# What is the Purpose of Neo-Logicism?

Philip A. Ebert & Marcus Rossberg
*Arché* – The AHRC Centre for the Philosophy of
Logic, Language, Mathematics, and Mind
University of St Andrews

October 19, 2006

## Introduction

This paper introduces and evaluates two contemporary approaches of *neo-logicism*. Our aim is to highlight the differences between these two neo-logicist programmes and clarify what each projects attempts to achieve. To this end, we first introduce the programme of the *Scottish* school – as defended by Bob Hale and Crispin Wright[1] which we believe to be a form of epistemic foundationalism in which logic is intended to play a foundational rôle in resolving specific epistemic challenges, such as our knowledge of arithmetic and analysis. We contrast this with what we call the *Stanford/Edmonton* school whose project is put forth and defended by Bernard Linsky and Edward N. Zalta.[2] This latter approach is a form of axiomatic metaphysics,

We would like to thank Bernard Linsky and Edward Zalta for many interesting discussion and helpful comments on earlier drafts of this paper. Earlier versions of this paper were presented at the British Logic Colloquium meeting in Bristol, 2005, the conference of the British Society for the Philosophy of Science in Southampton, 2006, and in the *Arché* philosophy of mathematics research seminar in St Andrews; we are indebted to the audiences. We would also like to thank Patrick Greenough, Hannes Leitgeb, Øystein Linnebo, Paul McCallion, Stewart Shapiro, and Crispin Wright for helpful comments and criticisms, and to Lindsay Duffield for literary advice. We are grateful for the funding of our research by the AHRC and the Leverhulme Trust.

[1]See (Wright, 1983), (Hale, 1987), (Hale and Wright, 2001); see also (MacBride, 2003).

[2]See (Linsky and Zalta, 1995), (Zalta, 2000), (Linsky, 2005), and (Linsky and Zalta, 2006).

which, if successful, achieves a different aim. Having offered an outline of the general outlook of these two schools we discuss what Frege took to be the *purpose* of his logicism. In the light of this discussion we aim to highlight why we think that the *Scottish* school is not only closer to Frege's own project but also draw attention to some inherent shortcomings of what can be achieved if one pursues the programme of the *Stanford/Edmonton* school.

# 1 Neo-Logicism: Two Schools

In this section we outline two schools of thought in the philosophy of mathematics, both of which claim logicist roots and consider themselves *neo-logicist* programmes. We focus on how, in general, the two schools attempt to recover arithmetic, analysis and set theory, or even the whole of mathematics. We first discuss the *Scottish* school, which is commonly known as the *Neo-Fregean* programme or *Abstractionism.*

## 1.1 The Scottish School

The neo-logicism of the Scottish school is to be considered a form of epistemic foundationalism. It aims to explain knowledge of arithmetic and possibly the whole of classical mathematics by appeal to what is called the context principle, certain basic principles – so-called *abstraction principles* – and standard second-order logic. It is with this trinity that they aim to resolve Benacerraf's well-known dilemma concerning mathematical knowledge by offering a platonist route to mathematical knowledge.

Roughly speaking, the function of the context principle is to guarantee that mathematical singular terms indeed refer, and so refer to abstract objects. The theory of abstraction principles aims firstly, to introduce mathematical singular terms and secondly, to offer a "epistemically tractable" way of how a subject can come to know basic mathematical principles. Lastly, second-order logic is adopted in order to generate the theorems of mathematics.

In this brief exposition we focus on the second and third component – abstraction principles and second-order logic – underlying the approach of the *Scottish* school. Generally speaking, this approach is a piecemeal approach to mathematics. That is, it is concerned with specific abstraction principles – in the case of arithmetic the so-called *Hume's Principle* – and evaluates whether these abstraction principles qualify as "epistemically tractable" principles that can found knowledge of arithmetic. In order to extend mathematical knowledge to other parts of mathematics, say, analysis or set theory, a similar investigation has to take place concerning other abstraction principles that introduce the notion of a real number or set. Let us briefly explain how, by using *Hume's Principle*, the Neo-Fregean story is meant to go for arithmetic. *Hume's Principle* (HP) can be formulated as follows:

$$\text{(HP)} \qquad \forall F \forall G(Nx : Fx = Nx : Gx \equiv F \approx G)$$

where '$Nx : Fx$' stands for 'the (cardinal) number of the $F$s' and '$\approx$' expresses a one-to-one correspondence.[3] Thus, the principle claims that the cardinal number belonging to the concept $F$ is identical to the cardinal number belonging to the concept $G$ if, and only if, there is a one-to-one correspondence between the objects falling under $F$ and those falling under $G$.

This abstraction principle, so the Neo-Fregean claims, can be put forth as an implicit definition. That is, the intention is to stipulate this principle as true[4] which thereby introduces a new expression: '$Nx :\ x$'. Assuming for the moment that this idea is legitimate and such stipulations are knowledge-conferring, the question arises how we can acquire knowledge of the right-hand side. This is exactly where the third claim gets its grip, and it is also why Neo-Fregeans consider themselves neo-*logicists*. For the claim is that it is a matter of logic that there are true instances of the right-hand side of HP. The Neo-Fregean argues that in order to see this one just has to note that it is a logical truth that the instances of the concept *being non-self-identical* can trivially be put into a one-to-one correlation with themselves. This true instance of the right-hand side of HP, suffices – assuming that HP is true – to yield a true identity statement about numbers on the left hand side. More formally this can be expressed as follows:

[3]The claim for the existence of a one-to-one correspondence can be formulated in purely (second-order) logical vocabulary. In full detail Hume's Principle is the following statement:

$$\forall F \forall G[Nx : Fx = Nx : Gx \equiv \exists R(\forall x[Fx \supset \exists y(Gy \wedge Rxy \wedge \\ \forall z(Gz \wedge Rxz \supset z = y))] \wedge \forall y[Gy \supset \exists x(Fx \wedge Rxy \wedge \forall z(Fz \wedge Rzy \supset z = x))])]$$

[4]The Neo-Fregean also grants that not every stipulation is successful – more on this below.

*Step 1*

$(Nx : x \neq x = Nx : x \neq x) \equiv (x \neq x) \approx (x \neq x)$

The right-hand side of this statement is a logical truth. Assuming the truth of HP we can discharge the right-hand side and derive:

*Step 2*

$(Nx : x \neq x = Nx : x \neq x)$

Assuming that number-terms are singular terms we can, by adopting the context principle, infer the claim that there is an object to which the singular term refers. We can thus existentially quantify into this formula (reading the existential quantifier objectually and ontologically committing):

*Step 3*

$\exists y(y = Nx : x \neq x)$

In addition, having the formal result in place that the second-order version of the Peano-Dedekind axioms for arithmetic can be deduced in second-order logic from Hume's Principle – a result which is called *Frege's Theorem*[5] – the Neo-Fregean can justifiably claim that knowledge of logic leads, merely through the stipulation of HP, to knowledge about numbers as objects and to knowledge of arithmetic. Since, the proponent of the Scottish school regards knowledge of arithmetic as a priori, he also embraces the additional claim that basic mathematical principles can be known a priori and that reasoning within second-order logic (which is needed to establish Frege's Theorem) preserves the epistemic status of the a priori knowable abstraction principle.

[5]This theorem was first explicitly noted by Parsons, in his (Parsons, 1965) and later independently "rediscovered" in (Wright, 1983), pp. 158–169. More recent presentations of the proof can be found in (Boolos, 1987) (discursive), (Boolos, 1990b) (rigourous), (Boolos, 1995), and (Boolos, 1996). Note that even a weaker version of Hume's Principle – *Finite Hume* – suffices for this derivation; see (Heck, 1997). Second-order logic is required for this proof. A relatively moderate portion of second-order logic suffices, however: $\Pi^1_1$ comprehension is enough. (Linnebo, 2004) has shown that Frege's Theorem cannot be proven in predicative second-order logic. (Heck, 2006) provides a proof that ramified second-order logic suffices. For formulations of the respective fragments of second-order logic see (Church, 1956), §58, and (Shapiro, 1991), chapter 3. For a general overview of the technical details of Fregean arithmetic, see (Burgess, 2005).

According to the Scottish school, some, but not all, abstraction principles have the status of being meaning-constitutive of the expression it is meant to introduce. It is claimed that this rôle as a meaning-constitutive principle endows them with an epistemic dimension: namely, in the best case, it provides part of the justification for holding these abstraction principles as true.[6] However, the Neo-Fregeans do not regard abstraction principles *per se* as having this special epistemic status, and also do not insist that they are *logical* principles, but merely that they are *analytically* true, since they are meaning-constituting.[7]

In this respect they depart from Frege's logicism. Frege, at least initially, regarded the ill-fated Basic Law V – which would be construed as an abstraction principle in the Neo-Fregean programme – as logical.

Nevertheless, logic does play an important epistemic rôle within the Scottish school. Since it is knowledge of logic that is needed to justifiably discharge the right-hand side of Hume's Principle, the Neo-Fregean believes that logical knowledge, plus knowledge of certain abstraction principles, suffices to account for mathematical knowledge.

## 1.2 The Stanford/Edmonton School

Bernard Linsky and Edward Zalta's approach to neo-logicism is based on so-called *Object Theory* (OT), a theory that was first introduced in (Zalta, 1983). The higher-order modal[8] theory contains some interesting features. One of them is that it does not have one, but two modes of predication. The first is the ordinary form of predication, which is referred to as *exemplification*. This predication is formalised as, for example, '$Fa$' and read '$a$ exemplifies $F$'. The second mode of predication is called

[6]The account is more complicated than this. Hale and Wright have also defended the idea that Hume's Principle (and presumably other abstraction principles) do not have *direct* ontological commitments which makes them particularly suitable for direct stipulations; see (Hale and Wright, 2000). For criticism of their approach using meaning-constituting principles see (Ebert, 2005).

[7]Exactly, what makes an abstraction principle a meaning-constituting principle which can underwrite knowledge is a notoriously hard questions and finding the right criteria is an ongoing research project.

[8]Linsky and Zalta typically claim that their approach to neo-logicism is non-modal, see e.g. (Linsky and Zalta, 2006, p. 88). This is not entirely correct, however: the possibility operator '$\Diamond$' occurs in the definition of the abstractness predicate, and the necessity operator '$\Box$' occurs in the definition of identity between abstracta – see below.

'*encoding*'. To distinguish the two, the order of the predicate letter and the name is switched around: '$aF$', read '$a$ encodes $F$'.

While any object whatsoever can exemplify properties, only *abstract* objects encode properties. Moreover, encoding a property does not entail exemplifying it: '$xF \nvdash Fx$', but encoding properties entails being abstract. A predicate '$A!$' stands for 'is abstract'. It is not taken as primitive, however, but defined with the help of the primitive predicate 'is concrete', '$E!$'. Being an ordinary object, '$O!$' is also defined:

$$A!x \ =_{df} \ \neg\Diamond E!x$$

$$O!x \ =_{df} \ \Diamond E!x$$

Abstract objects are those, that are not possibly concrete; and ordinary objects are those that are possibly concrete. The notion of an ordinary object allows Zalta in other projects to propose a theory of merely possible and also of fictional objects.[9] This, however, will be of no concern here.

Abstract objects enter OT via a comprehension schema for abstract objects (OC):

(OC) $\exists x(A!x \wedge \forall F(xF \equiv \varphi))$, where '$x$' is not free in $\varphi$

This axiom schema asserts that for any formula $\varphi$ (minding the restriction on free variables), there exists an abstract object that encodes all and only those properties $F$ that satisfy $\varphi$; or, expressed in a more sloppy way, for any collection of properties, there is an abstract object encoding them.

OC guarantees that any (abstract) object that is described by an expression of the form '$\imath x(A!x \wedge \forall F(xF \equiv \varphi))$' exists (where there is no free '$x$' in $\varphi$). So, there is, for example, an abstract object that encodes the property of *being Zalta* (or *being*

[9]OT was originally developed as a formal theory of fictional, abstract, and intensional objects inspired by the work of Meinong's student Ernst Mally: see (Zalta, 1983).

*identical to Zalta*):[10]

$$\imath x(A!x \wedge \forall F(xF \equiv \ \forall y(Fy \equiv y = \text{Zalta})))$$

An abstract object that encodes *being either Linsky or Zalta*:

$$\imath x(A!x \wedge \forall F(xF \equiv \ \forall y(Fy \equiv (y = \text{Linsky} \vee y = \text{Zalta}))))$$

An abstract object that encodes all the properties Zalta has:

$$\imath x(A!x \wedge \forall F(xF \equiv F(\text{Zalta})))$$

Note that Zalta himself is not identical to any of these objects (since he is concrete and not abstract). He exemplifies, rather than encodes the respective properties. Sherlock Holmes, on the other hand, is an abstract object, viz. the abstract object that encodes all the properties that (the fictional character) Sherlock Holmes has according to the stories by Arthur Conan Doyle. (The devise for fomalising this will be introduced below in the discussion of mathematical theories.) There is also an abstract object that encodes being a square circle:

$$\imath x(A!x \wedge \forall F(xF \equiv \forall y(Fy \equiv (y \text{ is a circle} \wedge y \text{ is square}))))$$

Moreover, there is an abstract object that encodes being a set that contains all and only those sets that do not contain themselves. In order to avoid inconsistency, the second-order comprehension schema for predicates:[11]

$$\exists X \forall x(Xx \equiv \varphi(x)), \text{ where } X \text{ is not free in } \varphi$$

[10] All of the following examples are, of course, dependent on the English names and predicates entering the formal language in some way. How this is done for mathematical terms is described below. Moreover, identity is a defined notion in OT. So, strictly speaking, one would have to specify that the identity relation referred to in our examples is identity between concrete, rather than abstract, objects.

[11] For simplicity's sake we only give the comprehension schema for monadic second-order variables. The restrictions apply in the same way for the general formulation for polyadic variables. We here use the common formulation of second-order logic introduced in (Church, 1956); the current bible of second-order logic is (Shapiro, 1991). Linsky and Zalta use an equivalent formulation that employs $\lambda$-conversion, which requires an analogous restriction.

(and likewise the third- and higher-order comprehension schemata) has to be restricted. It has to be demanded of the standardly unrestricted second-order comprehension schema that $\varphi$ does not contain any descriptions or "encoding subformulae". So, the fully explicit formulation of $\varphi$ must not contain subformulae of the form $\ulcorner xY \urcorner$, i.e. subformulae containing the encoding mode of predication.[12]

Identity between abstracta, '$=_A$', is a defined relation. Two abstract objects are identical if, and only if, they necessarily encode the same properties:

$$x =_A y \ =_{df} \ A!x \wedge A!y \wedge \Box\forall F(xF \equiv yF)$$

With this criterion for identity at hand, we can see that the abstract object introduced above which encodes *being Zalta* is distinct from the object encoding all of Zalta's properties: the latter encodes *using a Mac* while the former does not.

So much for the formal background. Linsky and Zalta now suggest that mathematical theories can be identified as those abstract objects, that encode all the mathematical propositions that are true according to them.[13] This needs some unpacking. First, encoding was introduced as a mode of predication, i.e. a second-level relation that holds between an object and a property. In order for mathematical theories to be able to encode *propositions*, they are handled as zero-place properties. Any proposition $p$ thus gives rise to a property *being such that* $p$; using the notation of $\lambda$-conversion, this can be expressed as: '$[\lambda y\, p]$'.

Being true according to a mathematical theory $\textsc{t}$ can then be characterised using the resources of OT: it is simply defined as $\textsc{t}$ encoding that particular truth:

$$\textsc{t} \models p \ =_{df} \ \textsc{t}[\lambda y\, p]$$

Note that '$\models$' does not denote a semantic consequence relation here, but merely abbreviates the encoding formulae on the right-hand side of the definition. There is, however, a *rule of closure* that guarantees that mathematical theories are deductively closed. Whenever a proposition is a proof-theoretic consequence[14] of some proposi-

[12]One might complain that object comprehension, OC, is suspect on the grounds that with the introduction of OC the well established second-order comprehension schema, considered logical by many, needs to be restricted to avoid inconsistency. We will not follow this criticism here.

[13]See (Linsky and Zalta, 1995), pp. 538–539, and (Linsky and Zalta, 2006), pp. 89–90.

[14]One might quibble whether this should mean a consequence according to the proof theory of

tions that are true according to the theory – i.e. that are encoded by the theory, which means in particular the axioms – then the theory also encodes this proposition:

> *Rule of Closure*
>
> If $p_1, ..., p_n \vdash q$ and $\textsc{t} \models p_1$ and ... and $\textsc{t} \models p_n$, then $\textsc{t} \models q$.

The theoretical terms of a mathematical theory can also be imported into OT in the following way: Take a term $\kappa$ of the mathematical theory $\textsc{t}$ in question and index it with the name of the theory. OC will then guarantee that there is a corresponding abstract object:

$$\kappa_{\textsc{t}} \ =_{df} \ \imath x(A!x \wedge \forall F(xF \equiv \textsc{t} \models F\kappa_{\textsc{t}}))$$

Generalising OC in the obvious way to also yield objects of higher types, i.e. properties and relations, we can import in an analogous way the properties and relations $\Pi$ of mathematical theories into OT (the bold-face letters stand for third-order predicates and variables):

$$\Pi_{\textsc{t}} \ =_{df} \ \imath R(\mathbf{A}!R \wedge \forall \mathbf{F}(R\mathbf{F} \equiv \textsc{t} \models \mathbf{F}\Pi_{\textsc{t}}))$$

Membership in Zermelo-Fraenkel set theory (ZF), for example, can thus be defined in OT as:

$$\in_{\textsc{zf}} \ =_{df} \ \imath R(\mathbf{A}!R \wedge \forall \mathbf{F}(R\mathbf{F} \equiv \textsc{zf} \models \mathbf{F}\in_{\textsc{zf}}))$$

With these items available, all propositions of the mathematical theory in question can be added to OT as (arguably) *analytic truths*, 'In theory $\textsc{t}$, $p$', in this way:

> Add to OT sentences of the form $\ulcorner \textsc{t} \models \varphi^* \urcorner$, where $\varphi$ is an axiom of the $\textsc{t}$ and $\varphi^*$ is arrived at by indexing all well-defined terms and predicates of theory $\textsc{t}$ as belonging to $\textsc{t}$.

The rule of closure will then take care of all the theorems of the mathematical theory in question. To use $\textsc{zf}$ as an example again, the existence of a set without members according to $\textsc{zf}$ can be expressed in OT as:

$$\textsc{zf} \models \exists x \neg \exists y \ y \in_{\textsc{zf}} x$$

---

the mathematical theory in question, or according to whatever metatheory Linsky and Zalta use for OT. (Zalta, 2000), p. 232, suggests that it is the proof-theoretic consequence relation of OT; it is unlikely that constructivist theories are faithfully represented in this way, not to speak of paraconsistent mathematics (see, for instance, (Priest, 1994)).

Linsky and Zalta take all the resulting sentences to be analytic, since all these sentences say, once they are imported into OT, is that a given mathematical theory affirms this-and-that. Moreover, since the theoretical terms of the mathematical theories are imported into OT, too, and OC guarantees a corresponding object, so to speak, OT also directly delivers the ontology to satisfy the *imported* sentences.[15]

In this way, *any possible* mathematical theory can be imported into OT. Linsky and Zalta write:

> [O]ur program ... takes as data any arbitrary mathematical theory that mathematicians may formulate, and provides a more general explanation and analysis as a whole. ((Linsky and Zalta, 2006), p. 89)

This, according to Linsky and Zalta, "constitutes a form of neologicism" since it is a weakening of the logicist claim that mathematics is reducible to logic alone. Being a weakening of this logicist claim is what Linsky and Zalta identify as a hallmark of neo-logicism. They write:

> Our claim is:
>
> Third-order object theory is a neologicism because it reduces (in the sense just described) all of mathematics to 'third-order' logic[16] and some analytic truths. ((Linsky and Zalta, 2006), p. 91)

Let us briefly come back to the importing of mathematical theories into OT. Are we not going in a circle here? Mathematical theories are supposed to enter OT as those abstract objects that encode all propositions true according to them, but importing these propositions into OT involves mentioning the mathematical theory. The crux is that it is not the imported mathematical statements that identify the

[15]It cannot, however, deliver the ontology that the original theories are intended to be about. The intended model of real analysis, for example, has an uncountable domain, but the technique described above will only ever deliver countably many objects, since the language is countable.

[16]Linsky and Zalta comment on this: "By quoting the phrase 'third-order', we are calling attention to the fact that the theory is weaker than full third-order logic. Though our theory is most naturally formulated using third-order syntax its logical strength is no greater than multi-sorted first-order logic." (*ibid.*) The "analytic truth" not only include the mathematical statements that are to be imported into OT, but also the comprehension schema for abstract objects, OC, along with the notion of encoding. It should also be noted that modal operators figure in some of the definitions; see footnote 8 above.

mathematical theory in OT. The definition, rather, *quantifies* over propositions, and it is outside of OT that we are to decide what propositions are true according to a given mathematical theory. In other words, mathematicians, or mathematical practice will tell us.

Formally, Linsky and Zalta can thus state that if $\text{T}$ is a mathematical theory, then it is that abstract object that encodes all its theorems:[17]

$$\begin{aligned}&\mathit{MathTheory}(\text{T}) \supset \\ &\text{T} = \imath x(A!x \wedge \forall F(xF \equiv \exists p(\text{T} \models p \wedge F = [\lambda y\, p])))\end{aligned}$$

This, however, leaves open what the antecedent actually says. The answer is to be found in (Zalta, 2000, pp. 229–230). First, take '$\mathit{Math}(p)$' as a primitive notion, with the intended meaning 'is a purely mathematical proposition'. Zalta asks us to rely upon the "pretty good pretheoretic grasp" we have to decide this predicate. Then, add another primitive predicate, namely '$Axy$' for '$x$ authored $y$', or, equivalently, '$y$ is an author of $x$'.[18] Now '$\mathit{MathTheory}(x)$' can be defined:

$$\begin{aligned}&\mathit{MathTheory}(x) \;=_{\mathit{df}} \\ &\forall F(xF \supset \exists p(\mathit{Math}(p) \wedge F = [\lambda y\, p])) \wedge \exists y(E!y \wedge Ayx)\end{aligned}$$

Thus, mathematical theories are a particular kind of story, akin to fiction in many ways. Mathematical theories have to be authored: there are no mathematical theories (yet) that have not been written or authored in some other way (yet).[19] This

[17]Identity between properties (for all types or orders) is defined as necessary co-encoding:

$$F = G \;=_{\mathit{df}}\; \Box\forall x(xF \equiv xG)$$

This is another place where the modal operator comes in; see footnote 8. Similarly, identity between propositions is defined as:

$$p = q \;=_{\mathit{df}}\; [\lambda y\, p] = [\lambda y\, q]$$

which unpacks as:

$$p = q \;=_{\mathit{df}}\; \Box\forall x(x[\lambda y\, p] \equiv x[\lambda y\, q])$$

See (Zalta, 2000), p. 224, fn. 9.

[18]We omit the type specification here for simplicity's sake. For a typed version see (Zalta, 2000), pp. 228ff.

[19]In (Zalta, 2000), p. 230, anticipates possible criticisms regarding this point. He suggests that possible authorship might be sufficient. Formally, this would be to introduce a possibility operator, '$\Diamond$', in front of the second existential quantifier.

definition also seems to put mathematical theory ontologically on a par with pieces of fiction: for example, Peano Arithmetic is ontological on one level with the Brothers Grimm's *Rumpelstielzchen*, metaphysically speaking. We will not further dwell on potential problems of the resulting ontology here.[20] Instead, let us briefly consider a short story (which, as such, should be taken with a grain of salt). It is meant to highlight the differences between the two schools and will be used to support our final conclusions.

## 2 Interlude: The Travels of Hero and Hera

The twins Hero and Hera had always been inseparable. After school they both went to Ohio State University, and together they discovered higher-order logic there. Each excelled at their new favourite subject, passing the final exams with flying colours. Before long, both Hero and Hera had applications for graduate programmes winging away in the post. Hero was awarded AHRC funding to go to St Andrews, whilst Hera gained a scholarship to study in Stanford.[21] And so, the time came for the twins to part company, as each budding young philosopher embarked on a PhD in logic at different universities, in different countries, and under different supervisors. Of course, being the diligent and obedient students they inevitably were, both Hero and Hera unquestioningly accepted every principle their respective supervisors (Crispin Wright and Edward Zalta) presented them with.

Hero's first week at St Andrews was a good one. In addition to testing the water (and a few other choice beverages) he learnt all about Hume's Principle, the abstraction principle which states that the number of the $F$s equals the number of $G$s if, and only if, there is a bijection between them. By his second week, Hero learnt the abstraction principle for real numbers. And, by the third week Hero had really begun to settle into the St Andrews lifestyle, to learn New V and Newer V, the two consistent restrictions of the inconsistent Basic Law V.

By this point, by way of deduction in second-order logic, Hero had acquired arithmetic from Hume's Principle and real analysis from the abstraction principle for

[20] We are, however, discussing these and other further issues in our (Ebert and Rossberg, 2006, in preparation).

[21] Hera briefly considered going to Edmonton instead, but the cold winters put her off.

the reals. Admittedly he didn't get much set theory from New V or Newer V, but towards the end of his final year Hero was lucky enough to learn that Julius Caesar is not a number, so all was well. He was able to deduce a set theory that interprets (most of) ZF from a version of Hume's Principle[22], New V, and Newer V, along with the solution to the Caesar Problem.[23]

Meanwhile, across the pond, life was different for Hera. In her first week she learnt all about a new (primitive) form of predication, called 'encoding', and that abstract objects *encode* properties much in the same way that concrete objects *have* properties. In her second week, Hera learnt where these abstract objects come from, viz. from *Object Comprehension*. By the third week, Hera's supervisor had explained to her that mathematical theories are just examples of some of those abstract objects whose existence is given by Object Comprehension, and also that one can add any statement of the form 'In theory T, $p$' to the system – which she learnt is (analytically) true if $p$ is a theorem of T. She also learnt that all the entities that these mathematical theories talk about exist as well: they are also given by Object Comprehension.

However, all that was only the beginning for Hera. She was quickly packed off to the mathematics department to learn all possible mathematical theories whose existence is guaranteed by Object Comprehension, and whose analytic truths ('In T, $p$') can be added to the system. Amidst all that, Hera was reminded that

> "Indeed, a unique feature of our program is that it yields no proper mathematics on its own, and so makes no judgments about which parts of mathematics are philosophically justified! Instead, it takes as data any arbitrary mathematical theory that mathematicians may formulate." ((Linsky and Zalta, 2006), p. 30)

As one might expect from the amount of studying involved, Hero was going to graduate sooner, because Hera's course took longer to complete – after all she did have

[22]Namely: *Finite Hume*, as introduced in (Heck, 1997).

[23]A careful study of (Cook, 2003) revealed to Hero that the following principles are enough to prove all axioms of ZF except Foundation. These are, New V, Newer V, the Size-Restricted Ordinal Abstraction Principle (SOAP) and the existence of infinitely many non-sets. From footnote 30 of (Cook, 2003) Hero learnt that, strictly speaking, SOAP can be dispensed with if slightly reformulated versions of New V and Newer V are adopted. Hero then realised (on his own) that the existence of infinitely many non-sets follows from Hume's Principle provided that the Caesar Problem is resolved in such a way as to yield that numbers are not sets, i.e. the problems raised in (Cook and Ebert, 2005) are resolved.

all possible mathematical theories to learn. (Hera certainly realised that she was not, strictly speaking, required to go and learn *all* possible mathematical theories – this is not part of the programme. Since she set out to acquire mathematical knowledge, however, she decided to make full use of the possibilities of Object Theory.) All Hero had to learn, on the other hand, was four abstraction principles (and that Julius Caesar is not a number).

Despite the physical and philosophical separation of Hero and Hera since their undergrad years, a remarkable coincidence conspired to unite them again in a bizarre way as each approached the ultimate end of their studies. In their final examinations, both Hero and Hera were confronted with the same questions:

1. Proof that $2 + 2 = 4$!

2. How do you know that it is true?

   Allowed materials: The principles your supervisor taught you.

Hero tackled the challenge in the following elegant way: He swiftly derived the Peano-Dedekind axioms of arithmetic from Hume's Principle in second-order logic, and used them to prove that $2 + 2 = 4$. For the second question, Hero simply claimed that he was allowed to take Hume's Principle as (analytically) true, since it is a meaning-constituting principle, and also since pure (second-order) logic can be used to derive the statement in question, he comes to know it simply by way of deduction from Hume's Principle. Although, his external examiners were not entirely convinced that the mere meaning-constituting character of an abstraction-principle will be enough to secure its truth and so account for his knowledge of Hume's Principle, and also had some misgivings about the adoption of second-order logic, they were sufficiently impressed by his story (and the presentation thereof) to award him a PhD. Yet they hoped he would – in the near future – say why exactly some abstraction principle succeed in founding knowledge while others fail.

For Hera, though, the challenge was considerably more daunting. After a little initial hesitation, she quickly proved that $2 + 2 = 4$ in Peano Arithmetic, then in Robinson Arithmetic, and then in real as well as complex analysis, followed by a proof in the system of *Principia Mathematica* and then in ZF (plus suitable definitions). Her examiners stopped her just before she started the proof in Aczel's anti-foundational set theory.[24]

"But you didn't tell me what '2' and '4' and '+' and '=' you were talking about," she protested. "So I just started with some common theories. I can also prove it in Priest's paraconsistent arithmetic if that's better."[25] Looking a bit sheepish as she said it, Hera added that she also knew a few disproofs, if the examiners would like. The examiners assured her that they wouldn't like that, and asked her to move on to the second question. Alas, Hera was stumped on that one. Eventually, she reluctantly said that she knows it is a theorem of various mathematical theories. She knew, for example, that in PA, $2 + 2 = 4$, but without clarifying which '2' and '4' and '+' and '=' is meant, she wasn't sure what she was meant to show.[26]

Her examiners were intrigued about her responses and awarded her the degree for her stimulating views in the philosophy of mathematics, her sophisticated axiomatic metaphysics, and her heroic attempt to account for *any* possible mathematical theory.

# 3 Axiomatic Metaphysics vs Epistemic Foundationalism: the Purpose of Neo-Logicism?

The idea in this section is not to pinpoint specific problematic issues that threaten the tenability of either of the programmes. Rather, assuming that each project is tenable and internally consistent, we aim to tackle the question what the purpose is of pursuing either of these two projects. That is, what is the philosophical payback from pursuing either the Scottish school, i.e. epistemic foundationalism, or the Stanford/Edmonton school, i.e. axiomatic metaphysics. We hope that the story highlighting the achievements of Hero and Hera will help to identify the differences between the two approaches. By appealing to what Frege thought what the aim of logicism is,

[24] A consistent set theory that dispenses with the Axiom of Foundation, and allows sets to contain themselves; see (Aczel, 1988).

[25] See (Priest, 1994).

[26] (Andersen and Zalta, 2004) present a different neo-logicist programme in the framework of second-order modal OT which allows for the derivation of '$2+2 = 4$' as a categorical statement, since this approach allows the derivation of some modest parts of mathematics as non-hedged statements from some additional assumption; see also (Zalta, 1999), and the discussion in (Linsky and Zalta, 2006), §4.2. It is argued in (Linsky and Zalta, 2006), §5, that the approach presented here is to be preferred to the Andersen and Zalta project.

we hope to show that the purpose of pursuing the Stanford/Edmonton school does not fit the bill.

The original logicist programme was clearly epistemological in spirit, as Frege writes:

> "The problem becomes, in fact, that of finding the proof of the sentence, and of following it up right back to the *primitive truths*. If in carrying out this process, one comes only to *general logical laws* and definitions, then the truth is an analytic one. [...] [If the] proof can be derived exclusively from general laws, which themselves neither need nor admit proof, then the truth is a priori." ((Frege, 1884), p. 4, our italics and translation)

and later in the Grundgesetze he writes the following:

> "In virtue of the gaplessness of the chain of inferences it is achieved that each axiom, each presupposition, hypotheses, or however else one might want to call that which a proof rests upon, is brought to light; and thus one gains a *foundation for the assessment of the epistemological nature* of the proven law." ((Frege, 1893), p. XXVI, our italics and translation)

These quotations provide a good indication that Frege's logicist project was foundationalist in nature. He aimed to identify a few select general logical laws, or basic laws, that were needed to provide an epistemic foundation: namely, mathematical knowledge was meant to "flow" from those basic principle and (what is now called) second-order logic. In addition, Frege also thought of the basic principles as providing an ontological foundation. Basic Law V was meant to identify the logical objects (extensions) by means of which numbers could then be defined. For Frege, logic was the most general of all sciences and concerned with the laws of thought. He considered it to be objectively valid independent of any thinker. Moreover, mathematical statements derived from these logical principles using second-order logic were also considered objective (and so independent of anyone "authoring" them) and the underlying objects were considered to exist mind-independently. There could not be two different yet equally acceptable logics, and there could not be different and incompatible theories of numbers.

The Scottish school is squarely in line with this epistemic foundationalist approach of Frege's. The aim is to select a few principles (which are, however, not regarded purely logical) and then to explain how Hero can, by means of grasping these principle come to know mathematics. The resulting theory explains (assuming it works) how mathematical knowledge can flow from basic principles and second-order knowledge. In addition, the objects these principles are purportedly about are considered to exist, and exist mind-independently. Mathematics and logic are considered objective and not as a mere game or fiction: the statements Hero knows are categorical statements involving a distinct ontology. Thus, we think, the Scottish school neatly fits the general methodology and the aims of Frege's logicist project and should be labelled *neo-logicist*.

In contrast, the Stanford/Edmonton school is an enterprise in axiomatic metaphysics. It aims to select a few metaphysical principles and then provides the tools for Hera to re-interpret any mathematical theory within the new metaphysical framework. Her mathematical knowledge does not flow from some basic mathematical or logical principles. Rather she knows how any mathematical statement, or any mathematical theory for that matter, can be re-interpreted within object-theory.

This is a difference worth emphasising: while the Stanford/Edmonton school wants to account for any mathematical theory, the Neo-Fregean, like Frege, believes that there are mathematical principles, and so mathematical theories, that are better than others.[27]

So, for Hera, every mathematical statements will be true provided it is bound by the respective 'In theory $\mathrm{T}$'-operator. Her mathematical knowledge thus reduces to knowledge of these hedged statements within the new metaphysical framework and, hence, is not categorical.[28] Also, since mathematical theories and with them

[27]The Neo-Fregean does not aim at an epistemic foundation of inconsistent theories, for example. We expand on this in our (Ebert and Rossberg, 2006).

[28]There is a way of simulating categorical statements in OT (and there is, as mentioned in footnote 26 above, also the approach presented in (Andersen and Zalta, 2004)). The statement that expresses that, according to ZF, the empty set does not have any members, is represented as a hedged OT sentence like this:

$$\mathrm{ZF} \models \neg\exists x(x \in_{\mathrm{ZF}} \emptyset_{\mathrm{ZF}})$$

Since for this we already have to import the terminology of ZF into OT, using the technique described in section 1.2 above, one can now also directly express a related statement: while it is not provable in OT that the (ZF) empty set has no (ZF) member, in the sense of it *exemplifying* the property of having no (ZF) members, there is a sentence that is a theorem of OT which asserts that the (ZF) empty set *encodes* having no (ZF) members; formally, that looks something like this:

$$\emptyset_{\mathrm{ZF}}[\lambda y(\neg\exists x(x \in_{\mathrm{ZF}} y))]$$

mathematical objects depend (in an ontological sense) on authorship[29], they cannot be regarded to exist mind- or subject-independently either.

Hence both, the methodology involved in positing an axiomatic metaphysics and the aims of the Stanford/Edmonton school, are very distinct from Frege's original logicist project. The Stanford/Edmonton programme aims to account for all possible mathematical theories, while the attention of the Neo-Fregean is restricted to classical number theory and set theory. Thus, once Hero has learnt the right abstraction principles (however hard these might be to identify), all his mathematical knowledge in these areas can be arrived at by way of inferring it from these principles using second-order logic. While, strictly speaking, Hera did not have to go and learn all possible mathematical theories, she nevertheless had to go and study mathematical theories to get mathematical knowledge: no mathematical knowledge is provided by the Stanford/Edmonton programme on its own. Although it surely has its own intellectual merits and interest, we believe that it fails to fulfil the purpose of logicism and so should not be regarded a form of neo-logicism.

While it might not be considered a major blow that according to these considerations Linsky and Zalta's proposal should be denied the (largely honorific) label 'neo-logicism', we nevertheless want to maintain that their claim that this project "*constitutes an epistemic foundation*, in the sense that it shows how we can have knowledge of mathematical claims"[30] cannot be upheld; and we also have to disagree

The trouble is that as soon as the OT-defined terms, like '$\emptyset_{\text{ZF}}$' or '$\in_{\text{ZF}}$', are unpacked, the hedged statements appear again; recall, for instance, the OT definition of ZF-membership:

$$\in_{\text{ZF}} =_{df} \ \imath R(\mathbf{A}!R \wedge \forall \mathbf{F}(R\mathbf{F} \equiv \text{ZF} \models \mathbf{F}\in_{\text{ZF}}))$$

Intuitively, while the sentences about what properties are encoded by these mathematical objects appear to be categorical (in OT), the identification of the mathematical objects and theories goes via the hedged sentences again, i.e. via statements about what is true according to this-or-that mathematical theory. Thus, also these "categorical" sentences express no more than what is the case according to a certain theory, and, hence, should not count as properly categorical in our opinion.

[29]Or *possible* authorship. Perhaps adopting the modal strategy mentioned in footnote 19 above addresses this concern: if a case can be made that dependence on merely possible authorship is consistent with the relevant notion of (mind-)independent existence. How attractive this approach is, however, requires further discussion: for example, since the Barcan Formula, '$\Diamond\exists x\varphi \supset \exists x\Diamond\varphi$', is a theorem of OT (see (Linsky and Zalta, 1995), p. 543, fn. 24), mathematical theories that are not actually authored commit us to the existence of authors who are bare possibilia.

[30](Linsky and Zalta, 2006), p. 61, our italics.

with their conclusion that their project “best addresses the underlying motives of the early logicists.”[31] The “principal driving force of the early logicists”, Linsky and Zalta suggest (correctly, as we think), were “epistemological concerns about how we can have knowledge of mathematics” (*ibid.*). We argued that the epistemic concerns of logicism are not addressed by this programme.[32]

## References


Aczel, P. (1988). *Non-Well-Founded Sets*. CSLI, Stanford.

Andersen, D. J. and Zalta, E. N. (2004). Frege, Boolos, and Logical Objects. *Journal of Philosophical Logic*, 33:1–36.

Black, M., editor (1965). *Philosophy in America*. Cornell University Press, Ithaca, NY.

Boghossian, P. and Peacocke, C., editors (2000). *New Essays on the A Priori*. Clarendon Press, Oxford.

Boolos, G. (1987). The Consistency of Frege’s Foundation of Arithmetic. In (Thomson, 1987), pages 3–20. Reprinted in (Boolos, 1998), pages 183–201.

Boolos, G., editor (1990a). *Meaning and Method: Essays in Honour of Hilary Putnam*. Cambridge University Press, Cambridge.

Boolos, G. (1990b). The Standard of Equality of Numbers. In (Boolos, 1990a), pages 261–277. Reprinted in (Boolos, 1998), pages 202–219.

Boolos, G. (1995). Frege’s Theorem and the Peano Postulates. *The Bulletin of Symbolic Logic*, 1:317–326. Reprinted in (Boolos, 1998), pages 291–300.

Boolos, G. (1996). On the Proof of Frege’s Theorem. In (Morton and Stich, 1996), pages 143–159. Reprinted in (Boolos, 1998), pages 275–290.

Boolos, G. (1998). *Logic, Logic, and Logic*. Harvard University Press, Cambridge, MA.

Burgess, J. P. (2005). *Fixing Frege*. Princeton University Press, Princeton.


[31] (Linsky and Zalta, 2006), p. 95.

[32] Further worries concerning the resulting ontology and other epistemological issues are discussed in (Ebert and Rossberg, 2006).

Church, A. (1956). *Introduction to Mathematical Logic*. Princeton University Press, Princeton.

Cook, R. T. (2003). Iteration one more time. *Notre Dame Journal of Formal Logic*, 44(2):63–92.

Cook, R. T. and Ebert, P. A. (2005). Abstraction and Identity. *Dialectica* [special issue on the *Julius Caesar Problem*], 59(2):1–19.

Ebert, P. A. (2005). Transmission of Warrant-Failure and the Notion of Epistemic Analyticity. *Australsian Journal of Philosophy*, 83(4):505–521.

Ebert, P. A. and Rossberg, M. (2006). What Neo-Logicism Could Not Be. In preparation.

Frege, G. (1884). *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*. Wilhelm Koebner, Breslau. English translation: (Frege, 1950).

Frege, G. (1893). *Grundgesetze der Arithmetik: Begriffsschriftlich abgeleitet*. I. Band. Verlag H. Pohle, Jena. Partial English translation in (Frege, 1964).

Frege, G. (1950). *The Foundations of Arithmetic*. Translated by J. L. Austin. Blackwell, Oxford.

Frege, G. (1964). *The Basic Laws of Arithmetic: Exposition of the System*. Translated and edited by M. Furth. University of California Press, Berkeley.

Hale, B. (1987). *Abstract Objects*. Basil Blackwell, Oxford.

Hale, B. and Wright, C. (2000). Implicit Definition and the A Priori. In (Boghossian and Peacocke, 2000), pages 286–319. Reprinted in (Hale and Wright, 2001), pages 117–150.

Hale, B. and Wright, C. (2001). *The Reason's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics*. Clarendon Press, Oxford.

Heck, R. G. (1997). Finitude and Hume's Principle. *Journal of Philosophical Logic*, 26:589–617.

Heck, R. G. (2006). Ramified Frege Arithmetic. *Journal of Philosophical Logic*. Forthcoming; available online: <http://phonline.org/paper.php?keynum=522>.

Linnebo, Ø. (2004). Predicative Fragments of Frege Arithmetic. *Bulletin of Symbolic Logic*, 10:153–174.

Linsky, B. (2005). Remarks on Platonized Naturalism. *Croatian Journal of Philosophy*, 5:3–15.

Linsky, B. and Zalta, E. N. (1995). Naturalized Platonism vs. Platonized Naturalism. *Journal of Philosophy*, 92:525–555.

Linsky, B. and Zalta, E. N. (2006). What is Neo-Logicism? *Bulletin of Symbolic Logic*, 12:60–99.

MacBride, F. (2003). Speaking with Shadows: A Study of Neo-Fregeanism. *British Journal for the Philosophy of Science*, 54:103–163.

Morton, A. and Stich, S., editors (1996). *Benacerraf and his Critics*. Blackwell, Oxford.

Parsons, C. (1965). Frege's Theory of Numbers. In (Black, 1965), pages 180–203.

Priest, G. (1994). Is Arithmetic Inconsistent? *Mind*, 103:337–349.

Shapiro, S. (1991). *Foundations Without Foundationalism: A Case For Second-Order Logic*. Oxford University Press, Oxford.

Thomson, J. J., editor (1987). *On Being and Saying: Essays for Richard Cartwright*. MIT Press, Cambridge, MA.

Wright, C. (1983). *Frege's Conception of Numbers as Objects*. Aberdeen University Press, Aberdeen.

Zalta, E. N. (1983). *Abstract Objects: An Introduction to Axiomatic Metaphysics*. Reidel, Dordrecht.

Zalta, E. N. (1999). Natural Number And Natural Cardinals As Abstract Objects: A Partial Reconstruction Of Frege's *Grundgesetze* In Object Theory. *Journal of Philosophical Logic*, 28:619–660.

Zalta, E. N. (2000). Neo-Logicism? An Ontological Reduction of Mathematics to Metaphysics. *Erkenntnis*, 53:219–226.