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Acceleration of femtosecond pulses to superluminal velocities by Gouy phase shift

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Abstract. The phase and the group velocities are calculated in a three-dimensional neighbourhood of the focus of an aberration-free lens illuminated by a spatially Gaussian beam. The Gouy phase shift caused by the diffraction results in superluminal pulse propagation on the optical axis within the Rayleigh range.

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The study of the intensity and phase distribution of a converging spherical light wave passing through a circular aperture has been undertaken by several authors [1–3]. One of the most interesting results of these treatments is that the phase behaviour near the focus differs from that of a converging perfect spherical wave. This phase difference discovered by Gouy is called phase anomaly. The predictions of the theoretical treatments have been verified by experiments both with light and with microwaves [2–5]. For microwaves, the phase fronts can be measured directly [4, 5]. The experimental results show that the scalar diffraction theory can be used if $\lambda \ll a \ll f$, where λ is the wavelength, a is the radius of the diffracting aperture and f is focal length measured from the aperture. The scalar treatment yields a reasonably accurate description of the optical images up to Numerical Apertures (NA) as high as 0.6 [6].

Because of the phase anomaly, we expect that the phase velocity could differ from the phase velocity of a perfect converging spherical wave which has a phase velocity of c, where c is the velocity of light in the infinite free space. Since the phase distribution (and thus the phase velocity) depends on the wavelength a difference between the phase and the group velocity is expected. There is a strong analogy between the pulse propagation in dispersive medium and the diffraction of light caused by a hole in the opaque screen [7] and it is known that the group velocity in dispersive medium can exceed c [8]. So the following question arises: could the diffraction by a circular aperture cause such a phase distribution in space which results in a superluminal pulse propagation. In the rest of this paper, we show that the phase anomaly in the vicinity of the focus leads to superluminal pulse propagation.

1 Phase anomaly of a focused Gaussian beam in the vicinity of the focus

Consider a monochromatic beam focused by an aberration-free thin lens which fills a circular aperture of radius a in an opaque screen. For reasons of simplicity and applicability to laser beams we assume that the incoming beam is a spatially Gaussian beam having a beam waist of w_0. The phase anomaly for a Gaussian beam is much more well behaved than the one for a spatially homogeneous wave [9]. We also assume that the truncation of the Gaussian beam is weak and the focal length of the lens is much greater than the radius of the aperture, that is $w_0 \ll a \ll f$, where f is the focal length of the lens. In this approximation, the focused beam behind the lens remains Gaussian but the beam waist w_f is changed [10]. If the waist of the incoming beam is situated on the lens, the beam waist of the focused beam is located at a distance d from the lens given by (Fig. 1)

$$w_f^2 = \frac{w_0^2}{1 + \left(\frac{z_0}{f}\right)^2}, \qquad d = f - \frac{f}{1 + \left(\frac{z_0}{f}\right)^2}, \tag{1}$$

where $z_0 = \pi w_0^2/\lambda$ is the Rayleigh length of the input beam and λ is the wavelength. Then $E(r, z^*) = E_0 \exp[-r^2/w^2(z^*)] \exp[-\mathrm{i}\Phi(r, z^*)]/w(z^*)$ describes the space-dependent part of the electric field behind the lens, where z^* is a coordinate along the optical axis measured from the beam waist, and $w^2(z^*) = w_f^2[1 + (z^*/z_f)^2]$ gives the spot size at a point z^* on the optical axis, and

$$\Phi(r, z^*) = kz^* + \frac{kr^2}{2R(z^*)} - \arctan\frac{z^*}{z_f} \tag{2}$$

is the phase of the electric field ($k = 2\pi/\lambda$ is the wave number). Here $z_f = \pi w_f^2/\lambda$ is the Rayleigh length of the

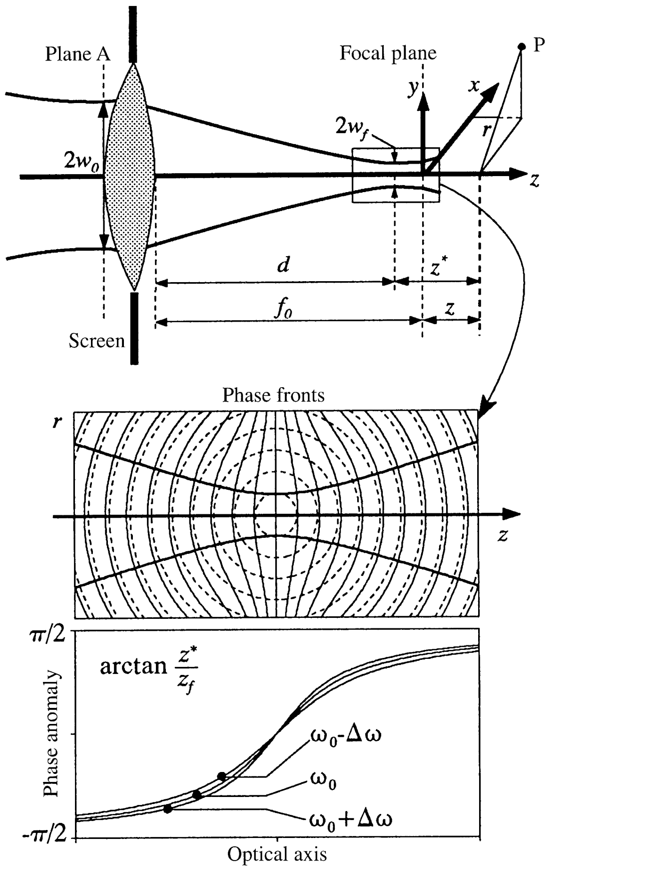

Fig. 1. The phase fronts of a focused Gaussian beam (solid lines) differ from the phase fronts of a perfect spherical wave (dashed lines). This phase difference discovered by Gouy is called as phase anomaly. The optical axis (z) is given by $\arctan(z/z_f)$, where z_f is the Rayleigh length of the focused beam. The distance between two neighbouring spherical wave fronts (dashed lines) is λ. The phase anomaly and the intensity distribution depend on the frequency. This dispersion causes that the phase velocity differs from the group velocity.

focused beam and $R(z^*) = z^*[1 + (z_f/z^*)^2]$ is the curvature of the phase front at a point z^* on the optical axis.

Equation (2) shows that the phase of a Gaussian beam differs from a converging perfect spherical wave. In Fig. 1 the phase fronts of a focused Gaussian beam are plotted with solid lines. The dashed lines show the phase fronts of a perfect spherical wave. The distance between two neighbouring spherical phase fronts is λ. The phase difference (i.e. the Gouy phase shift) on the optical axis is $\arctan(z^*/z_f)$. Since z_f depends on λ the phase anomaly also depends on λ. Because of this dispersion, a difference between the phase and the group velocities is expected.

In order to study the pure effects caused by the Gouy phase shift, we assume that the lens is free of chromatic error, that is $f(\lambda) = f_0$. For practical cases $(z_0/f_0)^2 \gg 1$, for example, if $\lambda = 0.5\ \mu\text{m}$, $w_0 = 2\ \text{mm}$ and $f_0 = 40\ \text{mm}$, $(z_0/f_0)^2 = (2\pi \cdot 10^2)^2 \approx 4 \cdot 10^5$. Then the small shift in position between the geometrical focus and the beam waist is negligible [(1)], that is $d = f_0$ and so $z^* = z$, where z is a coordinate along the optical axis measured from the focus (Fig. 1). In this approximation the beam waist of the focused beam is given by

$$w_f(\omega) = \frac{w_0 f_0}{z_0} = \frac{2c f_0}{\omega w_0}, \tag{3}$$

where $\omega = c \cdot k$. Then (2) becomes

$$\Phi(r, z) = \frac{\omega}{c} z + \frac{\omega}{2c} \frac{(K\omega r)^2 K\omega z}{1 + (K\omega z)^2} - \arctan(K\omega z), \tag{4}$$

where $K = w_0^2/(2cf_0^2)$.

2 Phase velocity

The phase velocity is defined by $v_p = \omega/|\nabla\Phi|$ [11]. Because of the axial symmetry of Φ it takes the form

$$v_p = \omega \Bigg/ \sqrt{\left(\frac{\partial\Phi}{\partial r}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2}. \tag{5}$$

Substituting (4) into (5) one can obtain the phase velocity at a point P by

$$v_p(\rho, \zeta) = c \frac{1 + (\vartheta^2\zeta)^2}{\sqrt{(\vartheta^2\zeta \cdot \vartheta^2\rho)^2 + \left(\frac{1}{2}(\vartheta^2\rho)^2 \frac{1 - (\vartheta^2\zeta)^2}{1 + (\vartheta^2\zeta)^2} + 1 + (\vartheta^2\zeta)^2 - \frac{\vartheta^2}{2}\right)^2}} \tag{6}$$

where ρ and ζ are dimensionless variables defined by

$$\rho = \pi\frac{r}{\lambda} = \frac{\omega r}{2c}, \qquad \zeta = \pi\frac{z}{\lambda} = \frac{\omega z}{2c}, \tag{7}$$

and $\vartheta = w_0/f_0$ is the divergence of the focused beam. On the optical axis (i.e. $\rho = 0$) (6) leads to

$$v_p(0, \zeta) = c \frac{1 + (\vartheta^2\zeta)^2}{1 + (\vartheta^2\zeta)^2 - \frac{1}{2}\vartheta^2}. \tag{8}$$

Figure 2a shows the phase velocity on the optical axis for different values of the divergence. It is easy to see from (8) that the phase velocity on the optical axis is larger than c and it reaches its maximum at the focus given by $v_{p,\max} = c/[1 - \vartheta^2/2] \approx c/\cos\vartheta$, which is the sweep speed of a pulse on the optical axis falling under the angle of divergence ϑ.

The position of the phase front on the optical axis in a moment t is determined by $t = \Phi(0, z)/\omega = z/c - \arctan(K\omega z)/\omega$. Of course, one can obtain this equation by integration using (8), too. For $|z| \gg z_f$ it has an asymptotic form: $z = ct + \text{sign}(z)\,\lambda/4$. This means that far from the focus if $z < 0$, the phase front lags behind, while if $z > 0$, it precedes the phase front of a perfect spherical wave with a distance $\lambda/4$ as it is shown in Fig. 1. This is why the phase velocity is larger than c.

In the focal plane (i.e. $\zeta = 0$ in (6)) one can calculate the phase velocity by

$$v_p(\rho, 0) = \frac{c}{\frac{1}{2}(\vartheta^2\rho)^2 + 1 - \frac{1}{2}\vartheta^2}. \tag{9}$$

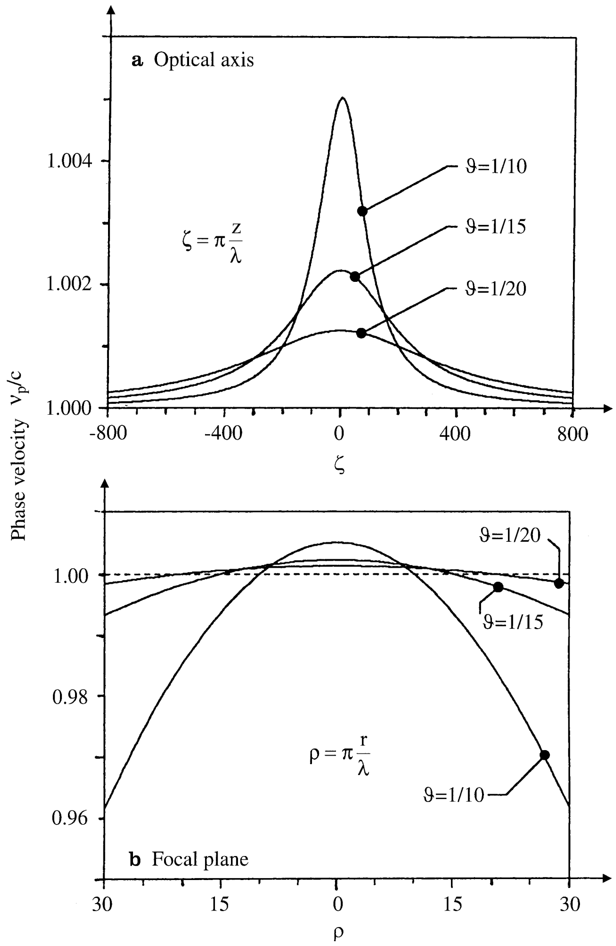

Fig. 2. **a** The phase velocity on the optical axis (z-axis) for different values of the divergence of the focused beam. The maximum is given by $v_{p,\max} = c/\cos\vartheta$. **b** The phase velocity in the focal plane. $1/\vartheta$ corresponds to the beam waist. $v_p > c$ inside the beam waist and $v_p < c$ outside.

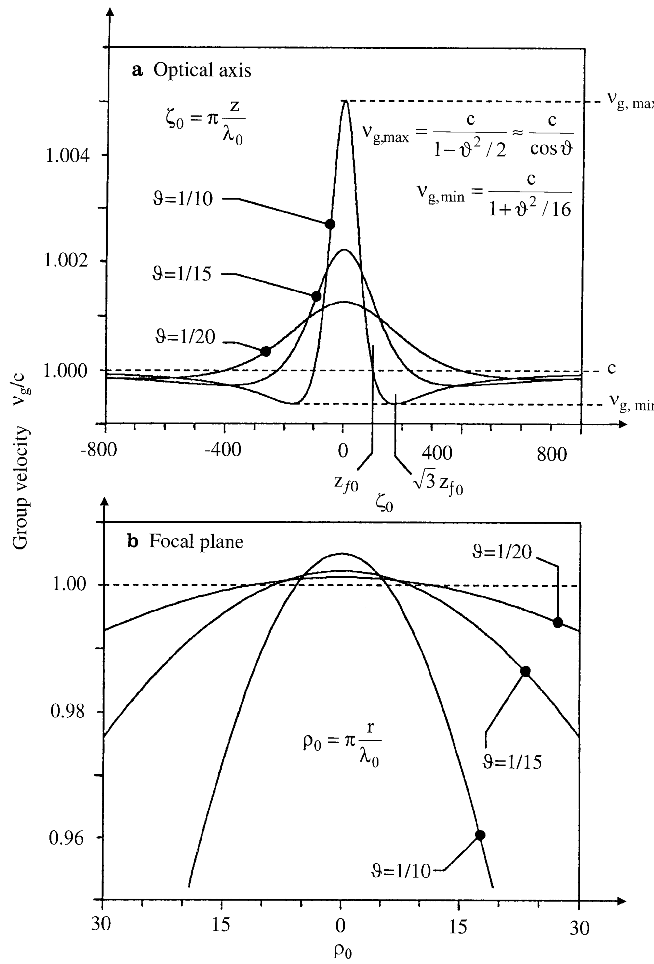

Fig. 3. **a** The group velocity on the optical axis (z-axis) for different values of ϑ ($\vartheta = w_0/f_0$ is the divergence of the focused beam). The maximum is given by $v_{g,\max} = c/\cos\vartheta$. $\zeta_0 = 1/\vartheta^2$ corresponds to $z = z_{f0}$, where z_{f0} is the Rayleigh length for the central frequency. $v_g > c$ inside the Rayleigh range and $v_g < c$ outside. The average of the group velocity is c. **b** The group velocity in the focal plane decreases faster than the phase velocity (Fig. 2b).

The graph of (9) is plotted in Fig. 2b. It follows from (9) that $v_p > c$ if $\rho < 1/\vartheta$, $v_p = c$ if $\rho = 1/\vartheta$ and $v_p < c$ if $\rho > 1/\vartheta$. Since $\rho = 1/\vartheta$ corresponds to $r = w_f$ [(3, 7)] the phase velocity in the focal plane is larger than c within the beam waist and it is less than c outside the beam waist.

3 Group velocity

The group velocity is defined by [12]

$$v_g = 1\Bigg/\left|\nabla \frac{\partial \Phi}{\partial \omega}\bigg|_{\omega_0}\right|, \tag{10}$$

where ω_0 is the central frequency of the pulse. From (4) we have

$$\frac{\partial \Phi}{\partial \omega} = \frac{z}{c}\left(\frac{1}{2}(K\omega r)^2 \frac{3 + (K\omega z)^2}{[1 + (K\omega z)^2]^2} + 1 - \frac{Kc}{1 + (K\omega z)^2}\right). \tag{11}$$

Inserting (11) into (10) the group velocity is given by

$$v_g(\rho_0, \zeta_0) = c[1 + (\vartheta^2\zeta_0)^2]^2 \left[[\vartheta^2\zeta_0\, \vartheta^2\rho_0(3 + (\vartheta^2\zeta_0)^2)]^2 \right.$$

$$+ \left(\frac{1}{2}(\vartheta^2\rho_0)^2\left(3 - (\vartheta^2\zeta_0)^2 - \frac{8(\vartheta^2\zeta_0)^2}{1 + (\vartheta^2\zeta_0)^2}\right)\right.$$

$$\left.\left. + [1 + (\vartheta^2\zeta_0)^2]^2 - \frac{\vartheta^2}{2}[1 - (\vartheta^2\zeta_0)^2]\right)^2\right]^{-1/2}, \tag{12}$$

where $\rho_0 = \rho(\omega_0)$ and $\zeta_0 = \zeta(\omega_0)$. On the optical axis (i.e. $\rho_0 = 0$), one can calculate the group velocity by

$$v_g(0, \zeta_0) = c\frac{[1 + (\vartheta^2\zeta_0)^2]^2}{[1 + (\vartheta^2\zeta_0)^2]^2 - \frac{\vartheta^2}{2}[1 - (\vartheta^2\zeta_0)^2]}. \tag{13}$$

Equation (13) is plotted in Fig. 3a for different values of divergence ϑ. Analysing (13) one can conclude that the group velocity on the optical axis equals c if $\zeta_0 = \pm 1/\vartheta^2$. According to (7) $\zeta_0 = \pm 1/\vartheta^2$ corresponds to $z = \pm z_{f0}$, where $z_{f0} = z_f(\omega_0)$ that is the Rayleigh length at ω_0. The group velocity on the optical axis is larger than

c within the interval $(-z_{f0}, z_{f0})$ and less than c outside. It reaches its maximum in the focus ($\zeta_0 = 0$) given by $v_{g,\max} = c/[1 - \vartheta^2/2] \approx c/\cos\vartheta$. The minimum is achieved in $z = \pm\sqrt{3}z_{f0}$ and it can be calculated by $v_{g,\min} = c/[1 + (\vartheta/4)^2]$.

The position of the pulse front on the optical axis in a moment t is determined by $t = \partial\Phi(0, z)/\partial\omega = z/c - Kz/[1 + (K\omega_0 z)^2]$. The asymptotic form of this equation is $z = ct$, which means that far from the focus the pulse front moves on the optical axis as the phase front of a spherical wave does (dashed lines in Fig. 1) and the average of the group velocity equals c.

The group velocity in the focal plane (i.e. $\zeta_0 = 0$) is given by

$$v_g(\rho_0, 0) = \frac{c}{\frac{3}{2}(\vartheta^2\rho_0)^2 + 1 - \frac{1}{2}\vartheta^2}. \tag{14}$$

It differs from the phase velocity (9) in the factor 3 of $(\vartheta^2\rho_0)^2$ only, which results in that the group velocity decreases faster than the phase velocity as a function of the radius in the focal plane. It follows from (14) that $v_g > c$ if $\rho_0 < \rho_g$, $v_g = c$ if $\rho_0 = \rho_g$ and $v_g < c$ if $\rho_0 > \rho_g$, where $\rho_g = 1/(\sqrt{3}\vartheta)$. Since ρ_g corresponds to the radius of $r = w_{f0}/\sqrt{3}$ the group velocity in the focal plane is larger than c within this radius and it is less than c outside, where $w_{f0} = w_f(\omega_0)$ is the beam waist at the central frequency. Fig. 3b shows the group velocity in the focal plane for different values of ϑ.

4 Conclusions

It has been shown that the Gouy phase shift causes abnormal pulse propagation in the neighbourhood of the focus of an aberration-free lens. It should be noted that the cause of the superluminality described in [13–17] has a completely different origin because in that case the superluminality is caused by the dispersion of the medium where the waves propagates in. In the case discussed above, the phase structure in the medium is induced by the diffraction and it arises in non-dispersive medium, even in vacuum as it was assumed during this calculation. Since the Gouy phase shift is generated by the diffraction it does not depend on the dispersion of the lens material. As far as we know, this kind of superluminal pulse propagation was not recognized before. Our results do not violate the theory of special relativity because the group velocity is the velocity of the peak of the pulse and an intensity maximum could move faster than c; it does not carry information.

In order to obtain analytical expressions, we used several approximations: we assumed that the light can be described by a scalar quantity and we considered paraxial approximation; besides, the small shift in position between the geometrical focus and the beam waist is omitted. As it has been mentioned if $\lambda \ll a \ll f$, the vector and scalar diffraction theory leads to the same result, so the scalar treatment of the light is appropriate in our case. Using the method described in [18] which is still paraxial approximation but where the focal shift is taken into account, (13) has been confirmed numerically for a 6 fs long Gaussian pulse with 620 nm central wavelength. Numerical evaluation of the generalized Kirchhoff integral [19] has been performed [20]. These calculations confirm the predictions of the paraxial approximation derived above. The phase and the group velocity in the focus exceeds c but it is smaller than the one predicted by the paraxial approximation. Assuming $\lambda = 0.5$ µm, $w_0 = 2$ mm, $a = 8$ mm ($a/w_0 = 4$ i.e. the truncation of the input Gaussian beam is weak) the relative errors caused by our approximations are less than 0.0144%, 0.0029% and 0.0009% for $\vartheta = w_0/f_0$ of $\frac{1}{10}$, $\frac{1}{15}$ and $\frac{1}{20}$, respectively.

The phase anomaly in the vicinity of the focus appears in case of homogeneous illumination, but the phase behaviour is more complicated. So similar abnormal pulse propagation can occur in that case as well. A superluminal pulse propagation described above might have a considerable effect on the strong-field high-harmonics-generation experiments [21] since the propagation speeds of the harmonics are affected by the wavelength as well.

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