Central Polygonal Numbers

Henrik Helsen

Central polygonal numbers, informally known as "the lazy caterer's sequence", is a sequence which shows the most number of pieces a circle or disk can be divided into with a given number of straight cuts. For example, a circle can be cut into 2 pieces with one cut, but it can be cut into 4 pieces with another cut, and into 7 pieces with yet another cut. The first ten numbers in the sequence are 2, 4, 7, 11, 16, 22, 29, 37, 46 and 56.

Formula

The formula of this sequence is given as:

$$p = \frac{n^2 + n + 2}{2}$$

Where p is the maximum number of pieces produced and n is the number of cuts, and the number of cuts is more than or equal to 0.

Proof[1]

When a circle is cut a certain number of times to produce the highest possible number of pieces, represented as $p = f(n)$ (where n shows the number of cuts), the cut must be considered. The number of pieces before the last cut is $f(n - 1)$, whereas the number of pieces added by the last cut is n.

To reach the highest possible number of pieces, the nth cut line should cross all the other previous cut lines inside the circle, but the circle cannot cut any intersection of previous cut lines. So, the nth line is cut in $n - 1$ places (in other words, every cut other than itself, because it cannot cross itself), and into n line segments. Each part divides 1 piece of the $(n - 1)$ cut into 2 parts, adding exactly n to the number of pieces. The new line can't have any more segments since it can only cross each previous line once. A cut line can cross over all previous cut lines, as rotating the knife at a small angle around a point that is not an existing intersection will - if the angle is small enough - intersect all the previous lines, including the last one added.

Thus, the total number of pieces after n cuts is:

$$p = n + f(n - 1)$$

If $f(n - 1)$ is expanded one term, the formula becomes:

$$p = n + (n - 1) + f(n - 2)$$

Expansion of $f(n - 2)$ can continue until the last term is reduced to $f(0)$, where only the original circle without cuts is left, which leaves:

$$p = n + (n - 1) + (n - 2) + 1 + f(0)$$

---

[1] Wikipedia, The Free Encyclopaedia. Last edited on 21 January 2021. [https://en.wikipedia.org/wiki/Lazy_caterer%27s_sequence/; last accessed on 25 March 2021]

Since $f(0) = 1$, because there is 1 piece before any cuts are made, this can be rewritten as:

$$p = 1 + (1 + 2 + 3 + 4 + 5 + 6 \ldots + n)$$

Which simplifies to:

$$p = 1 + \frac{n(n + 1)}{2}$$

Which is then equal to:

$$p = \frac{n^2 + n + 2}{2}$$

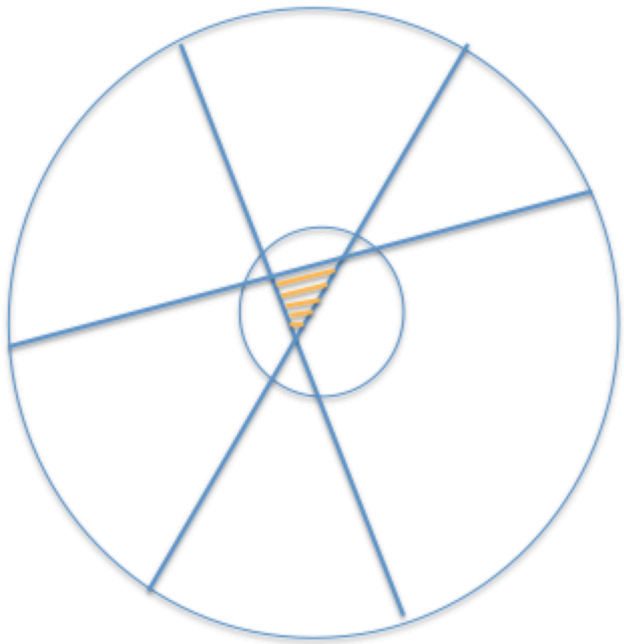
Although this sequence is usually used in the context of a circle, it will apply to any polygon regardless of the side length, as the *n*th cut will always make *n* segments regardless of how many sides the shape has, and this is reflected in the proof, which does not at any time make reference to the sides of the shape.

The Sequence with a Doughnut

Now, we will show how the sequence changes if there is a hole in the shape. A doughnut has been used as an example for simplicity; however, as explained above, the overall shape could be any polygon. In this example, the first two cuts made on the polygon will not make a difference, as they will produce 4 pieces, regardless of where they are.

Regardless of how the cuts are made, 4 pieces will always be produced, as shown above. This forms the baseline, and all scenarios will be taken from the point where there are 2 cuts on the doughnut.
We can now establish that in a doughnut, the maximum number of pieces created with a given number of straight cuts is *at most* equal to the maximum number of pieces created

in a normal circle; some "pieces" are lost as they are completely within the hole of the doughnut, as shown below:



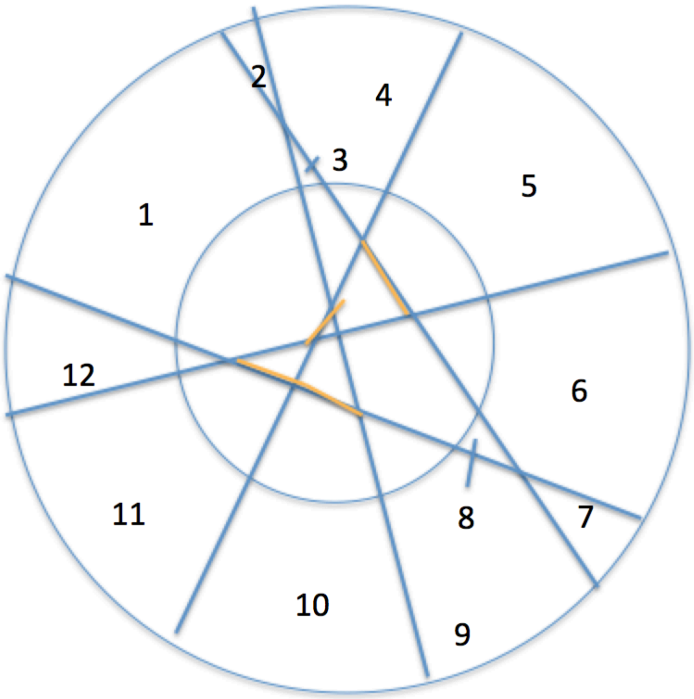
The piece shaded in orange cannot be counted, as in "real life" it would be in the hole of the dougnut and thus it would not exist. Therefore, the number of pieces created by the cuts in a doughnut will sometimes be less than in an ordinary circle.

In this case, a segment is a section of a cut created by the *n*th cut crossing all the previous (n - 1) cuts. So the *n*th cut has *n* segments.

Therefore to determine the number of pieces created with a certain number of cuts, you must subtract the number of segments created fully inside the hole of the doughnut, as for every segment inside the doughnut hole you add, you will also add a "piece" fully inside the doughnut hole, which cannot be counted as it does not exist. Therefore the formula for the maximum number of pieces ***p*** that you can create is:

$$p = \frac{n^2 + n + 2}{2} - k$$

where ***k*** is the number of segments fully inside the hole. For example, with 5 cuts and 4 segments outside the hole, the formula is 16 - 4 = maximum 12 pieces that can be made, as shown below:



5 cuts, ordinarily giving 16 pieces - 4 segments, highlighted in orange = 12 cuts.

However, if all the cuts always meet within the hole, then the formula for the number of pieces is ***2n***. This can be proven as follows:

In this case:

$$k = 1 + 2 + 3 \dots + n - 2$$

$$k = \frac{n(n+1)}{2}$$

$$k = \frac{n - 2(n - 2 + 1)}{2}$$

$$k = \frac{(n-2)(n-1)}{2}$$

$$k = \frac{n^2 - 3n + 2}{2}$$

The general formula is:

$$\frac{n^2 + n + 2}{2} - k$$

So we can now substitute in for ***k***:

$$p = \frac{n^2 + n + 2}{2} - \frac{n^2 - 3n + 2}{2}$$

$$p = \frac{n^2 + n + 2 - n^2 - 3n + 2}{2}$$

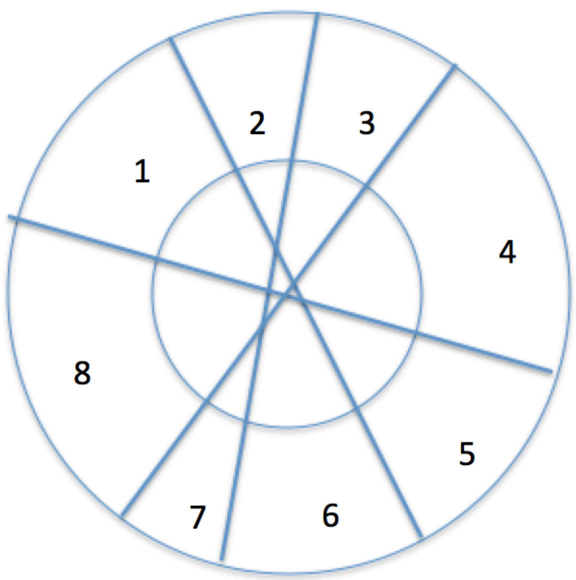
$$p = \frac{n + 2 + 3n + 2}{2}$$

$$p = \frac{n + 3n}{2}$$

$$p = \frac{4n}{2}$$

$$p = 2n$$

Below is a simple diagram to show how this is true with a doughnut.



The doughnut has 4 cuts, all of which meet in the hole, and so there are 8 pieces created.