$, and is closed under the rules of Substitution, Modus Ponens and Necessitation $\varphi/\Box_i\varphi$, for $i < 2$. Given a class \mathcal{C} of 2-frames, we always obtain a logic by taking

$$\text{Log } \mathcal{C} = \{\varphi : \varphi \text{ is a bimodal formula valid in every member of } \mathcal{C}\}.$$

We say that $\text{Log } \mathcal{C}$ is *determined by* \mathcal{C} , and call such a logic *Kripke complete*. Given a bimodal logic L and a set Σ of bimodal formulas, we say that Σ *axiomatises* L if L is the smallest bimodal logic containing Σ .

The usual operations on unimodal frames can be defined on 2-frames as well. In particular, given two 2-frames $\mathfrak{F} = (F, R_0^{\mathfrak{F}}, R_1^{\mathfrak{F}})$ and $\mathfrak{G} = (G, R_0^{\mathfrak{G}}, R_1^{\mathfrak{G}})$, a function $f : F \rightarrow G$ is called a *p-morphism from* \mathfrak{F} *to* \mathfrak{G} if it satisfies the following conditions, for all $u, v \in F$, $y \in G$, $i < 2$:

- $uR_i^{\mathfrak{F}}v$ implies $f(u)R_i^{\mathfrak{G}}f(v)$ (that is, f is a *homomorphism*),
- $f(u)R_i^{\mathfrak{G}}y$ implies that there is some $v \in F$ such that $f(v) = y$ and $uR_i^{\mathfrak{F}}v$ (the *backward condition*).

If f is onto then we say that \mathfrak{G} is a *p-morphic image* of \mathfrak{F} . \mathfrak{F} is a *subframe* of \mathfrak{G} if $F \subseteq G$ and $R_i^{\mathfrak{F}} = R_i^{\mathfrak{G}} \cap (F \times F)$, for $i < 2$. \mathfrak{F} is a *generated subframe* of \mathfrak{G} , if it is a subframe and we have $v \in F$, whenever $uR_i^{\mathfrak{G}}v$ for some $u \in F$, $v \in G$, $i < 2$. In particular, given some $x \in G$, the *subframe* \mathfrak{G}^x *of* \mathfrak{G} *generated by point* x is the subframe of \mathfrak{G} with the following set G^x of points:

$$G^x = \{x\} \cup \{y \in G : y \text{ is accessible from } x \text{ along the transitive closure of } R_0^{\mathfrak{G}} \cup R_1^{\mathfrak{G}}\}.$$

A 2-frame \mathfrak{G} is called *rooted* if $\mathfrak{G} = \mathfrak{G}^r$ for some point r . Similarly to the unimodal case, validity of bimodal formulas in 2-frames is preserved under taking p-morphic images and generated subframes.

Next, let us introduce some special, ‘two-dimensional’, 2-frames. Given unimodal Kripke frames $\mathfrak{F}_0 = (W_0, R_0)$ and $\mathfrak{F}_1 = (W_1, R_1)$, their *product* is defined to be the 2-frame

$$\mathfrak{F}_0 \times \mathfrak{F}_1 = (W_0 \times W_1, \bar{R}_0, \bar{R}_1),$$

where $W_0 \times W_1$ is the Cartesian product of W_0 and W_1 and, for all $u, u' \in W_0$, $v, v' \in W_1$,

$$\begin{aligned} (u, v)\bar{R}_0(u', v') &\quad \text{iff} \quad uR_0u' \text{ and } v = v', \\ (u, v)\bar{R}_1(u', v') &\quad \text{iff} \quad vR_1v' \text{ and } u = u'. \end{aligned}$$

2-frames of this form will be called *product frames* throughout. Now, for $i < 2$, let L_i be a Kripke complete unimodal logic in the language with \Box_i and \Diamond_i . The *product* of L_0 and L_1 is defined as the (Kripke complete) bimodal logic

$$L_0 \times L_1 = \text{Log } \{\mathfrak{F}_0 \times \mathfrak{F}_1 : \mathfrak{F}_i \text{ is a frame for } L_i, \text{ for } i < 2\}.$$

Note that product logics always have ‘non-standard’ frames, that is, 2-frames that are not isomorphic to product frames.

Notation. In this paper we are interested in product logics with a ‘linear’ first component, that is, where frames for L_0 are frames for **K4.3**. To emphasise this fact, the transitive and weakly connected relations in the 2-frames we will be dealing with will always be denoted by \leq_a , for some subscript a . This will not necessarily mean that \leq_a is reflexive. However, we will use the following notation:

$$u <_a v \quad \text{iff} \quad u \leq_a v \text{ and } v \not\leq_a u.$$

3 Main results

Throughout, an *irreflexive ω -fan* is a unimodal Kripke frame isomorphic to $\mathfrak{H}_\omega = (\omega + 1, R)$, where $R = \{(\omega, i) : i < \omega\}$. Similarly, a *reflexive ω -fan* is any frame isomorphic to $\mathfrak{H}_\omega^+ = (\omega + 1, R^+)$, where $R^+ = R \cup \{(i, i) : i \leq \omega\}$.

Theorem 1. *Let L be any bimodal logic such that*

- *L contains $\mathbf{K4.3} \times \mathbf{K}$, and*
- *the product of (ω, \leq) and an (irreflexive or reflexive) ω -fan is a frame for L .*

Then L is not axiomatisable using finitely many propositional variables.

Well-known examples of unimodal logics having an ω -fan among their frames are \mathbf{K} , $\mathbf{K4}$, $\mathbf{S4}$, Gödel–Löb logic \mathbf{GL} (the logic of irreflexive and transitive frames without infinite ascending chains), and Grzegorzcyk logic \mathbf{Grz} (the logic of reflexive and transitive frames without infinite ascending chains of distinct points). So we have the following:

Corollary 1.1. *Let L_0 be any of the logics $\mathbf{K4.3}$, $\mathbf{S4.3}$, $\text{Log}\{(\omega, \leq)\}$, and L_1 be any of the logics \mathbf{K} , $\mathbf{K4}$, $\mathbf{S4}$, \mathbf{GL} , \mathbf{Grz} . Then $L_0 \times L_1$ is not axiomatisable using finitely many propositional variables.*

Note that both $\mathbf{K4.3} \times \mathbf{K}$ and $\mathbf{K4.3} \times \mathbf{K4}$ are known to be recursively enumerable [10], $\mathbf{K4.3} \times \mathbf{K}$ is even decidable [9, 24]. (The same hold for reflexive versions.)

Our next result shows that some of these possible axiomatisations should also have a different kind of infinity. We define the *vertical depth* $vd(\varphi)$ of a bimodal formula φ inductively by taking

$$\begin{aligned} vd(p) &= 0, \\ vd(\psi_1 \wedge \psi_2) &= \max(vd(\psi_1), vd(\psi_2)), \\ vd(\neg\psi) &= vd(\psi), \\ vd(\diamond_0\psi) &= vd(\psi), \\ vd(\diamond_1\psi) &= vd(\psi) + 1. \end{aligned}$$

Theorem 2. *Let L be a bimodal logic such that*

$$\mathbf{K4.3} \times \mathbf{K} \subseteq L \subseteq \text{Log}\{(\omega, \leq)\} \times \mathbf{K}.$$

Then every axiomatisation of L must contain formulas of arbitrarily large vertical depth.

4 Infinitely many propositional variables are needed

In this section we prove Theorem 1. So let L be any bimodal logic containing $\mathbf{K4.3} \times \mathbf{K}$ such that the product of (ω, \leq) and an ω -fan is a frame for L . In order to show that L is not axiomatisable using finitely many propositional variables, we plan to proceed as follows. Given $m < \omega$, we call a Kripke model $\mathfrak{M} = (\mathfrak{F}, \vartheta)$ *m -generated* if there are at most m different propositional variables p such that $\vartheta(p) \neq \emptyset$. For every $0 < k < \omega$, we will define a 2-frame \mathfrak{F}_k such that:

(a) \mathfrak{F}_k is not a frame for $\mathbf{K4.3} \times \mathbf{K}$.

(b) If $k > 2^{4m} + 1$ then $\mathfrak{M} \models L$, for every m -generated model \mathfrak{M} based on \mathfrak{F}_k .

This will prove Theorem 1 because of the following. Suppose that Σ axiomatises L and Σ contains m propositional variables, for some $m < \omega$. Let $k > 2^{4m} + 1$ and take a 2-frame \mathfrak{F}_k satisfying (b). Let \mathfrak{M} be an arbitrary model based on \mathfrak{F}_k . Let \mathfrak{M}_m be another model over \mathfrak{F}_k that is the same as \mathfrak{M} on propositional variables occurring in Σ , and \emptyset otherwise. Then \mathfrak{M}_m is clearly m -generated and $\mathfrak{M}_m \models \Sigma$ iff $\mathfrak{M} \models \Sigma$. So by (b), we have $\mathfrak{M}_m \models L$. As $\Sigma \subseteq L$, we obtain $\mathfrak{M}_m \models \Sigma$, and so $\mathfrak{M} \models \Sigma$. This holds for any model \mathfrak{M} over \mathfrak{F}_k , so \mathfrak{F}_k is a frame for Σ . Therefore, $\text{Log}\{\mathfrak{F}_k\}$ is a bimodal logic containing Σ , and so we have that \mathfrak{F}_k is a frame for L . As $\mathbf{K4.3} \times \mathbf{K} \subseteq L$, this implies that \mathfrak{F}_k is a frame for $\mathbf{K4.3} \times \mathbf{K}$, contradicting (a).

We fix some $0 < k < \omega$, and begin with the definition of $\mathfrak{F}_k = (W, \leq_h, R_v)$, for the case when $(\omega, \leq) \times \mathfrak{H}_\omega$ is a frame for L :

$$W = \{y\} \cup \{x_i, u_i, v_i, w_i, z_i : i < k\},$$

\leq_h is the reflexive and transitive closure of

$$\{(u_i, v_i), (v_i, w_i), (w_i, z_i) : i < k\} \cup \{(x_i, x_j), (x_i, y), (y, x_i) : i, j < k\},$$

$$R_v = \{(x_i, u_j), (x_i, z_j) : i, j < k\} \cup \{(x_i, v_j) : i, j < k, i \neq j\} \cup$$

$$\{(y, u_i), (y, w_i), (y, z_i) : i < k\},$$

see Fig. 1. Note that in Fig. 1 (as well as in further figures) the reflexive, transitive and weakly connected \leq_h is depicted by ‘horizontal’ arrows and its clusters by ‘horizontal’ ellipses, and R_v by kind of ‘vertical’ arrows.

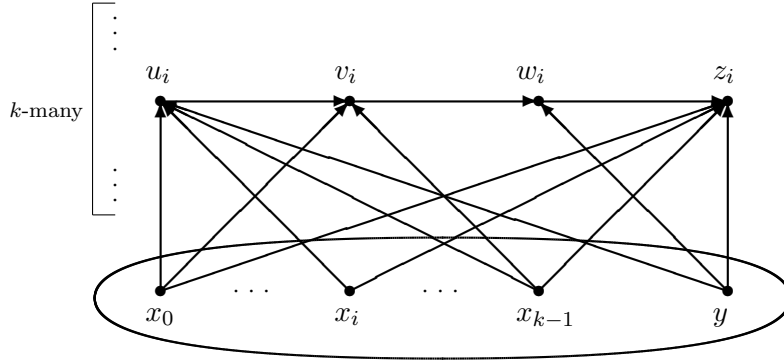


Figure 1: The frame \mathfrak{F}_k .

If L is such that $(\omega, \leq) \times \mathfrak{H}_\omega$ is not a frame for L , but $(\omega, \leq) \times \mathfrak{H}_\omega^+$ is, then we should add the pairs $\{(w, w) : w \in W\}$ to R_v . From now on, we discuss in detail the vertically irreflexive case only. The very similar proof of the reflexive case is left to the reader.

First, we prove (a). Let us begin with showing a general property of p -morphic images of weakly connected frames.

CLAIM 3. *Let f be a p -morphism from some weakly connected frame $\mathfrak{G}_0 = (W_0, \leq_0)$ onto a frame $\mathfrak{G}_1 = (W_1, \leq_1)$. For all $a, b \in W_0$, $x \in W_1$, if $a \leq_0 b$ and $f(a) \leq_1 x <_1 f(b)$ then there*

exists $c \in W_0$ such that $a \leq_0 c <_0 b$ and $f(c) = x$. Moreover, if $f(a) <_1 x <_1 f(b)$ then c can be chosen such that $a <_0 c <_0 b$.

Proof. Take some $a, b \in W_0$, $x \in W_1$ such that $a \leq_0 b$ and $f(a) \leq_1 x <_1 f(b)$. By the backward condition on f , there exists $c \in W_0$ such that $a \leq_0 c$ and $f(c) = x$. Moreover, as f is a homomorphism, if $f(a) <_1 x$ then $a <_0 c$. As \leq_0 is weakly connected, we have either $c = b$, or $b \leq_0 c$, or $c \leq_0 b$. But $f(c) <_1 f(b)$, so the first two cases cannot hold. Therefore, $c <_0 b$ follows. \square

As being transitive and weakly connected is first-order definable, the class of all frames for **K4.3** is closed under ultraproducts. As **K4.3** is a modal logic, its class of frames is also closed under point-generated subframes. So, by [17, Thm.2.10], we obtain:

CLAIM 4. *For every finite rooted 2-frame \mathfrak{F} , \mathfrak{F} is a frame for **K4.3** \times **K** iff \mathfrak{F} is a p-morphic image of a product frame for **K4.3** \times **K**.*

Therefore, in order to prove (a) it is enough to show the following:

LEMMA 5. *\mathfrak{F}_k is not a p-morphic image of a product frame for **K4.3** \times **K**.*

Proof. Suppose that there is a p-morphism f from a product frame $\mathfrak{H} = (U, \leq_0, R_1)$ with transitive and weakly connected \leq_0 onto $\mathfrak{F}_k = (W, \leq_h, R_v)$. Take some $i_0 < k$. As $x_{i_0} \leq_h yR_v w_{i_0}$, there are $a_0, b, c_0 \in U$ such that $a_0 \leq_0 bR_1 c_0$, $f(a_0) = x_{i_0}$, $f(b) = y$, and $f(c_0) = w_{i_0}$. As \mathfrak{H} is a product frame, there exists $d_0 \in U$ such that $a_0 R_1 d_0 \leq_0 c_0$. As f is a p-morphism, we must have that $f(d_0) = u_{i_0}$. As $u_{i_0} <_h v_{i_0} <_h w_{i_0}$, by Claim 3, there exists $e_0^1 \in U$ such that $d_0 <_0 e_0^1 <_0 c_0$ and $f(e_0^1) = v_{i_0}$. As \mathfrak{H} is a product frame, there exists $a_1 \in U$ such that $a_1 R_1 e_0^1$. As f is a p-morphism, we must have that $f(a_1) = x_{i_1}$, for some $i_1 < k$, $i_1 \neq i_0$.

Next, as $yR_v w_{i_1}$, there exists $c_1 \in U$ such that $bR_1 c_1$ and $f(c_1) = w_{i_1}$. As \mathfrak{H} is a product frame, there exists $d_1 \in U$ such that $a_1 R_1 d_1 \leq_0 c_1$. As f is a p-morphism, we must have that $f(d_1) = u_{i_1}$. As $u_{i_1} <_h v_{i_1} <_h w_{i_1}$, by Claim 3, there exists $e_1^2 \in U$ such that $d_1 <_0 e_1^2 <_0 c_1$ and $f(e_1^2) = v_{i_1}$. As \mathfrak{H} is a product frame, there exist $e_0^2, a_2 \in U$ such that $e_0^1 <_0 e_0^2 <_0 c_0$, $a_2 R_1 e_0^2$, and $a_2 R_1 e_1^2$. As f is a p-morphism, we must have the following:

$$v_{i_0} \leq_h f(e_0^2) \leq_h w_{i_0}, \quad (1)$$

$$f(a_2)R_v f(e_0^2), \quad (2)$$

$$f(a_2)R_v v_{i_1}. \quad (3)$$

These imply that $f(e_0^2) = v_{i_0}$ and $f(a_2) = x_{i_2}$, for some $i_2 < k$, $i_2 \notin \{i_0, i_1\}$. Indeed, by (3), we have that

$$f(a_2) = x_{i_2}, \text{ for some } i_2 < k, i_2 \neq i_1. \quad (4)$$

By (1), either $f(e_0^2) = v_{i_0}$ or $f(e_0^2) = w_{i_0}$. Now, by (2) and (4), $f(e_0^2) = v_{i_0}$ follows, and so $i_2 \neq i_0$ must also hold (see Fig. 2).

And so on, using that $yR_v w_{i_j}$ for all $j < k - 1$, after $k - 1$ steps we end up having $a_{k-1}, e_0^{k-1}, \dots, e_{k-2}^{k-1} \in U$ and pairwise distinct $i_0, \dots, i_{k-1} < k$ such that $f(a_{k-1}) = x_{i_{k-1}}$, $a_{k-1} R_1 e_j^{k-1}$, and $f(e_j^{k-1}) = v_{i_j}$, for all $j < k - 1$. Now, as $yR_v w_{i_{k-1}}$, there exists $c_{k-1} \in U$ such that $bR_1 c_{k-1}$ and $f(c_{k-1}) = w_{i_{k-1}}$. As \mathfrak{H} is a product frame, there exists $d_{k-1} \in U$ such that $a_{k-1} R_1 d_{k-1} \leq_0 c_{k-1}$. As f is a p-morphism, we must have that $f(d_{k-1}) = u_{i_{k-1}}$. As $u_{i_{k-1}} <_h v_{i_{k-1}} <_h w_{i_{k-1}}$, by Claim 3, there exists $e_{k-1} \in U$ such that $d_{k-1} <_0 e_{k-1} <_0 c_{k-1}$,

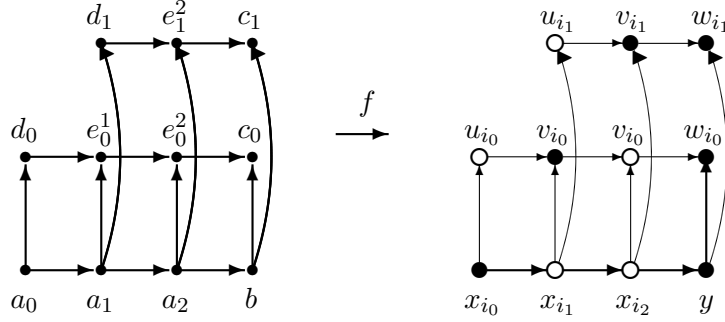


Figure 2: Building a p-morphism from a product frame to \mathfrak{F}_k : the first steps.

and $f(e_{k-1}) = v_{i_{k-1}}$. As \mathfrak{H} is a product frame, there exist $e_0, \dots, e_{k-2}, a_k \in U$ such that $e_j^{k-1} <_0 e_j <_0 c_j$ for all $j < k-1$, and $a_k R_1 e_j$ for all $j < k$. As f is a p-morphism, we must have the following:

$$v_{i_j} \leq_h f(e_j) \leq_h w_{i_j}, \text{ for all } j < k-1, \quad (5)$$

$$f(a_k) R_v f(e_j), \text{ for all } j < k-1, \quad (6)$$

$$f(a_k) R_v v_{i_{k-1}}. \quad (7)$$

By (7), we have that

$$f(a_k) = x_\ell, \text{ for some } \ell < k, \ell \neq i_{k-1}. \quad (8)$$

By (5), for each $j < k-1$, either $f(e_j) = v_{i_j}$ or $f(e_j) = w_{i_j}$. Now, by (6) and (8), $f(e_j) = v_{i_j}$ follows, for every $j < k-1$. Therefore, the index $\ell < k$ in (8) must also be different from each of i_0, \dots, i_{k-2} . But there is no such ℓ , a contradiction (see Fig. 3). \square

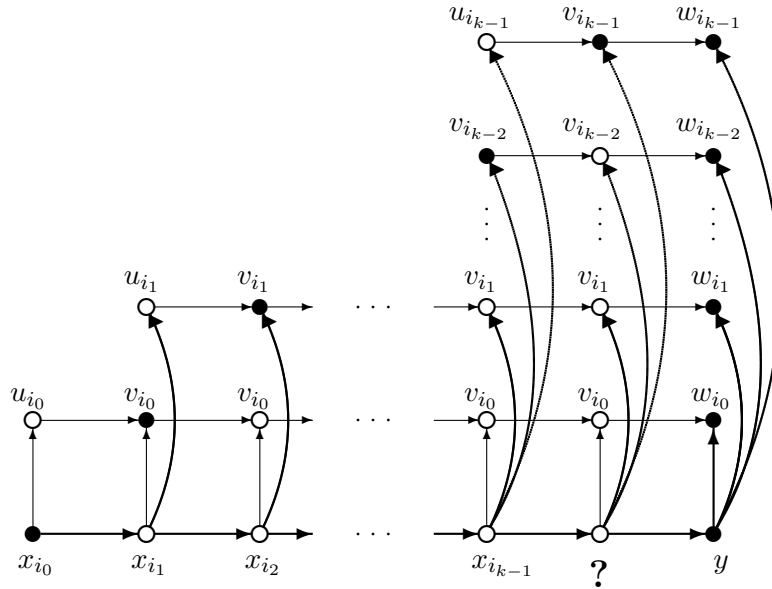


Figure 3: Building a p-morphism from a product frame to \mathfrak{F}_k : the contradiction.

Remark 6. Instead of proving that the existence of a p-morphism from a product frame for $\mathbf{K4.3} \times \mathbf{K}$ onto \mathfrak{F}_k leads to a contradiction, we could have proved Lemma 5 by playing a two-player ‘p-morphism game.’ Versions of such games are widely used in connection with axiomatisation problems in algebraic logic and many-dimensional modal logics, see e.g. [12, 15]. In this game, two players \exists and \forall are building step-by-step homomorphisms from larger and larger product frames for $\mathbf{K4.3} \times \mathbf{K}$ to a finite or countably infinite 2-frame \mathfrak{F} . At each step, \forall can choose a ‘defect’ showing that the actual homomorphism is not a p-morphism: an instance of points failing the backward condition. If \exists can reply with a larger homomorphism ‘fixing’ the chosen defect, then the game goes on, otherwise \forall wins the game. If \exists can always go on infinitely long, no matter what are \forall ’s choices, then we say that \exists *has a winning strategy in the ω -step game over \mathfrak{F}* . It is not hard to prove, using Claim 3, that \exists has such a winning strategy iff \mathfrak{F} is a p-morphic image of a product frame for $\mathbf{K4.3} \times \mathbf{K}$. Figures 2 and 3 can be read as descriptions of a particular play of this game over \mathfrak{F}_k , won by \forall : The black dots show the choices of \forall , and the empty circles are the points in possible replies of \exists , until she fails to continue.

Let us now turn to the proof of (b). We define a new frame $\mathfrak{G}_k = (V, \preceq_h, S_v)$ by adding some points and arrows to \mathfrak{F}_k :

$$V = W \cup \{x, u, u', v, w, z\},$$

\preceq_h is the reflexive and transitive closure of

$$\leq_h \cup \{(x, y), (y, x), (u, v), (u', v), (v, w), (w, z)\},$$

$$S_v = R_v \cup \{(x, u), (x, u'), (x, v), (x, z), (y, u), (y, u'), (y, w), (y, z)\} \cup \{(x, u_i), (x, v_i), (x, z_i), (x_i, u), (x_i, u'), (x_i, v), (x_i, z) : i < k\},$$

see Fig. 4.

LEMMA 7. \mathfrak{G}_k is a p-morphic image of $(\omega, \leq) \times \mathfrak{H}_\omega$.

Proof. Let \mathfrak{F}_ω be the irreflexive ω -fan with r as its root and $\omega \times (k+2)$ as its set of leaves. We define a function f from $(\omega, \leq) \times \mathfrak{F}_\omega$ to \mathfrak{G}_k . To begin with, for all $n < \omega$, we let

$$f(n, r) = \begin{cases} x_i, & \text{if } n = 3(\ell \cdot k + i), \ell < \omega, i < k, \\ x, & \text{if } n = 3\ell + 1, \ell < \omega, \\ y, & \text{if } n = 3\ell + 2, \ell < \omega. \end{cases}$$

Then, for all $n, m < \omega, i < k+2$, we define $f(n, (m, i))$ by taking

$$\begin{array}{ll} u_i, & \text{if } i < k, n < 3m - 3, \\ u_i, & \text{if } i < k, m = \ell \cdot k + i + 1 \text{ for some } \ell < \omega, n = 3m - 3, \\ v_i, & \text{if } i < k, m \neq \ell \cdot k + i + 1 \text{ for any } \ell < \omega, n = 3m - 3, \\ v_i, & \text{if } i < k, n = 3m - 2, \\ w_i, & \text{if } i < k, n = 3m - 1, \\ z_i, & \text{if } i < k, n \geq 3m, \\ u, & \text{if } i = k, n < 3m - 3, \\ u', & \text{if } i = k + 1, n < 3m - 3, \\ v, & \text{if } i = k \text{ or } k + 1, n = 3m - 3 \text{ or } 3m - 2, \\ w, & \text{if } i = k \text{ or } k + 1, n = 3m - 1, \\ z, & \text{if } i = k \text{ or } k + 1, n \geq 3m. \end{array}$$

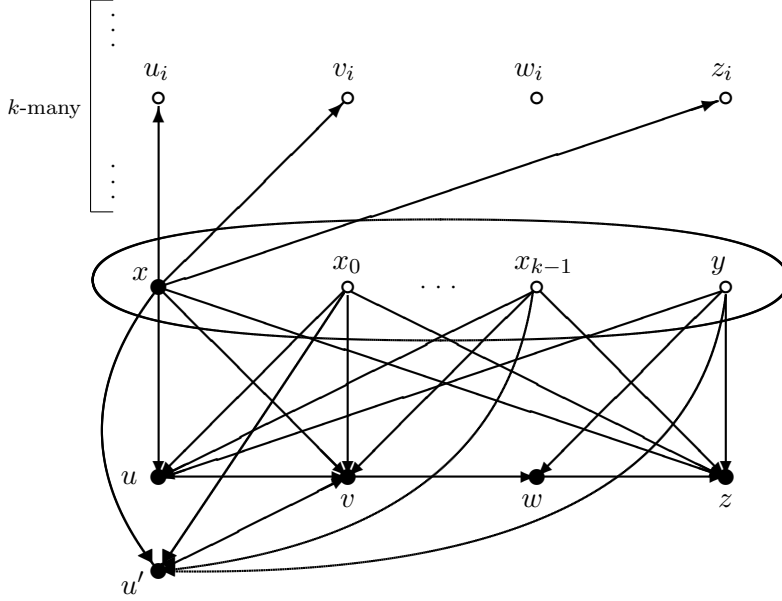


Figure 4: The new points and arrows of frame \mathfrak{G}_k .

It is tedious but straightforward to check that f is an onto p-morphism. Here is the trickiest case only. Take some $n < \omega$ such that $f(n, r) = x_i$, for some $i < k$. (Then $n = 3(\ell \cdot k + i)$ for some $\ell < \omega$.) We need to show that (i) for all $m < \omega$, $j < k + 2$, we have $x_i R_v f(n, (m, j))$, and (ii) for all a in \mathfrak{G}_k , if $x_i R_v a$, then there exist some $m < \omega$, $j < k + 2$ with $a = f(n, (m, j))$.

To begin with, the cases when $j = k$ or $j = k + 1$ in (i), and $a \in \{u, u', v, z\}$ in (ii) are straightforward. For the rest, we claim that the following holds, for every $j < k$:

$$\{f(n, (m, j)) : m < \omega\} = \begin{cases} \{u_j, v_j, z_j\}, & \text{if } j \neq i, \\ \{u_i, z_i\}, & \text{if } j = i. \end{cases} \quad (9)$$

Indeed, let us see the $j \neq i$ case first. On the one hand, we have \subseteq , as n is divisible by 3. For \supseteq : First, for any $m > \frac{n+3}{3}$, we have $f(n, (m, j)) = u_j$. Second, for any $m \leq \frac{n}{3}$, we have $f(n, (m, j)) = z_j$. Finally, observe that $\frac{n+3}{3} = \ell \cdot k + i + 1 \neq \ell' \cdot k + j + 1$ for any $\ell' < \omega$, as $i, j < k$ and $i \neq j$. So if $m = \frac{n+3}{3}$, then $f(n, (m, j)) = v_j$. This last observation also shows that $f(n, (m, i)) \neq v_i$ for any $m < \omega$, completing the proof of (9). \square

LEMMA 8. *Let $k > 2^{4m} + 1$, and let \mathfrak{M} be an m -generated model over \mathfrak{F}_k . Then there is a model \mathfrak{N} over \mathfrak{G}_{k-2} that is a p-morphic image of \mathfrak{M} .*

Proof. Let $\mathfrak{M} = (\mathfrak{F}_k, \vartheta)$ be a model such that $\vartheta(p_j) = \emptyset$ for every propositional variable p_j with $j \geq m$. We define an equivalence relation \sim_m on the set $\{0, 1, \dots, k-1\}$ by taking, for all $i, j < k$,

$$i \sim_m j \iff \forall \ell < m \left((x_i \in \vartheta(p_\ell) \leftrightarrow x_j \in \vartheta(p_\ell)) \wedge (v_i \in \vartheta(p_\ell) \leftrightarrow v_j \in \vartheta(p_\ell)) \wedge (w_i \in \vartheta(p_\ell) \leftrightarrow w_j \in \vartheta(p_\ell)) \wedge (z_i \in \vartheta(p_\ell) \leftrightarrow z_j \in \vartheta(p_\ell)) \right).$$

As there are 2^{4m} many \sim_m -classes on $\{0, 1, \dots, k-1\}$ and $k > 2^{4m} + 1 > 2^{4m}$, by the pigeonhole principle, there exist $i < j < k$, such that $i \sim_m j$.

Let n_0, \dots, n_{k-3} be an enumeration of $\{\ell < k : \ell \neq i, j\}$. We define a function f from \mathfrak{F}_k to \mathfrak{G}_{k-2} by taking

$$\begin{aligned} f(a_{n_\ell}) &= a_\ell, & \text{for } a \in \{x, u, v, w, z\}, \ell < k-2, \\ f(y) &= y, \\ f(x_i) &= f(x_j) = x, \\ f(u_i) &= u, \\ f(u_j) &= u', \\ f(v_i) &= f(v_j) = v, \\ f(w_i) &= f(w_j) = w, \\ f(z_i) &= f(z_j) = z. \end{aligned}$$

Next, we define a model $\mathfrak{N} = (\mathfrak{G}_{k-2}, \theta)$ by taking, for all a in \mathfrak{G}_{k-2} , and for all $\ell < m$,

$$a \in \theta(p_\ell) \iff a = f(b) \text{ for some } b \in \vartheta(p_\ell),$$

and $\theta(p_\ell) = \emptyset$, for $\ell \geq m$. It is straightforward to check that f is an onto p-morphism from \mathfrak{M} to \mathfrak{N} , as required. \square

Now we can complete the proof of **(b)**. Let L be a bimodal logic such that $(\omega, \leq) \times \mathfrak{H}_\omega$ is a frame for L , and let $k > 2^{4m} + 1$. Then, by Lemma 8, for every m -generated model \mathfrak{M} over \mathfrak{F}_k there is a model over \mathfrak{G}_{k-2} that is a p-morphic image of \mathfrak{M} . By Lemma 7, \mathfrak{G}_{k-2} is a p-morphic image of $(\omega, \leq) \times \mathfrak{H}_\omega$. Therefore, $\mathfrak{M} \models L$ follows.

5 Arbitrarily large vertical depth is needed

In this section we prove Theorem 2. Let L be a bimodal logic such that

$$\mathbf{K4.3} \times \mathbf{K} \subseteq L \subseteq \text{Log}\{(\omega, \leq)\} \times \mathbf{K}.$$

In order to show that every axiomatisation of L must contain formulas of arbitrarily large vertical depth, let us introduce some notions. Given a 2-frame $\mathfrak{G} = (W, R_h, R_v)$ and $x, y \in W$, a *path in \mathfrak{G} from x to y* is a sequence of points w_0, \dots, w_n in W such that $w_0 = x$, $w_n = y$, and for each $i < n$, either $w_i R_h w_{i+1}$ or $w_i R_v w_{i+1}$. The *vertical length* of such a path is the number of R_v -edges in it. Now, for every $x \in W$ and every $k < \omega$, we define

$$\begin{aligned} W^{x,k} &= \{x\} \cup \{y \in W : \text{there is a path in } \mathfrak{G} \text{ from } x \text{ to } y \text{ of vertical length } \leq k\}, \\ (\mathfrak{G})^{x,k} &= (W^{x,k}, R'_h, R'_v), \end{aligned}$$

where R'_h and R'_v are the respective restrictions of R_h and R_v to $W^{x,k}$ (that is, $(\mathfrak{G})^{x,k}$ is the subframe of \mathfrak{G} having $W^{x,k}$ as its universe). Clearly, for every bimodal formula φ with $vd(\varphi) \leq k$,

$$\text{if } \varphi \text{ is valid in } (\mathfrak{G})^{x,k} \text{ for every } x \in W, \text{ then } \varphi \text{ is valid in } \mathfrak{G}. \quad (10)$$

Now we plan to proceed as follows. For every $0 < k < \omega$, we will define a 2-frame \mathfrak{G}_k such that:

- (a) \mathfrak{G}_k is not a frame for $\mathbf{K4.3} \times \mathbf{K}$.
- (b) For every point x in \mathfrak{G}_k , $(\mathfrak{G}_k)^{x,k}$ is a frame for $\text{Log}\{(\omega, \leq)\} \times \mathbf{K}$.

This will prove Theorem 2 because of the following. Suppose that Σ axiomatises L , and there is some $k < \omega$ such that $vd(\varphi) \leq k$ for every φ in Σ . Take a 2-frame \mathfrak{G}_k satisfying (b). As $\Sigma \subseteq L \subseteq \text{Log}\{(\omega, \leq)\} \times \mathbf{K}$, for every point x in \mathfrak{G}_k , $(\mathfrak{G}_k)^{x,k}$ is a frame for Σ . Thus by (10), \mathfrak{G}_k is a frame for Σ . Therefore, $\text{Log}\{\mathfrak{G}_k\}$ is a bimodal logic containing Σ , and so we have that \mathfrak{G}_k is a frame for L . As $\mathbf{K4.3} \times \mathbf{K} \subseteq L$, this implies that \mathfrak{G}_k is a frame for $\mathbf{K4.3} \times \mathbf{K}$, contradicting (a).

Let us begin with the definition of $\mathfrak{G}_k = (W, \leq_h, R_v)$, for $0 < k < \omega$:

$$W = \{u_1, u_2, u_3, v_1, v_2, v_3, v_4\} \cup \{w_i^1, w_i^2, w_i^3 : 1 \leq i \leq k\},$$

\leq_h is the reflexive and transitive closure of

$$\{(u_1, u_2), (u_2, u_3), (v_1, v_2), (v_2, v_3), (v_3, v_4)\} \cup \{(w_i^1, w_i^2), (w_i^2, w_i^3), (w_i^3, w_i^1) : 1 \leq i \leq k\},$$

$$R_v = \{(u_1, w_1^1), (u_2, w_1^2), (u_3, w_1^3), (u_3, w_1^2), (u_3, w_1^1)\} \cup \\ \{(w_k^1, v_1), (w_k^1, v_4), (w_k^2, v_1), (w_k^2, v_3), (w_k^2, v_4), (w_k^3, v_1), (w_k^3, v_2), (w_k^3, v_4)\} \cup \\ \{(w_i^1, w_{i+1}^1), (w_i^2, w_{i+1}^2), (w_i^3, w_{i+1}^3) : 1 \leq i < k\},$$

see Fig. 5. Note that a vertically reflexive version of \mathfrak{G}_k would also do.

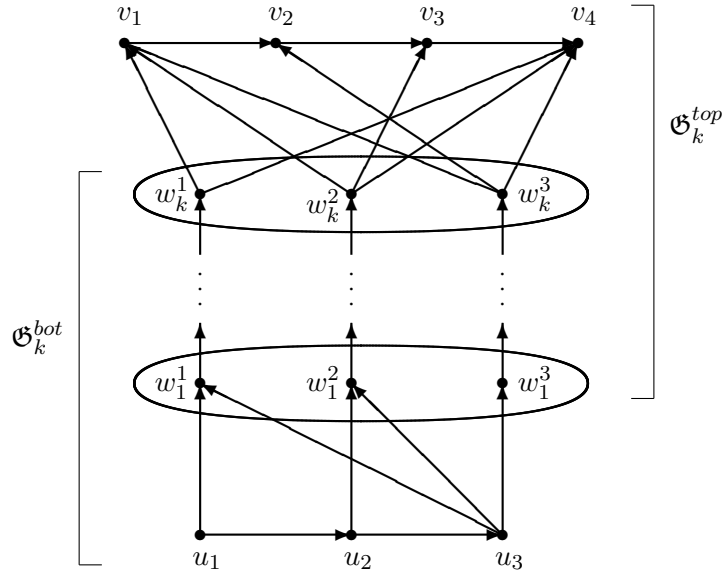


Figure 5: The frames \mathfrak{G}_k , \mathfrak{G}_k^{bot} , and \mathfrak{G}_k^{top} .

Next, let us prove (a). Actually, we give two different proofs for \mathfrak{G}_k not being a frame for $\mathbf{K4.3} \times \mathbf{K}$: one using Claim 4 above, and another, more ‘direct’ one. For the first: As \mathfrak{G}_k is a finite rooted 2-frame, by Claim 4, \mathfrak{G}_k is a frame for $\mathbf{K4.3} \times \mathbf{K}$ iff \mathfrak{G}_k is a p-morphic image of a product frame for $\mathbf{K4.3} \times \mathbf{K}$. Therefore, it is enough to show the following:

LEMMA 9. \mathfrak{G}_k is not a p-morphic image of a product frame for $\mathbf{K4.3} \times \mathbf{K}$.

Proof. Suppose that there is a p-morphism f from a product frame $\mathfrak{H} = (U, \leq_0, R_1)$ with transitive and weakly connected \leq_0 onto $\mathfrak{G}_k = (W, \leq_h, R_v)$. As $u_1 \leq_h u_2 R_v w_1^2 R_v \dots R_v w_k^2 R_v v_3$, there exist $a, b, c_1, \dots, c_k, c \in U$ such that $a \leq_0 b R_1 c_1 R_1 \dots R_1 c_k R_1 c$, $f(a) = u_1$, $f(b) = u_2$,

$f(c) = v_3$, and $f(c_i) = w_i^2$, for all $1 \leq i \leq k$. As \mathfrak{H} is a product frame, there exist points $d_1, \dots, d_n, d \in U$ such that $aR_1d_1R_1 \dots R_1d_kR_1d \leq_0 c$. As f is a p-morphism, we must have that $f(d) = v_1$. As $v_1 <_h v_2 <_h v_3$, by Claim 3, there exists $e \in U$ such that $d <_0 e <_0 c$ and $f(e) = v_2$. As \mathfrak{H} is a product frame, there exist $e_1, \dots, e_k, x \in U$ such that $d_i <_0 e_i <_0 c_i$ for $1 \leq i \leq k$, $a <_0 x <_0 b$, and $xR_1e_1R_1 \dots R_1e_kR_1e$. As f is a p-morphism, we must have that $f(e_i) = w_i^3$, for all $1 \leq i \leq k$. Further, $f(x)$ should be such that $u_1 \leq_h f(x) \leq_h u_2$ and $f(x)R_vw_1^3$. But there is no such $f(x)$, a contradiction. \square

As a second proof for (a), observe that in fact the proof of Lemma 9 shows that the following first-order (Π_2) sentence Φ_k fails in \mathfrak{G}_k :

$$\Phi_k : \quad \forall a, b, c \left[a \leq_h bR_v^{k+1}c \rightarrow \exists d \left(aR_v^{k+1}d \leq_h c \wedge \right. \right. \\ \left. \left. \forall e \left(d \leq_h e <_h c \rightarrow \exists x \left(a \leq_h x \leq_h b \wedge xR_v^{k+1}e \right) \right) \right) \right]$$

(throughout, we use $uR_v^{k+1}w$ as a shorthand for $\exists w_1 \dots w_k (uR_vw_1R_v \dots R_vw_kR_vw)$, see Fig. 6. (Note that Fig. 6 also shows the steps in a particular play of the p-morphism game over \mathfrak{G}_k that player \forall wins, cf. Remark 6.)

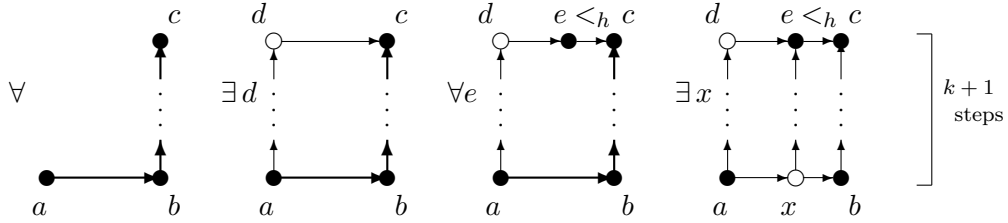


Figure 6: The property Φ_k .

It is straightforward to see that Φ_k holds in every product frame for $\mathbf{K4.3} \times \mathbf{K}$. Now we will show that Φ_k is definable in the bimodal language, giving us an explicit bimodal formula showing that \mathfrak{G}_k is not a frame for $\mathbf{K4.3} \times \mathbf{K}$. To this end, for every $n < \omega$ and bimodal formula ψ , we define $\diamond_1^n \psi$ by taking

$$\diamond_1^0 \psi = \psi \quad \text{and} \quad \diamond_1^{n+1} \psi = \diamond_1 \diamond_1^n \psi.$$

CLAIM 10. *There is a bimodal formula*

$$\varphi_k : \quad \diamond_0(p \wedge \diamond_1^{k+1}(r \wedge q \wedge \square_0 q)) \wedge \square_0(\diamond_0 p \rightarrow \square_1^{k+1} q) \rightarrow \diamond_1^{k+1}(\diamond_0 r \wedge \square_0 q)$$

such that, for all 2-frames $\mathfrak{F} = (W, \leq_0, R_1)$ with weakly connected \leq_0 , φ_k is valid in \mathfrak{F} iff Φ_k holds in \mathfrak{F} .

Proof. Let $\mathfrak{F} = (W, \leq_0, R_1)$ be a 2-frame such that \leq_0 is weakly connected.

\Leftarrow : Let \mathfrak{M} be a model based on \mathfrak{F} and suppose that, for some $a \in W$,

$$\begin{aligned} \mathfrak{M}, a \models \diamond_0(p \wedge \diamond_1^{k+1}(r \wedge q \wedge \square_0 q)) \quad \text{and} \\ \mathfrak{M}, a \models \square_0(\diamond_0 p \rightarrow \square_1^{k+1} q). \end{aligned} \tag{11}$$

So, there exist $b, c \in W$ such that $a \leq_0 bR_1^{k+1}c$ and

$$\mathfrak{M}, b \models p \quad (12)$$

$$\mathfrak{M}, c \models r \quad (13)$$

$$\mathfrak{M}, c \models q \wedge \Box_0 q. \quad (14)$$

Therefore, by Φ_k , there exists $d \in W$ such that $aR_1^{k+1}d \leq_0 c$ and for all $e \in W$, if $d \leq_0 e <_0 c$ then there exists $x \in W$ with $a \leq_0 x \leq_0 b$ and $xR_v^{k+1}e$. Then, by (13), we have $\mathfrak{M}, d \models \Diamond_0 r$. We claim that $\mathfrak{M}, d \models \Box_0 q$ also holds. Indeed, take some $e \in W$ with $d \leq_0 e$. As \leq_0 is weakly connected, either $e = c$, or $c \leq_0 e$, or $e \leq_0 c$. In the first two cases, $\mathfrak{M}, e \models q$ holds by (14). If $e <_0 c$ then there is some $x \in W$ with $a \leq_0 x \leq_0 b$ and $xR_v^{k+1}e$. So $\mathfrak{M}, e \models q$ holds by (11) and (12).

\Rightarrow : Suppose that Φ_k fails in \mathfrak{F} , that is, there exists $a, b, c \in W$ such that $a \leq_0 bR_1^{k+1}c$ and all $d \in W$ are ‘bad.’ We define a model $\mathfrak{M} = (\mathfrak{F}, \vartheta)$ by taking

$$\begin{aligned} \vartheta(p) &= \{b\}, \\ \vartheta(q) &= \{c\} \cup \{e : c \leq_0 e\} \cup \{e : \exists x (a \leq_0 x \leq_0 b \wedge xR_1^{k+1}e)\}, \\ \vartheta(r) &= \{c\}. \end{aligned}$$

Then clearly $\mathfrak{M}, a \models \Diamond_0(p \wedge \Diamond_1^{k+1}(r \wedge q \wedge \Box_0 q))$. We claim that $\mathfrak{M}, a \models \Box_0(\Diamond_0 p \rightarrow \Box_1^{k+1}q)$. Indeed, if $a \leq_0 x$ and $\mathfrak{M}, x \models \Diamond_0 p$, then $a \leq_0 x \leq_0 b$ should hold, and so $\mathfrak{M}, x \models \Box_1^{k+1}q$ follows. Next, we claim that $\mathfrak{M}, a \models \Box_1^{k+1}(\Diamond_0 r \rightarrow \neg \Box_0 q)$. Indeed, take some d such that $aR_1^{k+1}d$ and $\mathfrak{M}, d \models \Diamond_0 r$. Then $d \leq_0 c$ and, by assumption, there is some e such that $d \leq_0 e <_0 c$ and $xR_1^{k+1}e$ holds for no x with $a \leq_0 x \leq_0 b$. Therefore, $\mathfrak{M}, e \not\models q$ and $\mathfrak{M}, d \not\models \Box_0 q$ follows, as required. \square

Let us now turn to the proof of (b). We define two special subframes of \mathfrak{G}_k :

- Let \mathfrak{G}_k^{top} be the subframe of \mathfrak{G}_k with universe $W^{top} = W - \{u_1, u_2, u_3\}$, and
- let \mathfrak{G}_k^{bot} be the subframe of \mathfrak{G}_k with universe $W^{bot} = W - \{v_1, v_2, v_3, v_4\}$,

see Fig. 5. It is straightforward to see the following:

CLAIM 11. *For every point x in \mathfrak{G}_k , either $(\mathfrak{G}_k)^{x,k}$ is a generated subframe of \mathfrak{G}_k^{top} , or it is a generated subframe of \mathfrak{G}_k^{bot} .*

We complete the proof by showing that both \mathfrak{G}_k^{top} and \mathfrak{G}_k^{bot} are frames for L .

LEMMA 12. *\mathfrak{G}_k^{top} is a p -morphic image of $(\omega, \leq) \times \mathfrak{F}$, for some frame \mathfrak{F} .*

Proof. See Fig. 7 for a function f from $(\omega, \leq) \times \mathfrak{F}$, where \mathfrak{F} is a kind of ‘rake:’ an irreflexive and intransitive $k - 1$ -long path, followed by an irreflexive ω -fan. (In Fig. 7 each point of $(\omega, \leq) \times \mathfrak{F}$ is labelled by its f -image.) It is not hard to check that f is a p -morphism from $(\omega, \leq) \times \mathfrak{F}$ onto \mathfrak{G}_k^{top} . \square

LEMMA 13. *\mathfrak{G}_k^{bot} is a p -morphic image of $(\omega, \leq) \times \mathfrak{F}$, for some frame \mathfrak{F} .*

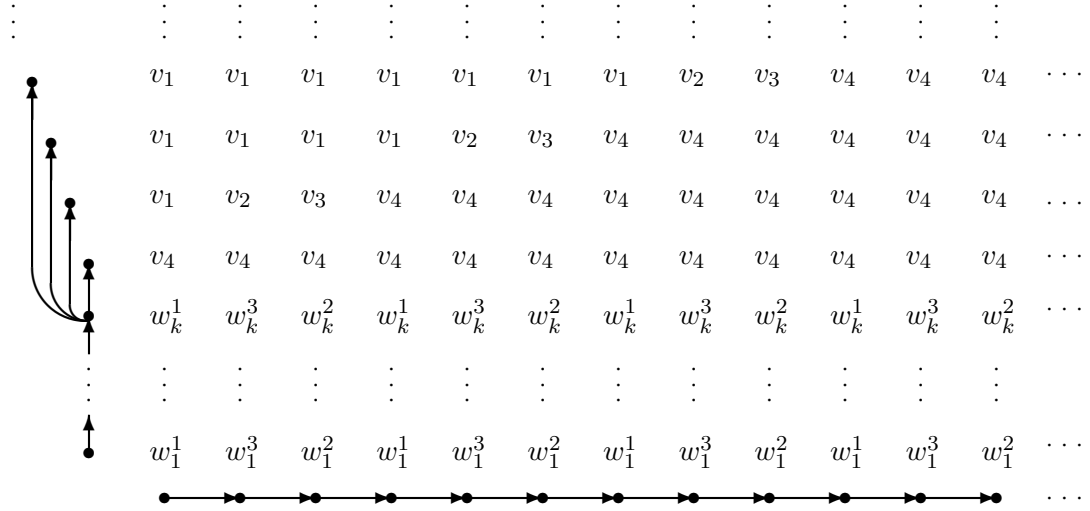


Figure 7: The p-morphism f from $(\omega, \leq) \times \mathfrak{F}$ onto \mathfrak{G}_k^{top} .

Proof. See Fig. 8 for a function g from $\mathfrak{F}_0 \times \mathfrak{F}_1$, where \mathfrak{F}_0 is a ‘balloon’ (a two-point reflexive linear order, followed by a 3-point cluster), and \mathfrak{F}_1 is a kind of ‘comb’: a root seeing three irreflexive and intransitive k -long branches. (In Fig. 8 each point of $\mathfrak{F}_0 \times \mathfrak{F}_1$ is labelled by its g -image.) It is not hard to check that g is a p-morphism from $\mathfrak{F}_0 \times \mathfrak{F}_1$ onto \mathfrak{G}_k^{bot} . As \mathfrak{F}_0 is a p-morphic image of (ω, \leq) , the lemma follows. \square

6 Discussion

We conclude the paper with a few open problems about axiomatising two-dimensional product logics.

(I) According to our present knowledge, a (recursively enumerable) product logic is either non-finitely axiomatisable or product-matching. It would be interesting to find some finitely axiomatisable but not product-matching product logic.

(II) Theorems 1 and 2 above do not apply to product logics $\mathbf{K4.3} \times L$, where L has no ω -fan among its frames. Important ‘standard’ logics of this kind are $\mathbf{K4.3}$ and $\mathbf{S5}$. So the questions whether the recursively enumerable logics $\mathbf{K4.3} \times \mathbf{K4.3}$ and $\mathbf{K4.3} \times \mathbf{S5}$ are finitely axiomatisable remain open. (The same applies to products with $\mathbf{S4.3}$.)

(III) $\mathbf{K4.3} \times \mathbf{K4.3}$ is known to be not product-matching (see (3) in Section 1). Here we present a 2-frame $\mathfrak{F} = (W, \leq_0, \equiv_1)$ showing that neither $\mathbf{K4.3} \times \mathbf{S5}$ nor $\mathbf{S4.3} \times \mathbf{S5}$ are product-matching: In Fig.9, the horizontal arrows and ellipses represent the reflexive, transitive and weakly connected \leq_0 , and the boxes, triangles and circles the \equiv_1 -equivalence classes. It is not hard to check that \leq_0 and \equiv_1 commute. (In case of a symmetric second relation, the Church–Rosser property follows from commutativity.) On the other hand, it is straightforward to see that property Φ_0 (see Fig. 6) fails in \mathfrak{F} .

(IV) As (ω, \leq) is not a frame for $\mathbf{Log} \{(\omega, <)\}$, Theorems 1 and 2 do not apply to product logics of the form $\mathbf{Log} \{(\omega, <)\} \times L$. Some logics of this form, such as $\mathbf{Log} \{(\omega, <)\} \times \mathbf{K}$ and $\mathbf{Log} \{(\omega, <)\} \times \mathbf{S5}$, are known to be recursively enumerable, even decidable [24], [9, Thms.6.33,

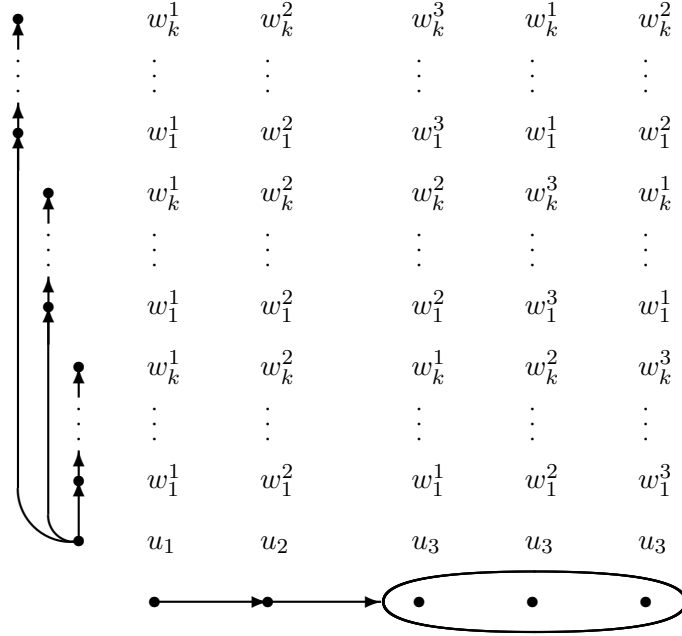


Figure 8: The p-morphism g from $\mathfrak{F}_0 \times \mathfrak{F}_1$ onto \mathfrak{G}_k^{bot} .

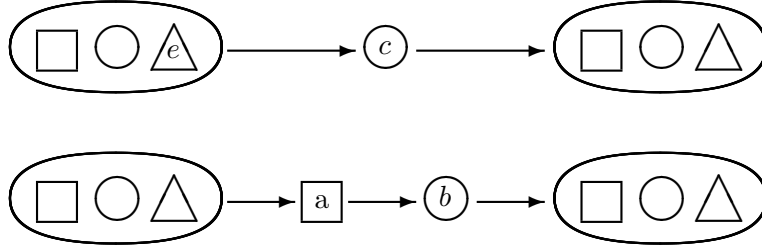


Figure 9: The frame \mathfrak{F} showing that $\mathbf{S4.3} \times \mathbf{S5}$ is not product-matching.

6.60]. We do not know whether $\text{Log}\{(\omega, <)\} \times \mathbf{K}$ or $\text{Log}\{(\omega, <)\} \times \mathbf{S5}$ is finitely axiomatisable. However, there is a related positive axiomatisation result. As is shown in [25] (see also [9, Thm.11.78]), if we have in the language of the first component logic not only \Box_0 and \Diamond_0 but also a next time operator X_0 , then the resulting logic $\text{Log}_{\Box, X}\{(\omega, <, +1)\} \times \mathbf{S5}$ is kind of product-matching: One needs to take the formulas axiomatising the components, plus the formula $X_0\Box_1 p \leftrightarrow \Box_1 X_0 p$, describing that X_0 and the $\mathbf{S5}$ -box \Box_1 commute. Note that the proof of this axiomatisation result uses the fact that any rooted frame for $\text{Log}_{\Box, X}\{(\omega, <, +1)\}$ is a p-morphic image of its ‘standard’ frame $(\omega, <, +1)$. However, the similar statement about arbitrary frames of $\text{Log}\{(\omega, <)\}$ is not true.

(V) As is shown in [17, Thm.2.10], if each L_i , $i < 2$, is a logic such that its class of frames is closed under ultraproducts, then $L_0 \times L_1$ is a canonical bimodal logic. So, say, $\mathbf{K4.3} \times \mathbf{K}$, $\mathbf{K4.3} \times \mathbf{K4}$, $\mathbf{K4.3} \times \mathbf{K4.3}$, and $\mathbf{K4.3} \times \mathbf{S5}$ are such. However, the following questions are open:

- Does any of these product logics have a canonical axiomatisation?

- Is the class of *all* frames for any of these product logics closed under ultraproducts?

It is quite difficult to think about modally expressible properties that do not have first-order correspondents. So answers to the above would directly be relevant to finding explicit, possibly infinite, axiomatisations for the logics in question. In case of higher dimensional product logics (and of algebras of relations) similar questions have negative answers [13, 14, 17]. It is not known, however, whether the techniques used to achieve these results are applicable to two-dimensional cases.

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