Chapter 3
Rotational Flows:
Circulation and Turbulence

Next we will consider rotating flows and the role friction plays in a fluid. As we shall see when we study Kelvin's theorem, friction is the source of circulation in fluids. When you stir your morning coffee with a spoon, if it were not for friction between the surface of the spoon and the coffee, it would not be possible to stir your coffee. Friction plays an all important role in fluid dynamics and in understanding something as exotic as the difference between storms in the atmospheres of Jupiter and Neptune or something so common as how to throw a curve ball. And finally, you will be able to understand why the Tacoma Narrows bridge began to oscillate wildly.

3.1 Circulation.

Consider a pure circulating field, a fluid that rotates as a rigid body. What quantities are useful to describe such a system? First if the fluid is rotating as a rigid body, then it must have a constant angular speed ω where

$$\omega = \frac{d\theta}{dt} \tag{3.1}$$

and the velocity is tangent to a circle

$$\vec{v} = r\,\omega\,\hat{\theta} \tag{3.2}$$

and must increase linearly with r so the field lines must become denser as r increases. A field for rigid body flow is shown in Figure 3.1.
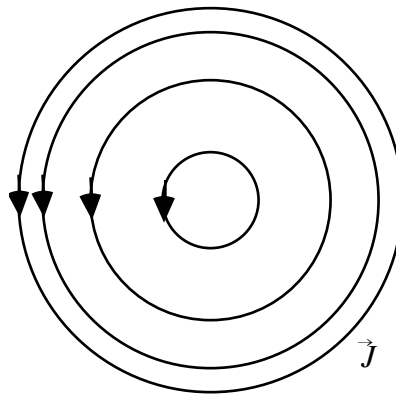

Figure 3.13
The current density for a fluid rotating as a rigid body.
The angular speed is constant and the linear speed
increases linearly with radius.

Since the field lines close on themselves, there are no ends and there can thus be no flux through any closed surface. So flux will not help us describe this field. Instead we might try summing the field along a circular path that follows the field line, i.e., use a path integral. We will define the

path integral around a closed curve C to be the *circulation*.

$$circulation = \oint_C \vec{J} \cdot \vec{dl} \qquad (3.3)$$

How is the circulation connected to the rotational properties of the fluid? The clearest example is for a fluid undergoing rigid body flow. To evaluate a path integral, we must always choose a path. Suppose we chose a circular path C along one of the field lines. Then dl becomes the arc length

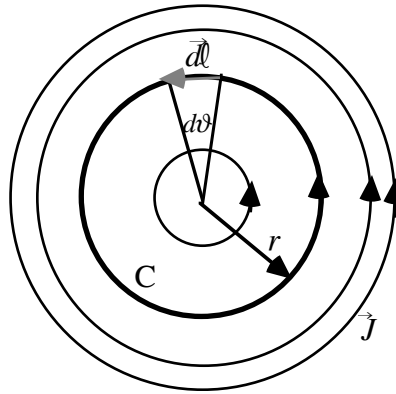$$dl = r \, d\theta \qquad (3.4)$$

shown in Figure 3.2.

Figure 3.14
The circulation is evaluated along a curve that lies along one of the field lines for a rigid body flow.

The dot product between $\vec{J}$ and $\vec{dl}$ gives

$$\vec{J} \cdot \vec{dl} = J \, r \, d\theta . \qquad (3.5)$$

Since $J = \rho v = \rho \, r\omega$, J is constant so long as r is constant. Thus, J may be pulled out of the integral, and

$$\oint_C \vec{J} \cdot \vec{dl} = \int_0^{2\pi} \rho r\omega \, r d\theta$$

$$\oint_C \vec{J} \cdot \vec{dl} = 2\pi \, \rho r^2 \, \omega. \qquad (3.6)$$

This is an amazing result. Recall that for a mass dm moving in a circle of radius r the angular momentum is

$$dL = (dm) \, r^2 \, \omega \qquad (3.7)$$

so that the angular momentum per unit volume is

$$dL/dV = \rho \, r^2 \, \omega \qquad (3.8)$$

Thus for a fluid under rigid body flow the circulation is

$$\oint_C \vec{J} \cdot \vec{d\ell} = 2\pi \ dL/dV \ , \qquad\qquad (3.9)$$

i.e., *the circulation in a fluid is proportional to the angular momentum per unit volume.* Thus, the circulation is the appropriate quantity to characterize rotating fluids. A direct consequence of this is that:

Kelvin's Theorem
If angular momentum is conserved, then the circulation is constant in time.

This is known as Kelvin's theorem. As with the relation between the force and the pressure gradient, the circulation is expressed in terms of the angular momentum density. It is now possible to define an irrotational fluid. An *irrotational fluid is one for which* $\oint_C \vec{J} \cdot \vec{d\ell} = 0$, *i.e., it has no circulation.*

This connection between the circulation and the angular momentum density provides us with a visual aid to determine when circulation is present in a vector field. Think of a small imaginary paddle wheel dropped into the fluid so that it flows along with the fluid and samples the angular momentum of a small volume of the fluid. If the paddle rotates, then that volume has angular momentum and thus circulation. If the paddle wheel does not rotate, there is no circulation. Obviously, a uniform flow (Figure 3.3(a)) will have no circulation. For rigid body flow a paddle wheel will turn through one complete rotation for every revolution about the center of flow.
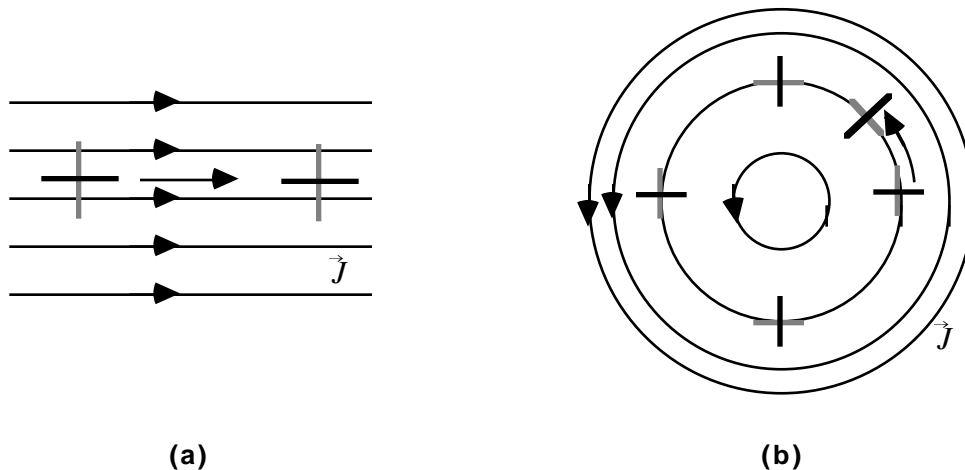

(a)   (b)

Figure 3.3
(a) An irrotational flow. (b) In rigid body flow, a paddle wheel will spin through one complete revolution for every complete revolution about the center of flow.

Notice that the paddle wheel always keeps the same side towards the center of flow. This is characteristic of rigid body flow.

Of course a fluid need not flow as a rigid body. Adjacent layers could slip against one another and layers further from the center might move more slowly as shown if Figure 3.4. Friction between adjacent layers would cause such an effect.
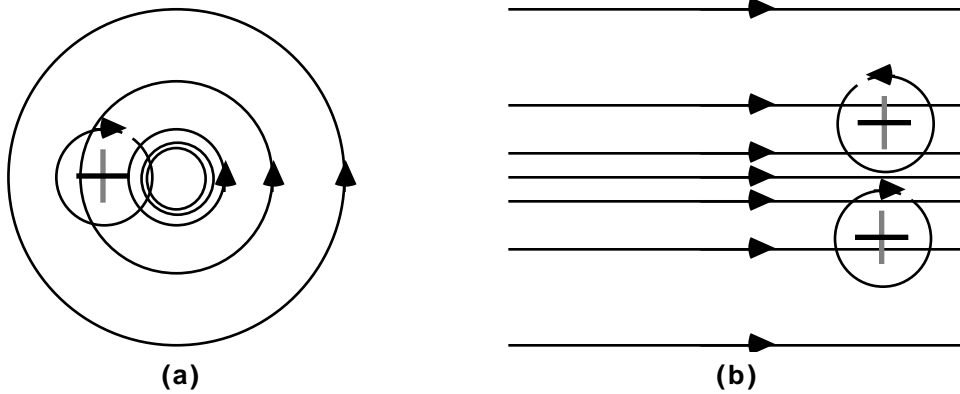
Figure 3.16
Two surprizing cases. a) The circulation need not be in the same sense as the rotation of the field. b) The fluid need not be rotating to have circulation.

If the flow decreases quickly enough, the outside edge of the paddle wheel will experience a flow that is so much slower than the inside edge that the inside edge will move with the flow while the outside edge moves against the flow. Thus, the paddle wheel can actually rotate in the opposite sense of the flow!

Note also that just having a non–uniform velocity distribution like that shown in Figure 3.4(b) can cause a paddle wheel to rotate. The flow in Figure 3.4(b) is not rotating at all yet it has non zero circulation. We can prove this by calculating the circulation around the curve shown in Figure 3.5.
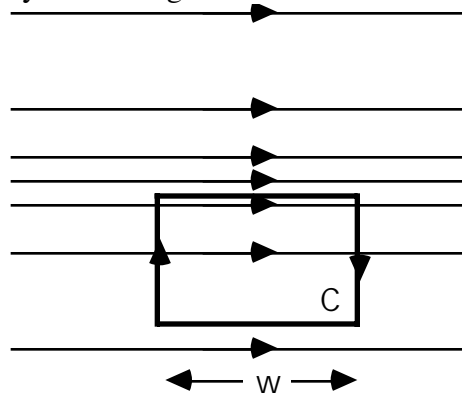

Figure 3.5
Use the path C to calculate the circulation for this flow.

Example 3.1

Calculate the circulation around the path in Figure 3.5 given the density of the fluid ρ, the velocity at the top $\vec{v}_t$, the velocity at the bottom $\vec{v}_b$, and the width of the path w.

Solution: We can divide the path integral into four segments

$$\oint \vec{J} \cdot d\vec{\ell} = \int_{top} \vec{J} \cdot d\vec{\ell} + \int_{right} \vec{J} \cdot d\vec{\ell} + \int_{bottom} \vec{J} \cdot d\vec{\ell} + \int_{left} \vec{J} \cdot d\vec{\ell}. \tag{3.10}$$

The integral over the right and left sides are zero because $\vec{J}$ is perpendicular to $d\vec{\ell}$ along those sides thus making the dot product zero. Recall that $\vec{J} = \rho \vec{v}$ so that performing the dot product

$$\oint \vec{J} \cdot \overrightarrow{d\ell} = \rho \int v_t \, d\ell - \rho \int v_b \, d\ell \qquad (3.11)$$

where along the bottom $\vec{v}$ is opposite to $\overrightarrow{d\ell}$ making the dot product negative. Since the flow is uniform along both the top and bottom paths, the velocities may be pulled out of the integral and if each path has a width w,

$$\oint \vec{J} \cdot \overrightarrow{d\ell} = \rho w(v_t - v_b). \qquad (3.12)$$

Since the velocity of the flow is greater along the top path

$$\oint \vec{J} \cdot \overrightarrow{d\ell} > 0.$$

Thus, the circulation is indeed non zero. This example is important for flows inside pipes where the friction between the walls and the fluid slow the fluid and create a current density like that shown in Figure 3.5.

3.2 Cylindrical Flows: Vortices and Tornados

We have seen how the circulation is related to the angular momentum density (angular momentum per unit volume) when a fluid rotates as a rigid body. Now we will explore some other properties of the circulation. First let's explore the sense of the circulation. The circulation mathematically is defined in terms of a path integral around a closed path. The path must have a sense of direction associated with it being either clockwise or anti–clockwise because of the dot product involved in the path integral. In the next section when we introduce the curl, we'll be able to precisely define this loose sense of rotation, but it is instructive to consider the sense of the circulation here.

Consider first a cylindrical incompressible flow which does not rotate as a rigid body, but rather, has the speed decrease as $1/r$ as the distance r from the center of rotation increases and is depicted in Figure 3.6. Since the magnitude of the velocity decreases with increasing r, the field lines become further apart.
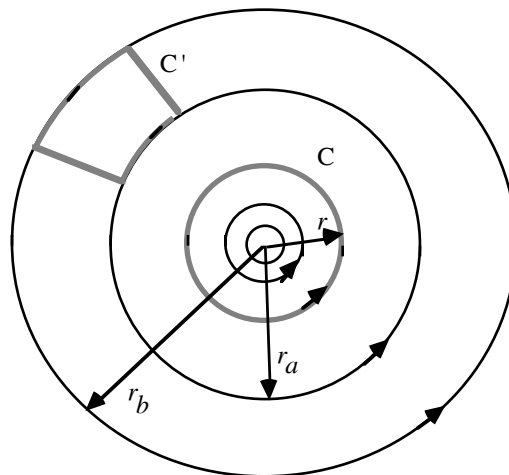

Figure 3.6
A cylindrical flow in which the speed varies inversely with r.

Since the velocity decreases with r the outer layers of fluid will not be moving as fast as the inner layers so adjacent layers must, therefore, slip against one another. Cases where the flow occurs in

nice ordered layers is called laminar flow. In contrast to laminar flow in turbulent flow, the field lines from different layers can mix in a whirlpool or can break up into a disordered mess.

For the flow shown in Figure 3.6, if we attempt to calculate the circulation about the two paths, C and C', shown, we find different results. Of course we should expect to find different results because the circulation depends on path, but the result is curious nonetheless. Given

$$\vec{J} = \frac{k}{r}\hat{\theta} \qquad (3.13)$$

where k is a positive constant, the circulation about path C is

$$\oint_C \vec{J} \cdot d\vec{\ell} = \int_0^{2\pi} \frac{k}{r} \, r d\theta \qquad (3.14)$$

where $d\ell$ is an arc length of the circular path. The radii cancel leaving

$$\oint_C \vec{J} \cdot d\vec{\ell} = 2\pi k. \qquad (3.15)$$

For the path C' there is no contribution for the radial legs because $\vec{J}$ is perpendicular to $d\vec{\ell}$, but there is for the two sections of arc:

$$\oint_C \vec{J} \cdot d\vec{\ell} = \int_{\theta_1}^{\theta_2} \frac{k}{r_a} r_a d\theta + \int_{\theta_2}^{\theta_1} \frac{k}{r_b} r_b d\theta = 0. \qquad (3.16)$$

What is so surprizing is that any path that does not include the origin will have zero circulation, but any path that does include the origin will have $2\pi k$ circulation! One would think that the circulation is either zero or not zero. What is going on?

To attempt to solve the apparent paradox, consider two examples. First a cylindrical incompressible flow which does not rotate as a rigid body, but rather, has the speed remain constant as the distance r from the center of rotation increases is depicted in Figure 3.7. Since the magnitude of the velocity is constant, the field lines are evenly spaced. Recall that for rigid body flow, a paddle wheel placed in the flow always keeps its same side towards the center of flow, i.e., the paddle spins once in one complete revolution about the center of flow.
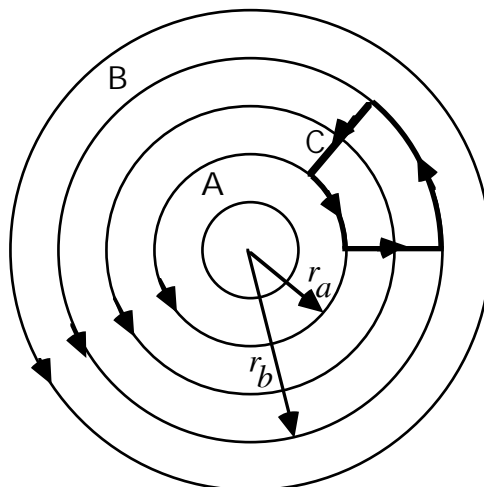
Figure 3.7
A cylindrical flow in which the speed is constant.

If the linear speed is the same in lines A and B, the fluid in line B will take much longer to complete one rotation than A because of the longer distance the fluid must travel for B. The layers of fluid must, again, slip against one another with the outer layers rotating more slowly (slower angular speed but same linear speed). If the layers slip, then a paddle wheel placed in this flow and free to move with the flow will not completely spin once for one revolution. The question is which way will it rotate, in the same sense as the fluid, or in the opposite sense?

If the magnitude of the current density is $|\vec{J}| = k$, then the current density is

$$\vec{J}' = k\,\hat{\theta}. \qquad\qquad (3.17)$$

Since the circulation is

$$\oint \vec{J}\cdot\vec{dl} = 2\pi\, L/V \qquad\qquad (3.18)$$

where L/V is the angular momentum density, by using the circulation we can answer this question. Let's assume that the paddle wheel will rotate in the same sense as the flow in Figure 3.7, i.e., counter–clockwise so we choose the circulation about a counter clockwise path C.

$$\oint \vec{J}\cdot\vec{dl} = \int_0^\theta k\, r_b d\theta + \int_\theta^0 k\, r_a d\theta = k\,\theta\,(r_b - r_a) \qquad\qquad (3.19)$$

Since $r_b > r_a$, the circulation is positive. This means the paddle wheel will rotate counter–clockwise, the same as it would if it were placed at the center of the flow.

Finally consider what would happen if the flow decreased more strongly with r, say as an inverse square field so that

$$\vec{J}\,'' = \frac{k}{r^2}\hat{\theta}. \qquad\qquad (3.20)$$

Then the circulation about a path such as C'' would be

$$\oint \vec{J}\cdot\vec{dl} = \int_0^\theta \frac{k}{r_b^2} r_b d\theta + \int_\theta^0 \frac{k}{r_a^2} r_a d\theta = k\theta\left(\frac{1}{r_b} - \frac{1}{r_a}\right). \qquad\qquad (3.21)$$

Since $r_b > r_a$, the circulation is negative! This means the paddle wheel placed inside the path C'' will rotate clockwise, opposite to a paddle wheel placed at the center. We thus see an important result. The case $J \propto 1/r$ is the critical case between $J = $ constant where the off–center paddle wheel will rotate in the same sense as the fluid, and $J \propto 1/r^2$ where the paddle wheel will rotate in the opposite sense to the flow, i.e., when placed in a counter–clockwise flow, the paddle wheel will rotate clockwise. For $J \propto 1/r$ the paddle wheel must not spin at all as it rotates about the fluid, hence it has no angular momentum. The ideal is that a small paddle wheel senses the local angular momentum, and hence, it senses the local circulation.
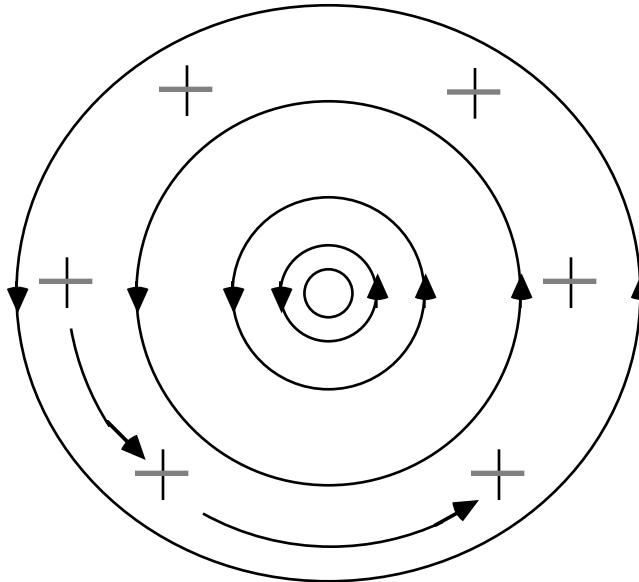
Figure 5.3
**For an inverse r cylindrical flow, the paddle wheel will not
spin as it flows about the center.**

This is important in the following example familiar to people who take baths or live in the midwest plain states of the United States.

Example 3.2

Construct a simple model of the whirlpool formed in the water of a draining bathtub finding the velocity as a function of position, and the circulation. Describe the important physics.

Solution: A model is a simple description often ignoring most complications and concentrating on one or two dominant effects. For our model we will ignore the asymmetrical shape of the typical bath tub, and we'll also ignore complicating effects like friction and surface tension.

Consider a cylinder of fluid which we give angular velocity $\vec{\omega}$ by rotating it. Suppose that the fluid has some small, but finite viscosity so that it will rotate with the cylinder If we then open a drain, a whirlpool will form, and a given particle of the fluid will move in a spiral during the process of approaching the drain as illustrated in Figure 3.8
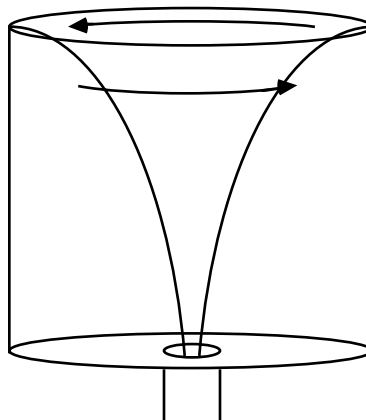

Figure 3.8
**A whirlpool will form in a circulating flow with a drain. A given particle
of the fluid will move in a spiral as it approaches the drain.**

If there are no significant viscous forces, then there will be no significant torques so angular momentum will approximately be conserved in our model. The angular momentum density is given by

$$|\vec{L}/V| = \rho\,|\vec{r} \times \vec{v}| = \rho\, r\, v\,.\qquad(3.22)$$

Bince$L$ is e a constant, the linear speed must fall off as $1/r$ from the center of the whirlpool.

$$v = L/(\rho r V)\qquad(3.23)$$

We have already calculated the circulation for this flow in equation (3.15).

$$\oint \rho\vec{v}\cdot d\vec{\ell} = \int_0^{2\pi} \rho\left(\frac{L}{\rho r V}\right) r\, d\theta = 2\pi L/V\qquad(3.24)$$

a familiar result which says that the circulation about the center is proportional to the angular momentum density about the center.

Now for a point not at the center, the circulation on any path not including the center within its boundary is zero as shown in equation (3.16). If you carefully observe a small paddle wheel in the flow, the paddle wheel will not spin as it rotates about the center of the whirlpool. The flow is irrotational. A paddle wheel set in the center of the whirlpool will rotate at the same speed as the fluid near the center. There is a wonderful demonstration of this at the exploratorium in San Francisco. The variations in the vortex are caused by interial effects. It has applications to some of the deadliest storms in nature, although tornados are not irrotational, we have gained some insight into their nature.

3.3 The Curl and Stokes' Theorem.

As with divergence, if we really wish to get a handle on the source of the circulation, we must look at what is happening at different points in the fluid, i.e., our description would be better the smaller we make the paddle wheel. An infinitesimal size would be best so we will define a quantity called the curl which will be related to the circulation divided by the area bounded by the curve, in the limit that the area goes to zero. There is one additional complication, the direction. We have been dealing with clockwise and counter–clockwise rotations, but clockwise and counter–clockwise are not readily transferable to vector notation (it is one of the unfortunate properties of our system of vector notation). We have run into this problem before with angular momentum and torque where a right hand rule is used to give the angular momentum or torque a unique direction. An example is when a point mass on a string rotates in a counter–clockwise sense in the x–y plane. If you curl the fingers of your right hand in the direction the mass is rotating, your thumb will point along the z axis, the direction of the angular momentum. Likewise, a clockwise rotation will produce an angular momentum along the negative z axis. This is exactly the case we have here.   A counter–clockwise circulation in the x–y plane can be tied by a right hand rule to a vector pointing in the positive z direction as in Figure 3.9
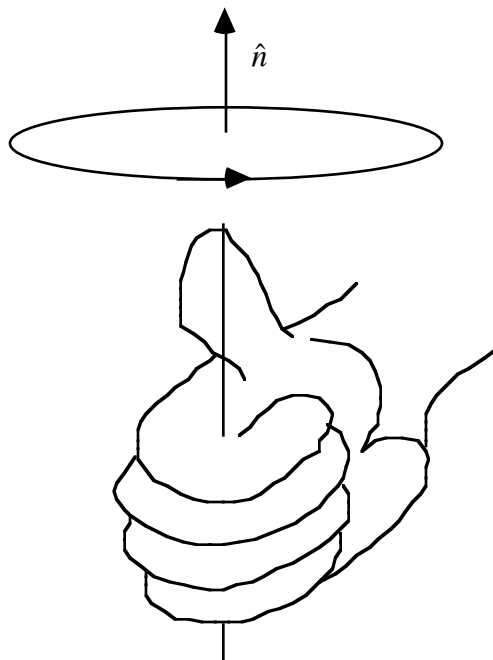

Figure 6.5
Using a right hand rule, a counter-clockwise circulation produces
a unique unit vector normal to the plane of circulation.

Thus, we need to give the curl a direction given by the right hand rule:

The Right Hand Rule for Curl

**Pick a path for which the circulation is positive. The direction of rotation will be
in the sense that a paddle wheel placed in the flow will rotate. Curl the fingers of
your right hand in the direction of the rotation of the fluid. Your thumb will give
the direction of the curl.**

If we call $\hat{n}$ the direction of the curl defined by the right hand rule above, then the definition of the curl of an arbitrary vector field $\vec{F}$ is

$$\hat{n} \cdot curl \, \vec{F} \;=\; \lim_{A \to 0} \frac{\oint_C \vec{F} \cdot d\vec{\ell}}{A} \qquad Curl \qquad (3.25)$$

where A is the area of the surface bounded by the curve C and $\hat{n}$ is unit normal to the area defined using the right hand rule.

Equation (3.25) is a formula that is impractical to use. It does provide an important geometrical interpretation for the curl. The curl is a *circulation density*, i.e., curl is the circulation per area. We can obtain a practical equation for the curl by working a rectangular circulation in the x–y plane, and applying the definition (3.25). Assuming a rectangular circulation will give us the curl in Cartesian coordinates, and easy system to use.

Apply the definition of the curl to the rectangular circulation shown in Figure 3.10. where the origin is at the arbitrary point *(x,y,z)*
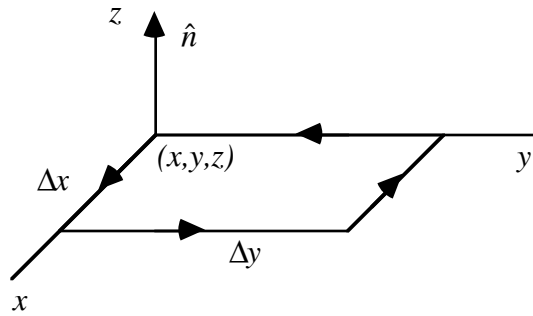

Figure 6.6
The curl of the rectangular current density shown will point along the z axis.

The sides of the rectangular path have lengths Δx and Δy respectively giving the curl in terms of the four path segments. In the limit, F can be taken as approximately constant along the sides:
$\hat{n} \cdot curl \, \vec{F} =$

$$\lim_{\Delta x \Delta y \to 0} \frac{1}{\Delta x \Delta y}\left(F_x(x,y,z)\Delta x + F_y(x+\Delta x,y,z)\Delta y - F_x(x,y+\Delta y,z)\Delta x - F_y(x,y,z)\Delta y \right)$$

where $F_y(x+\Delta x,y,z)$ means the y component of F evaluated at the point $(x+\Delta x,y,z)$. Rearranging terms we are left with

$$\hat{n} \cdot curl \, \vec{F} = \lim_{\Delta x \Delta y \to 0}\left(\frac{F_y(x+\Delta x,y,z) - F_y(x,y,z)}{\Delta x} - \frac{F_x(x,y+\Delta y,z) - F_x(x,y,z)}{\Delta y} \right)$$

which is the definition of the partial derivatives.

$$\hat{n} \cdot curl\ \vec{F} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}. \tag{3.26}$$

Note that the other components of the curl follow from an equally horrible calculation, that will not be reproduced here. Suffice it to say that the curl can be written quite compactly and elegantly by using the del operator and the cross product.

$$\nabla \times \vec{F}\ =\ curl\ \vec{F} \tag{3.27}$$

which can be evaluated using the determinant:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{3.28}$$

Equation (3.28) is the equation used to calculate the curl, but to interpret it geometrically, we use the definition given in equation (3.25). The direction of $\hat{n}$ is defined in a right hand sense by curling the fingers of your right hand in the direction of the path C. The curl, being a circulation density, has the same interpretation as the circulation except that the curl has a unique direction given by the right hand rule.

A simple example is of a conservative field which must have zero circulation, so that

$$\oint_C \vec{F} \bullet d\vec{\ell}\ =\ 0\ \ \text{for any path}\ \Rightarrow$$

$$\nabla \times \vec{F}\ =\ 0 \tag{3.29}$$

becomes an alternate definition of a conservative field. With the introduction of the curl the mathematics actually becomes easier. For example, for any scalar field f the identity

$$\nabla \times \nabla f = 0 \tag{3.30}$$

holds. It does not matter what scalar field we use because (6–34) is an identity which can be proven without knowing any details about f. This means that when we make the statement (3.29), it will automatically be true if

$$\vec{F}\ =\ -\ \nabla f. \tag{3.31}$$

So we can see that saying that the curl is zero automatically implies that the vector field is given by the gradient of a scalar field.

Because the definition of the curl in equation (3.25) does not refer to any particular coordinate system, it allows us to find the direction of the curl without having to calculate it using equation (3.28). It thus, allows us to avoid the messy details associated with any given coordinate system. When the curl is not zero, the direction can be found using the right hand rule. Place a paddle wheel in a flow and curl your fingers in the direction the paddle wheel rotates. Your thumb will give the direction of the curl. In (a) and (b) of Figure 3.11 the curl points out of the page and is

perpendicular to the page; however, for figure (c) the curl in into the page.

$$v \propto r^2 \qquad\qquad v \propto r^{-2}$$
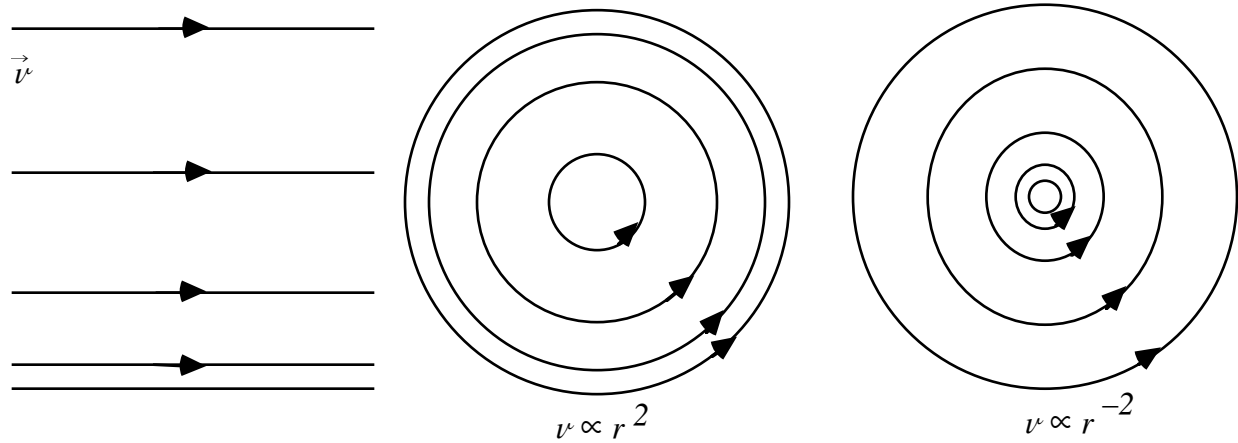
(a)   (b)   (c)

Figure 6.7
In (a) and (b) a paddle wheel will rotate
counter-clockwise giving a curl out of the page.
In (c) where the speed falls off as r -2, the curl is into the page.

The only cylindrical flow which does not have a curl is the inverse r flow where $v \propto 1/r$. All

other cylindrical flows have a curl. Generally for cylindrical flows where $v \propto r^\gamma$ if $\gamma > -1$, then the curl can be found by curling your fingers in the direction of the flow, and if
$\gamma < -1$, then the curl is found by curling your fingers in the opposite direction of the flow.

3.4 The Magnus Effect: Throwing A Curve Ball

Another illuminating example is the irrotational flow of an incompressible fluid around a sphere. This example is of interest to baseball pitchers who throw curve balls. Although air is not an incompressible fluid, to approximate air as incompressible sometimes works amazingly well. Conservation of mass means there must be no flux through a closed surface which means the divergence is zero. Conservation of angular momentum means the circulation must be zero; hence, the curl is zero. The equations for the velocity are

$$\nabla \cdot \vec{v} = 0$$

$$\nabla \times \vec{v} = 0. \qquad\qquad (3.32)$$

Two possible solutions which satisfy both of these equations are shown in Figure 3.12 where the circulating flow in Figure 3.12(b) is an inverse r field so curl is zero.
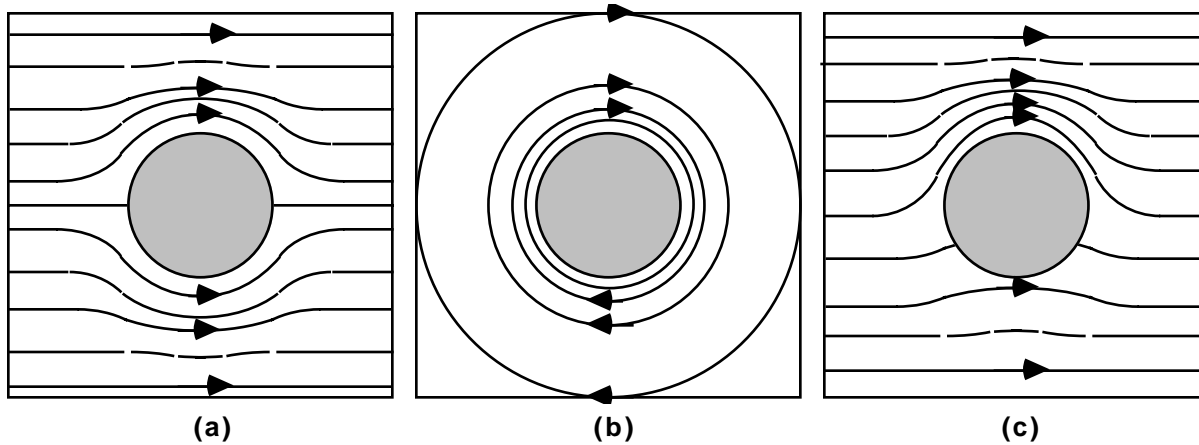
Figure 3.12
(a) and (b) are both solutions to Equations (3.32).
(b) is an inverse r field so the curl is zero.
(c) is the superposition of the two solutions.

Because the equations determining $\vec{v}$, (3.32), are linear in $\vec{v}$, the sum of any two solutions is also a solution. In general when ρ is not constant, the equations become highly nonlinear, and superposition does not hold. If we use superposition to add the solution in Figure 3.12(a) to that in (b) we obtain the flow shown in (c). Note that the velocity field lines in figure (c) are much denser above the sphere than below, giving rise to a large Bernoulli lifting force. This effect can cause a baseball to curve, and hopefully (from the pitcher's viewpoint) confound the batter.

The solution (b) is approximated by putting spin on the baseball, and (a) is caused by air flowing past the baseball as it hurtles towards the batter. This effect where a spinning ball experiences a Bernoulli force perpendicular to the flow is known as the Magnus effect, and happens in tennis and other sports besides baseball. The classic curve ball is actually thrown such that the Bernoulli force causes the ball to drop faster than gravity and the ball curves down as shown in Figure 3.13.
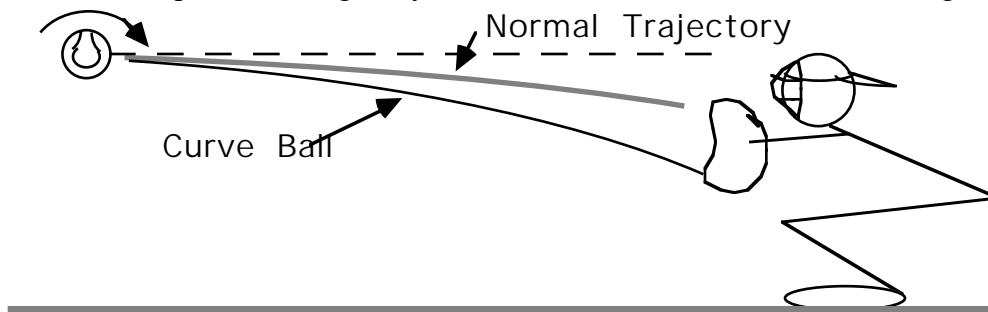

Figure 3.13
The classic curve ball is thrown causing it to curve down,
such that, it drops faster than expected.

 With a real baseball the picture is not this ideal. There are nonlinearities that play an important role. Windtunnel tests on spinning baseballs show that the baseball actually causes the air to deflect upwards as it passes the ball causing the ball to be driven downwards. Turbulence can also introduce anomalies and for smooth spheres can actually cause a reverse magnus effect. Rough spheres don't have a reverse magnus effect so the stitches on a baseball play an important role in making sure that the curve is in the correct direction. By using spin and turbulence in various ways it is also possible to get the ball to suddenly break in unusual directions. The effect is magnified if "foreign substances", i.e., gum or saliva are used and have been banned by the all the governing sports organizations in amature and professional baseball.

Not everyone throws a classic curve. Don Drysdale had a notorious side–arm curve that spun the ball about a vertical axis and caused the ball to deflect sideways.

3.5 Vorticity

Because a rotation in a fluid is often associated with a vortex, the curl of the velocity is called the vorticity, $\vec{\Omega}$.

$$\vec{\Omega} \; = \; \nabla \times \vec{v} \qquad\qquad \textit{vorticity} \qquad (3.33)$$

The vorticity possesses several peculiar properties. The first is that since the vorticity is a pure solenoidal field, it possesses no flux. This follows from another vector identity that is always true for any vector field.

$$\nabla \bullet (\nabla \times \vec{v}) \; = \; 0 \qquad\qquad (3.34)$$

Thus,

$$\nabla \cdot \vec{\Omega} \; = \; 0 \qquad\qquad (3.35)$$

always. This means vortex lines must circulate around and close upon themselves for if they were ever to end, they would cause a divergence. This property makes vortex lines very similar to magnetic field lines as we shall see later.

At first vorticity appears to be another complicated construction to learn, but after some thought you will find that it can make impossibly complicated problems tractable. Take for example the case of smoke rings. A smoke ring is produced by blowing a short puff of air and smoke through a circular aperture. A device for producing smoke rings can be made by cutting out one end of a large cylindrical can, stretching a sheet of rubber over the open end, and cutting a much smaller hole in the other end of the can as shown in Figure 6.8

Figure 3.14
A device for producing smoke rings. The ring is stable and can blow out a candle several feet away.

A sharp blow delivered to the rubber end cap will produce a ring of air that will be stable due to its angular momentum, and will be strong enough to blow out a candle a good distance away.

The smoke ring forms when a puff of air mixed with smoke passes through the hole. The air next to the edge of the hole is slowed down by drag of the surface of the hole. This drag causes the faster moving center air to roll around causing circulation. The trajectory of any given smoke particle as it loops around and flows along is actually quite complicated as shown in Figure 6.9.
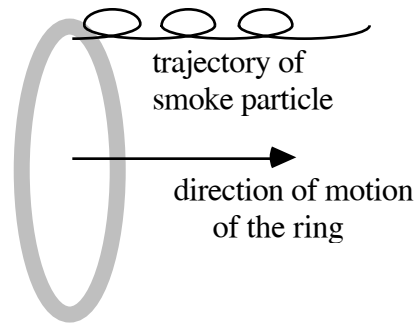
Figure 6.9
The actual trajectory of a smoke particle is a complicated looping trajectory.

By comparison, the structure of the vortex lines is much simpler than the velocity field. A cross section of a smoke ring is shown is Figure 6.10. The velocity field circulates around giving rise to vortex lines coming out of the page along the top, and into the page at the bottom. Since the vortex lines can have no ends, they must form closed circles mimicking the shape of the smoke ring itself.
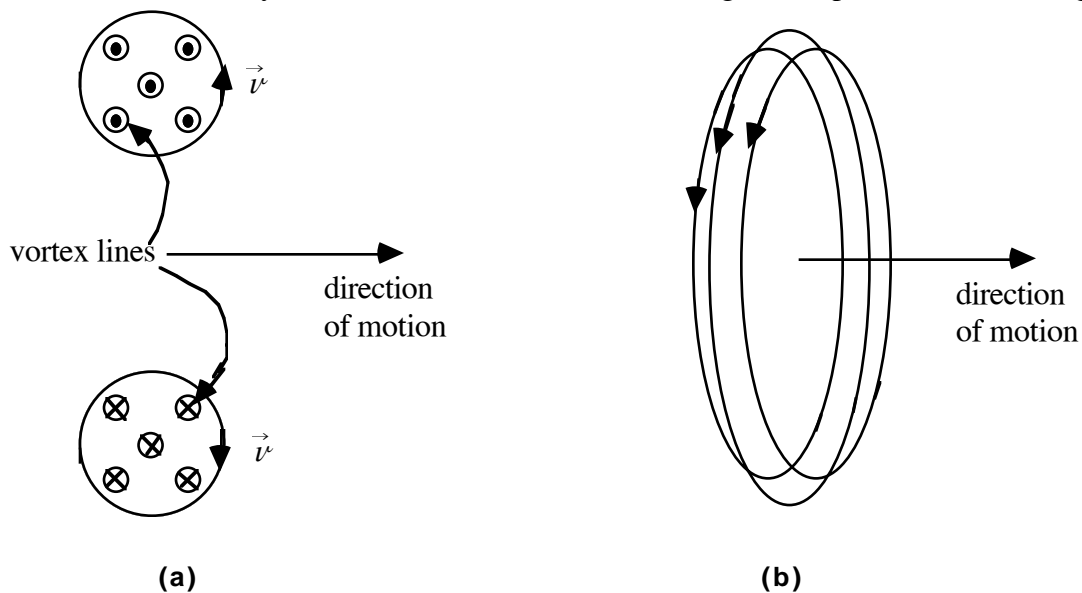

(a) **(b)**

Figure 6.10
The vortex lines given by the right hand rule from the velocity form closed loops that mimic the shape of the smoke ring and travel with the smoke ring. (a) A cross sectional view showing the velocity and vortex lines. (b) A view showing the complete vortex lines.

Note that there is no flow along vortex lines. These vortex lines represent lines of curl for the velocity field which is actually doing loop–the–loops. The vortex lines are much easier to visualize than the velocity, and thus, they give us a means to visualize what is going on in the smoke ring. In actuality, the structure of a smoke ring is more complicated as shown in Figure 3.15 in which the ring is not a true ring at all, but is a tightly wound toroidal spiral.
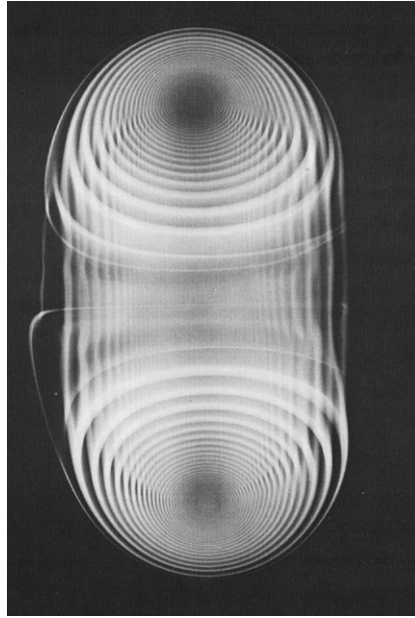
Figure 3.16
The structure of a smoke ring is actually a tightly wound toroidal spiral

3.6 Inviscid Flows, The Theory of "Dry Water".

We have treated conservation of mass and conservation of energy but we have not formally considered conservation of momentum nor angular momentum. We will not attempt to write the equations for momentum conservation owing to their mathematically abstract nature. We can see why they are so abstract. The momentum density $\rho\vec{v}$ is a vector, whereas for mass conservation, the mass density is a scalar. Mass conservation is

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}.\qquad(3.36)$$

To conserve momentum, the divergence of the momentum current density must give a vector to be equal to a vector momentum density time derivative. We have not yet seen how a dot product can result in a vector (remember, it's called the scalar product)! Thus, the momentum current density must be a higher rank object than is a vector. The momentum current density is actually a second rank tensor. Tensors are a bit above the level of this course so we will not write the conservation equations for momentum or angular momentum. Conservation of momentum and angular momentum are still very important, and we will discuss some implications.

For an inviscid incompressible fluid two of the equations that govern the velocity field are:

$$\nabla \cdot \vec{v} = 0$$

$$\nabla \times \vec{v} = \vec{\Omega}\qquad(3.37)$$

A problem arises because inviscid fluids do not have any friction, and thus, cannot support tangential forces within the fluid. A piece of an inviscid fluid only acts on another piece by applying a pressure. If there are no tangential forces within the fluid, then there is no way a piece of the fluid can place a force tangent to a surface and hence place a torque on another piece of the

fluid, i.e., within the fluid angular momentum is constant. From Kelvin's theorem, the circulation is thus a constant:

$$\oint \vec{J} \cdot \vec{dl} = 2\pi\, L/V = \text{const.} \qquad (3.38)$$

This means that at any time if the angular momentum in the fluid is zero, it must always be zero, and so too, must the vorticity always be zero.

This gives us a fatal result known as Lagrange's theorem.

Lagrange's Theorem

In an inviscid fluid if at any time $\vec{\Omega} = 0$, then $\vec{\Omega}$ must be zero for all time.

Lagrange's theorem points out just how bad the inviscid approximation is. There can be no vorticity unless the vorticity has been present right from the very beginning. Every morning when we stir our coffee we violate Lagrange's theorem. There is no way to stir an inviscid fluid. Lagrange's theorem lead John von Neumann to dub the theory of inviscid fluids as the study of "dry water".

3.7 Gradient, Divergence and Curl in Polar Coordinate Systems.

When the gradient, divergence, or curl must be explicitly calculated in a polar coordinate system, you cannot just write del in polar coordinates and perform the appropriate vector product. The entire operation must be transformed from its definition, e.g., Equation (6–1), div $\vec{J}$ = sdo6(A(lim,ς→0)) f(1,ς) $\oint$sdo8(S) $\vec{J} \cdot d\vec{A}$, or Equation (6–25). In Table 6.1 the gradient, divergence, and curl operations are written in Cartesian and polar coordinate systems

Table 6.1

Gradient, Divergence, & Curl in Polar Coordinates

Cylindrical (r,θ,z)

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial f}{\partial r} + \hat{k}\frac{\partial f}{\partial z} \qquad\qquad \nabla \cdot \vec{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \vec{F} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{k} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$$

Spherical (r,θ,ϕ)

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial f}{\partial r} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial f}{\partial \phi} \qquad \nabla\cdot\vec{F} = \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial\left(r^2 F_r\right)}{\partial r} + r\frac{\partial\sin\theta F_\theta}{\partial\theta} + r\frac{\partial F_\phi}{\partial\phi}\right)$$

$$\nabla\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta\,F_z \end{vmatrix}$$

An example shows why there are factors of r and $\sin\theta$ in strange places.

Example 3.3

A cylindrical flow has a vector field $\vec{F}$ given by

$$\vec{F}(r,\theta,z) = \frac{k}{r}\hat{\theta}.$$

Find the curl of this vector field.

Solution: We already know from previous discussions that an inverse r field should have zero curl except at the origin. Now let's prove it. Since this field is given in cylindrical coordinates, we use the cylindrical form for the curl:

$$\nabla\times\vec{F} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{k} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ 0 & r\dfrac{k}{r} & 0 \end{vmatrix}$$

$$\nabla\times\vec{F} = \frac{\hat{k}}{r}\left(\frac{\partial k}{\partial r}\right) = 0$$

Note that in the last line F_θ is multiplied by r before the derivative is taken. For an inverse r field this leaves the derivative of a constant k which is always zero.

3.8 Viscous Flows

Lagrange's theorem for inviscid fluids points out that circulation in a fluid is created by frictional nonconservative forces, i.e., friction is the source of circulation. Without friction fluids cannot be

stirred. For viscous flow $\nabla \times \vec{v}$ is seldom zero because any surface in contact with the fluid will exert a drag upon the fluid. The drag produces a no–slip condition at the surface, i.e., the fluid layer in contact with the surface must be at rest with respect to the fluid. This no slip condition is why fine dust does not get blown off fan blades, and also, why that fine layer of dirt on your car is not blown off by the wind as you drive your can down the freeway. Unfortunately, that dirt must be washed off by scrubbing. The no–slip condition at the surface is responsible for introducing vorticity into the fluid as the next example illustrates.

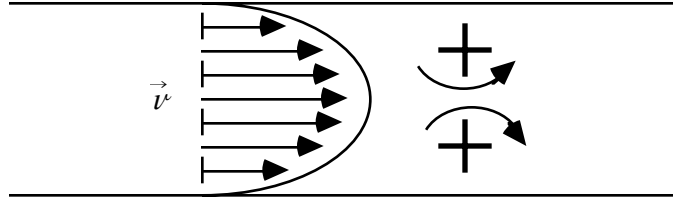Consider the flow of a fluid in a pipe as shown in cross section in Figure 3.17.

Figure 3.17
For a fluid flowing in a pipe, the no-slip condition
at the surface introduces vorticity into the flow.

The layer of fluid in contact with the pipe wall must be at rest. As we move towards the center of the pipe, each successive layer picks up speed. The flow at the center is moving the fastest.

The friction force in a fluid is proportional to the viscosity η, i.e., the more viscous a fluid is, the greater the frictional forces. For a fluid flowing in a pipe of radius R consider a cylinder of radius $r < R$. The friction force on this cylinder will be tangent to the $2\pi r \ell$ surface. It will cause a velocity gradient dv/dr in the pipe as shown in Figure 3.17. The friction force on the cylinder is thus:
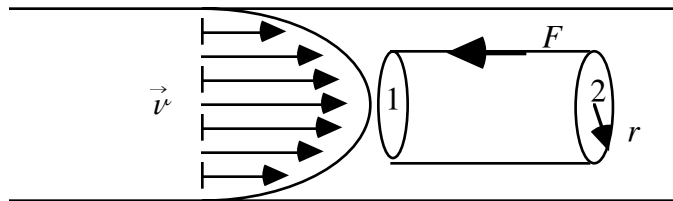
$$F_f = \eta A \frac{dv}{dr} \qquad (3.39)$$

Figure 3.18
Friction will drag on the fluid opposite to the flow.

In the fluid conservation of mass requires that along a streamline,

$$\rho v_1 A = \rho v_2 A \Rightarrow v_1 = v_2 \qquad (3.40)$$

There is another force which balances the friction force. This force must be due to drop in pressure along the length of our cylinder and this force must balance the friction force:

$$(P_1 - P_2)\pi r^2 = \eta(2\pi r\ell)\frac{dv}{dr} \qquad (3.41)$$

Separating variables and integrating:

$$(P_1 - P_2)\int_R^r rdr = \eta(2\ell)\int_0^v dv$$

$$v = \frac{\Delta P}{4\eta\ell}\left(R^2 - r^2\right) \qquad\qquad (3.42)$$

which shows that the velocity distribution is parabolic. We can calculate the mass flux in a pipe.

$$\int \rho\vec{v}\cdot d\vec{A} \;=\; \int_0^R \rho v(2\pi r dr)$$

where $dA = 2\pi r\, dr$ is the area of ring of thickness dr and circumference $2\pi r$. Using equation (3.42):

$$\int \rho\vec{v}\cdot d\vec{A} \;=\; \int_0^R \rho\frac{\Delta P}{4\eta\ell}\left(R^2 - r^2\right)(2\pi r dr) \;=\; \frac{\rho\Delta P}{8\eta\ell}R^4$$

This is known as Poiseuille's formula.   Note that the friction force causes a drop in pressure along the pipe, so if we were to place pressure measuring tubes in the pipe as in Figure 3.19, we would see the height of the fluid in the tubes decrease.
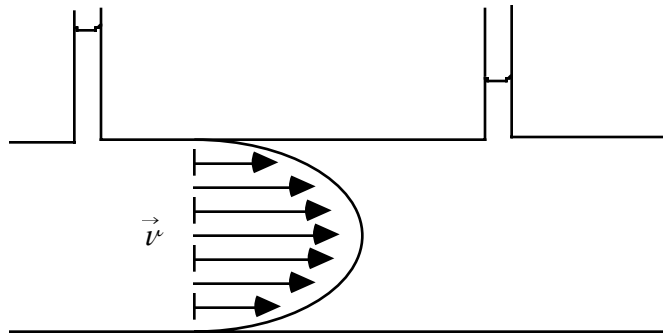

Figure 3.19
The pressure drops due to the friction force.

By considering the paddle wheels shown, the flow has vorticity out of the page in the top half and into the page in the bottom half. The vortex lines form concentric circles inside the pipe as shown in Figure 3.20.
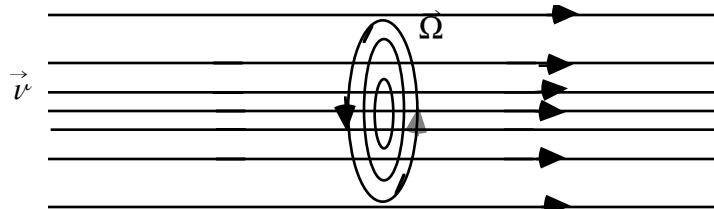

Figure 3.20
The vortex   lines form concentric circles with most of the vorticity near the pipe wall.

$$\nabla \times \vec{v} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{k} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ 0 & 0 & v \end{vmatrix} = -\,\hat{\theta}\,\frac{dv}{dr}$$

Note that by symmetry, v cannot vary with θ so the $dv/d\theta$ term is zero. Using equation (3.41)

$$\nabla \times \vec{v} = \frac{r\Delta P}{2\eta\ell}$$

Notice that the vorticity is greatest at the largest value of r which is at the pipe wall where friction effects are greatest. We have learned that friction causes vorticity.

3.9 Turbulence

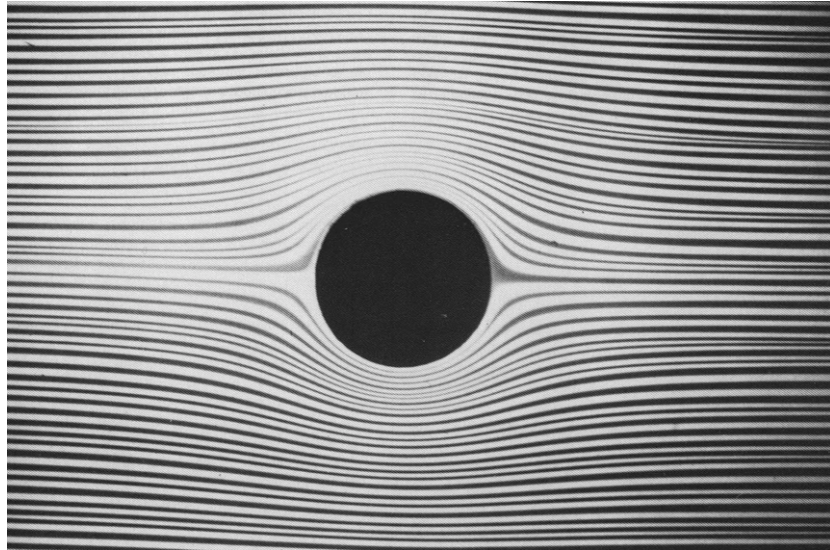For laminar flows there is tendency for the vorticity to be greatest near a surface where the drag is greatest;

Figure 3.21
Laminar flow past a cylinder for neglible Reynolds number.

however, for fast flows the vorticity can be large a good distance downstream from a surface as illustrated in Figure 3.22. In this case, the fluid cannot bend around the cylinder quickly enough so it is "torn" away from the surface sending the vorticity downstream. When this happens the flow becomes turbulent..
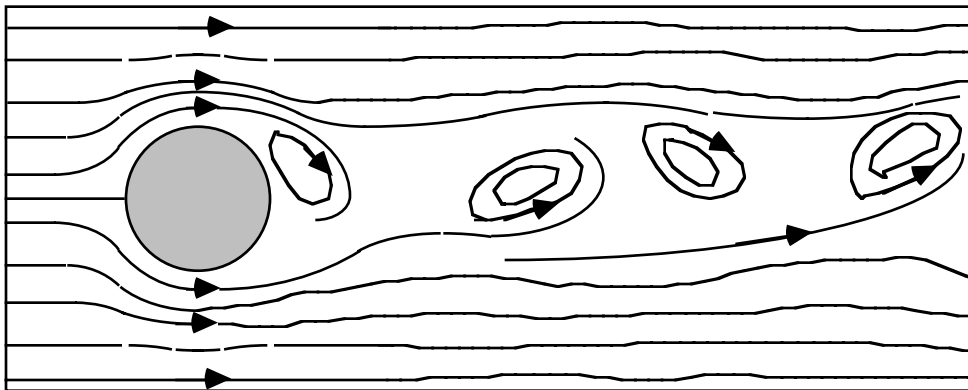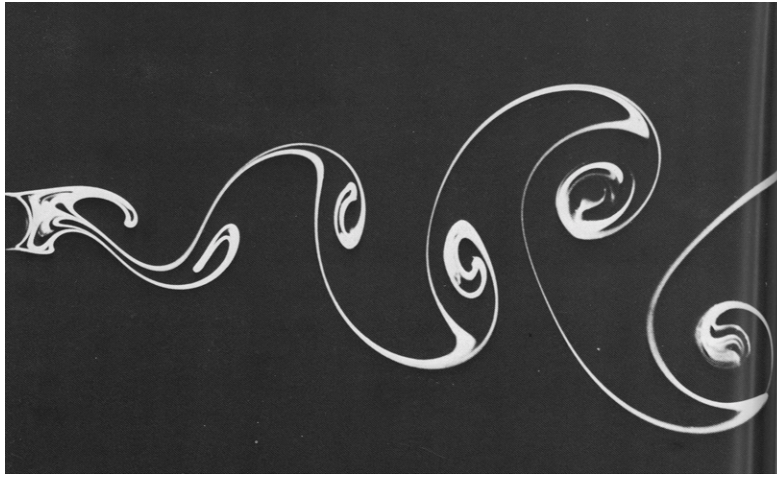
Figure 3.22
For large velocities the vorticity tears away from the surface
and flows downstream forming a turbulent wake (*R* = 140).

When vortices tear away from a body it is called *vortex shedding*. Turbulence is often described by the *Reynolds number*. If *D* is the diameter of the object, v is the velocity of the flow not in the immediate viscinity of the object, then

$$R = \frac{\rho D v}{\eta} \qquad\qquad \textit{Reynolds number} \qquad (3.43)$$

For some objects such as flat plates the flow becomes turbulent at a relatively low reynolds number. If you look at the Tacoma Narrows bridge, the flat girders on the sides meant that vortex shedding occurs in relatively low reynolds number flows. Vortex shedding gives rise to a periodic force on the bridge during high winds and led to the bridge's nickname, *Galloping Gertie*.
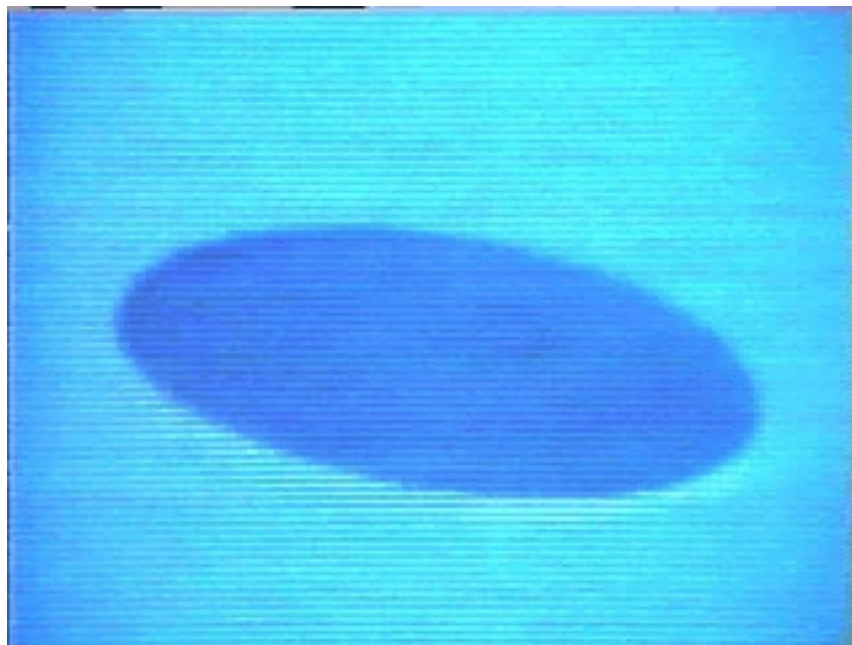

Figure 3.23
Tacoma Narrows bridge collapse. The periodic force on the
bridge was caused by vortex shedding.

Vortex shedding did not directly cause the collapse of the bridge on July 1, 1940, but it contributed to it. At first the bridge was in a purely linear osciallatory mode. Then the cables slipped at the point where they pass over the tower. This put the bridge in a torsional mode and the twisting ultimately lead to its distruction. It has been theorized that had the cables not slipped, the bridge might well have survived. It is also clear that had the support girder not been a solid piece, but had allowed air to flow through so that the vortex shedding had been much less, it would never have been galloping in the first place. The bridge was eventually rebuilt in exactly this way.

One of the most dramatic places you can see vortices and turbulence is in the planet Jupiter. The Great Red Spot, theorized to be a persistent storm, is surrounded by turbulence and vorticies.

(a)

(b)
Figure 3.24
(a) Great Red Spot on Jupiter. (b) Giant Dark Spot on Neptune.
Both are believed to be storms, but Neptune's atmosphere lacks friction.

It is interesting to compare this storm to the giant storm on Neptune. Neptune is much colder than Jupiter, so there is much less friction in its atomsphere. The Giant Dark Spot on Neptune is devoid of turbulence and vorticity.