

QUANTUM OPTICS IN ELECTRIC CIRCUITS  
763693S

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2017

## Abstract

With present nanofabrication methods it is possible to make such small electric circuits that quantum effects become essential. The circuits behave like artificial atoms and the methods to deal with them resemble those used in quantum optics and NMR rather than traditionally used by electrical engineers. This course is an introduction to the physics of such circuits. One major topic is how to include dissipation into quantum mechanics. This will be answered by deriving a master equation, and applying it to a harmonic oscillator and to a two-state system. The realization of the two-state system requires a nonlinear element, for which superconducting Josephson junctions are used. Another theme is different types of noise (thermal, shot, quantum). These can be derived by applying scattering formalism which considers electrons in a conductor like waves in a transmission line. We try to answer, among other things, if noise is present at zero temperature, is supercurrent noisy, and can zero-point fluctuations be measured.

Moderneilla valmistusmenetelmillä voidaan tehdä pienikokoisia sähköisiä piirejä, joissa kvanttimekaaniset ilmiöt ovat olennaisia. Nämä piirit toimivat kuin keinotekoiset atomit ja niiden kuvaamiseen käytetään menetelmiä, jotka ovat tutumpia kvanttioptiikassa ja ydinmagneettisessa resonanssissa kuin sähköopissa. Tämä kurssi on johdatus tällaisten piirien fysiikkaan. Yksi pääaihe on kuinka liittää häviölliset ilmiöt kvanttimekaniikkaan. Tämä tehdään johtamalla master-yhtälö, ja sitä sovelletaan harmoniseen oskillaattoriin ja kaksitasosysteemiin. Kaksitasosysteemin toteuttaminen edellyttää epälineaarista elementtiä, jona käytetään suprajohtavaa Josephson-liitosta. Toinen pääteema on erityyppiset kohinat kuten lämpö-, isku- ja kvanttikohina. Nämä voidaan johtaa käyttäen sirontaformalismia, missä elektroneja johteessa kuvataan kuin aaltoja aaltoputkessa. Tarkoitus on vastata mm. onko nollalämpötilassa kohinaa, kohiseeko supravirta ja voidaanko nollapistevärähtelyjä mitata.

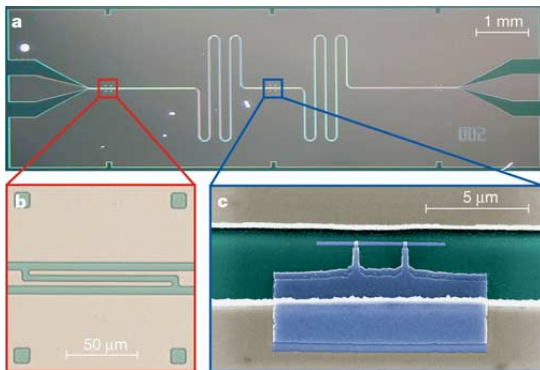


Figure: [A. Wallraff et al, *Nature* **431**, 162 (2004)]

Integrated circuit for cavity QED. a, The superconducting niobium coplanar waveguide. b, The capacitive coupling to the input and output lines. c, False

colour electron micrograph of a Cooper pair box (blue) fabricated onto the silicon substrate (green) into the gap between the centre conductor (top) and the ground plane (bottom) of a resonator (beige) using electron beam lithography and double angle evaporation of aluminium. The Josephson tunnel junctions are formed at the overlap between the long thin island parallel to the centre conductor and the fingers extending from the much larger reservoir coupled to the ground plane.

## Practicalities

Write your name, department/group, year class, and email address on the list.

The lectures are given in English if there is sufficient demand for it, otherwise in Finnish.

The web page of the course is <https://www.oulu.fi/tf/qo/index.html>

The web page contains the lecture material (these notes), the exercises and later also the solutions to the exercises. See the web page also for possible changes in lecture and exercise times.

## Time table 2017

Lectures: Tuesdays 14-16, room HU122 12. and 19.9, Thursday 12-14, room HU108 21.9, Tuesdays 14-16, room HU122 26.9.- 17.10. and 31.10.-5.12.

Exercises: Thursdays 8-10, 14.9.-19.10 and 2.11.-7.12, room HU122 except on 14.9. room HU109.

Examination: 14.12.2017

Teaching assistant: Iivari Pietikäinen

Doing exercises is essential for learning. In addition, showing completed exercised will affect your final evaluation. (You can improve by one, for example, from 3 to 4.) Start calculating (at home) before the exercise time, 2 hours is too short time to start from scratch.

# 1. Introduction

## Content

The following is only an appetizer, and details will be explained during the course.

### harmonic oscillator

electric elements, electric harmonic oscillator, stripline microwave oscillator, optical cavity, nanomechanical oscillator

familiar classical forced and damped harmonic oscillator solutions, quadrature variables, rotating wave approximation

$$\dot{\alpha} = -i\omega_0\alpha - \frac{\gamma}{2}\alpha + if. \quad (1)$$

quantum harmonic oscillator (no dissipation), energy eigenstates  $E = \hbar\omega_0(n + \frac{1}{2})$

general quantum state  $|\psi_i(t)\rangle$ , density operator  $\rho = \sum_i P_i |\psi_i\rangle\langle\psi_i|$

general derivation of master equation with dissipative terms, interaction picture, Markov approximation, rotating wave approximation, the master equation for harmonic oscillator

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar}[H_0 + H_d, \rho] \\ & + \frac{\gamma}{2}(N + 1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \\ & + \frac{\gamma}{2}N(2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger) \end{aligned} \quad (2)$$

thermal state solution, coherent states, decaying solution, driving terms, general solutions using translation operator.

for average values, the classical and quantum harmonic oscillators are identical.

statistical aspect, special case of fluctuation-dissipation theorem

### two-state system

qubit = two-state system, how to construct using Josephson junctions, circuit analysis with Josephson junctions.

master equation for two state system, rotating wave approximation, Bloch equations.

$$\begin{aligned} \dot{u} &= -\Delta v - \frac{u}{T_2'}, \\ \dot{v} &= \Delta u - \frac{v}{T_2'} + \kappa \mathcal{E} w, \\ \dot{w} &= -\frac{w - w_{\text{eq}}}{T_1} - \kappa \mathcal{E} v. \end{aligned} \quad (3)$$

harmonic oscillator coupled to a two-state system, Jaynes-Cummings model

### Electric transport and noise

transmission lines, transverse modes

temporal modes (wavelets), connection to information theory, Landauer formula

$$G = \frac{2e^2}{h}MT \quad (4)$$

$G$  conductance,  $M$  number of transverse modes,  $T$  transmission probability.

introduction to noise, spectral density  $S_{II}(\omega)$ ,

thermal noise  $S_{II}(\omega) = 2GkT$ ,

shot noise  $S_{II}(\omega) = e\langle I \rangle$ ,

quantum noise  $S_{II}(\omega) = 2G\hbar\omega\Theta(\omega)$

scattering states, derivation of thermal, shot, and quantum noise in scattering formalism

### Prerequisites

Prior knowledge on quantum mechanics is needed. The more you know the easier it is, but I try to explain the quantum mechanics that is used. Knowledge of classical Lagrangian and Hamiltonian mechanics is needed at some places.

### Books and articles

The first three are standard references in quantum optics:

D. F. Walls and G. J. Milburn, *Quantum optics* (Springer, Berlin, 1994, second edition 2008). Although not very pedagogical, I have been reading mainly this.

M. O. Scully and M. S. Zubairy, *Quantum optics* (Cambridge, 1997).

C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, *Atom-photon interactions, basic processes and applications* (Wiley, New York, 1992).

S. Haroche and J.-M. Raimond, *Exploring the quantum - atoms, cavities and photons* (Oxford 2006). Contains thorough introductions to various topics, experiments with cat states.

L. Allen and J. H. Eberly, *Optical resonance and two-level atoms* (Dover, New York, 1975). Good account of Bloch equations, but no master equation.

M. Devoret: Quantum fluctuations in electrical circuits, in *Les Houches, Session LXIII, 1995, Quantum Fluctuations*, (Elsevier 1997). p 351-386. Circuit theory with Josephson junctions. (download at <http://www.eng.yale.edu/qlab/archives.htm>)

S. Datta: *Electronic transport in mesoscopic systems* (Cambridge, 1995). Explains the scattering formulation and Landauer formula.

G B Lesovik, I A Sadovskyy: Scattering matrix approach

to the description of quantum electron transport, *Physics - Uspekhi* **54**, 1007-1059 (2011).

Ya. M. Blanter and M. Büttiker: Shot noise in mesoscopic conductors, *Phys. Rep.* **336**, 1-166 (2000). Scattering formulation of noise.

C. Beenakker and C. Schönberger: Quantum Shot Noise, *Physics Today*, May 2003, p. 37-42. A shorter discussion of shot noise.

### Practical applications of quantum electric circuits

No one knows yet

- quantum computation
- sensitive measuring devices

Here the main motivation is just to understand the phenomena!

### Notation

$e$  is the charge of an electron. I have tried to formulate everything so that it is irrelevant whether this quantity is negative or positive.

$\hbar = h/2\pi$  Planck's constant.

$a \ll b$  means that  $|a|$  is much smaller than  $|b|$ , and correspondingly for  $a \gg b$ .

$a/bc = a/(bc)$ .

The step function

$$\Theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0. \end{cases} \quad (5)$$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$ ,  $\sinh x = \frac{1}{2}(e^x - e^{-x})$ ,  
 $\tanh x = 1/\coth x = \sinh x/\cosh x$ .

## 2. Harmonic oscillator

Study oscillations around some equilibrium state. If the amplitude of the oscillations is small, nonlinear terms are not important, one is led to harmonic oscillator.

Examples of harmonic oscillator:

- mass in spring, mechanical pendulum (for small amplitudes)
- waves in string: each oscillation mode is a harmonic oscillator
- sound waves: every mode is a harmonic oscillator

electric analog - LC oscillator, modes in a transmission line, and modes of electromagnetic wave (any frequency: rf, microwave, infrared, light etc) are examples harmonic oscillators.

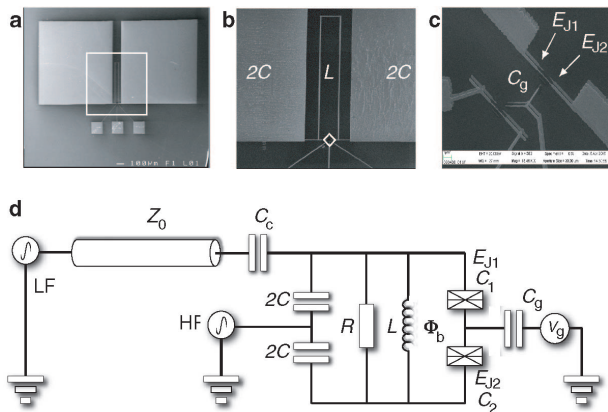


Figure: [D. Gunnarson et al, *Phys. Rev. Lett.* **101**, 256806 (2008)] LC oscillator coupled to superconducting single-electron transistor. The area of white rectangles in **a** and **b** is shown enlarged in **b** and **c**, respectively.

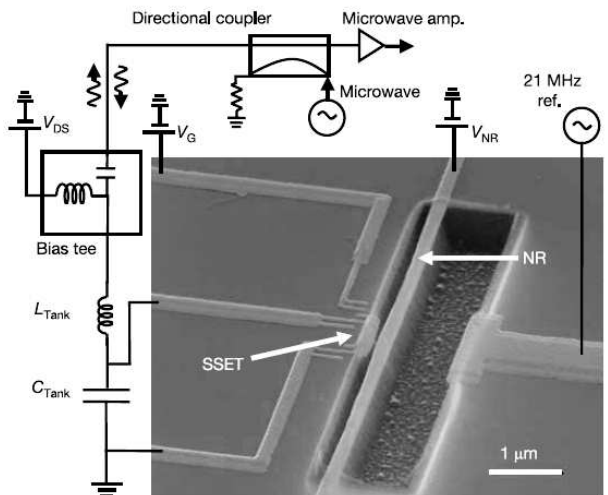


Figure: [A. Naik et al, *Nature* **443**, 193 (2006)] Scanning electron microscope image of the device: a 21.9-MHz, doubly clamped, SiN and Al nanomechanical resonator (NR) coupled to a superconducting single-electron transistor (SSET), with simplified measurement circuit

diagram. The SSET is read out by applying 1.17 GHz microwaves through the directional coupler, and monitoring the signal reflected off the LCR tank circuit formed by  $L_{\text{Tank}}$ ,  $C_{\text{Tank}}$  and the SSET.

## 2.1 Classical harmonic oscillator

Mechanical harmonic oscillator has the equation of motion

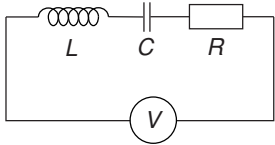
$$\dot{p} = m\ddot{q} = -kq - m\gamma\dot{q} + F \quad (6)$$

with spatial coordinate  $q$ , momentum  $p = m\dot{q}$ , mass  $m$ , friction constant  $m\gamma$ , spring constant  $k$ , external *driving force*  $F$  and time derivative marked with a dot. This can be written in the form

$$m\ddot{q} + m\gamma\dot{q} + kq = F. \quad (7)$$

In the limiting case  $k = 0$ , the oscillator equation (6) reduces to describe a free particle. Thus in this special limit the harmonic oscillator also applies to the description the free-electron model of solids (giving the Drude conductivity etc.).

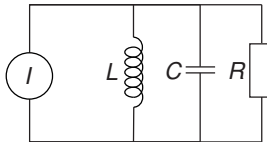
Consider an electric LC oscillator. Electric current  $I = \dot{Q}$ , where  $Q$  is the charge. The voltage  $V$  can be written  $V = \dot{\Phi}$  where  $\Phi$  is the magnetic flux. (This form will be convenient later, but not necessary at this point.) For the familiar electric elements we have the relations: capacitor  $Q = CV$ , inductor  $\Phi = LI$ , and resistor  $V = RI$ . We study first the series oscillator shown in the figure. It is driven by a voltage source (the voltage is independent of the current flowing through).



For this we deduce

$$V = \sum_i V_i = \frac{Q}{C} + R\dot{Q} + L\ddot{Q}. \quad (8)$$

Its dual, the parallel oscillator, is shown in the following figure. It is driven by a current source (the current is independent of the voltage over the source).



For this we find similarly

$$I = \sum_i I_i = C\ddot{\Phi} + \frac{\dot{\Phi}}{R} + \frac{\Phi}{L}. \quad (9)$$

(Work out the details as an exercise.)

The Maxwell equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{B} &= \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}. \end{aligned}$$

The fields  $\mathbf{E}(\mathbf{r}, t)$  and  $\mathbf{B}(\mathbf{r}, t)$  can be represented as sums of 3 vector components and spatial Fourier components. Consider a single *mode*, for example

$$\mathbf{E} = E(t)\hat{\mathbf{y}}e^{ikx}, \quad \mathbf{B} = iB(t)\hat{\mathbf{z}}e^{ikx}. \quad (10)$$

Assume there is some source current  $\mathbf{j}_0(\mathbf{r}, t)$ , for example, an oscillating charge. It has one Fourier component  $j_0(t)\hat{\mathbf{y}}e^{ikx}$  that couples to the mode above. In addition we assume some conductivity  $\sigma$  of the medium,  $\mathbf{j} = \mathbf{j}_0 + \sigma\mathbf{E}$ . (Alternatively, could consider dielectric loss.) Substituting to the Maxwell equations we get

$$\begin{aligned} \epsilon_0 \mu_0 \dot{E} &= kB - \mu_0 \sigma E - \mu_0 j_0, \\ \dot{B} &= -kE. \end{aligned} \quad (11)$$

This again is of the same form as the other oscillator equations above. Thus a mode of electromagnetic radiation can be considered as a harmonic oscillator. The whole electromagnetic field has a huge number of modes which all can be considered as independent harmonic oscillators.

We notice that all the equations of motion (7), (8), (9), and (11) are of the same form, so we need to study only one of them. Although we use the notation of mechanical oscillator in the following, it should be remembered that the analysis is valid for all kinds of harmonic oscillators.

We define  $\omega_0 = \sqrt{k/m}$ . Thus  $k = m\omega_0^2$  and the equations of motion can be written in the form

$$\begin{aligned} \dot{p} &= -m\omega_0^2 q - \gamma p + F \\ \dot{q} &= \frac{p}{m}. \end{aligned} \quad (12)$$

A more symmetric form is

$$\begin{aligned} \frac{\dot{p}}{\sqrt{m\omega_0}} &= -\omega_0 \sqrt{m\omega_0} q - \gamma \frac{p}{\sqrt{m\omega_0}} + \frac{F}{\sqrt{m\omega_0}} \\ \sqrt{m\omega_0} \dot{q} &= \omega_0 \frac{p}{\sqrt{m\omega_0}}. \end{aligned} \quad (13)$$

We define the complex parameters

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{2\hbar}} \left( \sqrt{m\omega_0} q + i \frac{p}{\sqrt{m\omega_0}} \right) \\ \alpha^* &= \frac{1}{\sqrt{2\hbar}} \left( \sqrt{m\omega_0} q - i \frac{p}{\sqrt{m\omega_0}} \right). \end{aligned} \quad (14)$$

Think here  $\hbar$  simply as some number, there is no quantum mechanics involved yet. In these variables the equations of motion (12) get an alternative form

$$\begin{aligned}\dot{\alpha} &= -i\omega_0\alpha - \frac{\gamma}{2}\alpha + \frac{\gamma}{2}\alpha^* + \frac{iF}{\sqrt{2\hbar m\omega_0}} \\ \dot{\alpha}^* &= i\omega_0\alpha^* - \frac{\gamma}{2}\alpha^* + \frac{\gamma}{2}\alpha - \frac{iF}{\sqrt{2\hbar m\omega_0}}.\end{aligned}\quad (15)$$

The latter equation is the complex conjugate of the former, and therefore one of them is sufficient. The solution in the absence of driving and damping is  $\alpha(t) = \alpha(0)e^{-i\omega_0 t}$ , i.e. a rotation in the complex  $\alpha$  plane with angular frequency  $\omega_0$ .

In the following we assume the driving force is harmonic,  $F(t) \propto \cos(\omega_d t)$ . Note that this is no real restriction, since any function  $F(t)$  can be represented as a Fourier series, and because the equation of motion (15) is linear, the solutions corresponding to each Fourier component can be solved separately, and finally summed together. For notational beauty we also write the prefactor in a convenient form. So we study a driving force of the form

$$\begin{aligned}F(t) &= \sqrt{2\hbar m\omega_0}f(t), \\ f(t) &= 2f_0 \cos(\omega_d t).\end{aligned}\quad (16)$$

The equations of motion (7) or (15) have an exact solution (Exercise). However, later we often need a *rotating wave approximation*, and it is appropriate to introduce it already here. The assumption is that the driving and dissipative terms are small relative to the first terms on the right hand side of (15). For that we transform to a frame rotating at the drive frequency by writing

$$\alpha(t) = \beta(t)e^{-i\omega_d t}.\quad (17)$$

Substitution gives

$$\begin{aligned}\dot{\beta} &= -i(\omega_0 - \omega_d)\beta - \frac{\gamma}{2}\beta + \frac{\gamma}{2}e^{2i\omega_d t}\beta^* \\ &\quad + if_0(1 + e^{2i\omega_d t}).\end{aligned}\quad (18)$$

If the driving frequency is near resonance,  $\omega_d \approx \omega_0$ , the first term on the right hand side is small, and supposedly  $\beta(t)$  changes slowly in time. In contrast, the two ‘‘counter rotating’’ terms contain a rapidly oscillating factor  $e^{2i\omega_d t}$ . The claim is that the effect of these average out on the time scale we are interested in, which is determined by ‘‘secular’’ non-oscillating terms on the right hand side. Therefore we simply drop out the rapidly oscillating terms. (The accuracy of the rotating wave approximation could be tested by comparison with the exact solution, or by doing perturbation theory with respect of the counter-rotating terms.) Transforming back to  $\alpha$  we have *the equation of motion for a classical harmonic oscillator in the rotating wave approximation*

$$\dot{\alpha} = -i\omega_0\alpha - \frac{\gamma}{2}\alpha + if_0e^{-i\omega_d t}.\quad (19)$$

The equation motion (19) can be solved without difficulties. The *steady state* solution is obtained by assuming  $\alpha = \text{constant} \times e^{-i\omega_d t}$ , and one finds

$$\alpha_{ss}(t) = \frac{f_0 e^{-i\omega_d t}}{\omega_0 - \omega_d - \frac{i}{2}\gamma}.\quad (20)$$

The general solution of an initial value problem is

$$\begin{aligned}\alpha(t) &= \alpha_{ss}(t) + [\alpha(0) - \alpha_{ss}(0)]e^{-i\omega_0 t - \frac{\gamma}{2}t} \\ &= \frac{f_0}{\omega_0 - \omega_d - \frac{i}{2}\gamma}(e^{-i\omega_d t} - e^{-i\omega_0 t - \frac{\gamma}{2}t}) \\ &\quad + \alpha(0)e^{-i\omega_0 t - \frac{\gamma}{2}t}.\end{aligned}\quad (21)$$

Exercise: plot the real and imaginary part of  $\alpha(t)$  as a function of time, and depict the trajectory of  $\alpha(t)$  in the complex plane for

- $f_0 = 0$ ,  $\alpha(0) = 1$ ,  $\omega_0 = \omega_d = 1$ ,  $\gamma = 0.1$ ;
- $f_0 = 1$ ,  $\alpha(0) = 0$ ,  $\omega_0 = \omega_d = 1$ ,  $\gamma = 0.1$ , and interpret the results.

Let us calculate the energy stored in the oscillator as the sum of kinetic and potential energies

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2.\quad (22)$$

Expressing the energy in terms of  $\alpha$  (14) gives the very suggestive form

$$E = \hbar\omega_0|\alpha|^2.\quad (23)$$

Note however that at this point  $\hbar$  is just some arbitrary normalization factor that also appears in the definition of  $\alpha$  (14) and thus it cancels out in the energy (23).

Calculating the time derivative of  $E$  (23) and using the equation of motion (19) one gets

$$\begin{aligned}\frac{dE}{dt} &= P_{\text{abs}} - P_{\text{diss}}, \\ P_{\text{abs}} &= 2\hbar\omega_0\Im(f_0 e^{i\omega_d t}\alpha), \\ P_{\text{diss}} &= \hbar\omega_0\gamma|\alpha|^2,\end{aligned}\quad (24)$$

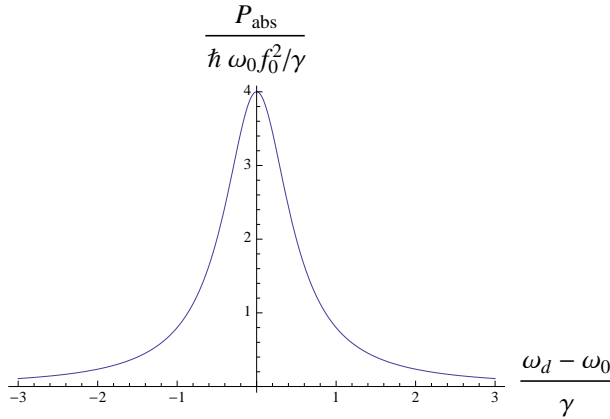
where  $\Im$  denotes the imaginary part. Here the first term on the right hand side can be interpreted as *absorption*, the power transferred from the driving force to the oscillator. The latter term, which always is negative, is the *dissipation*, the power lost in friction.

Let us calculate the absorption and dissipation in the steady state (20). We find that they are equal, and given by

$$P_{\text{abs}} = P_{\text{diss}} = \frac{\hbar\omega_0\gamma f_0^2}{(\omega_0 - \omega_d)^2 + \frac{1}{4}\gamma^2}.\quad (25)$$

One often thinks the absorbed power as function of the drive frequency,  $P_{\text{abs}}(\omega_d)$ . Its maximum corresponds to *resonance*, where the drive frequency equals the natural frequency of the oscillator,  $\omega_d = \omega_0$ . The form of  $P_{\text{abs}}(\omega_d)$

in Eq. (25) is called *Lorentzian line shape*. Its full width at half maximum (often abbreviated as FWHM) equals  $\gamma$ .



It is also instructive to look at

$$\begin{aligned} e^{i\omega_d t} \alpha_{ss}(t) &\equiv \alpha_{ss}(0) = \frac{f_0}{\omega_0 - \omega_d - \frac{i}{2}\gamma} \\ &= \frac{f_0(\omega_0 - \omega_d + \frac{i}{2}\gamma)}{(\omega_0 - \omega_d)^2 + \frac{1}{4}\gamma^2} = A e^{i\delta} \end{aligned} \quad (26)$$

in the complex plane. We decompose this complex number into real amplitude  $A$  and phase  $\delta$ . Here  $\delta$  is the phase shift between the oscillator spatial coordinate  $q$  and the driving force  $f$ . For small driving frequencies  $\delta \approx 0$ , i.e. the “particle” just tilts in the direction of the force [as is evident from (7) when the derivative terms can be neglected]. With increasing drive frequency  $\delta$  grows and gets the value  $\pi/2$  at resonance. It approaches  $\pi$  for  $\omega_d \rightarrow \infty$ , i.e. the particle tilts in the direction opposite to the force. Apart from constant factors, the imaginary part of  $\alpha_{ss}(0)$  is the *absorption* (25). The real part of  $\alpha_{ss}(0)$  makes the phase shift to differ from  $\pi/2$  and is called *dispersion*.

Exercise: plot the absorption (simply  $\Im[\alpha_{ss}(0)]$ ), dispersion and the phase shift  $\delta$  as a function of  $\omega_d$ , and depict the trajectory of  $\alpha_{ss}(0)$  in the complex plane for  $f_0 = 1$ ,  $\omega_0 = 1$ ,  $\gamma = 0.1$  and interpret the results.

One often used quantity is the *quality factor* or  $Q$ . It is a dimensionless number defined as the resonance angular frequency times the energy stored divided by the dissipated power. Using expressions above we calculate

$$Q = \frac{\omega_0 E}{P_{\text{diss}}} = \frac{\omega_0}{\gamma}. \quad (27)$$

The validity of the rotating wave approximation requires  $Q \gg 1$ .

## 2.2 Quantization

### Simple harmonic oscillator

Consider now the harmonic oscillator (12) but without damping and driving,

$$\dot{p} = -m\omega_0^2 q$$

$$\dot{q} = \frac{p}{m}. \quad (28)$$

Quantum mechanics makes  $p$  and  $q$  operators which are Hermitian,  $p^\dagger = p$ ,  $q^\dagger = q$ . They have to satisfy the commutation relations

$$[q, p] = i\hbar, \quad (29)$$

where  $[A, B] = AB - BA$  is a commutator. Notice that  $[A, A] \equiv 0$  and therefore  $[q, q] = [p, p] = 0$ .

We make the same change of variables as above (14), but instead of  $\alpha$  and  $\alpha^*$  we write  $a$  and  $a^\dagger$ ,

$$\begin{aligned} a &= \frac{1}{\sqrt{2\hbar}} \left( \sqrt{m\omega_0} q + i \frac{p}{\sqrt{m\omega_0}} \right) \\ a^\dagger &= \frac{1}{\sqrt{2\hbar}} \left( \sqrt{m\omega_0} q - i \frac{p}{\sqrt{m\omega_0}} \right). \end{aligned} \quad (30)$$

We see that  $a^\dagger$  is the Hermitian conjugate of  $a$  as  $i^\dagger = i^* = -i$ . By direct calculation we can now verify that

$$[a, a^\dagger] = 1. \quad (31)$$

From this relation it follows that the Hilbert space where  $a$  and  $a^\dagger$  operate is spanned by states of the form

$$|n\rangle, \quad n = 0, 1, 2, \dots, \infty, \quad (32)$$

and the following relations determine how  $a$  and  $a^\dagger$  operate on these,

$$\begin{aligned} a^\dagger a |n\rangle &= n |n\rangle, \\ a |n\rangle &= \sqrt{n} |n-1\rangle, \\ a^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle. \end{aligned} \quad (33)$$

Interpretation:  $a^\dagger a$  gives the number of *quanta*,  $a$  annihilates one quantum and  $a^\dagger$  creates one quantum. The states (32) are known as *number states* or *Fock states*.

For completeness, we give a proof of results (32) and (33).

**Proof.** (read at home) We study the operator

$$\tilde{n} = a^\dagger a. \quad (34)$$

It is Hermitian (verify). The eigenvalues of a Hermitian operator are real valued. Let us label the eigenstates of the operator  $\tilde{n}$  by using the eigenvalue  $n$  (a real number)

$$\tilde{n} |n\rangle = n |n\rangle. \quad (35)$$

We assume the eigenstates are normalized,  $\langle n | n \rangle = 1$ . We see that the eigenvalue  $n$  cannot be negative,

$$\begin{aligned} n &= \langle n | \tilde{n} | n \rangle = \langle n | a^\dagger a | n \rangle = \sum_m \langle n | a^\dagger | m \rangle \langle m | a | n \rangle \\ &= \sum_m |\langle m | a | n \rangle|^2 \geq 0. \end{aligned} \quad (36)$$

We easily calculate from (31) that

$$[\tilde{n}, a] = -a. \quad (37)$$

This implies

$$\tilde{n}(a|n\rangle) = a\tilde{n}|n\rangle - a|n\rangle = (n-1)(a|n\rangle), \quad (38)$$

and we see that  $a|n\rangle$  either is a state corresponding to the eigenvalue  $n-1$ , i.e.  $a|n\rangle = c|n-1\rangle$ , or else  $a|n\rangle = 0$ . The latter alternative implies  $\tilde{n}|n\rangle = 0$ , so it is possible only if  $n = 0$ . In the former case normalization gives

$$|c|^2 = \langle n|a^\dagger a|n\rangle = n, \quad (39)$$

and therefore we fix

$$a|n\rangle = \sqrt{n}|n-1\rangle. \quad (40)$$

If one operates sufficiently many times with  $a$ , one should arrive at negative eigenvalues, which is in contradiction with equation (36). The way out of this is that  $n$  is an integer, so that  $a|0\rangle = 0$ , and the process (40) ends. (Note the essential difference between the  $n = 0$  eigenstate  $|0\rangle$  and the zero of the linear space 0.) Correspondingly one can deduce results for  $a^\dagger$ , and we get the results (33). ■

The energy of the oscillator (22) is the same as the *Hamiltonian of the simple harmonic oscillator*

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2. \quad (41)$$

Writing it in terms of  $a$  and  $a^\dagger$  (14) gives

$$H = \hbar\omega_0 \left( a^\dagger a + \frac{1}{2} \right). \quad (42)$$

This allows us to easily find the energy eigenvalues of the oscillator using (33). The eigenstates are  $|n\rangle$  (32)

$$H|n\rangle = E_n|n\rangle \quad (43)$$

and the eigenenergies are

$$E_n = \hbar\omega_0 \left( n + \frac{1}{2} \right). \quad (44)$$

We see the energy is *quantized* to multiples of  $\hbar\omega_0$ .  $n$  is interpreted as *the number of quanta* in the oscillator. The lowest energy state  $|0\rangle$  satisfies  $a|0\rangle = 0$ . It corresponds to no quanta,  $n = 0$ , and *zero point energy*  $E = \hbar\omega_0/2$ . The higher eigenstates can be generated from this by repeated application of  $a^\dagger$ ,

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \quad (45)$$

Above we derived the energy eigenstates of the harmonic oscillator in the abstract form (45). The more usual Schrödinger approach also gives the explicit wave functions

$$\psi_n(q) = \langle q|n\rangle. \quad (46)$$

These are hardly needed in this course, but for completeness let us construct them as well.

In the Schrödinger approach one constructs explicit representation for the operators  $q$  and  $p$  so that the former is simply multiplication by  $q$  and the latter

$$p = -i\hbar \frac{d}{dq}. \quad (47)$$

Substituting these to  $a$  (30) and applying to  $a|0\rangle = 0$  gives

$$\left( \sqrt{m\omega_0}q + \frac{\hbar}{\sqrt{m\omega_0}} \frac{d}{dq} \right) \psi_0(q) = 0. \quad (48)$$

The normalized solution of this is

$$\psi_0(q) = \left( \frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{4}} \exp\left( -\frac{m\omega_0}{2\hbar} q^2 \right). \quad (49)$$

All higher states can be obtained by using (45)

$$\psi_n(q) = \frac{1}{\sqrt{n!}} \left( \sqrt{\frac{m\omega_0}{2\hbar}} q - \sqrt{\frac{\hbar}{2m\omega_0}} \frac{d}{dq} \right)^n \psi_0(q). \quad (50)$$

These consists of the exponential (49) multiplied by Hermite polynomials and normalization constants.

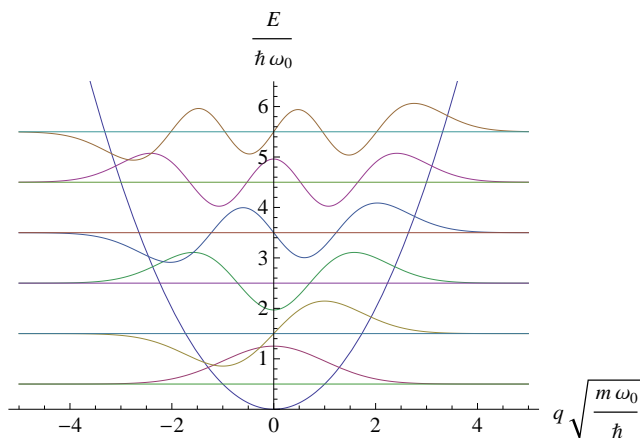


Figure: the parabolic potential energy, the 6 lowest energy eigenvalues (horizontal lines) and the corresponding wave functions (50).

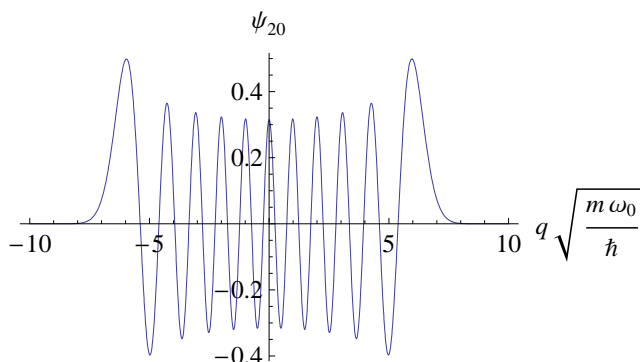


Figure: the wave function  $\psi_{20}(q)$ .

## Forced harmonic oscillator



We include the driving term by postulating a hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2q^2 - F(t)q. \quad (51)$$

Here the first two terms are the same as the energy of undriven oscillator (22), and the last one is a new term describing driving. Applying the classical Hamilton equations

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \quad (52)$$

we get the same equations of motion as above (12) except the damping term. This justifies the Hamiltonian (51).

Quantizing  $p$  and  $q$ , and making the change of variables (14) gives

$$H = \hbar\omega_0 \left( a^\dagger a + \frac{1}{2} \right) - \hbar(a + a^\dagger)f(t). \quad (53)$$

This form is exact. However, in the following we often make again the rotating wave approximation. This is best justified going back to the classical Hamiltonian (since we have not yet discussed the time dependence in the quantum case) by replacing  $a$  and  $a^\dagger$  by  $\alpha$  and  $\alpha^*$ . Using again  $\alpha(t) = \beta(t) \exp(-i\omega_0 t)$  with slow  $\beta$ , we see that the driving term in (53) is composed of two rapidly oscillating terms and two secular terms. Dropping the former and changing back to quantized notation gives

$$H = \hbar\omega_0 \left( a^\dagger a + \frac{1}{2} \right) - \hbar f_0(a^\dagger e^{-i\omega_0 t} + a e^{i\omega_0 t}). \quad (54)$$

We see that the forcing causes transitions between the eigenstates of the simple harmonic oscillator.

Although we could, we are not interested to work on this further before we also have the damping term.

### Including damping

The damping term  $-\gamma p$  in the equations of motion (12) cannot be reproduced by any simple term in a Hamiltonian. Therefore simple minded quantization of the damped harmonic oscillator fails.

To have success, we have to think what friction really is. It means coupling of macroscopic variables to some microscopic degrees of freedom. Energy is then transferred from the macroscopic variables to the microscopic ones. If the number microscopic degrees of freedom is large and they are sufficiently disordered, the energy will effectively not return back to the macroscopic degrees of freedom. Macroscopically this is seen as dissipation, loss of energy.

In order to describe dissipation in quantum mechanics, the obvious way is to include the microscopic degrees of freedom. In a strict sense this means that we have to know what they are. However, we could guess that the gross features of dissipation are independent of the details

of the microscopic world. Therefore we could use a model for them. The simplest model we know is the harmonic oscillator.

Thus we construct the Hamiltonian

$$H = H_0 + H_d + H_{\text{bath}} + H_{\text{int}}, \quad (55)$$

$$H_0 = \hbar\omega_0 \left( a^\dagger a + \frac{1}{2} \right), \quad (56)$$

$$H_d = -\hbar(a + a^\dagger)f(t), \quad (57)$$

$$H_{\text{bath}} = \sum_k \hbar\omega_k b_k^\dagger b_k, \quad (58)$$

$$H_{\text{int}} = -\hbar(a + a^\dagger)(\Gamma + \Gamma^\dagger), \quad \Gamma = \sum_k g_k b_k. \quad (59)$$

Here  $H_0$  is the Hamiltonian of our preferred harmonic oscillator,  $H_d$  is the forcing term (53),  $H_{\text{bath}}$  describes a set of microscopic harmonic oscillators, and  $H_{\text{int}}$  gives the coupling between the macroscopic and microscopic oscillators (we neglect the zero point energy). The operators  $b_k$  and  $b_k^\dagger$  are the annihilation and creation operators for the microscopic oscillators with frequencies  $\omega_k$  and satisfy

$$[b_k, b_l^\dagger] = \delta_{kl}, \quad [b_k, b_l] = [b_k^\dagger, b_l^\dagger] = 0, \quad (60)$$

i.e. all commutators having two different oscillators (including also the preferred one) vanish. The coupling between the oscillators is constructed in analogy with the driving term so that instead of a fixed function  $f(t)$  the time dependence comes from dynamical variables  $\Gamma + \Gamma^\dagger$ , which depend linearly on  $b_k$ 's and  $b_k^\dagger$ 's with coupling constants  $g_k$ . Such a bilinear coupling keeps the quadratic form of the Hamiltonian, which makes it not too difficult to solve.

While (55) should be considered as a model of friction in the case of a mechanical oscillator, it is more than that for an electromagnetic resonator. The preferred oscillator is a mode in the resonator, and it couples to a continuum of electromagnetic modes outside of the resonator.

The Hamiltonian looks complicated. There is an exact solution, at least in the absence of forcing: one can transform to normal coordinates (remember Analytic mechanics course). In these, the system just consists of oscillators that can be solved independently. This solution is unsatisfactory for several reasons. 1) the transformation to the normal coordinates and back is complicated, 2) the model contains a lot of microscopic details ( $\omega_k$ ,  $g_k$ ) and we would like to get rid of these as much as possible, 3) we want to apply the damping also to other systems than the harmonic oscillator.

In order to make progress, we study more quantum mechanics and introduce the density operator.

### 3. Quantum mechanics

#### 3.1 Vector spaces

Let us briefly remind about quantum mechanics using the Dirac notation. We already used the notation  $|n\rangle$  for the harmonic oscillator states. In general we assume a linear vector space formed by elements  $|a\rangle$  called *state vectors* or *states*. The assumption about linear vector space means that addition and multiplication by a scalar are defined

$$|a\rangle + |b\rangle = |c\rangle, \quad \lambda|d\rangle = |f\rangle, \quad (61)$$

with some elements  $|c\rangle$  and  $|d\rangle$  also belonging to the vector space, and  $\lambda$  is any complex number. With vector space we require the standard rules

$$\begin{aligned} |a\rangle + |b\rangle &= |b\rangle + |a\rangle, \\ (|a\rangle + |b\rangle) + |c\rangle &= |a\rangle + (|b\rangle + |c\rangle), \\ \lambda(\mu|d\rangle) &= (\lambda\mu)|d\rangle, \\ \lambda(|a\rangle + |b\rangle) &= \lambda|a\rangle + \lambda|b\rangle. \end{aligned} \quad (62)$$

In addition, we define another linear vector space formed by states  $\langle a|$  so that corresponding to any “ket” state  $|a\rangle$  there is a “bra” state  $\langle a|$ , and vice versa, satisfying

$$\begin{aligned} |a\rangle + |b\rangle = |c\rangle &\Leftrightarrow \langle a| + \langle b| = \langle c|, \\ \lambda|d\rangle = |f\rangle &\Leftrightarrow \lambda^*\langle d| = \langle f|. \end{aligned} \quad (63)$$

We see that the bra space is a copy of the ket space with the only difference that complex conjugation appears in the multiplication by a scalar. Using this, we further require the vector space to form a Hilbert space, so that there exist inner product of two state vectors  $\langle a|$  and  $|b\rangle$  and denoted by

$$\langle a|b\rangle \equiv \langle a||b\rangle = \lambda \quad (64)$$

with a complex number  $\lambda$ . With inner product one requires that

$$\begin{aligned} \langle a|(|b\rangle + |c\rangle) &= \langle a|b\rangle + \langle a|c\rangle, \\ \langle a|(\lambda|b\rangle) &= \lambda\langle a|b\rangle, \\ \langle a|b\rangle &= \langle b|a\rangle^*, \\ \langle a|a\rangle = 0 &\Leftrightarrow |a\rangle = 0. \end{aligned} \quad (65)$$

The first two mean linearity with respect to the ket state. Combined with the third these imply linearity also with respect to the bra state.

Exercise: check that the states

$$\sum_{n=0}^{\infty} \lambda_n |n\rangle, \quad (66)$$

where  $\lambda_n$  are complex and  $|n\rangle$  (32), form a linear vector space. Show that this is Hilbert space if one defines

$$\langle m|n\rangle = \delta_{m,n}. \quad (67)$$

Let  $A$  be a linear operator in the ket space,  $A|b\rangle = |c\rangle$ . The linearity means

$$\begin{aligned} A(|b\rangle + |c\rangle) &= A|b\rangle + A|c\rangle, \\ A(\lambda|b\rangle) &= \lambda A|b\rangle. \end{aligned} \quad (68)$$

The same operator can be made to operate “backwards” in the bra space,  $\langle c|A$ , by requiring

$$(\langle c|A)|b\rangle = \langle c|(A|b\rangle). \quad (69)$$

The Hermitian conjugate of  $A$  is denoted by  $A^\dagger$ . It has the property

$$\langle b|A|c\rangle = \langle c|A^\dagger|b\rangle^*. \quad (70)$$

The operator  $A$  is Hermitian if

$$\langle b|A|c\rangle = \langle c|A|b\rangle^*. \quad (71)$$

Exercise: check that the operators  $a$  and  $a^\dagger$  (33) are Hermitian conjugates and  $a^\dagger a$  is Hermitian.

If

$$A|b\rangle = \lambda|b\rangle, \quad (72)$$

it is said that  $|b\rangle$  is an eigenstate of  $A$  with eigenvalue  $\lambda$ . It can be shown that for a Hermitian operator all eigenvalues are real. Two states  $|b\rangle$  and  $|c\rangle$  having different eigenvalues  $\lambda \neq \mu$  with a Hermitian operator, are orthogonal,  $\langle c|b\rangle = 0$ .

Let  $|e_k\rangle$  be an orthonormal basis, i.e. any state vector can be represented as

$$|a\rangle = \sum_k c_k |e_k\rangle, \quad \langle e_l|e_k\rangle = \delta_{l,k}. \quad (73)$$

The coefficients are then given by  $c_k = \langle e_k|a\rangle$ . In the following we also assume  $|a\rangle$  is normalized,  $\langle a|a\rangle = 1$ .

Physical quantities correspond to operators in quantum mechanics. Suppose a measurement corresponding to operator  $B$ . The possible values obtained in a measurement are the eigenvalues  $\lambda$  of the operator  $B$ . Suppose we select the basis  $|e_k\rangle$  (73) from eigenstates of the operator  $B$ . Then the probability to find the eigenvalue  $\lambda$  is given by the sum of the squares of the coefficients,  $|c_k|^2$  of the eigenstates corresponding to the eigenvalue  $\lambda$ . The average value, denoted by  $\langle B \rangle$ , in state  $|a\rangle$  is given by

$$\langle B \rangle = \langle a|B|a\rangle. \quad (74)$$

Suppose that all eigenvalues of a Hermitian operator are different i.e. the spectrum is nondegenerate. This allows to label the eigenstates by the eigenvalue,  $|\lambda\rangle$ . It also allows to represent an arbitrary state  $|a\rangle$  by the set of numbers  $\langle \lambda|a\rangle$ . A special case of this is the wave function  $\psi_a(q) = \langle q|a\rangle$  already used above (46).

## 3.2 Time dependence

Based on direct correspondence with classical mechanics, the time dependence in quantum mechanics is given by

$$i\hbar \frac{dA}{dt} = [A, H] + i\hbar \frac{\partial A}{\partial t}. \quad (75)$$

Here  $H$  is the Hamiltonian operator and  $A = A(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n, t)$  is an arbitrary operator, which depends on the canonical operators  $q_i$  and  $p_i$  ( $i = 1, \dots, n$ ), and, in some cases, also can depend explicitly on time, giving rise to the second term on the right hand side of (75). An example of such an operator is the driving term in (51),  $A = -F(t)q$ .

As discussed above, the operators are not directly observable quantities, only their eigenvalues. This has the consequence that time dependence in quantum mechanics can be formulated in different ways called *pictures* so that all observable quantities are still the same. The equation of motion above (75) is valid in *Heisenberg picture*. In this picture the state vectors have no time dependence [as all is expected to be in (75)]. In alternative pictures part of the time dependence is transferred to the state vectors. For that purpose assume a unitary time-dependent operator  $U(t)$ . The unitarity means

$$UU^\dagger = 1 = U^\dagger U \quad (76)$$

with a unity operator 1. If we now define new operators and state vectors by

$$A' = U^\dagger A U, \quad |a'\rangle = U^\dagger |a\rangle, \quad (77)$$

we see that all expectation values are unchanged,

$$\langle b' | A' | c' \rangle = \langle b | A | c \rangle. \quad (78)$$

The same applies to the eigenvalues, i.e. there is no physical change. However, the equations for the new state vectors and operators are changed. Now it is most convenient to first postulate the following equations in the *Schrödinger picture*

$$i\hbar \frac{d|a\rangle}{dt} = H|a\rangle \quad (79)$$

$$i\hbar \frac{dA}{dt} = i\hbar \frac{\partial A}{\partial t}. \quad (80)$$

Here the state vectors have time dependence determined by the Hamiltonian  $H$ , but the operators only have the explicit time dependence. Making the transformation (77) we have

$$\begin{aligned} i\hbar \frac{d|a'\rangle}{dt} &= (U^\dagger H U + i\hbar \frac{dU^\dagger}{dt} U) |a'\rangle \\ i\hbar \frac{dA'}{dt} &= \left[ A', -i\hbar \frac{dU^\dagger}{dt} U \right] + U^\dagger i\hbar \frac{\partial A}{\partial t} U. \end{aligned} \quad (81)$$

(Exercise)

Let us fix  $U(t)$  by the condition

$$i\hbar \frac{dU}{dt} = H U, \quad (82)$$

and we define  $\partial A' / \partial t = U^\dagger (\partial A / \partial t) U$ . [In order to see that this is a reasonable definition consider the example case  $A = -F(t)q$ ]. Then (81) reduce to

$$i\hbar \frac{d|a'\rangle}{dt} = 0 \quad (83)$$

$$i\hbar \frac{dA'}{dt} = [A', H'] + i\hbar \frac{\partial A'}{\partial t}. \quad (84)$$

These are precisely the same as the Heisenberg picture equations discussed above (75). Thus this transformation also justifies the Schrödinger picture postulated above.

One more picture is useful in the following. We decompose the Hamiltonian

$$H = H_0 + H_1, \quad (85)$$

where  $H_0$  usually is some simple, time-independent part and  $H_1$  is more complicated “interaction” part of  $H$ . Let us fix  $U(t)$  by the simple part only

$$i\hbar \frac{dU}{dt} = H_0 U \Leftrightarrow U = \exp\left(-\frac{i}{\hbar} H_0 t\right). \quad (86)$$

Then (81) reduce to equations in the *interaction picture*

$$i\hbar \frac{d|a'\rangle}{dt} = H'_1 |a'\rangle \quad (87)$$

$$i\hbar \frac{dA'}{dt} = [A', H_0] + i\hbar \frac{\partial A'}{\partial t}, \quad (88)$$

where  $H'_0 = H_0$ .

As an example consider  $a$  and  $a^\dagger$  taking as  $H_0$  the simple harmonic oscillator Hamiltonian (42). Using (88) we find

$$a(t) = a e^{-i\omega_0 t}, \quad a^\dagger(t) = a^\dagger e^{i\omega_0 t}, \quad (89)$$

where the time dependence in  $a(t)$  and  $a^\dagger(t)$  denotes interaction picture.  $a$  and  $a^\dagger$  are the time-independent Schrödinger picture operators. (Exercise)

## 3.3 Density operator

In order to simplify the description of damping, we have to average over the environmental degrees of freedom. However, in quantum mechanics we cannot simply take an average over a wave function (needs justification). Instead, we first form a density operator, and this allows taking averages.

The *density operator* is the operator

$$\rho = \sum_k p_k |a_k\rangle \langle a_k|. \quad (90)$$

Here  $p_k$  is the probability that the system is in state  $|a_k\rangle$ . The state vectors  $|a_k\rangle$  are assumed to be normalized,

$\langle a_k | a_k \rangle = 1$ . Also the probabilities have to be normalized,  $\sum_k p_k = 1$ . The case that only one state is possible is called *pure state* and the density operator reduces to

$$\rho = |a\rangle\langle a|. \quad (91)$$

The nice thing with the density operator is that the expectation value of any operator  $B$  can be calculated as

$$\langle B \rangle = \text{Tr}(B\rho), \quad (92)$$

where the trace  $\text{Tr}$  means the sum over the basis vectors of an arbitrary orthonormal basis  $|e_l\rangle$ ,

$$\text{Tr}(\dots) = \sum_l \langle e_l | \dots | e_l \rangle. \quad (93)$$

In order to show (92) we calculate

$$\begin{aligned} \text{Tr}(B\rho) &= \sum_l \sum_k \langle e_l | B | a_k \rangle p_k \langle a_k | e_l \rangle \\ &= \sum_k p_k \sum_l \langle a_k | e_l \rangle \langle e_l | B | a_k \rangle = \sum_k p_k \langle a_k | B | a_k \rangle, \end{aligned} \quad (94)$$

which is the expectation value (74) each state being weighted by its probability  $p_i$ .

A useful property of the trace is that for any operators  $A$  and  $B$

$$\text{Tr}(AB) = \text{Tr}(BA). \quad (95)$$

Exercise: show that

$$\text{Tr} \rho = 1. \quad (96)$$

In order to get some insight, let us calculate the density operator corresponding to the state  $|a\rangle = a_1|e_1\rangle + a_2|e_2\rangle$  in the basis  $(|e_1\rangle, |e_2\rangle)$ . We get the density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} |a_1|^2 & a_1 a_2^* \\ a_1^* a_2 & |a_2|^2 \end{pmatrix}. \quad (97)$$

This pure state should be contrasted with a “mixture” of states  $|e_1\rangle$  and  $|e_2\rangle$ , where the two states appear with classical probabilities  $p_1$  and  $p_2$ . This gives

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}. \quad (98)$$

The essential difference is the presence of off-diagonal components  $\rho_{12}$  and  $\rho_{21}$  in the “coherent” superposition (97). (Exercise)

We wish to have dynamical equation for the density operator. Taking the derivative of  $\rho$  (90) and using the Schrödinger equation (79) we get

$$i\hbar \frac{d\rho}{dt} = [H, \rho]. \quad (99)$$

This is known as *von Neumann equation*. Note that the density operator is exceptional that it does not follow the general rule for operators in the Schrödinger picture (80). Rather the equation (99) resembles the Heisenberg-picture equation (84), but has the opposite sign. In the interaction picture (87) one gets

$$\begin{aligned} i\hbar \frac{d\rho'}{dt} &= [H'_1, \rho'] \\ H'_1 &= \exp\left(\frac{i}{\hbar} H_0 t\right) H_1 \exp\left(-\frac{i}{\hbar} H_0 t\right). \end{aligned} \quad (100)$$

(Exercise)

### 3.4 Equilibrium statistical mechanics

In equilibrium at temperature  $T$  the probability of each state is given by the Gibbs distribution

$$p_i = \frac{1}{Z} e^{-\beta E_i} \Leftrightarrow \rho = \frac{1}{Z} e^{-\beta H}. \quad (101)$$

Here  $\beta = 1/k_B T$  and  $Z$  is determined by the normalization condition  $\sum_i p_i = 1$  or (96). Applying this to the simple harmonic oscillator (42) gives that the average number of quanta is given by the Bose distribution

$$\langle n \rangle = \frac{1}{e^{\beta \hbar \omega_0} - 1} \equiv N \quad (102)$$

and the density operator is

$$\begin{aligned} \rho &= \sum_{n=0}^{\infty} (1 - e^{-\beta \hbar \omega_0}) e^{-\beta \hbar \omega_0 n} |n\rangle\langle n| \\ &= \sum_{n=0}^{\infty} \frac{1}{1 + N} \left( \frac{N}{1 + N} \right)^n |n\rangle\langle n|. \end{aligned} \quad (103)$$

(Exercise)

### 3.5 Correlation functions

Sometimes important information can be expressed by a *correlation function*

$$\langle A(t_2) B(t_1) \rangle, \quad (104)$$

where two different times appear. This notation implicitly says that the operators should be expressed in the Heisenberg picture.

Exercise: Calculate  $\langle a(t) \rangle$ ,  $\langle a^\dagger(t) \rangle$ , and the correlation functions  $\langle a(t) a^\dagger(0) \rangle$  and  $\langle a^\dagger(t) a(0) \rangle$  for simple harmonic oscillator in equilibrium at temperature  $T$  and try to interpret the result (for  $T = 0$  and  $T > 0$ ).

## 4. Master equation

### 4.1 Derivation

We are now ready to go back to the model of damping (55). Our program is to formulate a dynamical equation for the density operator of the whole system, solve it assuming weak coupling between the macroscopic and microscopic degrees of freedom, and then average over the microscopic ones. This should lead to a *master equation* for the density operator of the macroscopic system only. (WM, section 6.)

For generality, write

$$H = H_S + H_R + V \quad (105)$$

with  $H_S$  denoting system,  $H_R$  the reservoir and  $V$  their coupling. Let  $w(t)$  denote the density operator of the complete system. In the interaction picture it satisfies

$$i\hbar \frac{dw(t)}{dt} = [V(t), w(t)]. \quad (106)$$

The *reduced density operator* of the system is

$$\rho(t) = \text{Tr}_R[w(t)], \quad (107)$$

where  $\text{Tr}_R$  marks the trace over reservoir degrees of freedom only. We assume that at initial time  $t = 0$  the system and the reservoir are uncorrelated,

$$w(0) = \rho(0)\rho_R(0). \quad (108)$$

We integrate (106)

$$w(t) = w(0) - \frac{i}{\hbar} \int_0^t dt_1 [V(t_1), w(t_1)]. \quad (109)$$

Iterating this gives

$$w(t) = w(0) + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \times [V(t_1), [V(t_2), \dots [V(t_n), w(0)] \dots]]. \quad (110)$$

Taking the trace over the reservoir gives

$$\begin{aligned} \rho(t) &= \rho(0) + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \\ &\quad \times \text{Tr}_R[V(t_1), [V(t_2), \dots [V(t_n), \rho(0)\rho_R(0)] \dots]] \\ &= (1 + U_1(t) + U_2(t) + \dots)\rho(0) \\ &= U(t)\rho(0), \end{aligned} \quad (111)$$

where  $U_n(t)$  is an operator acting on  $\rho$ ,

$$\begin{aligned} U_n(t) &= \left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \\ &\quad \times \text{Tr}_R[V(t_1), [V(t_2), \dots [V(t_n), \rho_R(0)(\cdot)] \dots]]. \end{aligned}$$

(Exercise: commutator as a linear operator) We now take the time derivative

$$\begin{aligned} \frac{d\rho(t)}{dt} &= (\dot{U}_1(t) + \dot{U}_2(t) + \dots)\rho(0) \\ &= (\dot{U}_1(t) + \dot{U}_2(t) + \dots)U^{-1}(t)\rho(t) \\ &= l(t)\rho(t), \end{aligned} \quad (112)$$

where the time development operator

$$l(t) = (\dot{U}_1(t) + \dot{U}_2(t) + \dots)U^{-1}(t). \quad (113)$$

We assume that the interaction term is such that

$$\text{Tr}_R[V(t)\rho_R] = 0. \quad (114)$$

This implies that  $U_1 = 0$ . It follows that up to second order

$$\begin{aligned} l(t) &= \dot{U}_2(t) \\ &= -\frac{1}{\hbar^2} \int_0^t dt_1 \text{Tr}_R[V(t), [V(t_1), \rho_R(0)(\cdot)]], \end{aligned}$$

and thus

$$\frac{d\rho(t)}{dt} = -\frac{1}{\hbar^2} \int_0^t dt_1 \text{Tr}_R[V(t), [V(t_1), \rho_R(0)\rho(t)]]. \quad (115)$$

We now specify to the system-reservoir coupling (59).

First we change to the interaction picture. Secondly, we make again the rotating wave approximation. It will be seen that the most important case is that all frequencies are of similar magnitude  $\omega_d \sim \omega_0 \sim \omega_k$ . Thus we can drop terms of type  $ab_k$  and  $a^\dagger b_k^\dagger$ . We are left with the interaction

$$\begin{aligned} V(t) &= -\hbar(a^\dagger \Gamma(t)e^{i\omega_0 t} + a\Gamma^\dagger(t)e^{-i\omega_0 t}), \\ \Gamma(t) &= \sum_k g_k b_k e^{-i\omega_k t}. \end{aligned} \quad (116)$$

Substituting into (115) we get

$$\begin{aligned} \frac{d\rho(t)}{dt} &= -\int_0^t dt_1 \text{Tr}_R[(a^\dagger \Gamma(t)e^{i\omega_0 t} + a\Gamma^\dagger(t)e^{-i\omega_0 t}), \\ &\quad [(a^\dagger \Gamma(t_1)e^{i\omega_0 t_1} + a\Gamma^\dagger(t_1)e^{-i\omega_0 t_1}), \\ &\quad \rho_R(0)\rho(t)]]. \end{aligned} \quad (117)$$

Writing out also the double commutator, there will be 16 different terms in total. Let us look at one of them,

$$\begin{aligned} &(-1)^2 \int_0^t dt_1 \text{Tr}_R[a\Gamma^\dagger(t)e^{-i\omega_0 t} \rho_R(0)\rho(t)a^\dagger \Gamma(t_1)e^{i\omega_0 t_1}] \\ &= a\rho(t)a^\dagger \int_0^t dt_1 \text{Tr}_R[\Gamma^\dagger(t)\rho_R(0)\Gamma(t_1)]e^{-i\omega_0 t}e^{i\omega_0 t_1} \\ &= a\rho(t)a^\dagger \int_0^t dt_1 \text{Tr}_R[\rho_R(0)\Gamma(t_1)\Gamma^\dagger(t)]e^{-i\omega_0(t-t_1)} \\ &= a\rho(t)a^\dagger I, \end{aligned} \quad (118)$$

where we have used the cyclic property of trace (95). We see that the trace is the expectation value (92) in the reservoir and we write

$$\begin{aligned} I &= \int_0^t dt_1 \langle \Gamma(t_1) \Gamma^\dagger(t) \rangle e^{-i\omega_0(t-t_1)} \\ &= \int_0^t dt_1 \sum_{k,l} g_k g_l^* \langle b_k b_l^\dagger \rangle e^{i(-\omega_k t_1 + \omega_l t)} e^{-i\omega_0(t-t_1)}. \end{aligned}$$

Thus we have to evaluate the expectation value  $\langle b_k b_l^\dagger \rangle$  for the reservoir oscillators. Assuming they initially are in thermal equilibrium, we can use (103) for each oscillator. Also using (60) we get

$$\langle b_k b_l^\dagger \rangle = \delta_{k,l} [1 + N(\omega_k)] \quad (119) \quad \text{where}$$

$$I = \int_0^t dt_1 \sum_k |g_k|^2 [1 + N(\omega_k)] e^{i(\omega_k - \omega_0)(t-t_1)}.$$

Assuming the reservoir oscillators have continuous spectrum with density  $\varrho(\omega)$

$$I = \int_0^t dt_1 \int_0^\infty d\omega_1 \varrho(\omega_1) |g(\omega_1)|^2 [1 + N(\omega_1)] \times e^{i(\omega_1 - \omega_0)(t-t_1)}. \quad (120)$$

Changing the integration variables we have

$$\begin{aligned} I &= \int_0^t d\tau \int_0^\infty d\omega_1 \varrho(\omega_1) |g(\omega_1)|^2 \\ &\quad \times [1 + N(\omega_1)] e^{i(\omega_1 - \omega_0)\tau} \\ &= \int_0^t d\tau \int_{-\omega_0}^\infty d\epsilon \varrho(\omega_0 + \epsilon) |g(\omega_0 + \epsilon)|^2 \\ &\quad \times [1 + N(\omega_0 + \epsilon)] e^{i\epsilon\tau} \\ &= \int_0^t d\tau \int_{-\omega_0}^\infty d\epsilon f(\epsilon) e^{i\epsilon\tau} \end{aligned} \quad (121)$$

with  $f(\epsilon) = \varrho(\omega_0 + \epsilon) |g(\omega_0 + \epsilon)|^2 [1 + N(\omega_0 + \epsilon)]$ . The integrand oscillates which leads to cancellations. The main contribution comes from  $|\epsilon|\tau \lesssim 1$ . Assuming  $t\omega_0 \gg 1$ , we can then extend the integrations to infinity, but simultaneously we add a convergence parameter  $\eta > 0$  to guarantee that there is no contribution from  $\tau = \infty$ ,

$$I = \int_0^\infty d\tau \int_{-\infty}^\infty d\epsilon f(\epsilon) e^{(i\epsilon - \eta)\tau} \quad (122)$$

and in the end we take the limit  $\eta \rightarrow 0$ . Making the time integration gives

$$I = i \int_{-\infty}^\infty d\epsilon \frac{f(\epsilon)}{\epsilon + i\eta}. \quad (123)$$

We use the symbolic identity

$$\frac{1}{\epsilon \pm i\eta} = P \frac{1}{\epsilon} \mp i\pi \delta(\epsilon), \quad (124)$$

where  $P$  denotes the principal value defined by

$$\begin{aligned} P \int_{-\infty}^\infty d\epsilon \frac{f(\epsilon)}{\epsilon} \\ = \lim_{a \rightarrow 0} \left[ \int_{-\infty}^{-a} d\epsilon \frac{f(\epsilon)}{\epsilon} + \int_a^\infty d\epsilon \frac{f(\epsilon)}{\epsilon} \right] \end{aligned} \quad (125)$$

(Exercise). We get

$$\begin{aligned} I &= iP \int_{-\infty}^\infty d\epsilon \frac{f(\epsilon)}{\epsilon} + \pi f(0) \\ &= i\Delta + \frac{\gamma}{2} [N(\omega_0) + 1], \end{aligned} \quad (126)$$

$$\gamma = 2\pi \varrho(\omega_0) |g(\omega_0)|^2 \quad (127)$$

and

$$\begin{aligned} \Delta = P \int_{-\infty}^\infty d\epsilon \frac{1}{\epsilon} \varrho(\omega_0 + \epsilon) |g(\omega_0 + \epsilon)|^2 \\ \times [1 + N(\omega_0 + \epsilon)]. \end{aligned} \quad (128)$$

Physically  $\Delta$  is the frequency shift of the system caused by coupling to the reservoir. Usually this is a small effect, and we neglect it here. Later in studying a two-state system we see that this shift is similar to the *Lamb shift* observed in atoms.

One term in the master equation (117) is thus given by (118) and (126). The task that remains is to calculate the rest 15 of the 16 terms. Eight of the terms vanish because averages  $\langle b_k b_l \rangle$  and  $\langle b_k^\dagger b_l^\dagger \rangle$  vanish in a thermal state (103). The 7 remaining terms can be calculated with slight modifications of the above. After that we finally get the *master equation for damped harmonic oscillator in the interaction picture*

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{\gamma}{2} (N+1) (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \\ &\quad + \frac{\gamma}{2} N (2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger). \end{aligned} \quad (129)$$

Recall that  $N$  is the Bose occupation at the oscillator frequency (102), and  $\gamma$  is a new parameter describing damping, as soon will be seen. If the reservoir is at zero temperature,  $T = 0$ , we have  $N = 0$  and (129) reduces to

$$\frac{d\rho}{dt} = \frac{\gamma}{2} (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a). \quad (130)$$

If there are other ‘‘interaction’’ terms in the Hamiltonian of the system, they can be included according to the general rules of density operator in the interaction picture (100). In particular, the forcing term (57), after a rotating wave approximation, can be added to give the *master equation for forced and damped harmonic oscillator in the*

interaction picture

$$\begin{aligned}\frac{d\rho}{dt} &= -\frac{i}{\hbar}[H_{\text{dI}}, \rho] \\ &\quad + \frac{\gamma}{2}(N+1)(2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \\ &\quad + \frac{\gamma}{2}N(2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger),\end{aligned}\quad (131)$$

$$H_{\text{dI}} = -\hbar f_0 [a^\dagger e^{-i\omega a t} + a e^{i\omega a t}], \quad (132)$$

with  $a(t)$  and  $a^\dagger(t)$  in the interaction picture (89), although the time dependence is not marked explicitly.

Finally, we can also write the master equation in the Schrödinger picture

$$\begin{aligned}\frac{d\rho}{dt} &= -\frac{i}{\hbar}[H_0 + H_{\text{d}}, \rho] \\ &\quad + \frac{\gamma}{2}(N+1)(2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \\ &\quad + \frac{\gamma}{2}N(2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger)\end{aligned}\quad (133)$$

with (56) and (57). More explicitly

$$\begin{aligned}\frac{d\rho}{dt} &= -i\omega_0 [a^\dagger a, \rho] + i f_0 [a^\dagger e^{-i\omega a t} + a e^{i\omega a t}, \rho] \\ &\quad + \frac{\gamma}{2}(N+1)(2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \\ &\quad + \frac{\gamma}{2}N(2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger).\end{aligned}\quad (134)$$

Let us comment the derivation of the master equation. We assumed the interaction between system and reservoir as weak. The coupling was taken into account to second order in the interaction, which is known as *Born approximation*. Note that the second order approximation was made in expressing  $\dot{\rho}$  against  $\rho$  at the same instant (112). Compared to the system at the initial time  $t=0$  terms of all orders are included. (Even though the interaction is weak at each instant, it can have a huge effect over long time.) Another less explicit assumption is called *Markovian*, the time derivative  $\dot{\rho}(t)$  only depends on  $\rho(t)$ , not any longer history of  $\rho$ . Thus the reservoir is assumed to forget any effect caused upon it more rapidly than the system can change its state. We also used the rotating wave approximation. The form of the damping term in the master equations above is known as *Lindblad form*.

## 4.2 Applications

### Averages

As first step learning the master equation, let us calculate some expectation values. Consider the annihilation operator,

$$\frac{d\langle a \rangle}{dt} = \frac{d}{dt} \text{Tr}(a\rho). \quad (135)$$

This can be calculated in any picture and the result is the same. In the Schrödinger picture  $a$  is time independent and

$$\frac{d\langle a \rangle}{dt} = \text{Tr}(a \frac{d\rho}{dt}) \quad (136)$$

(Exercise). Substituting from (134) and using (31) and (95) one finds

$$\frac{d\langle a \rangle}{dt} = -i\omega_0 \langle a \rangle - \frac{\gamma}{2} \langle a \rangle + i f_0 e^{-i\omega a t}. \quad (137)$$

This equation is precisely the same as the classical equation (19) for  $\alpha$ . This allows to identify the coefficient  $\gamma$  in the master equation as the same damping coefficient as in the classical theory.

The results for  $\alpha$  can now be translated to  $\langle a \rangle$ . For example, in the absence of forcing,  $f_0 = 0$ , we have exponential decay

$$\langle a(t) \rangle = \langle a(0) \rangle e^{-i\omega_0 t - \frac{\gamma}{2} t}. \quad (138)$$

Let us apply the same prescription to the number operator  $n = a^\dagger a$ . We get

$$\begin{aligned}\frac{d\langle a^\dagger a \rangle}{dt} &= i f_0 (\langle a^\dagger \rangle e^{-i\omega a t} - \langle a \rangle e^{i\omega a t}) \\ &\quad - \gamma \langle a^\dagger a \rangle + \gamma N\end{aligned}\quad (139)$$

(Exercise). In the absence of forcing the solution is

$$\langle n(t) \rangle = N + (\langle n(0) \rangle - N) e^{-\gamma t}. \quad (140)$$

Even if initially  $\langle n \rangle = 0$ , the occupation will grow and approaches exponentially the Bose occupation  $N = (e^{\hbar\omega_0/k_B T} - 1)^{-1}$ . This can be understood that the reservoir at temperature  $T > 0$  has excitations, and these will excite the system as well, until a thermal equilibrium is reached so all oscillators are at the same temperature.

We see that  $\langle a \rangle$  vanishes in the thermal equilibrium. This can also be seen directly from the equilibrium  $\rho$  (103), which only has diagonal elements. This can be understood so that there are quanta present in the oscillator, but their phases are random, leading to vanishing  $\langle a \rangle$ . Classically we can think an ensemble of different  $\alpha_k = A_k e^{i\phi_k}$ . If their phases  $\phi_k$  are random we have  $\langle \alpha \rangle = 0$  but nonzero  $\langle |\alpha|^2 \rangle = \langle A^2 \rangle$ , which is the classical quantity corresponding to  $\langle n \rangle$ .

It is also possible to calculate correlation functions (104). The trick used to calculate correlations like

$$\langle A(t)B(0) \rangle, \quad (141)$$

is called *quantum regression theorem*. We write the full trace of the Heisenberg operators

$$\begin{aligned}\langle A(t)B(0) \rangle &= \text{Tr}[U^\dagger(t)A(0)U(t)B(0)w(0)] \\ &= \text{Tr}[A(0)U(t)B(0)w(0)U^\dagger(t)].\end{aligned}\quad (142)$$

The last form has Schrödinger picture time-independent operator  $A(0)$  and the time-dependent density matrix with the only difference that  $B(0)$  appears at initial time  $t = 0$ . In the derivation of the master equation we did not use any special property of the density operator. Thus we can calculate (142) as above but using the initial value  $B(0)\rho(0)$  for the density operator instead of  $\rho(0)$ . As an example, calculate

$$\begin{aligned} \frac{d\langle a^\dagger(t)a(0) \rangle}{dt} &= i\omega_0 \langle a^\dagger(t)a(0) \rangle - if_0 \langle a(0) \rangle e^{i\omega_0 t} \\ &\quad - \frac{\gamma}{2} \langle a^\dagger(t)a(0) \rangle. \end{aligned} \quad (143)$$

The solution in thermal equilibrium,  $f_0 = 0$  is

$$\langle a^\dagger(t)a(0) \rangle = N e^{i\omega_0 t - \frac{\gamma}{2}|t|}, \quad (144)$$

i.e. thermal equilibrium shows nonvanishing correlations that vanish in time exponentially.

### Number representation

Let us now study the full density matrix, not only the averages. One obvious possibility is to represent the density matrix in the number basis (32)

$$\rho = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{mn} |m\rangle \langle n|. \quad (145)$$

This is an infinite matrix, but often the matrix can be truncated to a finite size with some maximum  $m$  and  $n$ . The master equation in the number basis is

$$\begin{aligned} \frac{d\rho_{mn}}{dt} &= -i\omega_0(m-n)\rho_{mn} \\ &+ if_0 e^{-i\omega_0 t} (\sqrt{m}\rho_{m-1,n} - \sqrt{n+1}\rho_{m,n+1}) \\ &+ if_0 e^{i\omega_0 t} (\sqrt{m+1}\rho_{m+1,n} - \sqrt{n}\rho_{m,n-1}) \\ &+ \gamma(N+1) \\ &\times [\sqrt{(m+1)(n+1)}\rho_{m+1,n+1} - \frac{1}{2}(m+n)\rho_{mn}] \\ &+ \gamma N [\sqrt{mn}\rho_{m-1,n-1} - \frac{1}{2}(m+n+2)\rho_{mn}]. \end{aligned} \quad (146)$$

We notice, that only the driving terms contain nondiagonal elements of  $\rho_{mn}$ . Assuming  $f_0 = 0$  we can thus write the equation for  $P(n) = \rho_{nn}$  as

$$\begin{aligned} \frac{dP(n)}{dt} &= t_+(n-1)P(n-1) + t_-(n+1)P(n+1) \\ &\quad - [t_+(n) + t_-(n)]P(n), \end{aligned} \quad (147)$$

where

$$\begin{aligned} t_+(n) &= \gamma N(n+1) \\ t_-(n) &= \gamma(N+1)n \end{aligned} \quad (148)$$

can be interpreted as *transition probabilities* from state  $n$  to states  $n+1$  and  $n-1$ . In the steady state they satisfy *detailed balance* conditions

$$t_+(n)P(n) = t_-(n+1)P(n+1). \quad (149)$$

Verify as an exercise that the Bose distribution (103) satisfies the detailed balance condition and thus is a steady state solution of the master equation (147).

The physical interpretation of the transition probabilities will be discussed more later in the connection of a two-state system.

As one particular solution we can consider the temperature relaxation (140) starting from the ground state  $P(n) = \delta_{n,0}$  (Exercise). Substituting the Bose distribution (103) into the master equation (147) we see that the solution has the Bose distribution at all times with the average number of quanta satisfying

$$\langle n(t) \rangle = N(1 - e^{-\gamma t}) \quad (150)$$

and thus corresponding to temperature

$$T(t) = \frac{\hbar\omega_0}{k_B} \frac{1}{\ln[1 + N^{-1}(1 - e^{-\gamma t})^{-1}]} \quad (151)$$

with  $N$  evaluated at the reservoir temperature.

### Glauber states

The following operator is of particular interest

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a), \quad (152)$$

where  $\alpha$  is a complex-valued parameter. This is called *displacement operator*, as will be evident from relation (156) below. Using the relation  $e^{A+B} = e^A e^B e^{-[A,B]/2}$ , which is valid if  $[A, [A, B]] = [B, [A, B]] = 0$ , the displacement operator has the alternative forms

$$\begin{aligned} D(\alpha) &= \exp(\alpha a^\dagger - \alpha^* a) \\ &= \exp(-|\alpha|^2/2) \exp(\alpha a^\dagger) \exp(-\alpha^* a) \\ &= \exp(|\alpha|^2/2) \exp(-\alpha^* a) \exp(\alpha a^\dagger). \end{aligned} \quad (153)$$

The exponentials (or any function) of an operator should be understood as Taylor expansions

$$\exp(\beta A) = \sum_{k=0}^{\infty} \frac{\beta^k A^k}{k!}. \quad (154)$$

We verify the properties

$$D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha) \quad (155)$$

$$\begin{aligned} D^\dagger(\alpha) a D(\alpha) &= a + \alpha \\ D^\dagger(\alpha) a^\dagger D(\alpha) &= a^\dagger + \alpha^* \end{aligned} \quad (156)$$

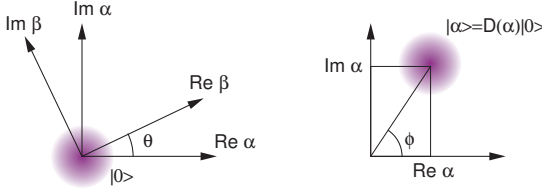
because  $[a, f(a^\dagger)] = f'(a^\dagger)$ , as follows from the commutator (31).

Using the translation operator we construct the following states

$$|\alpha\rangle = D(\alpha)|0\rangle. \quad (157)$$



These are called *Glauber states* or *coherent states*. (The latter may be confusing since “coherent” can mean also other things.) In order to understand the meaning of these states, let us consider the following.



The Hamiltonian of the simple harmonic oscillator has circular symmetry in the  $q - p$  variables when expressed in the  $\alpha$  variables (23). Since the number states are nondegenerate, the wave functions (50) are the same when looked in any coordinates  $\Re\beta$  (or  $\Im\beta$ ) rotated with respect to  $\alpha$  by arbitrary angle  $\theta$ , (except a possible phase shift). Thus the ground state  $|0\rangle$  can be represented by a circularly symmetric dot centered at  $\alpha = 0$  of width 1. Now the Glauber state  $|\alpha\rangle$  is represented by the same dot but displaced by  $\alpha = |\alpha|e^{i\phi}$ . This state is described by a wave function that is identical to  $|0\rangle$  except a coordinate shift, which for the  $\Re\beta$  axis would be  $|\alpha| \cos(\phi - \theta)$ .

Here are some more useful properties of Glauber states. The representation with number states

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (158)$$

and the property that they are eigenstates of the annihilation operator

$$a|\alpha\rangle = \alpha|\alpha\rangle. \quad (159)$$

The distribution of the photon number in Glauber states follows the Poissonian distribution

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \quad (160)$$

with the average and the variance

$$\langle \tilde{n} \rangle = |\alpha|^2, \quad \sigma_n^2 \equiv \langle \tilde{n}^2 \rangle - \langle \tilde{n} \rangle^2 = \langle \tilde{n} \rangle, \quad (161)$$

and standard deviation  $\sigma_n = \sqrt{\langle \tilde{n} \rangle}$  (Exercise). The Glauber states are not orthogonal,

$$|\langle \alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \quad (162)$$

but are approximately so if  $|\alpha - \beta| \gg 1$ . They satisfy the completeness relation

$$\frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d^2\alpha = 1. \quad (163)$$

### Elimination of driving terms

Let us try the transformation (77) with  $U = D(\alpha)$  on the full master equation with the drive (134). We get that the

effective Hamiltonian corresponding to  $H = H_0 + H_d$  is [neglecting constants and using (156)]

$$\begin{aligned} \tilde{H} &= D^\dagger(\alpha) H D(\alpha) + i\hbar \frac{\partial D^\dagger(\alpha)}{\partial t} D(\alpha) \\ &= \hbar\omega_0 (a^\dagger + \alpha^*)(a + \alpha) - \hbar f_0 [(a^\dagger + \alpha^*) e^{-i\omega_a t} \\ &\quad + (a + \alpha) e^{i\omega_a t}] + i\hbar [(-\frac{\alpha^*}{2} - a^\dagger) \dot{\alpha} + (\frac{\alpha}{2} + a) \dot{\alpha}^*] \\ &= \hbar\omega_0 a^\dagger a + \hbar a^\dagger [\omega_0 \alpha - f_0 e^{-i\omega_a t} - i\dot{\alpha}] \\ &\quad + \hbar a [\omega_0 \alpha^* - f_0 e^{i\omega_a t} + i\dot{\alpha}^*] + \text{constant}. \end{aligned} \quad (164)$$

The transformed density matrix obeys (we drop tilde)

$$\begin{aligned} \frac{d\rho}{dt} &= -\frac{i}{\hbar} [\tilde{H}, \rho] + \frac{\gamma}{2} (N+1) [2(a + \alpha)\rho(a^\dagger + \alpha^*) \\ &\quad - (a^\dagger + \alpha^*)(a + \alpha)\rho - \rho(a^\dagger + \alpha^*)(a + \alpha)] \\ &\quad + \frac{\gamma}{2} N [2(a^\dagger + \alpha^*)\rho(a + \alpha) - (a + \alpha)(a^\dagger + \alpha^*)\rho \\ &\quad - \rho(a + \alpha)(a^\dagger + \alpha^*)] \\ &= -\frac{i}{\hbar} [\tilde{H}, \rho] + \frac{\gamma}{2} (N+1) (2a\rho a^\dagger - a^\dagger a\rho \\ &\quad - \rho a^\dagger a - \alpha[a^\dagger, \rho] + \alpha^*[a, \rho]) \\ &\quad + \frac{\gamma}{2} N (2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger + \alpha[a^\dagger, \rho] - \alpha^*[a, \rho]) \\ &= -i\omega_0 [a^\dagger a, \rho] + \frac{\gamma}{2} (N+1) (2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \\ &\quad + \frac{\gamma}{2} N (2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger) \\ &\quad - i[a^\dagger, \rho] (\omega_0 \alpha - f_0 e^{-i\omega_a t} - i\dot{\alpha} - i\frac{\gamma}{2} \alpha) \\ &\quad - i[a, \rho] (\omega_0 \alpha^* - f_0 e^{i\omega_a t} + i\dot{\alpha}^* + i\frac{\gamma}{2} \alpha^*). \end{aligned} \quad (165)$$

Now as long as  $\alpha$  satisfies the classical equation (19), the last two terms vanish. This means that the drive completely drops out of the transformed master equation (165). Based on this we can now construct several possible solutions for the driven and damped harmonic oscillator.

- 1) At  $T = 0$ , the equilibrium solution for the transformed master equation (165) is the ground state  $|0\rangle$ . Transforming back to the original representation, *the full solution is a Glauber state*  $|\alpha\rangle$  (157) with  $\alpha$  satisfying the classical equation (19).
- 2) Starting from an arbitrary initial state but reservoir at  $T = 0$ , the transformed master equation (165) gives relaxation to the ground state, and the solution after initial relaxation is the Glauber state solution above.
- 3) At general  $T$ , the equilibrium solution for the transformed master equation (165) is the thermal equilibrium state (103). The full solution is the thermal state with “center of mass”  $\alpha$  satisfying the classical equation (19).
- 4) Any thermal relaxation in the transformed master equation (165) takes place independently of the classical motion (19) of “center of mass”  $\alpha$ . Thus one can have, for example, initially a coherent state experiencing the

thermal relaxation (150) simultaneously as  $\alpha$  changes according to the classical equation (19).

### Cat states

Consider a coherent superposition of two Glauber states,

$$|\psi\rangle = \mathcal{N}(|\alpha\rangle + |\beta\rangle). \quad (166)$$

Here  $\mathcal{N}$  is a normalization constant, which is very close to  $1/\sqrt{2}$  when  $|\alpha - \beta| \gg 1$ , because of (162). The density operator corresponding to such a state is

$$\rho = \mathcal{N}^2 (|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| + |\alpha\rangle\langle\beta| + |\beta\rangle\langle\alpha|). \quad (167)$$

It is of fundamental interest to study how such an initial state evolves with time. In particular, the last two terms are off-diagonal and contain information about quantum coherence between states  $|\alpha\rangle$  and  $|\beta\rangle$ . That such an interference exists on microscopic states is an essential part of the success of the quantum mechanics in explaining physics. However, when applied to macroscopic systems, such coherence has not been observed. A famous discussion of this was given by Schrödinger, who considered a cat being either alive or dead as the two states, and according to naive extrapolation of quantum mechanics, one could expect to see oscillations between the alive and dead states of the cat. With the Glauber states we can now test this expectation since the Glauber states can be macroscopic, there is no upper limit on the average number of quanta (161).

In order to study this, we first want to verify that

$$\rho_a = ||\alpha\rangle\langle\beta||e^{-\alpha\beta^*} \quad (168)$$

is a solution of the master equation (130) if  $\alpha$  and  $\beta$  satisfy the corresponding classical equation. (For simplicity assume  $T = 0$  and  $f_0 = 0$ .) Here we have used unnormalized Glauber states defined by

$$||\alpha\rangle \equiv e^{|\alpha|^2/2} |\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (169)$$

and we need the following properties

$$\begin{aligned} a||\alpha\rangle &= \alpha||\alpha\rangle, & \langle\alpha||a^\dagger &= \langle\alpha||\alpha^*, \\ a^\dagger||\alpha\rangle &= \frac{d||\alpha\rangle}{d\alpha}, & \langle\alpha||a &= \frac{d\langle\alpha||}{d\alpha^*}. \end{aligned} \quad (170)$$

After some calculation one can now see that (168) indeed satisfies the master equation (Exercise). All that remains is to multiply  $\rho_a$  (168) with an appropriate constant in order to produce any of the four terms in (167). For the third term we find the time evolution

$$\begin{aligned} \rho_3 &= \mathcal{N}^2 e^{\frac{1}{2}(|\alpha|^2 + |\beta|^2 - |\alpha_0|^2 - |\beta_0|^2)} e^{\alpha_0\beta_0^* - \alpha\beta^*} |\alpha\rangle\langle\beta| \\ &= \mathcal{N}^2 e^{-\frac{1}{2}(|\alpha_0|^2 - 2\alpha_0\beta_0^* + |\beta_0|^2)(1 - e^{-\gamma t})} |\alpha\rangle\langle\beta| \\ &= \mathcal{N}^2 e^{-|\alpha_0|^2(1 - e^{i\phi})(1 - e^{-\gamma t})} |\alpha\rangle\langle\beta|, \end{aligned} \quad (171)$$

where  $\alpha_0$  and  $\beta_0$  refer to the initial values of  $\alpha$  and  $\beta$ , and in the last line we have assumed  $\beta_0 = e^{-i\phi}\alpha_0$ . For short times and  $\phi = \pi$  the exponential factor is  $e^{-2\gamma\langle n\rangle t}$ . For large  $\langle n\rangle$  this decays very rapidly. The time constant  $(2\gamma\langle n\rangle)^{-1}$  for the decay is half of the time constant it takes to emit one quantum to the reservoir. Therefore coherent oscillations between macroscopic states can only take place during a very short time, if at all.

The result above can also be interpreted that the reservoir is continuously watching the system. Rather than “reservoir”, we should speak about the “environment” of the system, which contains everything except the system itself. Once the system emits one quantum to the environment, the phase of that photon contains information about the system. Thus it acts like a measurement, independently of whether any human watches it or not. This kills the coherence between the states, and leaves only the classical probability [the first two terms in (167)]. Which of these two is realized in any particular experiment, can be determined if the information contained in the emitted photon can be revealed. If not, one has to keep both classical alternatives.

The Glauber states are robust against the effect of the environment. Such states are called *pointer states*, i.e. they correspond to some value of a pointer in some classical measuring device. Any coherent superposition of Glauber states is fragile, i.e. it rapidly decays to a classical probability distribution of different Glauber states. The same in other words, the pointer in the measuring device can have only classical probability, it cannot exist in any coherent superposition (except of very short times).

Note that also the number states  $|n\rangle$  are fragile. In the presence of dissipation a high number state  $n \gg 1$  rapidly decomposes to an incoherent probability distribution of Glauber states  $|\alpha\rangle$  with different phases  $\phi$ , where  $\alpha = Ae^{i\phi}$  and  $A \approx \sqrt{n}$ .

Indeed, the equilibrium density matrix (103) can also be represented using Glauber states

$$\rho = \frac{1}{\pi N} \int d^2\alpha e^{-|\alpha|^2/N} |\alpha\rangle\langle\alpha|. \quad (172)$$

The identity of this expression with (103) can be verified by straightforward calculation. In light of the previous discussion, the interpretation of  $\rho$  in terms of the Glauber states is more natural than in terms of the numbers states, but because there is no difference in  $\rho$ , there is no measurable difference.

There are real experiments with the cat states, see HR.

## 5. Two-state system

In the following we wish to consider the case that the energy eigenvalues are not equally spaced. In this case, one can often approximate the system by two states only. A *two-state system* means that one can excite the system from the lower to the higher state, but any third state is sufficiently off resonance that its presence can be neglected. A two-state system is often called a *qubit*, a short for quantum bit.

In atomic and molecular spectra non-equally spaced energy eigenvalues occur as a rule. However, the electric elements we have considered above (capacitors, inductors and resistors) are all linear. Any circuit made of such elements is linear (equivalently the Hamiltonian is quadratic, at most), and has states that are equally spaced in energy. In order to make a two-state system, one needs a nonlinear element. In the following we study a *Josephson junction*. This is a *tunnel junction* between two *superconducting* metals. It is an ideal nonlinear element. The practical drawback is that it works only at low temperatures, where superconductivity appears. In practice this limitation is not serious because in order to see quantum effects in electric circuits, one has to cool to low temperatures anyway to get rid of noise.

### 5.1 Circuit analysis

Before going to Josephson junctions, let us exercise a bit circuit analysis using fluxes and Lagrangian approach. The general rules are given by Devoret. Here is an attempt to reproduce them.

A circuit consists of branches. To each branch  $b$  we associate a potential difference  $V_b$  and a current  $I_b$ . To be precise, these could be defined

$$\begin{aligned} V_b(t) &= \int_{\text{beginning of } b}^{\text{end of } b} \mathbf{E} \cdot d\mathbf{s}, \\ I_b(t) &= \frac{1}{\mu_0} \oint_{\text{around } b} \mathbf{B} \cdot d\mathbf{s}. \end{aligned} \quad (173)$$

About directions we follow the standard but confusing way of counting  $V_b$  positive in the opposite direction as the positive  $I_b$  direction. (A more systematic procedure would be to write Ohm's law as  $I = -V/R$ , the potential decreases in the direction of the current.)

We define branch fluxes and charges by

$$\begin{aligned} \Phi_b(t) &= \int_{-\infty}^t V_b(t') dt', \\ Q_b(t) &= \int_{-\infty}^t I_b(t') dt'. \end{aligned} \quad (174)$$

The integration is started at some time past where no voltages and charges were present. The energy stored in a branch is obtained by calculating

$$E_b(t) = \int_{-\infty}^t V_b(t') I_b(t') dt'. \quad (175)$$

A capacitor is described by the relation  $V_b = Q_b/C$ . Using  $I_b = \dot{Q}_b$  we get the energy

$$E_b = \frac{Q_b^2}{2C} = \frac{C \dot{\Phi}_b^2}{2}. \quad (176)$$

where we have used  $V_b = \dot{\Phi}_b$  in the latter form. An inductor is described by the relation  $\Phi_b = LI_b$ . Using the same relations we get the energy

$$E_b = \frac{\Phi_b^2}{2L} = \frac{L \dot{Q}_b^2}{2}. \quad (177)$$

In the following we wish to construct the Lagrangian, which is defined as kinetic energy minus potential energy. In using the flux variables, we can associate capacitive energies as kinetic (because they depend on  $\dot{\Phi}$ ) and inductive energies as potential (because they depend on  $\Phi$ ).

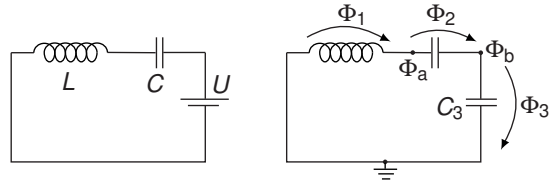
A dc voltage source can be represented by adding a fake capacitor to the circuit, and then taking the limit  $C \rightarrow \infty$  and  $Q \rightarrow \infty$  while  $V = Q/C$  remains constant. Correspondingly, a dc current source can be represented by an inductor, and an ac source by an  $LC$  circuit.

Not all branch variables are independent variables, they are constrained by Kirchhoff rules. They say that the voltages around each closed loop must sum to zero, and the the currents arriving to each node must also sum to zero. In terms of flux and charge variables these are

$$\sum_{\text{all } b \text{ around } l} \Phi_b = \tilde{\Phi}_l \quad (178)$$

$$\sum_{\text{all } b \text{ arriving at } n} Q_b = \tilde{Q}_n, \quad (179)$$

where  $\tilde{\Phi}_l$  and  $\tilde{Q}_n$  are constants. [The node rule (179) is not explicitly used in the following. It is automatically satisfied by the procedure that is used below.]



Let us consider an example, the circuit in the figure on the left. The right hand figure describes how we can assign branch ( $\Phi_1$ ,  $\Phi_2$  and  $\Phi_3$ ) and node fluxes ( $\Phi_{\text{ground}} = 0$ ,  $\Phi_a$  and  $\Phi_b$ ) and change the battery to a capacitor.

The loop rule (178) gives  $\Phi_1 + \Phi_2 + \Phi_3 = \tilde{\Phi}$ . We express the branch fluxes in terms of the node ones,

$$\begin{aligned} \Phi_1 &= \Phi_a \\ \Phi_2 &= \Phi_b - \Phi_a \\ \Phi_3 &= \tilde{\Phi} - \Phi_b. \end{aligned} \quad (180)$$

The next step is to write the Lagrangian as a function of the node fluxes. We get

$$\begin{aligned} \mathcal{L}(\Phi_a, \Phi_b, \dot{\Phi}_a, \dot{\Phi}_b) \\ = \frac{C(\dot{\Phi}_b - \dot{\Phi}_a)^2}{2} + \frac{C_3 \dot{\Phi}_b^2}{2} - \frac{\Phi_a^2}{2L}. \end{aligned} \quad (181)$$

We can now take the limit  $C_3 \rightarrow \infty$ , fixing  $\dot{\Phi}_b = U$ . Neglecting constants we have

$$\mathcal{L}(\Phi_a, \dot{\Phi}_a) = \frac{C(\dot{\Phi}_a - U)^2}{2} - \frac{\Phi_a^2}{2L}. \quad (182)$$

Using the standard procedure of analytical mechanics, one defines the canonical momentum corresponding to the node flux  $\Phi_a$  as

$$Q_a = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a} = C(\dot{\Phi}_a - U). \quad (183)$$

In general this can be identified as the node charge, the sum of the charges on the capacitor plates attached to the node. Now there is only one such capacitor, which contributes the whole  $Q_a$ . The Hamiltonian is calculated using the standard rule

$$H = \sum_{\text{nodes } i} \dot{\Phi}_i Q_i - \mathcal{L}. \quad (184)$$

In the present case this gives (neglecting constant again)

$$H = \dot{\Phi}_a Q_a - \mathcal{L} = \frac{C\dot{\Phi}_a^2}{2} + \frac{\Phi_a^2}{2L}. \quad (185)$$

Writing it as a function of the canonical variables gives finally

$$H(\Phi_a, Q_a) = \frac{(Q_a + CU)^2}{2C} + \frac{\Phi_a^2}{2L}. \quad (186)$$

Except the shift  $CU$  caused by the battery, the result is the same as obtained above (41), taking into account the correspondence of the constants in (7) and (9).

## 5.2 Josephson junctions

In the following we use a simple model of the Josephson junction. In superconductivity the electrons form pairs, which are called *Cooper pairs*. Here we consider only the Cooper pairs, and neglect any unpaired electrons. This is a reasonable approximation as long as the voltage  $V$  is small compared to the energy gap  $\Delta$  of the superconductor

$$V \ll \frac{\Delta}{e}. \quad (187)$$

Often one uses aluminum, which becomes superconducting below  $T_c = 1.2$  K. The energy gap at  $T \ll T_c$  is approximately 0.2 meV (corresponding to 2 K in temperature units, i.e.  $\Delta/k_B$ ). Thus we can neglect

unpaired electrons for  $T \ll 1$  K and  $V \ll 0.2$  mV. For Nb the conditions are somewhat more relaxed.

There is another limitation to be encountered soon. We need a metallic island that is coupled to the environment so that it matters if there is one more or less electron on the island. This means that the capacitive energy of one electron  $E_c = e^2/2C$  should exceed the thermal energy  $k_B T$ . With nanofabrication methods one can make such small structures that  $C \sim 10^{-15}$  F, meaning that  $T < 1$  K.

We now define a Josephson element by the energy

$$E = -E_J \cos \frac{2\pi\Phi}{\Phi_0}. \quad (188)$$

Here  $\Phi_0 = h/2e$  is the flux quantum and  $E_J$  is a constant (coupling energy). Although the cosine form (188) may look strange, a very simple interpretation of this is given below in connection of equation (200). Using the fact that  $\dot{E} = IV = I\dot{\Phi}$  we can find the current

$$I = I_c \sin \frac{2\pi\Phi}{\Phi_0} = I_c \sin \phi. \quad (189)$$

The latter form follows because we define *phase*

$$\phi = \frac{2\pi}{\Phi_0} \Phi = \frac{2e}{\hbar} \Phi. \quad (190)$$

The relation (189) is known as *dc Josephson effect* saying that at  $V = 0$  there is a constant *supercurrent* that depends sinusoidally on  $\Phi$  (or  $\phi$ ). The maximum supercurrent  $I_c$  is given by

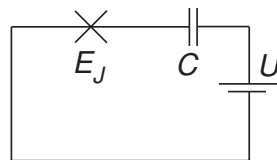
$$I_c = \frac{2\pi E_J}{\Phi_0} = \frac{2e E_J}{\hbar}. \quad (191)$$

We also see the *ac Josephson effect* saying that at constant  $V \neq 0$  there is alternating current as  $\Phi(t) = Vt + \text{constant}$ . The frequency of the alternating current is

$$\omega = \frac{2e}{\hbar} V. \quad (192)$$

At voltage 0.1 mV this gives the frequency  $\nu = \omega/2\pi = 48$  GHz.

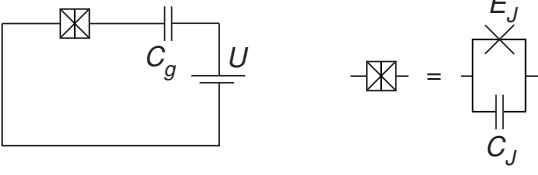
## 5.3 Cooper pair box



Let us now repeat the same circuit calculation as above but replacing the inductor with a Josephson junction. Going through the calculation one notices that there is not much change, and the result is

$$H(\Phi, Q) = \frac{(Q + CU)^2}{2C} - E_J \cos \frac{2\pi\Phi}{\Phi_0}. \quad (193)$$

(We have dropped the subindex  $a$  as unnecessary.)



Let us add one more realistic feature. A real Josephson junction always can store charge on the metal electrodes. This can be represented as an additional capacitance  $C_J$  parallel to the Josephson element (figure). Do the analysis for this circuit diagram (exercise) to find the result

$$H(\Phi, Q) = \frac{(Q + Q_0)^2}{2C_\Sigma} - E_J \cos \frac{2\pi\Phi}{\Phi_0}. \quad (194)$$

This is the Hamiltonian of a *Cooper pair box*. We have defined  $Q_0 = C_g U$ ,  $C_\Sigma = C_g + C_J$ . The lead from the battery to the capacitance  $C_g$  is called “gate” and the electrode between the Josephson junction and the gate capacitance is called “island” or “box”.

Until now the circuit analysis has been classical. We can shift to quantum theory using the standard canonical quantization procedure. For a mechanical system  $q$  and  $p$  are canonical conjugate variables, and in quantum theory they become operators satisfying the commutation relation (29). For the electric circuit considered, the canonical variables are  $\Phi$  and  $Q$ , and thus analogously they become operators satisfying

$$[\Phi, Q] = i\hbar. \quad (195)$$

In Schrödinger picture  $\Phi$  corresponds to the spatial coordinate  $q$  (or  $x$ ) and  $Q$  is then the analog of  $p$  (47) operator

$$Q = -i\hbar \frac{d}{d\Phi}. \quad (196)$$

The Hamiltonian (194) corresponding to the Schrödinger eigenvalue equation for the wave function  $u(\Phi)$  is

$$\frac{1}{2C_\Sigma} \left( -i\hbar \frac{d}{d\Phi} + Q_0 \right)^2 u(\Phi) - E_J \cos \frac{2\pi\Phi}{\Phi_0} u(\Phi) = E u(\Phi). \quad (197)$$

Alternatively, this can be written using  $\phi$  (190)

$$4E_c \left( -i \frac{d}{d\phi} + \frac{Q_0}{2e} \right)^2 u(\phi) - E_J \cos \phi u(\phi) = E u(\phi), \quad (198)$$

where  $E_c = e^2/2C_\Sigma$  is the *single electron charging energy*. We see that this Schrödinger equation has potential  $V = -E_J \cos \phi$  which is periodic.

Before analyzing the solutions of the Schrödinger equation (198), it is useful (if not necessary) to discuss

the representation of the same equation in the charge basis. The eigenstates of the charge operator (196) are plane waves

$$\psi_Q(\Phi) = e^{iQ\Phi/\hbar}. \quad (199)$$

Operating with the Cooper pair box Hamiltonian on this gives

$$H(\Phi, -i\hbar \frac{d}{d\Phi}) \psi_Q(\Phi) = e^{iQ\Phi/\hbar} \left[ \frac{(Q + Q_0)^2}{2C_\Sigma} - \frac{E_J}{2} (e^{i2e\Phi/\hbar} + e^{-i2e\Phi/\hbar}) \right]. \quad (200)$$

The first term is the capacitive energy times the original charge eigenstate, as one should expect. The second, Josephson term gives two different charge eigenstates. Their charges differ from  $Q$  by  $\pm 2e$ . Thus we see a simple interpretation of the Josephson energy term (188): it transfers one Cooper pair across the junction, from side 1 to side 2, or vice versa.

Take a look on the circuit diagram of the Cooper pair box above. In the absence of the Josephson element, the island is coupled to the rest of the circuit by capacitors only. Thus the charge on the island cannot change. Introducing the Josephson element changes this situation so that the charge can be an integral multiple  $n$  of  $2e$ , i.e. the possible values of charge

$$Q = Q_1 + 2en \quad (201)$$

with a constant  $Q_1$ . Let us denote a charge eigenstate having  $n$  Cooper pairs on the island by  $|n\rangle$ . Thus the Hamiltonian (194) can be written in the charge representation as

$$H = \sum_n \left[ 4E_c \left( n + \frac{Q_0}{2e} \right)^2 |n\rangle\langle n| - \frac{E_J}{2} (|n+1\rangle\langle n| + |n-1\rangle\langle n|) \right]. \quad (202)$$

(Here we have neglected  $Q_1$  since it can be eliminated by redefinition of  $Q_0$ .) Note that in this respect the Josephson element is simpler than an inductor. An inductor can pass an arbitrary charge and thus  $Q_a$  in (186) can vary continuously and a countable charge basis as in (202) is insufficient.

What is the meaning of the discrete charge (201) in the flux/phase space? The most general allowed wave function is a discrete superposition of the plane waves (199),

$$u(\phi) = \sum_n u_n e^{in\phi}, \quad (203)$$

which is now simpler in terms of phase  $\phi = 2e\Phi/\hbar$  (190) and we neglect  $Q_1$ . But this condition is the same as

$$u(\phi + 2\pi) = u(\phi), \quad (204)$$

i.e. the wave function  $u$  in (198) has to be periodic and thus carries independent information only in range  $-\pi < \phi \leq \pi$ , say. Note that  $u$  and the potential  $V = -E_J \cos \phi$  have the same period.

It is now convenient to define a new wave function by the transformation

$$\psi(\phi) = e^{iQ_0\phi/2e} u(\phi), \quad \psi(\phi + 2\pi) = e^{i\pi Q_0/e} \psi(\phi). \quad (205)$$

Substituting into (198) gives

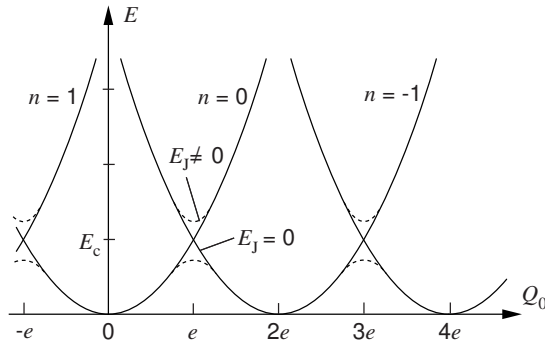
$$-4E_c \frac{d^2}{d\phi^2} \psi(\phi) - E_J \cos \phi \psi(\phi) = E \psi(\phi). \quad (206)$$

Thus the dependence on the “off-set charge”  $Q_0$  has moved from the Schrödinger equation to the boundary conditions.

The equation (206) has exact solution in terms of Mathieu functions. We learn more by considering two limiting cases.

1)  $E_c \ll E_J$ , the kinetic energy (of lowest states) is small compared to the potential. Close to the minimum at  $\phi = 0$  the cosine potential is approximately harmonic. Thus the lowest energy eigenvalues are harmonic oscillator like, with a small correction coming from the  $\phi^4$  term in the Taylor series of  $-\cos \phi$ .  $Q_0$  has negligible effect on the states since  $\psi(\pm\pi) \approx 0$ .

2)  $E_c \gg E_J$ , called charge limit. Consider first  $E_J = 0$ . As discussed above the solutions in the phase representation are plane waves. The energies are  $4E_c(n + Q_0/2e)^2$ .



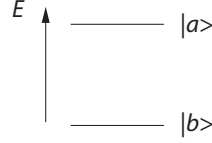
For any given  $Q_0$ , the capacitive energy is minimized by the integer  $n$  that is closest to  $Q_0/2e$ . This leads to the set of parabolas as a function of  $Q_0$  shown in the figure. The lowest state is non-degenerate except at  $-e, e, 3e$  etc, where the energies cross. If one now turns on a small  $E_J$ , the degeneracy is lifted: the system is in a superposition of states where there are either  $n$  or  $n + 1$  Cooper pairs on the island. As depicted in the figure, there are two states that are closely spaced in energy, and all others are at much higher energies. The energies depend essentially on  $Q_0$  as there is not much effect of  $E_J$  except near the charge degeneracy points.

As a summary, we see that a two-state system is formed in circuits containing a Josephson junction. We have

discussed here so-called *charge qubit*. Also two other types of qubits (“flux” and “phase”) can be constructed using Josephson junctions.

## 5.4 Rabi oscillations

We now consider a general two-state system. We label the two states as  $a$  and  $b$ , where  $b$  is the lower energy state.



The energy separation of the states is denoted by  $\hbar\Omega_0$ , where  $\Omega_0 \geq 0$ . The general wave function can be written

$$\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \quad \text{or} \quad |\psi\rangle = \psi_a |a\rangle + \psi_b |b\rangle. \quad (207)$$

In the matrix representation the unperturbed Hamiltonian can be written

$$H_0 = \frac{\hbar\Omega_0}{2} \sigma_z, \quad (208)$$

where we use the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (209)$$

It is also convenient to define

$$\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (210)$$

These are the same as  $a$  and  $a^\dagger$  operating on  $|1\rangle$  and  $|0\rangle$  (33) with the exception that all higher states are cut off,

$$\begin{aligned} \sigma_- |a\rangle &= |b\rangle, & \sigma_+ |a\rangle &= 0, \\ \sigma_- |b\rangle &= 0, & \sigma_+ |b\rangle &= |a\rangle. \end{aligned} \quad (211)$$

Let us consider a driven (not damped) qubit. With the analogy to the harmonic oscillator (53) the Hamiltonian is

$$H = H_0 + H_d = \frac{\hbar\Omega_0}{2} \sigma_z - 2\hbar f_0 \sigma_x \cos \omega_d t. \quad (212)$$

The Schrödinger equation

$$\begin{aligned} & \begin{pmatrix} \dot{\psi}_a \\ \dot{\psi}_b \end{pmatrix} \\ &= -\frac{i}{2} \begin{pmatrix} \Omega_0 & -4f_0 \cos(\omega_d t) \\ -4f_0 \cos(\omega_d t) & -\Omega_0 \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}. \end{aligned} \quad (213)$$

We can try to remove the time dependence of  $H$  by making transformation (77) with

$$U = \exp(-i\omega_d t \sigma_z / 2) = \begin{pmatrix} e^{-i\omega_d t / 2} & 0 \\ 0 & e^{i\omega_d t / 2} \end{pmatrix} \quad (214)$$

and get

$$\tilde{H} = \frac{\hbar}{2} \begin{pmatrix} \Omega_0 - \omega_d & -4f_0 \cos(\omega_d t) e^{i\omega_d t} \\ -4f_0 \cos(\omega_d t) e^{-i\omega_d t} & -(\Omega_0 - \omega_d) \end{pmatrix}. \quad (215)$$

If we are interested only in a slowly moving solution in this frame, we may neglect the fast oscillating terms, and get a time independent Hamiltonian

$$\tilde{H}_{\text{rwa}} = \frac{\hbar}{2} \begin{pmatrix} \Omega_0 - \omega_d & -2f_0 \\ -2f_0 & -(\Omega_0 - \omega_d) \end{pmatrix}. \quad (216)$$

The eigenvalues of this are given by

$$E_{\pm} = \pm \frac{\hbar}{2} \sqrt{4f_0^2 + (\Omega_0 - \omega_d)^2}. \quad (217)$$

At resonance ( $\omega_d = \Omega_0$ ) the eigenvalues are  $\pm \hbar f_0$ , and the solution is

$$\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = A \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-if_0 t} + B \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{if_0 t}. \quad (218)$$

Assuming  $b$  as the initial state at  $t = 0$  gives

$A = B = 1/2$  and thus  $\psi_a = i \sin f_0 t$  and  $\psi_b = \cos f_0 t$ . We find the density matrix

$$\begin{aligned} \rho &= \begin{pmatrix} |\psi_a|^2 & \psi_a \psi_b^* \\ \psi_b \psi_a^* & |\psi_b|^2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 - \cos(2f_0 t) & i \sin(2f_0 t) \\ -i \sin(2f_0 t) & 1 + \cos(2f_0 t) \end{pmatrix}. \end{aligned} \quad (219)$$

Thus the system oscillates between the ground and excited states at the angular frequency  $\Omega_{R0} = 2f_0$ , known as the Rabi frequency. In the general (nonresonant) case there is oscillation at the (generalized) Rabi frequency  $\Omega_R = \sqrt{4f_0^2 + (\Omega_0 - \omega_d)^2}$ . Rabi oscillations were first observed by Rabi in atoms (1937). In a Josephson-junction qubit they were seen by Nakamura et al (1999).

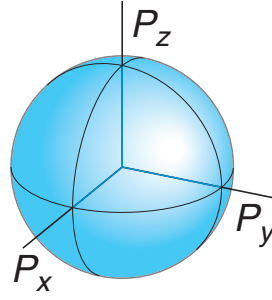
In the following it is useful to introduce a representation of the density matrix in terms of a polarization vector  $\mathbf{P}$ . One can also think it as a fictitious spin vector. Any  $2 \times 2$  matrix can be written in terms of unit matrix  $I$  and the three Pauli matrices (209). Taking into account that  $\text{Tr} \rho = 1$ , we can write it as

$$\rho = \frac{1}{2} (I + \mathbf{P} \cdot \boldsymbol{\sigma}) = \frac{1}{2} \begin{pmatrix} 1 + P_z & P_x - iP_y \\ P_x + iP_y & 1 - P_z \end{pmatrix}. \quad (220)$$

The inverse of this is

$$\begin{aligned} P_x &= \rho_{ab} + \rho_{ba} \\ P_y &= i(\rho_{ab} - \rho_{ba}) \\ P_z &= \rho_{aa} - \rho_{bb}. \end{aligned} \quad (221)$$

Because  $\rho$  is Hermitian [ $\rho^\dagger = \rho$ , which follows from (90)],  $\mathbf{P}$  is real valued vector. It can be represented by a point in the Bloch sphere depicted below.



Note that the representation (220) is valid for arbitrary statistical mixtures. For a pure state  $\mathbf{P}$  is normalized to 1 corresponding to points on the surface of the Bloch sphere. For incoherent mixtures  $P_x^2 + P_y^2 + P_z^2 < 1$  and they correspond to states inside of the Bloch sphere.

Consider now an arbitrary  $2 \times 2$  matrix Hamiltonian  $H = cI + (\hbar/2)\boldsymbol{\Omega} \cdot \boldsymbol{\sigma}$ . Substituting this and  $\rho$  (220) into the von Neumann equation (99) gives the equation of motion (exercise)

$$\frac{d\mathbf{P}}{dt} = \boldsymbol{\Omega} \times \mathbf{P}. \quad (222)$$

Thus the vector  $\boldsymbol{\Omega}$  is the angular velocity of polarization vector  $\mathbf{P}$ . Thinking  $\mathbf{P}$  as a spin, we can interpret  $\boldsymbol{\Omega}$  as the direction of the magnetic field around which the spin precesses.

It is of interest to interpret the previous Rabi oscillation solution in this picture. First, the static magnetic field makes the spin to rotate around  $z$  by  $\Omega_0$ . The transformation (214) corresponds to changing to a frame rotating around  $z$  by  $\omega_d$ . If  $\omega_d = \Omega_0$ , the only motion in this frame is around the driving field  $\boldsymbol{\Omega} = -2f_0 \hat{x}$  (in RWA), and thus causes the Rabi oscillation. If  $\omega_d \neq \Omega_0$  there is rotation around  $\boldsymbol{\Omega} = -2f_0 \hat{x} + (\Omega_0 - \omega_d) \hat{z}$ .

We see that with proper driving fields, the qubit can be brought to any orientation. This is the purpose in quantum engineering. The problem is that there are unwanted couplings to environment, whose effect we will study next.

## 5.5 Bloch equations

In order to get the damping terms we resort to the master equation. The master equation for the two-state system density matrix is analogous to the harmonic oscillator (133) with substitutions  $a \rightarrow \sigma^-$ ,  $a^\dagger \rightarrow \sigma^+$  and thus

$$\begin{aligned} \frac{d\rho}{dt} &= -\frac{i}{\hbar} [H_0 + H_d, \rho] \\ &\quad + \frac{\gamma}{2} (N + 1) (2\sigma^- \rho \sigma^+ - \sigma^+ \sigma^- \rho - \rho \sigma^+ \sigma^-) \\ &\quad + \frac{\gamma}{2} N (2\sigma^+ \rho \sigma^- - \sigma^- \sigma^+ \rho - \rho \sigma^- \sigma^+). \end{aligned} \quad (223)$$

Here  $H_0 + H_d$  is two-state Hamiltonian (212). The boson occupation (102) has to be evaluated at the energy

separation  $\hbar\Omega_0$ , i.e.

$$N = \frac{1}{\exp(\beta\hbar\Omega_0) - 1} = \frac{1}{2} \left( \coth \frac{\beta\hbar\Omega_0}{2} - 1 \right), \quad (224)$$

where the latter form is sometimes more convenient.

Let us study the master equation first without driving. Calculating (223) for the  $2 \times 2$  matrix components we get

$$\begin{aligned} \dot{\rho}_{aa} &= -\gamma(N+1)\rho_{aa} + \gamma N \rho_{bb} \\ \dot{\rho}_{ab} &= -i\Omega_0 \rho_{ab} - \gamma(N + \frac{1}{2})\rho_{ab} \\ \dot{\rho}_{ba} &= i\Omega_0 \rho_{ba} - \gamma(N + \frac{1}{2})\rho_{ba} \\ \dot{\rho}_{bb} &= \gamma(N+1)\rho_{aa} - \gamma N \rho_{bb}. \end{aligned} \quad (225)$$

We notice that the diagonal and off-diagonal elements of  $\rho$  develop independently. The diagonal elements conserve the normalization  $\rho_{aa} + \rho_{bb} = 1$  because  $\dot{\rho}_{bb} = -\dot{\rho}_{aa}$ . We also can give physical interpretation to the three terms contributing to

$$\dot{\rho}_{aa} = -\gamma N \rho_{aa} - \gamma \rho_{aa} + \gamma N \rho_{bb}. \quad (226)$$

The term  $\gamma N \rho_{bb}$  corresponds to *absorption* of a quantum from the environment. The term  $-\gamma N \rho_{aa}$  corresponds to *stimulated emission* of a quantum to the environment. Finally, the term  $-\gamma \rho_{aa}$  corresponds to *spontaneous emission* of a quantum to the environment. The interpretation of absorption and emission comes because these processes increase or decrease the energy of the qubit, and from energy conservation. We notice that absorption and stimulated emission are possible only if  $N \neq 0$ , i.e. the environment is at finite temperature,  $T > 0$ .

One may ask how can the spontaneous emission take place by coupling the qubit to the vacuum only (thinking the reservoir oscillators as electromagnetic modes). One could interpret that even vacuum is not completely empty since there are zero-point oscillations, and these could kick the qubit from the upper to the lower state. (We return to this soon.)

Similarly to the case of harmonic oscillator, the equilibrium state of the qubit is determined by the detailed balance condition

$$\dot{\rho}_{aa} = 0 \Leftrightarrow \gamma(N+1)\rho_{aa} = \gamma N \rho_{bb}. \quad (227)$$

This leads to the Gibbs distribution of the qubit

$$\rho_{aa} = e^{-\hbar\Omega_0/k_B T} \rho_{bb}. \quad (228)$$

In deriving the master equation we assumed the coupling (59) or  $V(t) = -\hbar[\sigma^-(t) + \sigma^+(t)][\Gamma(t) + \Gamma^\dagger(t)]$  between the system and the reservoir. In addition, we could also consider a coupling of the form  $V(t) = \hbar\tilde{\Gamma}(t)\sigma_z$ , where  $\tilde{\Gamma}(t)$  is some interaction-picture Hermitian operator acting in the reservoir. [Note that the interaction picture operator  $\sigma_z(t) \equiv \sigma_z$ .] The derivation of the master

equation in this case is much simpler and one finds on the right hand side of the master equation (223) an additional term

$$-\frac{\gamma_\phi}{4} [\sigma_z, [\sigma_z, \rho]], \quad (229)$$

where

$$\gamma_\phi = 4 \int_0^\infty dt (\tilde{\Gamma}(t)\tilde{\Gamma}(0)). \quad (230)$$

Adding this to (225) gives

$$\begin{aligned} \dot{\rho}_{aa} &= -\gamma(N+1)\rho_{aa} + \gamma N \rho_{bb} \\ \dot{\rho}_{ab} &= -i\Omega_0 \rho_{ab} - \gamma(N + \frac{1}{2})\rho_{ab} - \gamma_\phi \rho_{ab} \\ \dot{\rho}_{ba} &= i\Omega_0 \rho_{ba} - \gamma(N + \frac{1}{2})\rho_{ba} - \gamma_\phi \rho_{ba} \\ \dot{\rho}_{bb} &= \gamma(N+1)\rho_{aa} - \gamma N \rho_{bb}. \end{aligned} \quad (231)$$

We see that there is no effect on the diagonal terms, but the off-diagonal ones acquire additional damping. This can be understood that the longitudinal coupling induces fluctuations into the energy separation  $\Omega_0$ . Thus different realizations rotate at different speeds, and lead to diminishing of the average coherence. Note that the essential quantity to this dephasing is fluctuations at  $\omega \approx 0$  (230), whereas for the mixing terms ( $\propto \gamma$ ) the fluctuations at  $\omega \approx \Omega_0$  are dominant. To avoid such fluctuations is one of the main problems in realizing quantum operations in electric circuits.

Let us now represent the damping terms using the representation (220). Combining everything gives

$$\begin{aligned} \frac{dP_x}{dt} &= (\mathbf{\Omega} \times \mathbf{P})_x - \frac{1}{T_2} P_x \\ \frac{dP_y}{dt} &= (\mathbf{\Omega} \times \mathbf{P})_y - \frac{1}{T_2} P_y \\ \frac{dP_z}{dt} &= (\mathbf{\Omega} \times \mathbf{P})_z - \frac{1}{T_1} (P_z - P_{z0}). \end{aligned} \quad (232)$$

Here we assume the main static part of  $\mathbf{\Omega}$  equals  $\Omega_0 \hat{z}$ , and

$$\begin{aligned} P_{z0} &= -\frac{1}{2N+1} = -\tanh \frac{\hbar\Omega_0}{2kT} \\ \frac{1}{T_1} &= \gamma(2N+1) = \gamma \coth \frac{\hbar\Omega_0}{2kT} \\ \frac{1}{T_2} &= \frac{1}{2T_1} + \gamma_\phi, \end{aligned} \quad (233)$$

where  $P_{z0}$  is the equilibrium value of the polarization and  $T_1$  and  $T_2$  are two relaxation times. Equations (232) are *Bloch equations* that have extensively been used in nuclear magnetic resonance and in optics.

In the literature the Bloch equations appear in various notations. Allen&Eberly write the Bloch equations in the rotating frame in the form

$$\begin{aligned} \dot{u} &= -\Delta v - \frac{u}{T_2} \\ \dot{v} &= \Delta u - \frac{v}{T_2} + \kappa \mathcal{E} w \\ \dot{w} &= -\frac{w - w_{\text{eq}}}{T_1} - \kappa \mathcal{E} v. \end{aligned} \quad (234)$$



Here  $u$ ,  $v$  and  $w$  are the three components of  $\mathbf{P}$ ,  $w_{\text{eq}} = P_{z0}$ ,  $\Delta = \Omega_0 - \omega_d$ , and  $\kappa\mathcal{E} = 2f_0$  is the driving field. The steady state solution for constant  $\kappa\mathcal{E}$  is

$$\begin{aligned} u &= -w_{\text{eq}} \frac{(\Delta T_2)(\kappa\mathcal{E}T_2)}{1 + (\Delta T_2)^2 + T_1 T_2 (\kappa\mathcal{E})^2} \\ v &= w_{\text{eq}} \frac{\kappa\mathcal{E}T_2}{1 + (\Delta T_2)^2 + T_1 T_2 (\kappa\mathcal{E})^2} \\ w &= w_{\text{eq}} \frac{1 + (\Delta T_2)^2}{1 + (\Delta T_2)^2 + T_1 T_2 (\kappa\mathcal{E})^2}. \end{aligned} \quad (235)$$

The last one can also be written as

$$w - w_{\text{eq}} = -w_{\text{eq}} \frac{T_1 T_2 (\kappa\mathcal{E})^2}{1 + (\Delta T_2)^2 + T_1 T_2 (\kappa\mathcal{E})^2}. \quad (236)$$

Note that the last of equations (234) is an energy equation (apart from constant factor) similar to (24) for harmonic oscillator. Thus the absorption is proportional to  $v$ , the dissipation is proportional to  $w - w_{\text{eq}}$ , and in the steady state they are equal. For small drives these reduce to the same as for a harmonic oscillator. At higher drives one gets ‘‘saturation’’, where the absorption approaches a maximum. Another effect is ‘‘resonance fluorescence’’ that the emitted radiation is not at the same frequency as the drive, but has three different peaks centered at  $\omega_d$  and  $\omega_d \pm \Omega_{R0}$ . This spectrum can be determined by studying a Fourier transform of the correlation function (104).

## 5.6 Fluctuation-dissipation theorem

Let us have an interlude to study in more detail the force  $f_r$  exerted by the reservoir on the system. By the analogy of (57) and (59), this force is the operator  $f_r = \Gamma + \Gamma^\dagger$ . This force cannot be characterized by its mean value since it vanishes. Instead, we can use *spectral density*. For later use we give here full definition of this quantity for an arbitrary quantity  $f(t)$ . We first define  $\delta f = f - \langle f \rangle$ , the deviation of  $f$  from its mean value. Secondly we define the autocorrelation function  $R_f$  of  $f$  by

$$R_f(\tau) = \langle \delta f(t) \delta f(t - \tau) \rangle. \quad (237)$$

(In some cases an additional average over time  $t$  may be needed.) As  $f$  may be an operator (which is assumed to be Hermitian), the order of  $f(t)$  and  $f(t - \tau)$  is important. The spectral density of  $f(t)$  is defined as

$$S_f(\omega) = \int_{-T_w/2}^{T_w/2} d\tau e^{i\omega\tau} R_f(\tau), \quad (238)$$

where  $T_w$  gives the width of the time window used in the Fourier transform. (More on Fourier transform on page 28.)

Returning to  $f_r = \Gamma + \Gamma^\dagger$  we have

$$S_{f_r}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle \delta f_r(t) \delta f_r(t - \tau) \rangle$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle f_r(\tau) f_r(0) \rangle \\ &= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle \Gamma(\tau) \Gamma^\dagger(0) + \Gamma^\dagger(\tau) \Gamma(0) \rangle \\ &= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \sum_{i,j} g_i g_j (\langle b_i b_j^\dagger \rangle e^{-i\omega_i \tau} + \langle b_i^\dagger b_j \rangle e^{i\omega_i \tau}) \\ &= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \sum_i g_i^2 [(N+1)e^{-i\omega_i \tau} + N e^{i\omega_i \tau}] \\ &= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \int_0^{\infty} d\omega_1 \rho(\omega_1) g^2(\omega_1) \\ &\quad \times [(N(\omega_1) + 1)e^{-i\omega_1 \tau} + N(\omega_1)e^{i\omega_1 \tau}] \\ &= 2\pi\rho(|\omega|)g^2(|\omega|) \\ &\quad \times [(N(\omega) + 1)\Theta(\omega) + N(-\omega)\Theta(-\omega)] \end{aligned} \quad (239)$$

because

$$\int_{-\infty}^{\infty} d\tau e^{i\omega\tau} = 2\pi\delta(\omega) \quad (240)$$

and we have used the step function  $\Theta(x)$  (5). We see that the frequency  $\omega$  in  $S_{f_r}(\omega)$  can be either negative or positive. Using (102) we can write both parts in the same expression as

$$S_{f_r}(\omega) = \frac{\text{sgn}(\omega)\gamma(\omega)}{1 - e^{-\hbar\omega/k_B T}}, \quad (241)$$

where  $\text{sgn}(\omega) = \omega/|\omega|$  and  $\gamma$  was defined above (127) for positive frequencies and we extend it symmetrically for negative ones,

$$\gamma(\omega) = 2\pi\rho(|\omega|)g^2(|\omega|). \quad (242)$$

Using the form (16) we can write the spectral density for the unreduced force  $F$  in (7), where  $\omega_0 = \omega$  is the angular frequency at which we are interested to analyze  $F(t)$ . We get

$$S_F(\omega) = \frac{2\hbar\omega m\gamma(\omega)}{1 - e^{-\hbar\omega/k_B T}}. \quad (243)$$

This result is known as *fluctuation-dissipation theorem*. On the left hand side  $S_F$  represents *fluctuations* or *noise*, the random force exerted by the reservoir on the system. On the right hand side  $\gamma$  represent *dissipation*, the frictional force caused by the reservoir on the system. This theorem says that these are related, one cannot have dissipation without fluctuations, if  $T > 0$ , and vice versa.

Of special interest is the classical limit of (243), meaning  $k_B T \gg \hbar\omega$ . Expanding the exponential factor we get

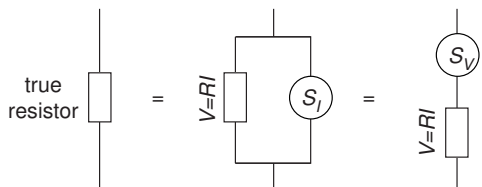
$$S_F(\omega) = 2k_B T m \gamma(\omega). \quad (244)$$

Although we have been using the notation of mechanical oscillator, we must remember that all results are valid for all harmonic oscillators, only the constants have different names. In particular, comparing (7) with the electrical

oscillators equations (8) or (9), we see that voltage and current fluctuations are related to the resistance

$$\begin{aligned} S_V(\omega) &= 2k_B T R, \\ S_I(\omega) &= 2k_B T / R. \end{aligned} \quad (245)$$

The physics is that a resistor not only dissipates energy, but also acts as a source of current and voltage fluctuations.



The famous relation (245) was derived by Nyquist (1928) and was motivated by the experimental results by Johnson (1928). [Electrical engineers usually define  $S$  differently from the present physicist's way so that they have a factor 4 instead of 2 in (245).]

As an ideal circuit element, the resistance  $R$  of a resistor is independent of frequency. This certainly is an idealization, but is approximately true for real resistors in a rather wide range of frequencies. (At least this is limited by  $\omega \ll 1/\tau$  where  $\tau \sim 10^{-14}$  s is the Drude relaxation time of electrons in a metal). Thus the noise spectral density (245) is independent of frequency. Such noise is called “white”. In particular, it is also symmetric,  $S_V(-\omega) = S_V(\omega)$ . This is natural for classical noise since any asymmetry must arise from non-commutation of  $f(t)$  and  $f(t - \tau)$  in (237).

Let us now return back to the general expression (243), or equivalently

$$S_V(\omega) = \frac{2\hbar\omega R}{1 - e^{-\hbar\omega/k_B T}}. \quad (246)$$

What is the meaning of the frequency asymmetry  $\omega \leftrightarrow -\omega$  in this expression? For that it is better to go back to analyze equations (239). We see that for a given  $\omega$ , an excitation is created that has energy  $\Delta E = \hbar\omega$  relative the equilibrium state. For positive  $\omega$  this means energy increase, that is absorption, in the reservoir. (Note that “reservoir” is now the system we aim to study, and what we considered as the system is now a detector, a device used to measure the reservoir.) The absorption in the reservoir is possible only if the detector initially is in an excited state. Conversely for negative  $\omega$  the energy decreases, that is emission, in the reservoir. This process is possible only if the reservoir is at a finite temperature,  $T > 0$ . Note that absorption in the reservoir means emission in the detector and vice versa.

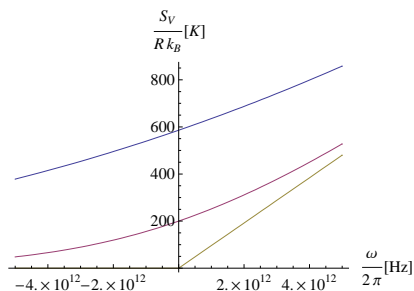


Figure: Noise in a resistor as a function of frequency at temperatures  $T = 293$  K,  $T = 100$  K and  $T = 0$  (from top to down). We see that the classical result (245) is an approximation that works reasonably in a limited frequency range around  $\omega = 0$ . (The range is quite large,  $\omega/2\pi \lesssim 10^{12}$  Hz, compared to typical frequencies in standard electronics.) For large positive  $\omega$  the noise increases meaning that at large energies zero point fluctuations, causing spontaneous emission at the detector, become the dominant cause of emission compared to thermal excitations. For large negative  $\omega$  the noise goes small as the reservoir cannot give off energy to excite the detector.

## 5.7 Coupled qubit and harmonic oscillator

Consider a two-state system coupled to a *single* harmonic oscillator. Two realizations of such systems were shown in the beginning of this course (A. Wallraff et al and D. Gunnarson et al). The hamiltonian for this system is again obtained by adapting our previous model of one plus a reservoir of oscillators (55)-(59). The simplest way is to cut the reservoir to a single two-state system, and we get

$$H = \frac{\hbar\Omega_0}{2}\sigma_z + \hbar\omega_0(a^\dagger a + \frac{1}{2}) - \hbar g\sigma_x(a + a^\dagger). \quad (247)$$

In the following we are interested in the case that the energy separations are not much different,  $|\Omega_0 - \omega_0| \ll \Omega_0 + \omega_0$  [the reason also being that there is no exact solution of (247)]. This allows to use the rotating wave approximation, and we get the *Jaynes-Cummings (JC) model*

$$H_{JC} = \frac{\hbar\Omega_0}{2}\sigma_z + \hbar\omega_0(a^\dagger a + \frac{1}{2}) - \hbar g(\sigma^+ a + \sigma^- a^\dagger), \quad (248)$$

which we are now going to study.

The space where the JC Hamiltonian (248) operates is a direct product of the qubit space and the harmonic oscillator space. The former has basis  $|a\rangle$  and  $|b\rangle$ , and the latter  $|n\rangle$  with  $n = 0, 1, 2, \dots$ . Thus the JC space has basis formed by  $|a, n\rangle$  and  $|b, n\rangle$ . One can now notice that  $H_{JC}$  couples the state  $|a, n\rangle$  only to itself and to the state  $|b, n + 1\rangle$ . To help to visualize that it may be useful to

write Hamiltonian in the form

$$H_{JC} = \hbar \begin{pmatrix} \omega_0(a^\dagger a + \frac{1}{2}) + \frac{\Omega_0}{2} & -ga \\ -ga^\dagger & \omega_0(a^\dagger a + \frac{1}{2}) - \frac{\Omega_0}{2} \end{pmatrix}. \quad (249)$$

We can now search the eigenstates in the form

$$\begin{pmatrix} \psi_a|n\rangle \\ \psi_b|n+1\rangle \end{pmatrix}. \quad (250)$$

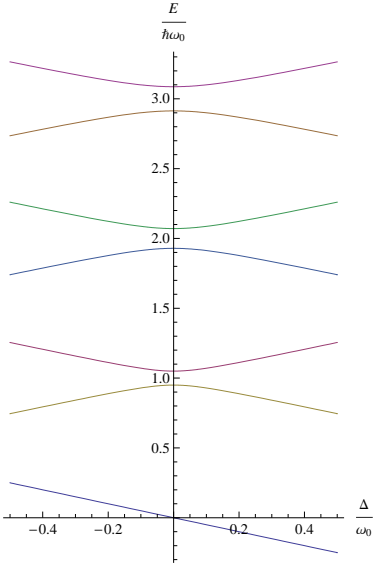
This reduces the problem to diagonalizing the matrix

$$H = \hbar \begin{pmatrix} \omega_0(n + \frac{1}{2}) + \frac{\Omega_0}{2} & -g\sqrt{n+1} \\ -g\sqrt{n+1} & \omega_0(n + \frac{3}{2}) - \frac{\Omega_0}{2} \end{pmatrix} \\ = (n+1)\hbar\omega_0 + \frac{\hbar}{2} \begin{pmatrix} \Delta & -2g\sqrt{n+1} \\ -2g\sqrt{n+1} & -\Delta \end{pmatrix},$$

where  $\Delta = \Omega_0 - \omega_0$ . The energies of the eigenstates are ( $\pm$  correspond to higher and lower energy states)

$$E_{\pm,n} = (n+1)\hbar\omega_0 \pm \frac{\hbar}{2} \sqrt{4g^2(n+1) + \Delta^2}, \quad (251)$$

where  $n = 0, 1, \dots$ . These states appear as pairs, but the ground state ( $n = -1$ ) is a single state  $(0, |0\rangle)^T$  with energy  $E = -\hbar\Delta/2$ .



The physical content is that the energy eigenstates are superpositions of states with qubit in the upper state and qubit in the lower state but one more quantum in the resonator (250). If we have an initial state  $|b, n\rangle$ , this is not an energy eigenstate, and there will be Rabi oscillations between states  $|b, n\rangle$  and  $|a, n-1\rangle$ . This is quite similar as we had for a driven two-state system above (218). The difference is that the driving field is now produced by quanta in the resonator, and we can see explicitly that when the qubit flips up, one quantum is removed from the resonator.

Another noteworthy point is the dispersive limit where  $\Delta \gg g$ . One way to study this is to expand the exact

solution above to leading powers in  $g/\Delta$ . Here we present the same result using a transformation directly on the JC Hamiltonian (248). Quite generally assume  $H = H_0 + H_1$  and try to eliminate  $H_1$  in first order. We make a transformation

$$\begin{aligned} \tilde{H} &= e^S H e^{-S} \\ &= (1 + S + \frac{S^2}{2} + \dots) H (1 - S + \frac{S^2}{2} + \dots) \\ &= H + [S, H] + \frac{1}{2}[S, [S, H]] + \dots \end{aligned} \quad (252)$$

Choosing  $[S, H_0] = -H_1$  cancels the leading order term giving

$$\tilde{H} = H_0 + \frac{1}{2}[S, H_1] + \dots \quad (253)$$

We see the second term is second order in  $H_1$ . We wish to use this to the interaction term in the JC model (248).

We guess  $S = \alpha(\sigma^+ a - a^\dagger \sigma^-)$ , and find

$$[\alpha(\sigma^+ a - a^\dagger \sigma^-), \hbar\omega_0 a^\dagger a] = \hbar\omega_0 \alpha(\sigma^+ a + a^\dagger \sigma^-)$$

$$[\alpha(\sigma^+ a - a^\dagger \sigma^-), \frac{1}{2}\hbar\Omega_0 \sigma_z] = -\hbar\Omega_0 \alpha(\sigma^+ a + a^\dagger \sigma^-),$$

which for  $\alpha = g/\Delta$  eliminates the coupling term in first order. The new second order term is

$$\begin{aligned} &\frac{1}{2}[\alpha(\sigma^+ a - a^\dagger \sigma^-), \hbar g(a^\dagger \sigma^- + \sigma^+ a)] \\ &= \hbar g \alpha [\sigma^+ a, a^\dagger \sigma^-] \\ &= \frac{\hbar g^2}{\Delta} (a^\dagger a \sigma_z + \frac{1}{2} \sigma_z + \frac{1}{2}). \end{aligned} \quad (254)$$

The transformed JC Hamiltonian, neglecting constants and corrections of order  $g^3/\Delta^2$ , is

$$\tilde{H} = \hbar \left( \omega_0 + \frac{g^2}{\Delta} \sigma_z \right) a^\dagger a + \frac{\hbar}{2} \left( \Omega_0 + \frac{g^2}{\Delta} \right) \sigma_z. \quad (255)$$

We see that this is diagonal in  $\sigma_z$  and in the number operator  $a^\dagger a$ , and the energy eigenvalues can directly be read from the Hamiltonian. But one should note that the operators  $a$ ,  $a^\dagger$  and  $\sigma_z$  are not the same as above, because of the transformation (252). Thus the eigenstates of  $\tilde{H}$  (255) are called *dressed* states, i.e. they are linear combinations of the original “bare” states  $|a, n\rangle$  and  $|b, n\rangle$ .

We note the first term of (255) is the harmonic oscillator, but the frequency is shifted depending on the state of the qubit. Alternatively, collecting together the terms proportional to  $\sigma_z$  one can interpret that the qubit energy difference has two shifts. The first shift  $2g^2n/\Delta$  is called ac Stark shift because it is analogous to the shift of atomic energy levels in alternating electric field (which has the same name). The second shift  $g^2/\Delta$  is independent on the photon number  $n$  and is called the Lamb shift, again with analogy to atoms. It is a shift of the qubit or the atom caused by coupling to the vacuum

of electromagnetic field. A similar shift was encountered above (128) but was neglected there.

The JC model is a good starting point for the analysis of the experiment by Wallraff et al mentioned in the introduction of this course.

## 6. Electron transport and noise in mesoscopic structures

### 6.1 Transverse modes in a wire

Consider electrons in an ideal conducting wire (Datta). For simplicity we consider electrons in two dimensions. The effective Schrödinger equation for the energy level of a single electron is

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y) + V(x,y)\psi(x,y) = E\psi(x,y), \quad (256)$$

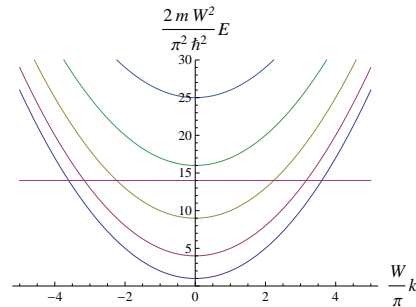
where  $m$  is the effective mass (as we do not include the periodic lattice potential in  $V$ ). The wire is in the  $x$  direction and has finite width  $W$  in the “transverse”  $y$  direction. Again, for simplicity assume the potential for electrons outside the wire is very high. Then we know that the possible wave functions in the  $y$  direction are  $\sin(\pi ny/W)$  with  $n = 1, 2, 3, \dots$  for a wire between  $y = 0$  and  $y = W$ . For a homogeneous wire [ $V(x,y) = V(y)$ ] the energy eigenstates are

$$\psi_{k,n}(x,y) = \sqrt{\frac{2}{LW}}e^{ikx}\sin\frac{\pi ny}{W}, \quad (257)$$

where  $k$  is the wave number in the  $x$  direction and  $L$  is the quantization length where we could require periodic boundary conditions. The energies of the levels (257) are

$$E_{k,n} = \frac{\hbar^2}{2m}\left[k^2 + \left(\frac{\pi n}{W}\right)^2\right]. \quad (258)$$

These are plotted in the figure below. The index  $n$  labels different (transverse) *modes*. Because of the electron spin, there are two *energy levels* corresponding to each value of  $k$  and  $n$ .



With each energy level we associate a velocity in the  $x$  direction

$$v = \frac{1}{\hbar} \frac{dE_{k,n}}{dk}. \quad (259)$$

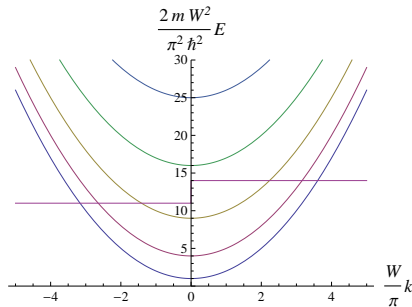
[This is obtained by calculating the current in the wave function (257). The result can also be seen as a Hamilton’s equation (52).] In our simple model (258) this agrees with the classical result  $v = p_x/m = \hbar k/m$ . In thermal equilibrium at temperature  $T$  the occupation of the levels is given by the Fermi distribution

$$f(E) = \frac{1}{e^{\beta(E-\mu)} + 1}, \quad (260)$$

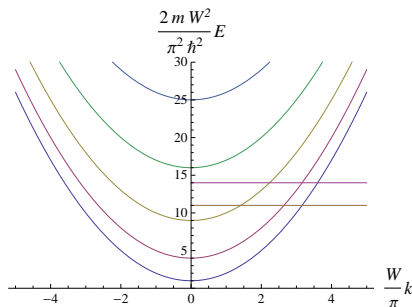
where  $\mu$  is the chemical potential and  $\beta = 1/k_B T$ . At zero temperature  $\mu$  is the same as Fermi energy, and all levels at lower energies are occupied,  $f = 1$ , and the levels at higher energies are empty,  $f = 0$ .

Assume  $T = 0$  and suppose the Fermi energy corresponds to the horizontal line in the previous figure. Then there are three conducting modes in the wire. The modes whose lowest energy is above the Fermi energy are empty, and thus do not contribute to electric conduction.

Next we want to determine the conductivity of the wire. We assume that the left end is connected to a reservoir of chemical potential  $\mu_L$  and the right end to a reservoir at chemical potential  $\mu_R$  and we assume they are in equilibrium at zero temperature,  $T = 0$ . Thus the potential difference  $V = (\mu_L - \mu_R)/e$ . We assume that the wire is fully coherent (no decoherence) and ballistic (no scattering) so that we can discuss the system in terms of freely propagating waves (257). Then the occupation of the levels in the wire is given in the following figure



Here the levels with  $k > 0$  propagate from left to right and they are filled up to the energy  $\mu_L$ . Conversely, the levels with  $k < 0$  propagate from right to left and they are filled up to the energy  $\mu_R$ . It is then clear that the net current is carried by the levels with  $k > 0$  between the energies  $\mu_R$  and  $\mu_L$ , i.e. the levels between the two horizontal lines in the following figure. At lower energies levels propagating at opposite velocities are equally occupied and thus do not carry any net current.



The current  $I$  and the conductance

$$G = \frac{1}{R} = \frac{I}{V} = \frac{eI}{\mu_L - \mu_R} \quad (261)$$

can be calculated using standard practices of condensed matter physics, see Datta. Instead, here we derive the same result using an approach that is close to information

theory. In information theory one can ask what is the maximum amount of information that can be transmitted in a channel within a given frequency interval (between frequencies  $\nu_2$  and  $\nu_1$ ) and time interval  $\Delta t$ . Our related question is, how many electrons can be transmitted in a wire within a given energy interval  $\mu_L - \mu_R$  and time interval  $\Delta t$ . It turns out that the result is different only in the respect that electrons are fermions whereas in information transfer one mostly uses fields that are bosons, like the electromagnetic field.

## 6.2 Wavelets

Consider functions of time,  $f(t)$ , and their Fourier transforms

$$\hat{f}(\omega) = \int_{-T_w/2}^{T_w/2} dt e^{i\omega t} f(t), \quad (262)$$

where  $T_w$  is the width of the time window used in the Fourier transform. Independent information of  $f(t)$  is contained only in  $f(\omega)$  at frequencies  $\omega = \omega_n = 2\pi n/T_w$  with  $n = 0, \pm 1, \pm 2, \dots$ . The inverse Fourier transform returns  $f(t)$  within the interval  $T_w$ ,

$$f(t) = \frac{1}{T_w} \sum_{n=-\infty}^{\infty} e^{-i\omega_n t} \hat{f}(\omega_n) \quad \text{for } -\frac{T_w}{2} < t < \frac{T_w}{2}. \quad (263)$$

If the function  $f(t)$  is sufficiently well behaved, one can take the limit  $T_w \rightarrow \infty$ , in which case the transform and its inverse relation take the forms

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} f(t), \quad (264)$$

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega). \quad (265)$$

Whenever there is doubt about using these, one can go back to the case of finite  $T_w$ , where possible problems are easier to solve. Instead of the angular frequency  $\omega$  one can also use the frequency  $\nu = \omega/2\pi$ , which sometimes is a more convenient variable.

Consider now a  $f(t)$  which is essentially different from zero only during a finite time, a pulse. From the formulas above we see that its Fourier transform is nonzero in a finite interval of frequencies, i.e. the pulse is formed by a wave packet, which contains waves of different frequencies. In more detail, one can prove the uncertainty relation

$$\sigma_t \sigma_\omega \geq \frac{1}{2} \Leftrightarrow \sigma_t \sigma_\nu \geq \frac{1}{4\pi}. \quad (266)$$

Here  $\sigma_t$  is the standard deviation of  $t$  in the probability distribution  $P(t) \propto |f(t)|^2$  and  $\sigma_\nu$  is the same for  $\nu$  in  $P(\nu) \propto |\hat{f}(\nu)|^2$ .

More important than the uncertainty relation (266) for us is the following question. Given a time window  $\Delta t$  and a frequency window  $\Delta\nu = \Delta\omega/2\pi$ , what is the number  $n$  of

mutually orthogonal functions it can support. To be precise, we expect the answer to be valid only asymptotically as  $\Delta t \Delta \omega \gg 1$ . The answer, as we discuss below, is

$$n = \frac{1}{2\pi} \Delta t \Delta \omega = \Delta t \Delta \nu. \quad (267)$$

We can interpret the result saying that each function takes a unit area in  $t - \nu$  plane. (This area is by factor  $4\pi$  larger than the minimum value of  $\sigma_t \sigma_\nu$ .) Notice also that the result (267) is independent of the location of the frequency interval, only its width enters the expression.

Instead of a full proof, we make the preceding statement plausible by the following calculation. We define the function

$$\phi(t) = \int_{\omega_1}^{\omega_2} d\omega e^{-i\omega t} \quad (268)$$

known as Shannon wavelet. We notice that in the  $\omega$  space it is non-zero only within the range  $\omega_1 < \omega < \omega_2$ . We calculate its overlap with the same function but translated in time by  $u$ ,

$$\begin{aligned} & \int_{-\infty}^{\infty} dt \phi^*(t) \phi(t-u) \\ &= \int_{-\infty}^{\infty} dt \int_{\omega_1}^{\omega_2} d\omega \int_{\omega_1}^{\omega_2} d\omega' e^{i\omega t} e^{-i\omega'(t-u)} \\ &= 2\pi \int_{\omega_1}^{\omega_2} d\omega e^{i\omega u} \\ &= \frac{2\pi}{iu} \int_{\omega_1}^{\omega_2} e^{i\omega u} \\ &= \frac{4\pi}{u} e^{i(\omega_1+\omega_2)u/2} \sin \frac{(\omega_2 - \omega_1)u}{2}. \end{aligned} \quad (269)$$

This vanishes for  $u = 2\pi n / (\omega_2 - \omega_1)$  for integers  $n \neq 0$ . This shows that the next orthogonal function can be placed at a distance  $u$  satisfying  $u(\omega_2 - \omega_1) = 2\pi$ . We can now tile the area in  $t - \nu$  plane in different ways, either selecting  $\omega_2$  and  $\omega_1$  as the end points of the range  $\Delta\omega$ , or by dividing this in several frequency intervals, and define the orthogonal wavelets for each subinterval as above. All choices lead to the same result (267) asymptotically for  $\Delta t \Delta \omega \gg 1$ .

## 6.3 Landauer formula

We can now apply the result (267) to calculate the conductivity of the wire. Consider first one transverse mode only. The electric current  $I = en_e / \Delta t$  is the electric charge  $e$  times the number of transmitted electrons  $n_e$  per time  $\Delta t$ . The wave functions that carry the current are in the energy range  $\Delta E = eV$  and thus frequency range  $\Delta \nu = eV/h$ . The number of transmitted orthogonal wave functions is given by  $n = \Delta t \Delta \nu$  (267). Since electrons are

fermions, there are two electrons (because of spin) per each wave function. Thus the current is

$$I = \frac{2e^2}{h} V \quad (270)$$

or the conductance

$$G = \frac{2e^2}{h}. \quad (271)$$

This is the *Landauer formula* in its simplest form (one perfectly transmitting mode at  $T = 0$ ). This is an amazingly simple result since it does not depend on any materials parameters, it contains only the fundamental constants  $e$  and  $h$ . (We return to the discussion soon.) The inverse of (271) is known as “quantum resistance”

$$R_Q = \frac{h}{2e^2} = 12.9 \text{ k}\Omega. \quad (272)$$

The simple form of the Landauer formula allows several generalizations. First, we can generalize to arbitrary number  $M$  of modes simply by multiplying the current by  $M$ . Second, we can generalize to imperfect wire. There the waves are transmitted with probability  $\mathcal{T}$  and reflected back by probability  $\mathcal{R} = 1 - \mathcal{T}$ . Since only transmitted waves contribute to the current, the current is given by  $I = (2e^2/h) M \mathcal{T} V$ . Third, we can generalize to arbitrary temperature by applying the argument leading to Landauer formula to arbitrary thin energy slices separately. The result is

$$I = \frac{2e}{h} \int dE M(E) \mathcal{T}(E) [f_L(E) - f_R(E)] \quad (273)$$

with integration over the energy band. Here  $f_L(E)$  and  $f_R(E)$  are the Fermi distribution functions on the left and right reservoirs. We have also included the possibility that both the transmission probability and the number of modes are functions of energy. Fourth, one can consider a general scattering from any incoming mode to any reflected or transmitted mode. Without going into details (see Datta) we claim that by making a unitary transformation on the modes, one can make the transmission probability diagonal, and the only effect in formula (273) is to replace  $M(E) \mathcal{T}(E) \rightarrow \sum_i \mathcal{T}_i(E)$ , where  $\mathcal{T}_i$  are the elements of the diagonalized transmission probability matrix. Also, we can argue that the Landauer formula is independent of the specific assumptions of effective mass approximation or the form of the potential in the effective Schrödinger equation (256). Thus we have essentially removed all assumptions made at the start except the one of the coherent transport.

Let us work out the current (273) in the case there is no energy dependence on  $M$  and  $\mathcal{T}$ . At any temperature the Fermi distribution gives

$$\int dE [f_L(E) - f_R(E)] = \mu_L - \mu_R = eV \quad (274)$$

and thus

$$G = \frac{2e^2}{h} M\mathcal{T}, \quad (275)$$

which is an often appearing form of the Landauer formula.

To confirm that all has been done right, let us still consider deriving the Landauer formula in  $x - p$  variables. The formula (267) in  $t - \omega$  space can be directly applied in the spatial coordinate vs. wave vector space,  $x - k$ , and number of orthogonal wave functions is thus  $n = \Delta x \Delta k / 2\pi$ . Remembering that  $p = \hbar k$  we can extend this to  $x - p$  phase space, implying that one wave function takes an area  $h$  in the phase space. This result  $n = \Delta x \Delta p / h$  is also known as the semiclassical quantization condition. The current is thus

$$I = \frac{en_e}{\Delta t} = \frac{2e\Delta x \Delta p}{h\Delta t} = \frac{2e}{h} v \Delta p = \frac{2e}{h} \Delta E. \quad (276)$$

This is the same result as obtained in the  $t - \omega$  space. Looking at the  $k - E$  pictures presented above, we see that different modes have different  $\Delta p$  but this effect is just cancelled by the differences in the velocity  $v$  (259) so that each mode contributes just proportional to  $v \Delta p = \Delta E$ . (Here the formulas assumed a small  $\Delta p$ . If that is not the case one should divide  $\Delta p$  into thinner slices and integrate, but the result is unchanged because  $\int v dp = \Delta E$ .)

Let us discuss the relation to information theory. For fermions each energy level can be either occupied or not. Thus the information that can be coded is one bit per level (two bits per one orthogonal wave function). For bosons there are more possibilities since one can have different numbers of bosons in one level. For example, resolving 8 different amplitude values, one can code 3 bits of data per level. The Shannon formula for channel capacity is

$$C = M \Delta \nu \log_2(1 + S/N), \quad (277)$$

where  $C$  is the maximum number of bits transferred per time and  $S/N$  is the “signal to noise ratio” that determines how many amplitude values can be resolved.

The Landauer formula is best demonstrated experimentally, if one has a method to change the mode number or the transmission. One system where this can be done is a semiconductor heterostructure, where one can adjust the width of the conducting region by gate electrodes.

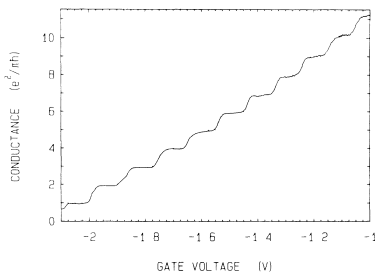


Figure: B. J. van Wees et al, Phys. Rev. Lett. **60**, 848 (1988). The conductance in a semiconductor heterostructure as a function of gate voltage. Quantized values of  $G$  are seen when there is one or more modes whose  $\mathcal{T}_i \approx 1$  and for other modes  $\mathcal{T}_i \approx 0$ .

Now that we roughly understand the plateaus in the plot, let us next concentrate on what happens at the steps between the plateaus.

## 6.4 Classical shot noise

Let us consider a mode with a transmission probability  $\mathcal{T} \sim \frac{1}{2}$ . This means that some of the incident electrons are transmitted and some are reflected back. Compared to the case of  $\mathcal{T} = 1$ , where a uniform flow of electrons takes place (as will be shown later), there are now fluctuations in the current. Such a noise is called *shot noise*. It was first analyzed by W. Schottky in electron tubes (1918).

Let us first study the shot noise using classical physics. We assume that electrons arrive randomly at a rate  $\lambda$  and the arrivals are instantaneous. Thus the current is a sum of delta functions

$$I(t) = e \sum_n \delta(t - t_n), \quad (278)$$

and the arrival times  $t_n$  have the probability distribution

$$P(t_n - t_{n-1}) = \lambda e^{-\lambda(t_n - t_{n-1})} \Theta(t_n - t_{n-1}), \quad (279)$$

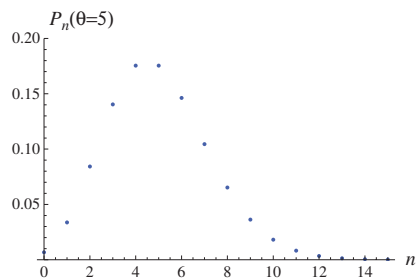
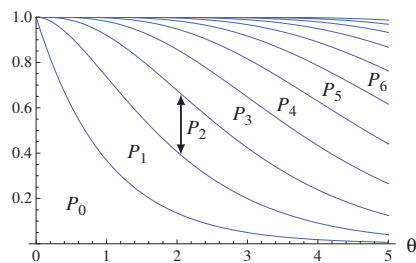
where  $\Theta$  is the step function (5), implying that  $t_n \geq t_{n-1}$ . We see that the average interval between two arrivals is  $1/\lambda$  and therefore the average current

$$\langle I \rangle = e\lambda. \quad (280)$$

Also one can show that charge  $Q = en$  collected in time  $\Delta t$  obeys Poisson distribution

$$P_n = e^{-\theta} \frac{\theta^n}{n!} \quad (281)$$

with the mean value  $\theta = \lambda \Delta t$  (exercise).



Next we want to calculate the autocorrelation function (237) for the electric current,

$$R_I(\tau) = \langle \delta I(t) \delta I(t - \tau) \rangle, \quad (282)$$

where  $\delta I = I - e\lambda$ . We argue physically that since the arrival times are random, the main contribution to  $R_I$  comes from the correlation of an electron with itself. Thus we claim that the main part comes from terms

$$\begin{aligned} R_I(\tau) &= \langle \delta I(t) \delta I(t - \tau) \rangle \\ &\approx e^2 \langle \sum_n \delta(t - t_n) \delta(t - \tau - t_n) \rangle \\ &= e^2 \langle \sum_n \delta(t - t_n) \rangle \delta(-\tau) \\ &= e^2 \lambda \delta(-\tau). \end{aligned} \quad (283)$$

In fact this result is exact, since the dropped terms can be shown to cancel each other. The spectral density (238) is then

$$\begin{aligned} S_I(\omega) &= \int_{-T_w/2}^{T_w/2} d\tau e^{i\omega\tau} R_I(\tau) = e^2 \lambda \\ &= e \langle I \rangle. \end{aligned} \quad (284)$$

This is the Schottky result that the shot noise spectral density is directly proportional to the current. This formula is also interesting because the constant of proportionality is the electron charge. Even though the effect of one single electron may be too small to resolve, the discreteness of the charge still could be seen in the fluctuations of the current.

Electron shot noise could be compared to rain falling on a roof. This makes noise because rain is not a continuous flow of water but consists of droplets. We hear the noise although it may be difficult to resolve the sound of the individual droplets.

We see that similarly as the thermal noise (245), the shot noise is white, i.e. independent of frequency. This result is supported by experiments in a range of frequencies, but there are deviations for both at very high frequencies and at low frequencies. At very high frequencies one should start to reveal the time structure of a single-electron transmission event, which here was idealized to be a delta function. At low frequencies the experiments reveal a noise that is larger than the Schottky result. This is called *1/f noise*, the name indicating that the spectral density increases at low frequencies approximately as 1/frequency. *1/f-noise* is interpreted so that systems are not constant in time. For example, an impurity atom in a resistor changes state and results in a change of the resistance. This noise occurs in all systems but its magnitude varies. It was first analyzed in electron tubes, see J. B. Johnson, Phys. Rev. **26**, 71-85 (1925) for a nice historic study.

## 6.5 Transmission and noise in scattering formalism

We return to consider the conducting wire. Now we make a quantum mechanical calculation using the field operators. The calculation is rather formal and therefore not very transparent. However, as the end result it gives several already familiar results, like the Landauer formula, thermal and shot noise and the fluctuation-dissipation theory in addition to new results like combinations of thermal, shot and quantum noise. The shot noise by this method was first calculated by G. Lesovik (1989). For further details see Blanter&Büttiker.

We think of dividing the wire into a scattering region and two “leads” on both sides of it. We label the leads by  $\alpha$  that is either left or right. Both leads are assumed to have a single mode only, for simplicity. In both leads we have a coordinate  $x_\alpha$  that is positive towards the scattering region. We consider the annihilation and creation operators for electron waves propagating in the leads. Here  $a_{\alpha k}$  are the annihilation operators for the waves moving in the positive  $x_\alpha$  direction (incoming waves) and  $b_{\alpha k}$  the annihilation operators for the waves moving in the negative  $x_\alpha$  direction (reflected and transmitted waves), both with wave number  $k$ . These are required to obey the anticommutator rules

$$\begin{aligned} \{a_{\alpha k}, a_{\alpha k'}^\dagger\} &= \delta_{k,k'}, & \{b_{\alpha k}, b_{\alpha k'}^\dagger\} &= \delta_{k,k'}, \\ \{a_{\alpha k}, a_{\alpha k'}\} &= \{a_{\alpha k}^\dagger, a_{\alpha k'}^\dagger\} = 0, \\ \{b_{\alpha k}, b_{\alpha k'}\} &= \{b_{\alpha k}^\dagger, b_{\alpha k'}^\dagger\} = 0, \end{aligned} \quad (285)$$

where  $\{A, B\} = AB + BA$ . These rules imply the Fermi statistics, where the only occupations allowed in one level are 0 and 1.

We define the field operator

$$\hat{\psi}_\alpha(x_\alpha, t) = \frac{1}{\sqrt{L}} \sum_{k>0} (e^{ikx_\alpha} a_{\alpha k} + e^{-ikx_\alpha} b_{\alpha k}) e^{-iE_{\alpha k}t/\hbar}, \quad (286)$$

where  $E_{\alpha k} = E_{\alpha 0} + \hbar^2 k^2 / 2m$ . The current operator in lead  $\alpha$  is defined as if  $\hat{\psi}_\alpha$  were a usual wave function,

$$\begin{aligned} \hat{I}_\alpha(t) &= \frac{ge\hbar}{2mi} \left[ \hat{\psi}_\alpha^\dagger(x_\alpha, t) \frac{d}{dx_\alpha} \hat{\psi}_\alpha(x_\alpha, t) \right. \\ &\quad \left. - \left( \frac{d}{dx_\alpha} \hat{\psi}_\alpha^\dagger(x_\alpha, t) \right) \hat{\psi}_\alpha(x_\alpha, t) \right]. \end{aligned} \quad (287)$$

Here  $g = 2$  in order to include the electron spin. The idea here is that applied to usual wave function  $\psi(x_\alpha)$ , Eq. (287) gives the usual expression for current, and the only generalization here is that  $a$  and  $b$  operators indicate which of these levels are occupied. We substitute the field operators. Using that  $k$  varies only in a small region compared to its value, we take  $k$  outside of the summation (but still preserve the difference in energies) and get

$$\hat{I}_\alpha(t) = \frac{ge\hbar}{2mi} \frac{1}{L} \sum_{k>0} \sum_{k'>0} 2ik e^{i(E_{\alpha k} - E_{\alpha k'})t/\hbar}$$



$$\begin{aligned}
& \times (a_{\alpha k}^\dagger a_{\alpha k'} - b_{\alpha k}^\dagger b_{\alpha k'}) \\
& = \frac{ge}{L} \sum_{k>0} \sum_{k'>0} \frac{\hbar k}{m} e^{i(E_{\alpha k} - E_{\alpha k'})t/\hbar} \\
& \quad \times (a_{\alpha k}^\dagger a_{\alpha k'} - b_{\alpha k}^\dagger b_{\alpha k'}). \quad (288)
\end{aligned}$$

Using

$$\int dt e^{i(\hbar\omega + E_k - E_{k'})t/\hbar} = T_w \delta_{\hbar\omega + E_k, E_{k'}} \rightarrow \hbar \delta(\hbar\omega + E_k - E_{k'}) \quad (289)$$

we get

$$\begin{aligned}
\hat{I}_\alpha(\omega) & = \int dt e^{i\omega t} \hat{I}_\alpha(t) \\
& = \frac{geh}{L} \sum_{k>0} \sum_{k'>0} \frac{\hbar k}{m} \delta(\hbar\omega + E_k - E_{k'}) \\
& \quad \times (a_{\alpha k}^\dagger a_{\alpha k'} - b_{\alpha k}^\dagger b_{\alpha k'}). \quad (290)
\end{aligned}$$

We now take the scattering into account by expressing the outgoing waves in terms of the incoming,  $b_{\alpha E} = r a_{\alpha E} + t a_{\beta E}$ . Here  $r$  and  $t$  are the reflection and transmission amplitudes. In order to conserve unitarity  $|t|^2 + |r|^2 = 1$ . This allows us to determine the reflection and transmission probabilities  $\mathcal{R} = |r|^2$  and  $\mathcal{T} = |t|^2$ . We define a short hand

$$\begin{aligned}
\hat{X}_{kk'} & = a_{\alpha k}^\dagger a_{\alpha k'} - b_{\alpha k}^\dagger b_{\alpha k'} \\
& = a_{\alpha k}^\dagger a_{\alpha k'} - (r^* a_{\alpha k}^\dagger + t^* a_{\beta k}^\dagger)(r a_{\alpha k'} + t a_{\beta k'}) \\
& = |t|^2 a_{\alpha k}^\dagger a_{\alpha k'} - |t|^2 a_{\beta k}^\dagger a_{\beta k'} \\
& \quad - r^* t a_{\alpha k}^\dagger a_{\beta k'} - r t^* a_{\beta k}^\dagger a_{\alpha k'}. \quad (291)
\end{aligned}$$

We calculate its expectation value assuming the particles arriving from the reservoirs are in thermal equilibrium, and get

$$\langle \hat{X}_{kk'} \rangle = \delta_{k,k'} |t|^2 [f_\alpha(E_k) - f_\beta(E_k)]. \quad (292)$$

Therefore

$$\begin{aligned}
\langle \hat{I}_\alpha(t) \rangle & = \frac{ge}{L} \sum_{k>0} \frac{\hbar k}{m} |t|^2 [f_\alpha(E_k) - f_\beta(E_k)] \\
& = \frac{ge}{h} \int_0^\infty dE |t|^2 [f_\alpha(E) - f_\beta(E)] = \frac{ge^2}{h} |t|^2 V \quad (293)
\end{aligned}$$

assuming  $t$  independent of energy (or a small  $V$  and  $T$ ) in the last equality. Thus we have Ohm's law  $I = GV$  with

$$G = \frac{ge^2}{h} |t|^2. \quad (294)$$

We have rederived the Landauer equation with the field-theory formalism.

Now consider the fluctuations. We are looking at a stationary process, and therefore can add to the autocorrelation function (237) an extra integration over time,

$$R_f(\tau) = \frac{1}{T_w} \int_{-T_w/2}^{T_w/2} dt \langle \delta f(t) \delta f(t - \tau) \rangle \quad (295)$$

and we would like to take the limit  $T_w \rightarrow \infty$ . For the power spectrum we get

$$\begin{aligned}
S_f(\omega) & = \int_{-T_w/2}^{T_w/2} d\tau e^{i\omega\tau} R_f(\tau) \\
& = \frac{1}{T_w} \int d\tau \int dt e^{i\omega\tau} \langle \delta f(t) \delta f(t - \tau) \rangle \\
& = \frac{1}{T_w} \langle \int dt \delta f(t) e^{i\omega t} \int d\tau \delta f(t - \tau) e^{-i\omega(t - \tau)} \rangle \\
& = \frac{1}{T_w} \langle \delta f(\omega) \delta f(-\omega) \rangle, \quad (296)
\end{aligned}$$

which is known as Wiener-Khinchin theorem. [In fact, one could define the spectral density  $S_f$  with the last form of (296), and then the Wiener-Khinchin theorem would be the same as our definition of  $S_f$  (238). This way of defining  $S_f$  is not favored here since  $\delta f(\omega)$  diverges in the limit  $T_w \rightarrow \infty$ , but  $S_f(\omega)$  is supposed to stay finite.] [The treatment of the integration limits in (296) is not completely satisfactory, but apparently the error is small in the limit  $T_w \rightarrow \infty$ .]

We calculate

$$\begin{aligned}
S_I(\omega) & = \frac{ge^2 h^2}{\Delta t L^2} \sum_{k>0} \sum_{k'>0} \sum_{q>0} \sum_{q'>0} \frac{\hbar^2 k q}{m^2} \\
& \quad \times \delta(\hbar\omega + E_k - E_{k'}) \delta(-\hbar\omega + E_q - E_{q'}) \langle \delta \hat{X}_{kk'} \delta \hat{X}_{qq'} \rangle. \quad (297)
\end{aligned}$$

For that we need the expectation value of for annihilation/creation operators, and we calculate

$$\langle a_1^\dagger a_2 a_3^\dagger a_4 \rangle = \langle a_1^\dagger a_2 \rangle \langle a_3^\dagger a_4 \rangle + \langle a_1^\dagger a_4 \rangle \langle a_2 a_3^\dagger \rangle, \quad (298)$$

$$\begin{aligned}
\langle \delta X_{kk'} \delta X_{qq'} \rangle & = \langle X_{kk'} X_{qq'} \rangle - \langle X_{kk'} \rangle \langle X_{qq'} \rangle \\
& = \langle (|t|^2 a_{\alpha k}^\dagger a_{\alpha k'} - |t|^2 a_{\beta k}^\dagger a_{\beta k'} - r^* t a_{\alpha k}^\dagger a_{\beta k'} - r t^* a_{\beta k}^\dagger a_{\alpha k'}) \\
& \quad \times (|t|^2 a_{\alpha q}^\dagger a_{\alpha q'} - |t|^2 a_{\beta q}^\dagger a_{\beta q'} - r^* t a_{\alpha q}^\dagger a_{\beta q'} - r t^* a_{\beta q}^\dagger a_{\alpha q'}) \rangle \\
& \quad - \langle X_{kk'} \rangle \langle X_{qq'} \rangle \\
& = \delta_{kq'} \delta_{k'q} \{ |t|^4 f_\alpha(E_k) [1 - f_\alpha(E_{k'})] \\
& \quad + |t|^4 f_\beta(E_k) [1 - f_\beta(E_{k'})] \\
& \quad + |rt|^2 f_\alpha(E_k) [1 - f_\beta(E_{k'})] \\
& \quad + |rt|^2 f_\beta(E_k) [1 - f_\alpha(E_{k'})] \} \\
& = \delta_{kq'} \delta_{k'q} |t|^2 \{ |t|^2 f_\alpha(E_k) [1 - f_\alpha(E_{k'})] \\
& \quad + |t|^2 f_\beta(E_k) [1 - f_\beta(E_{k'})] \\
& \quad + |r|^2 f_\alpha(E_k) [1 - f_\beta(E_{k'})] \\
& \quad + |r|^2 f_\beta(E_k) [1 - f_\alpha(E_{k'})] \}, \quad (299)
\end{aligned}$$

$$\begin{aligned}
S_I(\omega) & = \frac{ge^2 h^2}{\Delta t L^2} \sum_{k>0} \sum_{k'>0} \frac{\hbar^2 k^2}{m^2} \\
& \quad \delta(\hbar\omega + E_k - E_{k'}) \delta(-\hbar\omega + E_{k'} - E_k) |t|^2 \\
& \quad \times \{ |t|^2 f_\alpha(E_k) [1 - f_\alpha(E_{k'})] + |t|^2 f_\beta(E_k) [1 - f_\beta(E_{k'})] \}
\end{aligned}$$

$$\begin{aligned}
& +|r|^2 f_\alpha(E_k)[1 - f_\beta(E_{k'})] + |r|^2 f_\beta(E_k)[1 - f_\alpha(E_{k'})] \} \\
& = \frac{ge^2 \hbar}{L^2} \sum_{k>0} \sum_{k'>0} \frac{\hbar^2 k^2}{m^2} \delta(\hbar\omega + E_k - E_{k'}) |t|^2 \\
& \times \{ |t|^2 f_\alpha(E_k)[1 - f_\alpha(E_{k'})] + |t|^2 f_\beta(E_k)[1 - f_\beta(E_{k'})] \\
& + |r|^2 f_\alpha(E_k)[1 - f_\beta(E_{k'})] + |r|^2 f_\beta(E_k)[1 - f_\alpha(E_{k'})] \}.
\end{aligned}$$

Going to the continuum limit we get

$$\begin{aligned}
S_I(\omega) &= \frac{ge^2}{h} \int dE \int dE' \delta(\hbar\omega + E - E') |t|^2 \\
& \times \{ |t|^2 f_\alpha(E)[1 - f_\alpha(E')] + |t|^2 f_\beta(E)[1 - f_\beta(E')] \\
& + |r|^2 f_\alpha(E)[1 - f_\beta(E')] + |r|^2 f_\beta(E)[1 - f_\alpha(E')] \}.
\end{aligned} \quad (300)$$

An alternative form of the same result is

$$\begin{aligned}
S_I(\omega) &= \frac{ge^2}{h} \int dE \int dE' \delta(\hbar\omega + E - E') |t|^2 \\
& \times \{ f_\alpha(E)[1 - f_\alpha(E')] + f_\beta(E)[1 - f_\beta(E')] \\
& + |r|^2 [f_\alpha(E)f_\alpha(E') - f_\alpha(E)f_\beta(E')] \\
& + f_\beta(E)f_\beta(E') - f_\beta(E)f_\alpha(E')] \}.
\end{aligned} \quad (301)$$

Below we consider special cases of this formula.

1) Thermal equilibrium. We have  $f_\alpha(E) = f_\beta(E)$  and we are interested in  $\omega = 0$ . We get from (300)

$$S_I(0) = \frac{2ge^2}{h} |t|^2 \int dE f(E)[1 - f(E)]. \quad (302)$$

Using  $f(E)[1 - f(E)] = -k_B T df(E)/dE$  we get

$$S_I(0) = \frac{2ge^2}{h} |t|^2 k_B T = 2Gk_B T, \quad (303)$$

which is the Johnson-Nyquist noise (245). It is interesting to note that previously we derived the same formula by coupling the system to a set of harmonic oscillators, where the oscillation quanta are bosons. Here the coupling is to electrons, which are fermions, but the thermal noise is the same. We can interpret the result that at  $T > 0$  there are electron and hole excitations that can be transmitted through the junction causing the current that has nonzero fluctuations (303) but whose average vanishes (since  $V = 0$ ).

2) Shot noise. We assume  $\omega = 0$  and  $T = 0$  and get from (300)

$$\begin{aligned}
S_I(0) &= \frac{ge^2}{h} |t|^2 |r|^2 \int dE \{ f_\alpha(E)[1 - f_\beta(E)] \\
& + f_\beta(E)[1 - f_\alpha(E)] \} \\
& = \frac{ge^2}{h} |t|^2 (1 - |t|^2) |eV|.
\end{aligned} \quad (304)$$

In the limit  $|t|^2 \ll 1$  this reduces to the Schottky result  $S_I = |eI|$  (284). In the opposite limit  $|t|^2 = 1$  one gets the interesting result that the *shot noise vanishes for perfect*

*transmission*. Thus the current flow at temperature  $T = 0$  is “quiet” in a perfect conductor. This applies at the plateaus in the conductance figure above [after eq. (277)], whereas at the steps the current is noisy.

Shot noise is often quantified by “Fano factor”, which is the ratio  $F = S_I/e\langle I \rangle$ . Here we get  $F = 1 - |t|^2$ . For classical shot noise this is unity but it vanishes for perfect transmission.

3) General  $\omega = 0$  gives from (301)

$$\begin{aligned}
S_I(0) &= \frac{ge^2}{h} \int dE |t|^2 \{ f_\alpha(E)[1 - f_\alpha(E)] \\
& + f_\beta(E)[1 - f_\beta(E)] + |r|^2 [f_\alpha(E) - f_\beta(E)]^2 \}.
\end{aligned} \quad (305)$$

We calculate, aided by Mathematica,

$$\begin{aligned}
& \int dE [f(E + eV/2) - f(E - eV/2)]^2 \\
& = 2k_B T \left( \frac{eV}{2k_B T} \coth \frac{eV}{2k_B T} - 1 \right),
\end{aligned} \quad (306)$$

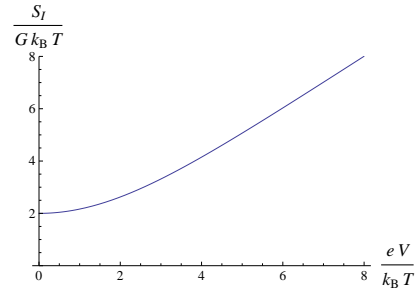
and therefore

$$S_I(0) = \frac{ge^2}{h} |t|^2 \left( 2k_B T |t|^2 + eV |r|^2 \coth \frac{eV}{2k_B T} \right). \quad (307)$$

For a tunneling barrier, where the transmission probability is small ( $|t|^2 \ll 1$ ), only the second term is important and

$$S_I(0) = \frac{ge^2}{h} |t|^2 eV \coth \frac{eV}{2k_B T}. \quad (308)$$

We see that the total noise is not a simple sum of thermal and shot noise.



4) General  $|t|$  independent of energy. We calculate, aided by Mathematica,

$$\begin{aligned}
\int dE f(E)[1 - f(E + \Delta E)] &= \Delta E \frac{e^{\beta \Delta E}}{e^{\beta \Delta E} - 1} \\
&= \frac{1}{2} \Delta E \left( 1 + \coth \frac{\Delta E}{2k_B T} \right),
\end{aligned} \quad (309)$$

$$\begin{aligned}
& \int dE \{ f(E)[1 - f(E + \hbar\omega + eV)] \\
& + f(E + eV)[1 - f(E + \hbar\omega)] \} \\
& = \frac{1}{2} (\hbar\omega + eV) \left( 1 + \coth \frac{\hbar\omega + eV}{2k_B T} \right)
\end{aligned}$$

$$\begin{aligned}
& +\frac{1}{2}(\hbar\omega - eV) \left(1 + \coth \frac{\hbar\omega - eV}{2k_B T}\right) \\
& = \hbar\omega + \frac{1}{2}(\hbar\omega + eV) \coth \frac{\hbar\omega + eV}{2k_B T} \\
& \quad + \frac{1}{2}(\hbar\omega - eV) \coth \frac{\hbar\omega - eV}{2k_B T}, \tag{310}
\end{aligned}$$

and we get from (300)

$$\begin{aligned}
S_I(\omega) & = \frac{ge^2}{h} \int dE |t|^2 \{ |t|^2 f_\alpha(E) [1 - f_\alpha(E + \hbar\omega)] \\
& \quad + |t|^2 f_\beta(E) [1 - f_\beta(E + \hbar\omega)] \\
& \quad + |r|^2 f_\alpha(E) [1 - f_\beta(E + \hbar\omega)] \\
& \quad + |r|^2 f_\beta(E) [1 - f_\alpha(E + \hbar\omega)] \} \\
& = \frac{ge^2}{h} |t|^2 \{ |t|^2 \hbar\omega (1 + \coth \frac{\hbar\omega}{2k_B T}) \\
& \quad + |r|^2 [\hbar\omega + \frac{1}{2}(\hbar\omega + eV) \coth \frac{\hbar\omega + eV}{2k_B T} \\
& \quad + \frac{1}{2}(\hbar\omega - eV) \coth \frac{\hbar\omega - eV}{2k_B T}] \} \\
& = \frac{ge^2}{h} |t|^2 \{ \hbar\omega + |t|^2 \hbar\omega \coth \frac{\hbar\omega}{2k_B T} \\
& \quad + |r|^2 [\frac{1}{2}(\hbar\omega + eV) \coth \frac{\hbar\omega + eV}{2k_B T} \\
& \quad + \frac{1}{2}(\hbar\omega - eV) \coth \frac{\hbar\omega - eV}{2k_B T}] \}. \tag{311}
\end{aligned}$$

Note that the first term is odd in  $\omega$  and the rest is even. At  $V = 0$  this reduces to

$$S_I(\omega) = G\hbar\omega (1 + \coth \frac{\hbar\omega}{2k_B T}) = \frac{2G\hbar\omega}{1 - e^{-\beta\hbar\omega}}, \tag{312}$$

which agrees with the fluctuation-dissipation theorem (246).

At finite  $V$  but temperature  $T = 0$ ,  $S_I$  (311) at a given  $\omega$  consists of three linear segments as a function of  $V$ . The same holds as a function of  $\omega$  at a fixed  $V$ . At finite  $T$  there the two corners are rounded. The shot noise (at  $T = 0$ ) is limited to frequencies less than  $\nu = eV/h$ . This is the rate of electrons (in one transverse mode) attempting transmission according to (267) for  $\Delta\nu = eV/h$ .

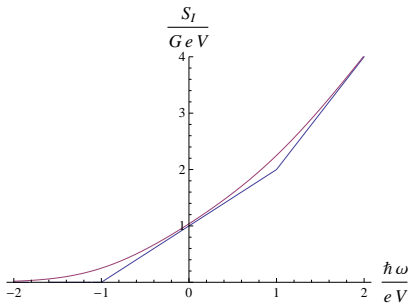


Figure: noise as a function of  $\omega$  at  $T = 0$  and  $T = eV/4k_B$  for small transmission.

5) Long conductor: a long conductor can be considered as consisting of several phase coherent systems in series. This leads to cancellation of shot noise, so that the thermal noise dominates in long conductors. More precisely, from (303) we get for thermal noise

$$S_V(0) = R^2 S_I(0) = 2Rk_B T, \tag{313}$$

and for shot noise (304)

$$S_V(0) = R^2 S_I(0) = eR^2 I = eRV. \tag{314}$$

Here the first is additive in resistance but the latter is not. The same conclusion holds also at finite frequencies, i.e. in an long conductor

$$S_V(\omega) = R\hbar\omega (1 + \coth \frac{\hbar\omega}{2k_B T}) = \frac{2R\hbar\omega}{1 - e^{-\beta\hbar\omega}}. \tag{315}$$

## 7. Conclusion

As a summary, we started by studying the driven and damped harmonic oscillator. In the quantized version, the damping has to be represented by a coupling to a bath, which was chosen to consist of harmonic oscillators. In addition to damping, the bath also produces fluctuations. The theory presented for harmonic oscillator can be simply generalized to two-state systems, and to more complicated systems. Finally, we studied the fluctuations caused by coupling to a bath of electrons. The hope is that this course forms a good starting point for understanding current research on topics that bridge quantum optics and electron transport in mesoscopic condensed matter structures.