EINSTEIN and the PHOTOELECTRIC EFFECT

$E_{\rm max} = hv - P$

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To Albert Einstein for his services to theoretical physics and especially for his discovery of the photoelectric effect.

Nobel citation – 1921





% The 1905 paper: what did Einstein do?



Development: where did he go from there?



Reaction: what did others think?





A little something from quantum optics

On a Heuristic Point of View about the Creation and Conversion of Light^{\dagger}

A. EINSTEIN

The wave theory of light which operates with continuous functions in space has been excellently justified for the representation of purely optical phenomena and it is unlikely ever to be replace by another theory. One should, however, bear in mind that optical observations refer to time averages and not to instantaneous values and notwithstanding the complete experimental verification of the theory of diffraction, reflexion, refraction, dispersion, and so on, it is quite conceivable that a theory of light involving the use of continuous functions in space will lead to contradictions with experience, if it is applied to the phenomena of the creation and conversion of light.

† Ann. Physik **17**, 132 (1905).

In fact, it seems to me that the observations on the "black-body radiation", photoluminescence, the production of cathode rays by ultraviolet light and other phenomena involving the emission or conversion of light can be better understood on the assumption that the energy of light is distributed discontinuously in space. According to the assumption considered here, when a light ray starting from a point is propagated, the energy is not continuously distributed over an ever increasing volume, but it consists of a finite number of energy quanta, localised in space, which move without being divided and which can be absorbed or emitted only as a whole.

In the following, I shall communicate the train of thought and the facts which lead me to this conclusion, in the hope that the point of view to be given may turn out to be useful for some research workers in their investigations.

1. On a Difficulty in the Theory of "Black-body Radiation"

To begin with, we take the point of view of Maxwell's theory and electron theory and consider the following case.....

- 1. On a Difficulty in the Theory of "Black-body Radiation"
- 2. On Planck's Determination of Elementary Quanta
- 3. On the Entropy of the Radiation
- 4. Limiting Law for the Entropy of Monochromatic Radiation for Low Radiation Density
- 5. Molecular–Theoretical Investigation of the Volume-dependence of the Entropy of Gases and Dilute Solutions
- 6. Interpretation of the Volume-dependence of the Entropy of Monochromatic Radiation according to Boltzmann's Principle
- 7. On Stokes' Rule
- 8. On the Production of Cathode Rays by the Illumination of Solids
- 9. On the Ionisation of Gases by Ultraviolet Light

The 1905 paper: what did Einstein do?



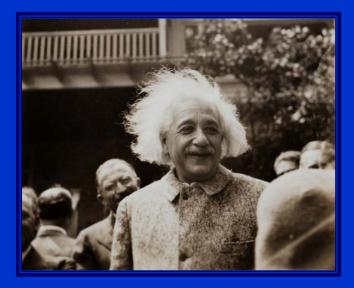
Development: where did he go from there?

Reaction: what did others think?





A little something from quantum optics



"I have thought a hundred times as much about the quantum problems as I have about general relativity."

recollection of Otto Stern
 Jost, R., 1977, letter to A. Pais, August 17

1907 – specific heats: Einstein, A., 1907, Ann. Phys. (Leipz.) 22, 180, 800

1909 – energy fluctuations, waves & particles: Einstein, A., 1909, Phys. Z. 10, 185, 817

1916 – dynamical equilibrium, A and B theory: Einstein, A., 1916, Verh. Dtsch. Phys. Ges. 18, 318 Einstein, A., 1916, Mitt. Ph. Ges. Zürich 16, 47 Einstein, A., 1917, Phys. Z. 18, 121

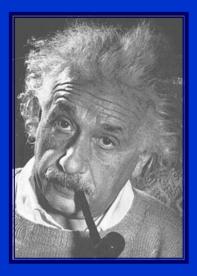
1925 – quantum gas, Bose-Einstein condensation: Einstein, A., 1925, Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl., p. 3

1935 – objective reality, incompleteness of QM: Einstein, A., B. Podolsky, and N. Rosen, 1935, Phys. Rev. 48, 73 1909-energy fluctuations, waves & particles: Einstein, A., 1909, Phys. Z. 10, 185, 817

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle + \langle n \rangle^2$$

"I already attempted earlier to show that our current foundations of the radiation theory have to be abandoned ... it is my opinion that the development of theoretical physics will bring us to a theory of light which can be interpreted as a kind of fusion of the wave and the emission theory ... [the] wave structure and [the] quantum structure ... are not to be considered mutually incompatible ... it seems to follow from the Jeans law that we will have to modify our current theories, not to abandon them completely."

 — summary by A. Pais of comments from the 1909 papers Pais, A., 1979, Rev. Mod. Phys. 51, 878



"All these 50 years of pondering have not brought me closer to answering the question 'What are light-quanta'."

letter to Michelle Besso, 1951
 Einstein, A., 1951, letter to M. Besso, December 12
 in Speziali, P., 1972, editor, *Albert Einstein Michelle Besso correspondence 1903-1955* (Herman, Paris), p.453

"It appears dubious whether a field theory can account for the atomistic structure of matter and radiation as well as the quantum phenomena. Most physicists will reply with a convinced 'No', since they believe that the quantum problem has been solved in principle by other means. However that may be, Lessing's comforting word stays with us: the aspiration to truth is more precious than its assured possession."

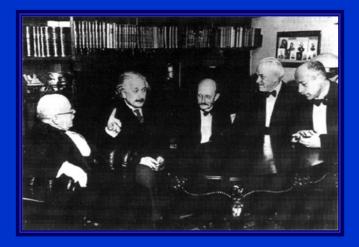
— last autobiographical sketch, Princeton, March 1955,
 Einstein, A., 1956, in *Helle Zeit, Dunkle Zeit*, edited by C. Seelig (Europa, Zürich)

The 1905 paper: what did Einstein do?

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A little something from quantum optics

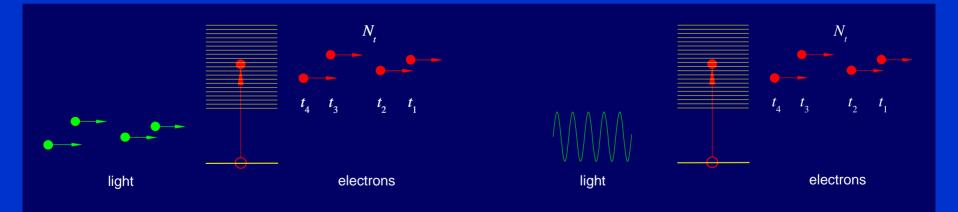


"In sum one can say that there is hardly one among the great problems, in which modern physics is so rich, to which Einstein has not made a remarkable contribution. That he may sometimes have missed the target in his speculations, as for example in his hypothesis of light-quanta, cannot really be held too much against him for it is not possible to introduce new ideas even in the most exact sciences without sometimes taking a risk."

— Planck, Nernst, Rubens, Warburg

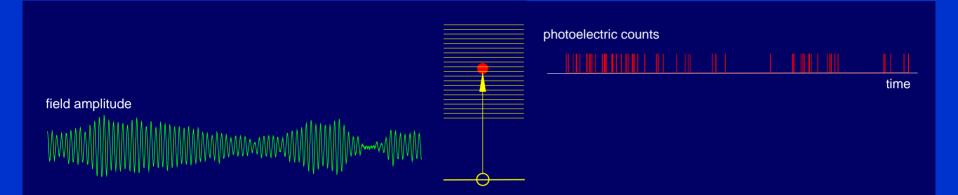
Bohr Kramers Slater





Bohr, N., H.A. Kramers, and J.C. Slater, 1924, Philos. Mag. 47, 785

1909 – energy fluctuations, waves & particles: Einstein, A., 1909, Phys. Z. 10, 185. 817



 $(\Delta n)^2 = \langle n \rangle + \langle n \rangle^2$ particle wave term term

The 1905 paper: what did Einstein do?

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X Do you know?

A little something from quantum optics

the electron is then $hv - \phi$ and this must be the kinetic energy of the emerging electron,⁴

 $K = hv - \phi$

Einstein's photoelectric equation

(12)

This is Einstein's photoelectric equation. It shows that the kinetic energy is indeed a linearly increasing function of the frequency, in agreement with the data of Figure 40.8. According to Eq. (12), the slope of the straight line in Figure 40.8 should equal Planck's constant.

Einstein's photoelectric equation was verified in detail by a long series of meticulous experiments by R. A. Millikan (the data in Figure 40.8 are due to him). In order to obtain reliable results, Millikan found it necessary to take extreme precautions to avoid contamination of the surface of the photosensitive electrode. Since the surfaces of metals exposed to air quickly accumulate a layer of oxide, he developed a technique for shaving the surfaces of his metals in a vacuum by means of a magnetically operated knife.

The results of these experiments gave strong support to the quantum theory of light. This success of Einstein's theory was all the more striking in view of the failure of the classical wave theory of light to an count for the features of the photoelectric effect. According to the wave theory, the crucial parameter that determines the ejection of a photoelectron should be the intensity of light. If an intense electromagnetic wave strikes an electron, it should be able to joli it loose from the metal, regardless of the frequency of the wave. Furthermore, the kinetic energy of the ejected electron should be a function of the intensity of the wave. The observational evidence contradicts these predictions of the wave theory: A wave with a frequency below the threshold frequency never ejects an electron, regardless of its intensity. And, furthermore, the kinetic energy depends on the frequency, and not on the intensity. High-intensity light ejects more photoelectrons, but does not give the individual electrons more kinetic energy.

Today, the photoelectric effect finds many practical applications in sensitive electronic devices for the detection of light. For instance, in a photomultiplier tube, an incident photon ejects an electron from an electrode; this electron is accelerated toward a second electrode (called a dynode; see Figure 40.9) where its impact ejects several secondary electrons; these, in turn, are accelerated toward a third electrode where their impact ejects tertiary electrons, etc. Thus, one electron from the first electrode generates an avalanche of electrons. In a highgain photomultiplier tube, a pulse of 10° electrons emerges from the last electrode, delivering a measurable pulse of current to an external circuit. In this way, the photomultiplier tube can detect the arrival of individual photons. Some sensitive television cameras, such as the image orthicon, rely on the same multiplier principle to convert the arrival of a photon at a photosensitive faceplate into a measurable pulse of current.

EXAMPLE 4. The work function for platinum is 9.9×10^{-19} J. What is the threshold frequency for the ejection of photoelectrons from platinum?

This success of Einstein's theory was all the more striking in view of the failure of the classical wave theory of light to account for the features of the photoelectric effect. According to the wave theory, the crucial parameter that determines the ejection of a photoelectron should be the intensity of light. If an intense electromagnetic wave strikes an electron, it should be able to jolt it loose from the metal, regardless of the frequency of the wave. Furthermore, the kinetic energy of the ejected electron should be a function of the intensity of the wave.

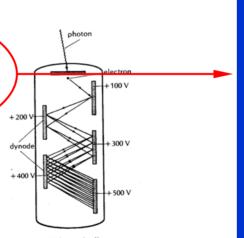


Fig. 40.9 Schematic diagram of a photomultiplier tube. The secondary electrodes are called dynodes. For the purpose of this diagram it has been assumed that each electron impact on a dynode releases two electrons. The arrows show an avalanche of electrons.

Again assume that only one of the eigenstates of H_0 is populated initially. Perturbation (5.6.39) is assumed to be turned on at t = 0, so

$$c_n^{(1)} = \frac{-i}{\hbar} \int_0^t \left(\mathscr{V}_{ni} e^{i\omega t'} + \mathscr{V}_{ni}^\dagger e^{-i\omega t'} \right) e^{i\omega_{ni}t'} dt'$$
$$= \frac{1}{\hbar} \left[\frac{1 - e^{i(\omega + \omega_{ni})t}}{\omega + \omega_{ni}} \mathscr{V}_{ni} + \frac{1 - e^{i(\omega_{ni} - \omega)t}}{-\omega + \omega_{ni}} \mathscr{V}_{ni}^\dagger \right]$$
(5.6.40)

where $\mathscr{V}_{ni}^{\dagger}$ actually stands for $(\mathscr{V}^{\dagger})_{ni}$. We see that this formula is similar to the constant perturbation case. The only change needed is

$$\phi_{ni} = \frac{E_n - E_i}{\hbar} \to \omega_{ni} \pm \omega.$$
(5.6.41)

So as $t \to \infty$, $|c_n^{(1)}|^2$ is appreciable only if

$$\omega_{ni} + \omega \simeq 0 \quad \text{or} \quad E_n \simeq E_i - \hbar \omega$$
 (5.6.42a)

$$\omega_{-i} - \omega \simeq 0 \quad \text{or} \quad E_n \simeq E_i + \hbar \omega.$$
 (5.6.42b)

Clearly, whenever the first term is important because of (5.6.42a), the second term is unimportant, and vice versa. We see that we have no energy-conservation condition satisfied by the quantum-mechanical system alone; rather the apparent lack of energy conservation is compensated by the energy given out to-or energy taken away from-the "external" potential V(t). Pictorially, we have Figure 5.9. In the first case (stimulated emission), the quantum-mechanical system gives up energy $\hbar\omega$ to V; this is clearly possible only if the initial state is excited. In the second case (absorption), the quantum-mechanical system receives energy $\hbar\omega$ from V and ends up as an excited state. Thus a time-dependent perturbation can be regarded as an inexhaustible source or sink of energy.

In complete analogy with (5.6.34), we have

$$w_{i \to [n]} = \frac{2\pi}{\hbar} \left| \overline{\varphi_{ni}} \right|^{2} \rho(E_{n}) \Big|_{E_{n} = E_{i} - \hbar\omega}$$

$$w_{i \to [n]} = \frac{2\pi}{\hbar} \left| \overline{\varphi_{ni}} \right|^{2} \rho(E_{n}) \Big|_{E_{n} = E_{i} - \hbar\omega}$$
(5.6.43)

or, more commonly,

$$\gamma_{i \to n} = \frac{2\pi}{\hbar} \begin{pmatrix} |\mathscr{Y}_n|^2 \\ |\mathscr{Y}_n^{\dagger}|^2 \end{pmatrix} \delta(E_n - E_i \pm \hbar\omega).$$
 (5.6.44)

Note also that

$$|\mathscr{V}_{ni}|^2 = |\mathscr{V}_{in}^{\dagger}|^2, \qquad (5.6.45)$$

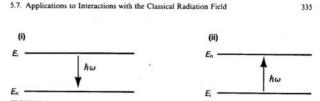


FIGURE 5.9. (i) Stimulated emission: Quantum-mechanical system gives up $\hbar\omega$ to V (possible only if initial state is excited). (ii) Absorption: Quantum mechanical system receives $\hbar\omega$ from V and ends up as an excited state.

which is a consequence of

DC

$$\langle i|\mathscr{V}^{\dagger}|n\rangle = \langle n|\mathscr{V}|i\rangle^{*}$$
 (5.6
(remember $\mathscr{V}^{\dagger}|n\rangle \leftrightarrow \langle n|\mathscr{V}\rangle$). Combining (5.6.43) and (5.6.45), we have

$$\frac{\text{emission rate for } i \to [n]}{\text{density of final states for } [n]} = \frac{\text{absorption rate for } n \to [i]}{\text{density of final states for } [i]},$$
(5.6.47)

where in the absorption case we let i stand for final states. Equation (5.6.47), which expresses symmetry between emission and absorption, is known as detailed balancing.

To summarize, for constant perturbation, we obtain appreciable transition probability for $|i\rangle \rightarrow |n\rangle$ only if $E_n \simeq E_i$. In contrast, for harmonic perturbation we have appreciable transition probability only if $E_n \approx E_i - \hbar\omega$ (stimulated emission) or $E_{\mu} \simeq E_{\mu} + \hbar \omega$ (absorption).

5.7. APPLICATIONS TO INTERACTIONS WITH THE CLASSICAL **RADIATION FIELD**

Absorption and Stimulated Emission

We apply the formalism of time-dependent perturbation theory to the interactions of atomic electron with the classical radiation field. By a classical radiation field we mean the electric or magnetic field derivable from a classical (as opposed to quantized) radiation field.

The basic Hamiltonian, with |A|² omitted, is

$$H = \frac{\mathbf{p}^2}{2m_e} + e\phi(\mathbf{x}) - \frac{e}{m_e c} \mathbf{A} \cdot \mathbf{p}, \qquad (5.7.1)$$

which is justified if

$$\nabla \cdot \mathbf{A} = 0; \tag{5.7.2}$$

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theory gives us

$$=\frac{1}{2\pi}\frac{\omega^2}{c}|A_0|^2,$$
 (5.7.11)

where we have used

$$\left(\frac{E_{\max}^2}{8\pi} + \frac{B_{\max}^2}{8\pi}\right) \tag{5.7.12}$$

for energy density (energy per unit volume) with

$$\mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{A}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{5.7.13}$$

Putting everything together, we get (remembering that $\hbar\omega$ = energy absorbed by atom for each absorption process)

$$\sigma_{abs} = \frac{\hbar\omega(2\pi/\hbar)(e^2/m_c^2 c^2)|A_0|^2|\langle n|e^{i(\omega/c)(\hbar\cdot\mathbf{x})}\hat{\mathbf{z}}\cdot\mathbf{p}|i\rangle|^2\delta(E_n - E_i - \hbar\omega)}{(1/2\pi)(\omega^2/c)|A_0|^2}$$
$$= \frac{4\pi^2\hbar}{m_{\omega}^2}\left(\frac{e^2}{\hbar c}\right)|\langle n|e^{i(\omega/c)(\hbar\cdot\mathbf{x})}\hat{\mathbf{z}}\cdot\mathbf{p}|i\rangle|^2\delta(E_n - E_i - \hbar\omega). \tag{5.7.14}$$

Equation (5.7.14) has the correct dimension $[1/(M^2/T)](M^2L^2/T^2)T = L^2$ if we recognize that $\alpha = e^2/\hbar c \approx 1/137$ (dimensionless) and $\delta(E_n - E_i - \hbar\omega) = (1/\hbar)\delta(\omega_{ni} - \omega)$, where $\delta(\omega_{ni} - \omega)$ has time dimension T.

Electric Dipole Approximation

The electric dipole approximation (E1 approximation) is based on the fact that the wavelength of the radiation field is far longer than the atomic dimension, so that the series (remember $\omega/c = 1/\lambda$)

$$e^{i(\omega/c)\mathbf{h}\cdot\mathbf{x}} = 1 + i\frac{\omega}{c}\,\mathbf{\hat{n}}\cdot\mathbf{x} + \cdots \qquad (5.7.15)$$

can be approximated by its leading term, 1. The validity of this for a light atom is as follows: First, the $\hbar\omega$ of the radiation field must be of order of atomic level spacing, so

$$\omega \sim \frac{Ze^2}{(a_0/Z)} \simeq \frac{Ze^2}{R_{\text{atom}}}.$$
 (5.7.16)

This leads to

$$\frac{c}{b} = \lambda \sim \frac{c\hbar R_{\text{atom}}}{Ze^2} \simeq \frac{137R_{\text{atom}}}{Z}.$$
(5.7.17)

In other words,

$$\frac{1}{R_{\text{atom}}} \sim \frac{Z}{137} \ll 1$$
 (5.7.18)

for light atoms (small Z). Because the matrix element of x is of order R_{atom} ,

specifically, we work with a monochromatic field of the plane wave for

$$\mathbf{A} = 2A_0\hat{\mathbf{c}}\cos\left(\frac{\omega}{c}\hat{\mathbf{n}}\cdot\mathbf{x} - \omega t\right) \tag{6.7.3}$$

where $\hat{\epsilon}$ and \hat{n} are the (linear) polarization and propagation direction. Equation (5.7.3) obviously satisfies (5.7.2) because $\hat{\epsilon}$ is perpendicular to the propagation direction \hat{n} . We write

$$\cos\left(\frac{\omega}{c}\mathbf{\hat{n}}\cdot\mathbf{x}-\omega t\right) = \frac{1}{2}\left[e^{i(\omega/c)\mathbf{\hat{n}}\cdot\mathbf{x}-i\omega t} + e^{-i(\omega/c)\mathbf{\hat{n}}\cdot\mathbf{x}+i\omega t}\right]$$
(5.7.4)

and treat $-(e/m_e c)\mathbf{A} \cdot \mathbf{p}$ as time-dependent potential, where we express A in (5.7.3) as

$$\mathbf{A} = A_0 \hat{\mathbf{\epsilon}} \Big[e^{i(\omega/c)\mathbf{h} \cdot \mathbf{x} - i\omega t} + e^{-i(\omega/c)\mathbf{h} \cdot \mathbf{x} + i\omega t} \Big].$$
(5.7.5)

Comparing this result with (5.6.39), we see that the $e^{-i\omega t}$ -term in

$$-\left(\frac{e}{m_e c}\right)\mathbf{A} \cdot \mathbf{p} = -\left(\frac{e}{m_e c}\right)A_0 \hat{\mathbf{e}} \cdot \mathbf{p} \left[e^{i(\omega/c)\mathbf{h} \cdot \mathbf{x} - i\omega t} + e^{-i(\omega/c)\mathbf{h} \cdot \mathbf{x} + i\omega t}\right]$$
(5.7.6)

is responsible for absorption, while the $e^{+i\omega t}$ -term is responsible for stimulated emission.

Let us now treat the absorption case in detail. We have

$$\mathscr{V}_{ni}^{\dagger} = -\frac{eA_0}{m_c c} \left(e^{i(\omega/c)(\mathbf{h}\cdot\mathbf{x})} \hat{\mathbf{\epsilon}} \cdot \mathbf{p} \right)_{ni}$$
(5.7.7)

and

$$w_{i\to n} = \frac{2\pi}{\hbar} \frac{e^2}{m_{cc}^2} |A_0|^2 |\langle n|e^{i(\omega/c)(\mathbf{h}\cdot\mathbf{x})} \hat{\mathbf{z}} \cdot \mathbf{p}|i\rangle|^2 \delta(E_n - E_i - \hbar\omega). \quad (5.7.8)$$

The meaning of the δ -function is clear. If $|n\rangle$ forms a continuum, we simply integrate with $\rho(E_n)$. But even if $|n\rangle$ is discrete, because $|n\rangle$ cannot be a ground state (albeit a bound-state energy level), its energy is not infinitely sharp; there may be a natural broadening due to a finite lifetime (see Section 5.8); there can also be a mechanism for broadening due to collisions. In such cases, we regard $\delta(\omega - \omega_{ni})$ as

$$\delta(\omega - \omega_{ni}) = \lim_{\gamma \to 0} \left(\frac{\gamma}{2\pi}\right) \frac{1}{\left[\left(\omega - \omega_{ni}\right)^2 + \gamma^2/4\right]}.$$
 (5.7.9)

Finally, the incident electromagnetic wave itself is not perfectly monochromatic; in fact, there is always a finite frequency width.

We derive an absorption cross section as

$$\frac{\text{(Energy/unit time) absorbed by the atom }(i \rightarrow n)}{\text{Energy flux of the radiation field}}$$
. (5.7.10)

For the energy flux (energy per area per unit time), classical electromagnetic

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Approximation Methods

that of x^2 , of order R^2_{atom} , and so on, we see that the approximation of replacing (5.7.15) by its leading term is an excellent one.

Now we have

$$\langle n|e^{i(\omega/\epsilon)(\mathbf{\hat{n}}\cdot\mathbf{x})}\hat{\mathbf{\hat{\epsilon}}}\cdot\mathbf{p}|i\rangle \rightarrow \hat{\mathbf{\hat{\epsilon}}}\cdot\langle n|\mathbf{p}|i\rangle.$$
 (5.7.19)

In particular, we take $\hat{\mathbf{e}}$ along the x-axis (and $\hat{\mathbf{n}}$ along the z-axis). We must calculate $\langle n | p_x | i \rangle$. Using

$$[x, H_0] = \frac{i\hbar p_x}{m}, \qquad (5.7.20)$$

we have

$$\langle n | p_x | i \rangle = \frac{m}{i\hbar} \langle n [x, H_0] | i \rangle$$

= $im\omega_{ni} \langle n | x | i \rangle.$ (5.7.21)

Because of the approximation of the dipole operator, this approximation scheme is called the **electric dipole approximation**. We may here recall [see (3.10.39)] the selection rule for the dipole matrix element. Since x is a spherical tensor of rank 1 with $q = \pm 1$, we must have $m' - m = \pm 1$, $|j' - j| = 0,1 (no 0 \rightarrow 0 \text{ transition})$. If $\hat{\mathbf{e}}$ is along the y-axis, the same selection rule applies. On the other hand, if $\hat{\mathbf{e}}$ is in the z-direction, q = 0; hence, m' = m.

With the electric dipole approximation, the absorption cross section (5.7.14) now takes a simpler form upon using (5.7.19) and (5.7.21) as

$$\sigma_{nk} = 4\pi^2 \alpha \omega_n |\langle n|x|i \rangle|^2 \delta(\omega - \omega_{ni}). \qquad (5.7.22)$$

In other words, σ_{abs} treated as a function of ω exhibits a sharp δ -functionlike peak whenever $\hbar\omega$ corresponds to the energy-level spacing at $\omega \approx (E_n - E_i)/\hbar$. Suppose $|i\rangle$ is the ground state, then ω_{ni} is necessarily positive; integrating (5.7.22), we get

$$\int \sigma_{abs}(\omega) \, d\omega = \sum_{n} 4\pi^2 \alpha \omega_{ni} |\langle n|x|i \rangle|^2. \tag{5.7.23}$$

In atomic physics we define oscillator strength, f_{ni} , as

$$T_{ni} = \frac{2m\omega_{ni}}{\hbar} |\langle n|x|i \rangle|^2.$$
 (5.7.24)

It is then straightforward (consider $[x, [x, H_0]]$) to establish the **Thomas-Reiche-Kuhn sum rule**,

$$\sum_{n} f_{ni} = 1.$$
 (5.7.25)

In terms of the integration over the absorption cross section, we have

$$\int \sigma_{abs}(\omega) \, d\omega = \frac{4\pi^2 \alpha \hbar}{2m_e} = 2\pi^2 c \left(\frac{e^2}{m_e c^2}\right). \tag{5.7.26}$$

5.7. Applications to Interactions with the Classical Radiation Field

Notice how \hbar has disappeared. Indeed, this is just the oscillation sum rule already known in classical electrodynamics (Jackson 1975, for instance). Historically, this was one of the first examples of how "new quantum mechanics" led to the correct classical result. This sum rule is quite remarkable because we did not specify in detail the form of the Hamiltonian.

Photoelectric Effect

We now consider the photoelectric effect—that is, the ejection of an electron when an atom is placed in the radiation field. The basic process is considered to be the transition from an atomic (bound) state to a continuum state E > 0. Therefore, $|i\rangle$ is the ket for an atomic state, while $|n\rangle$ is the ket for a continuum state, which can be taken to be a plane-wave state $|k_f\rangle$, an approximation that is valid if the final electron is not too slow. Our earlier formula for $\sigma_{abs}(\omega)$ can still be used, except that we must now integrate $\delta(\omega_{ni} - \omega)$ together with the density of final states $\rho(E_n)$.

Our basic task is to calculate the number of final states per unit energy interval. As we will see in a moment, this is an example where the matrix element depends not only on the final state energy but also on the momentum *direction*. We must therefore consider a group of final states with both similar momentum directions and similar energies.

To count the number of states it is convenient to use the box normalization convention for plane-wave states. We consider a plane-wave state normalized if when we integrate the square modulus of its wave function for a cubic box of side L, we obtain unity. Furthermore, the state is assumed to satisfy the periodic boundary condition with periodicity of the side of the box. The wave function must then be of form

$$\langle \mathbf{k} | \mathbf{k}_f \rangle = \frac{e^{i \mathbf{k}_f \cdot \mathbf{x}}}{L^{3/2}},$$
 (5.7.27)

where the allowed values of k_x must satisfy

0

$$k_x = \frac{2\pi n_x}{L}, ..., \tag{5.7.28}$$

with n_x a positive or negative integer. Similar restrictions hold for k_y and k_z . Notice that as $L \to \infty$, k_x , k_y , and k_z become continuous variables.

The problem of counting the number of states is reduced to that of counting the number of dots in three-dimensional lattice space. We define n such that

$$n^2 = n_x^2 + n_y^2 + n_z^2. (5.7.29)$$

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$$\mathbf{A} = 2A_0\hat{\mathbf{\epsilon}}\cos\left(\frac{\omega}{c}\hat{\mathbf{n}}\cdot\mathbf{x} - \omega t\right)$$
(5.7.3)

where $\hat{\epsilon}$ and $\hat{\mathbf{n}}$ are the (linear) polarization and propagation direction. Equation (5.7.3) obviously satisfies (5.7.2) because $\hat{\epsilon}$ is perpendicular to the propagation direction $\hat{\mathbf{n}}$. We write

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(5.7.5)

Comparing this result with (5.6.39), we see that the $e^{-i\omega t}$ -term in

$$-\left(\frac{e}{m_e c}\right)\mathbf{A} \cdot \mathbf{p} = -\left(\frac{e}{m_e c}\right)A_0 \hat{\mathbf{\epsilon}} \cdot \mathbf{p} \left[e^{i(\omega/c)\mathbf{\hat{n}}\cdot\mathbf{x} - i\omega t} + e^{-i(\omega/c)\mathbf{\hat{n}}\cdot\mathbf{x} + i\omega t}\right]$$
(5.7.6)

is responsible for absorption, while the $e^{+i\omega t}$ -term is responsible for stimulated emission.

Let us now treat the absorption case in detail. We have

$$\mathscr{V}_{ni}^{\dagger} = -\frac{eA_0}{m_c c} \left(e^{i(\omega/c)(\mathbf{\hat{n}}\cdot\mathbf{x})} \hat{\mathbf{\hat{c}}} \cdot \mathbf{p} \right)_{ni}$$
(5.7.7)

and

$$w_{i \to n} = \frac{2\pi}{\hbar} \frac{e^2}{m_e^2 c^2} |A_0|^2 |\langle n| e^{i(\omega/c)(\hbar \cdot \mathbf{x})} \hat{\mathbf{\epsilon}} \cdot \mathbf{p} |i\rangle |\delta(E_n - E_i - \hbar \omega). \quad (5.7.8)$$

The meaning of the δ -function is clear. If $|n\rangle$ forms a continuum, we simply integrate with $\rho(E_n)$. But even if $|n\rangle$ is discrete, because $|n\rangle$ cannot be a ground state (albeit a bound-state energy level), its energy is not infinitely sharp; there may be a natural broadening due to a finite lifetime (see Section 5.8); there can also be a mechanism for broadening due to collisions. In such cases, we regard $\delta(\omega - \omega_{ni})$ as

$$\delta(\omega - \omega_{ni}) = \lim_{\gamma \to 0} \left(\frac{\gamma}{2\pi}\right) \frac{1}{\left[\left(\omega - \omega_{ni}\right)^2 + \gamma^2/4\right]}.$$
 (5.7.9)

Finally, the incident electromagnetic wave itself is not perfectly monochromatic; in fact, there is always a finite frequency width.

We derive an absorption cross section as

(Energy/unit time) absorbed by the atom
$$(i \rightarrow n)$$
. (5.7.10)
Energy flux of the radiation field

For the energy flux (energy per area per unit time), classical electromagnetic

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spe

cifically, we work with a monochromatic near even with the form
$$\mathbf{A} = 2A_0 \hat{\mathbf{c}} \cos\left(\frac{\omega}{c} \hat{\mathbf{h}} \cdot \mathbf{x} - \omega t\right)$$
 (5.7.3)

where $\hat{\mathbf{\epsilon}}$ and $\hat{\mathbf{n}}$ are the (linear) polarization and propagation direction. Equation (5.7.3) obviously satisfies (5.7.2) because $\hat{\mathbf{\epsilon}}$ is perpendicular to the propagation direction $\hat{\mathbf{n}}$. We write

$$\cos\left(\frac{\omega}{c}\mathbf{\hat{n}}\cdot\mathbf{x}-\omega t\right) = \frac{1}{2}\left[e^{i(\omega/c)\mathbf{\hat{n}}\cdot\mathbf{x}-i\omega t} + e^{-i(\omega/c)\mathbf{\hat{n}}\cdot\mathbf{x}+i\omega t}\right]$$
(5.7.4)

and treat $-(e/m_e c)\mathbf{A} \cdot \mathbf{p}$ as time-dependent potential, where we express A in (5.7.3) as

$$\mathbf{A} = A_0 \hat{\boldsymbol{\varepsilon}} \left[e^{i(\omega/c)\mathbf{h} \cdot \mathbf{x} - i\omega t} + e^{-i(\omega/c)\mathbf{h} \cdot \mathbf{x} + i\omega t} \right].$$
(5.7.5)

Id of the plane wave for

.6)

Comparing this result with (5.6.39), we see that the $e^{-i\omega t}$ -term in

$$-\left(\frac{e}{m_e c}\right)\mathbf{A} \cdot \mathbf{p} = -\left(\frac{e}{m_e c}\right)A_0 \hat{\mathbf{\epsilon}} \cdot \mathbf{p}\left[e^{i(\omega/c)\mathbf{h}\cdot\mathbf{x}-i\omega t} + e^{-i(\omega/c)\mathbf{h}\cdot\mathbf{x}+i\omega t}\right]$$
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We derive an absorption cross section as

$$\frac{(\text{Energy/unit time) absorbed by the atom }(i \to n)}{\text{Energy flux of the radiation field}}.$$
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For the energy flux (energy per area per unit time), classical electromagnetic

W The 1905 paper: what did Einstein do?

Development: where did he go from there?

Reaction: what did others think?

Do you know?

A little something from quantum optics

1909 – energy fluctuations, waves & particles: Einstein, A., 1909, Phys. Z. 10, 185, 817

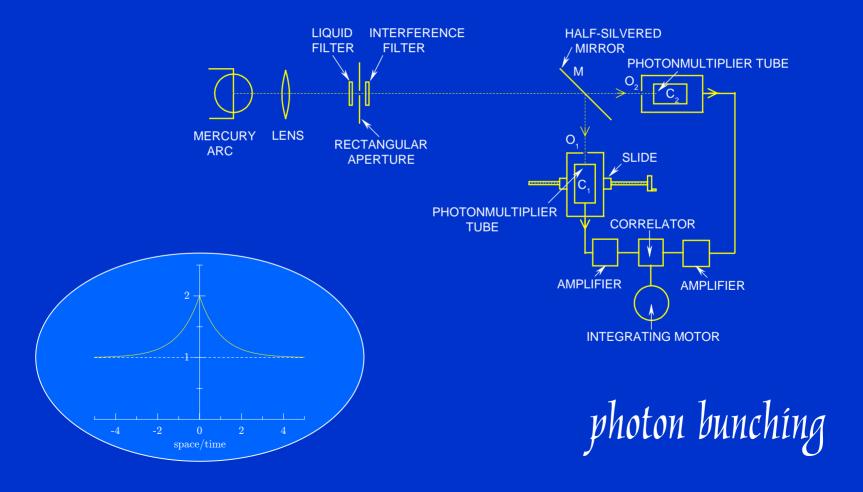
$$\langle \mathcal{E}^2(\nu,T) \rangle = \left(h\nu\rho + \frac{c^3}{8\pi\nu^2}\rho^2\right)Vd\nu$$

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle + \langle n \rangle^2$$

 $\langle n^2 \rangle - \langle n \rangle = \langle n \rangle^2 + \langle n \rangle^2 = 2 \langle n \rangle^2$
 $\frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2} = 2$ (for thermal light)

Hanbury Brown and Twiss light intensity interferometer

R. Hanbury Brown and R.Q. Twiss, Nature 177, 27 (1956) R.Q. Twiss, A.G. Little, and R. Hanbury Brown, Nature 180, 324 (1957)

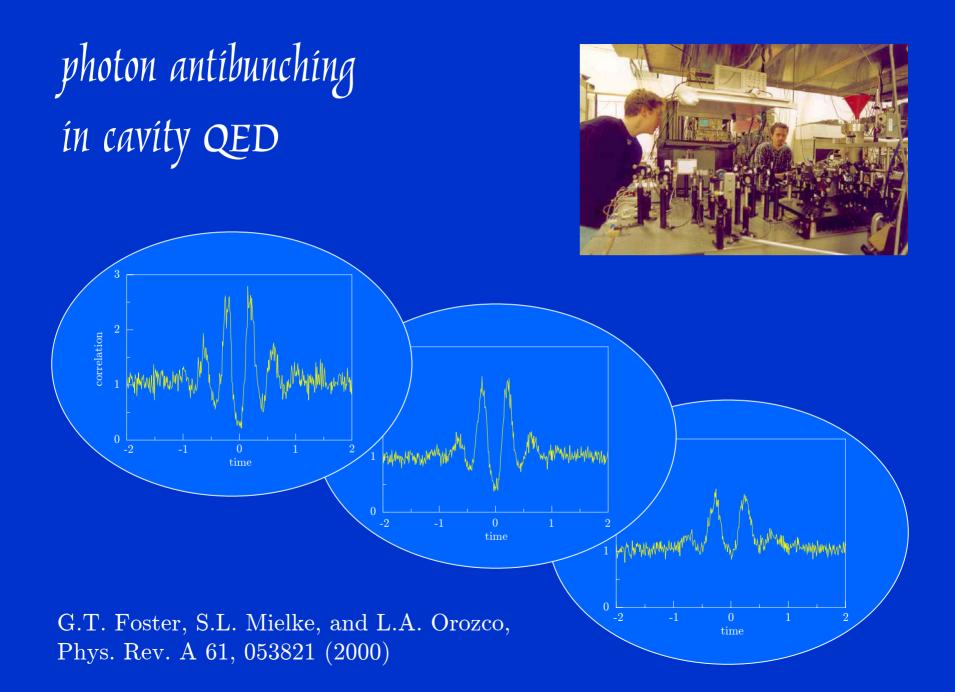


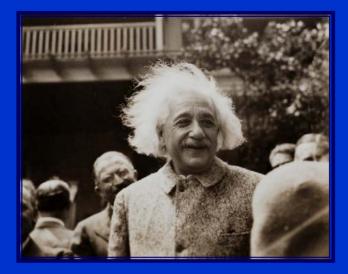
Bohr Kramers Slater

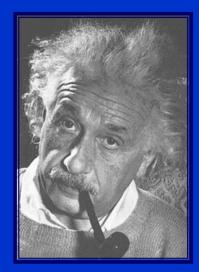


 $(\Delta n)^2 = \langle n \rangle + \langle n \rangle^2$ particle wave term term

 $rac{\langle n^2
angle - \langle n
angle}{\langle n
angle^2} = 1 + ext{variance of wave fluctuation}$







"All these 50 years of pondering have not brought me closer to answering the question 'What are light-quanta'."

letter to Michelle Besso, 1951
Einstein, A., 1951, letter to M. Besso, December 12
in Speziali, P., 1972, editor, *Albert Einstein Michelle Besso correspondence 1903-1955* (Herman, Paris), p.453