Chapter 12

Degenerate Ideal Fermi Gases

Applications of the properties of a degenerate fermion gas are relevant to (i) the electrons in a metal and (ii) the interiors of very dense stars (white dwarf and neutron stars).

12.1 The completely degenerate limit

In the completely degenerate limit, which corresponds to zero thermal motions, all the lowest energy states of the particles are occupied, up to a maximum energy, called the Fermi energy, and all the higher energy states are unoccupied. Let the spin degeneracy for each state be g, with $g = 2$ in all cases of interest to us. Then the occupation number for a completely degenerate Fermi gas is

$$n(\mathbf{p}) = \begin{cases} g & \text{for } p < p_F, \\ 0 & \text{for } p > p_F, \end{cases} \tag{12.1}$$

where p_F is the Fermi momentum, which is the momentium corresponding to the Fermi energy. Now consider the limit $T \to 0$ of the FD distribution (11.8), viz., $n(\varepsilon) = g/\{\exp[(\varepsilon - \mu)/kT] - 1\}$. This gives $n(\varepsilon) = g$ for $\varepsilon < \mu$ and $n(\varepsilon) = 0$ for $\varepsilon > \mu$. Comparison with (12.1) shows that the Fermi energy is equal to the chemical potential, $\mu = \varepsilon_F$, for a completely degenerate Fermi gas.

The relativistically correct relation between the energy and momentum is $\varepsilon = (m^2c^4 + p^2c^2)^{1/2}$. However, it is convenient to define the Fermi energy to be zero when the Fermi momentum is zero, so that (12.1) applies. This requires that one subtract the rest energy in writing

$$\varepsilon_F = (m^2c^4 + p_F^2c^2)^{1/2} - mc^2. \tag{12.2}$$

In order to treat both relativistic and nonrelativistic particles together, it is helpful to introduce hyperbolic functions by writing

$$p = mc \sinh \chi, \qquad \varepsilon = mc^2 \cosh \chi, \tag{12.3}$$

where the relation $\cosh^2\chi - \sinh^2\chi = 1$ is used. Let $\chi = \chi_F$ correspond to the Fermi energy. A more convenient quantity in the following is the parameter $\xi = 4\chi_F$. Then one has

$$p_F = mc\sinh\chi_F = mc\sinh(\xi/4), \qquad \varepsilon_F = mc^2[\cosh(\xi/4) - 1]. \tag{12.4}$$

The quantities N, U and P may be evaluated relatively simply for the distribution (12.1). The number of particle corresponding to the distribution (12.1) is found by integrating over all of the 6-dimensional phase space:

$$N = \int \frac{d^3\mathbf{x}d^3\mathbf{p}}{(2\pi\hbar)^3}\, n(\mathbf{p}) = \frac{4\pi gV}{3}\left(\frac{p_F}{2\pi\hbar}\right)^3 = \frac{4\pi gV}{3}\left(\frac{mc}{2\pi\hbar}\right)^3 \sinh^3(\xi/4). \tag{12.5}$$

The internal energy is given by a similar integral, with an extra factor $\varepsilon - mc^2$, corresponding to the kinetic energy per particle in the integrand. The integral in this case is most easily performed by changing the variable of integration from p to χ, with $dp = mc\cosh\chi\, d\chi$. One finds

$$\begin{aligned} \frac{U}{V} &= \frac{4\pi g}{(2\pi\hbar)^3}\int_0^{p_F} dpp^2\,[(m^2c^4 + p^2c^2)^{1/2} - mc^2] \\ &= \frac{4\pi g}{32(2\pi\hbar)^3}\, m^4c^5(\sinh\xi - \xi). \end{aligned} \tag{12.6}$$

The pressure may be determined from $\Omega = -PV$, as in (11.14). This gives

$$\Omega = -\frac{4\pi gV}{3(2\pi\hbar)^3}\int_0^{p_F} dpp^3\,\frac{d\varepsilon}{dp} = -\frac{4\pi g}{(2\pi\hbar)^3}\,\frac{m^4c^5}{32}\left[\frac{1}{3}\sinh\xi - \frac{8}{3}\sinh(\xi/2) + \xi\right]. \tag{12.7}$$

In the nonrelativistic (NR) and ultrarelativistic (UR) limits it is much simpler to perform the integrals, cf. Exercise 12.1). The result obtained, either directly, or from (12.7) with (12.5) is

$$P_0 = \begin{cases} \dfrac{\hbar^2}{5m}\left(\dfrac{6\pi^2}{g}\right)^{2/3}\left(\dfrac{N}{V}\right)^{5/3} & \text{NR}, \\[2ex] \dfrac{2\pi\hbar c}{4}\left(\dfrac{3}{4\pi g}\right)^{1/3}\left(\dfrac{N}{V}\right)^{4/3} & \text{UR}. \end{cases} \tag{12.8}$$

The thermodynamic potential has a numerical value determined by $\Omega_0 = -P_0V$, but it must be expressed as a function of μ. To write down the form of Ω_0 for a completely degenerate Fermi gas, one uses (12.5) and $\mu = \varepsilon_F$. Hence one finds

$$\Omega_0 = -\frac{4\pi gV}{3(2\pi\hbar)^3}\begin{cases} \dfrac{(2m\mu)^{5/2}}{5m} & \text{NR}, \\[2ex] \dfrac{\mu^4}{4c^3} & \text{UR}, \end{cases} \tag{12.9}$$

where the subscript 0 indicates that the result applies only at $T = 0$.

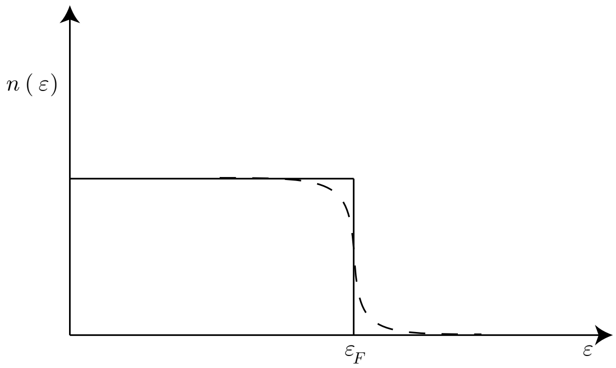

Figure 12.1: The occupation number for a completely degenerate Fermi gas is illustrated (solid line) and how it it modified in the almost degenerate limit (dashed line).

12.2 The almost degenerate limit

The completely degenerate limit corresponds to $T \to 0$. One may define a Fermi temperature in terms of the Fermi energy by writing $T_F = \varepsilon_F/k$. According to (12.5), p_F and ε_F are determined by the value of the number density $n = N/V$. It is convenient to assume that p_F, ε_F and T_F are defined in terms of n for any value of T. The degenerate limit corresponds to $T \ll T_F$, and the nondegenerate limit to $T \gg T_F$. In the almost degenerate limit one assumes $T \ll T_F$ and expands in powers of T/T_F. The distribution function is then of the form illustrated schematically in Figure 12.1.

For a Fermi gas at $T = 0$ the chemical potential is equal to the Fermi energy, $\mu = \varepsilon_F$. This is not the case for $T \neq 0$. As T increases at fixed n, μ decreases, passes through zero and becomes large and negative in the nondegenerate limit. In the almost degenerate limit, μ is approximately equal to ε_F and the actual expansion made is in powers of kT/μ.

In the almost degenerate limit we wish to expand an integral of the form

$$I = \int_0^\infty d\varepsilon \, \frac{F(\varepsilon)}{e^{(\varepsilon-\mu)/kT} + 1} \tag{12.10}$$

in powers of kT/μ. The first step is to change the variable of integration to $z = (\varepsilon - \mu)/kT$, so that one has $\varepsilon = \mu + kTz$, and then one assumes that kTz is a small correction, and one expands in terms of it. After some algebra one finds

$$I = \int_0^\infty d\varepsilon \, F(\varepsilon) + (kT)^2 F'(\mu) \int_0^\infty dz \, \frac{z}{e^z + 1} + \frac{1}{3}(kT)^4 F'''(\mu) \int_0^\infty dz \, \frac{z^3}{e^z + 1} + \cdots , \tag{12.11}$$

where a prime denotes a derivative. The integrals may be evaluated explicitly, cf. Exercise 12.2), to give

$$I = \int_0^\infty d\varepsilon\, F(\varepsilon) + \frac{\pi^2}{6}(kT)^2 F'(\mu) + \frac{7\pi^4}{360}(kT)^4 F'''(\mu) + \cdots, \tag{12.12}$$

which is the desired expansion.

In the case of the evaluation of the thermodynamic potential, Ω, the specific integrals that appear have $F(\varepsilon) = \varepsilon^{3/2}$ and $F(\varepsilon) = \varepsilon$ in the NR and UR cases, respectively. One finds

$$\Omega = \Omega_0 \begin{cases} 1 + \dfrac{5\pi^2}{8}\left(\dfrac{kT}{\mu}\right)^2 - \dfrac{7\pi^4}{384}\left(\dfrac{kT}{\mu}\right)^4 + \cdots & \text{NR,} \\ 1 + 2\pi^2\left(\dfrac{kT}{\mu}\right)^2 - \dfrac{7\pi^4}{40}\left(\dfrac{kT}{\mu}\right)^4 + \cdots & \text{UR.} \end{cases} \tag{12.13}$$

where Ω_0 is given by (12.9). The value of μ may be found by inserting (12.13) into $N = -\partial\Omega/\partial\mu$, expressing the resulting value of N/V in terms of ε_F using (12.5). This gives an expansion of ε_F in powers of μ, which may be inverted to find an expansion of μ in powers of ε_F, cf. Exercise 12.3).

12.3 The Richardson effect

Historically, one of the earliest successes of FD statistics relates to the theory of thermionic emission from metals. Let us compare the predictions of classical theory (MB statistics) with the theory assuming that the electrons in a metal constitute an almost degenerate Fermi gas.

Due to their thermal motions, some electrons can always escape from a metal. Inside the metal the electrons are at a uniform potential, $-W$, where W is the work function of the metal. Let the z-axis be normal to the surface. Then electrons with $p_z^2/2m > W$ should be able to escape. The number of electrons that leave unit area of the surface in unit time is ($g = 2$)

$$R = \int\limits_{p_z > (2mW)^{1/2}} \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\, \frac{p_z}{m}\, n(\mathbf{p}). \tag{12.14}$$

In the classical and Fermi cases one has

$$n(\mathbf{p}) = \begin{cases} 2e^{(\mu-\varepsilon)/kT} & \text{(MB),} \\ \dfrac{2}{e^{(\varepsilon-\mu)/kT} + 1} & \text{(FD).} \end{cases} \tag{12.15}$$

For $\varepsilon \geq W \gg kT$, which is the case of relevance in (12.14), the FD distribution is indistinguishable from the MB distribution. (That is, even for an almost degenerate Fermi gas, the high energy tail of the distribution is of the same

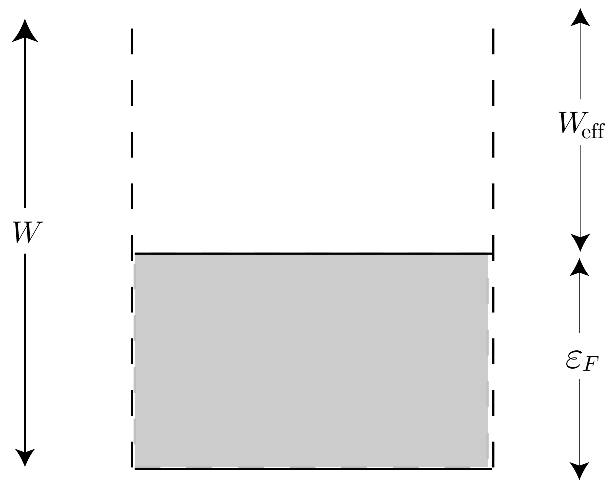

Figure 12.2: The work function for a metal, illustrating how the effective value $W_{\rm eff}$ in the degenerate Fermi case differs from the classical value, W.

form as for a classical thermal gas.) Hence, in either case the integral in (12.14) gives

$$R = \frac{4\pi m(kT)^2}{(2\pi\hbar)^3}\, e^{(\mu-W)/kT}. \tag{12.16}$$

The interpretation of (12.16) is quite different in the two cases. In the case of a classical gas one has, cf. (11.10),

$$e^{\mu/kT} = \frac{N\lambda_T^3}{2V}, \tag{12.17}$$

so that μ is large and negative, whereas in the degenerate Fermi case $\mu = \varepsilon_F$ is positive.

The work function, W, may be measured experimentally, by applying an electric field to the metal surface and observing how the rate, R, of thermionic emission changes as a function of the potential energy, $e\phi$, of this field. The prediction of the classical theory, $R \propto n_e T^{1/2} e^{-W/kT}$, is not compatible with the experimental data. The FD case implies a smaller effective work function, $W_{\rm eff} = W - \varepsilon_F$, than does the classical theory, as illustrated in Figure 12.2. The FD case also implies a different functional form $R \propto T^2 e^{-W_{\rm eff}/kT}$. It is found that ε_F can be a sizable fraction of W, so that the effective work function can be much less than classical theory would imply.

12.4 White dwarf stars

Another major success of the theory of degenerate Fermi gases was in the theory of white dwarf stars. White dwarf stars are supported against their self-gravity by degenerate electron pressure. The basic theory for white dwarf stars was worked out by Chandrasekhar in the 1930s, when he was a student on a ship from India to England to begin his PhD studies at Cambridge.

In any star, the gravitational force is balanced by a pressure gradient. In an ordinary star, called a "main sequence" star, thermal pressure balances gravity. Thermal pressure requires that the star be maintained at a high temperature, and because a hot body loses energy through its thermal ("black body") radiation, an energy source is required. That source is nuclear energy in a main sequence star. When its energy source is exhausted, a main sequence star must die, and leave some form of remnant star. The remnants is usually either a white dwarf or neutron star either of which are supported by degeneracy pressure, which does not require an ongoing energy source. It is accepted that stars less than about four times as massive as the Sun leave a white dwarf as a remnant. More massive stars explode as supernovas and leave behind a remnant that is usually a neutron star. The only other possibility is that the star goes into continuous collapse under its own self gravity, leading to a black hole.

The relation between the gravitational field and the mass density is of the same form for all stars. Let $\psi(r)$ be the gravitational potential of a (spherically symmetric) star, and let $\rho(r)$ be its mass density, where r is the radial coordinate. The radial component of the gravitational acceleration is then $-d\psi(r)/dr$. The contribution to this acceleration from the gravitation force due to a shell of matter of thickness dr, volume $4\pi r^2 dr$ and hence mass $4\pi\rho(r)r^2 dr$ is $-4\pi G\,\rho(r)r^2 dr$, where $G = 6.67\times 10^{-11}\,\mathrm{N\,m^2kg^{-2}}$ is Newton's constant. Hence, the gravitational potential is determined by

$$\nabla^2\psi(r) = \frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\psi(r)}{dr}\right] = -4\pi G\,\rho(r). \tag{12.18}$$

Equation (12.18) describes only how the gravitational field is related to the mass profile. A model for a main sequence star requires three other equations that relate the pressure, temperature and energy flux to the properties of the matter. One of these equation relates the pressure gradient to the gravitational force:

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \tag{12.19}$$

where

$$M(r) = 4\pi\int_0^r dr'\,r'^2\,\rho(r') \tag{12.20}$$

is the mass inside a radius r. The radius of the star is R and its mass is $M = M(R)$. The boundary conditions are $dP(r)/dr = 0$ at $r = 0$ and $P(R) = 0$. These equations also apply to a white dwarf star. Although (12.19) applies to a white dwarf star, it is simpler to impose the condition that the star be in force balance using a different argument.

In a white dwarf star the only important source of pressure is the completely degenerate electrons. The pressure is given by (12.7). We may express $P(r)$ in terms of the Fermi energy $\varepsilon_{\mathrm{F}}(r)$ by eliminating the variable ξ between (12.4) and

(12.7). At $T = 0$, $\varepsilon_{\rm F}$ may be interpreted as the local chemical potential. There is another contribution to the chemical potential. Recall that the chemical potential is the energy required to add a particle to the system. To add a particle at radius r one also needs to give it the gravitational potential energy required by a particle at that radius. Let this potential be $U(r)$, which is just the mass time $\psi(r)$. Each electron has a proton or another ion associated with it, such that the star is charge neutral. The appropriate mass associated with an electron added to the system is the mean mass per electron $\zeta_e m_H$, where m_H is the mass of a hydrogen atom. The chemical potential for electrons inside the star must be a constant in equilibrium:

$$\mu(r) = \varepsilon_{\rm F}(r) + \zeta_e m_H \psi(r) = \text{constant}. \tag{12.21}$$

The mass density is $\rho(r) = \zeta_e m_H n_e(r)$, with $n_e = N/V$. Equation (12.21) is more convenient than (12.19) for describing the equilibrium of a white dwarf star.

Using (12.21), one may express $\psi(r)$ in terms of ε_F on the left hand side of (12.18). The right hand side involves only constants and the number density of electrons, $n_e = N/V$, which may be written in terms of ε_F using (12.5). In this way (12.18) is reduced to a differential equation for ε_F as a function of r. It is convenient to introduce dimensionless variables by writing $\xi = r/R$ and combining the other constants to define an energy ε_c, and writing $\varepsilon_F(r) = \varepsilon_c f(\xi)$. In this way, (12.18) reduces to

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2 \frac{df(\xi)}{d\xi}\right] = \begin{cases} -[f(\xi)]^{3/2} & \text{NR}, \\ -[f(\xi)]^3 & \text{UR}, \end{cases} \tag{12.22}$$

with

$$\varepsilon_c = \begin{cases} \dfrac{3^2(2\pi\hbar)^6}{2^{13}\pi^4\zeta_e^4 m_H^4 m_e^{9/2} G^3 R^6} & \text{NR}, \\[2ex] \left[\dfrac{3(2\pi\hbar)^3 c^3}{32\pi^2\zeta_e^2 m_H^2 G^3 R^2}\right]^{1/2} & \text{UR}. \end{cases} \tag{12.23}$$

One needs to solve (12.22) numerically, with the boundary conditions $f'(0) = 0$ and $f(1) = 0$. Chandrasekhar found

$$f(0) = \begin{cases} 178.2 & \text{NR}, \\ 6.897 & \text{UR}, \end{cases} \qquad f'(1) = \begin{cases} -132.4 & \text{NR}, \\ -2.018 & \text{UR}. \end{cases} \tag{12.24}$$

The numerical solution may be integrated to find the mass of the star.

Apart from a numerical value, the mass of the star may be estimated from the central density of the star times its volume. The central density is $\zeta_e m_H \varepsilon_c f(0)$, and the volume is $4\pi R^3/3$. In the NR case one has $M \propto \varepsilon_c^{3/2} R^3 \propto 1/R^3$, and in the UR case one has $M \propto \varepsilon_c^3 R^3$ which is independent of R. This leads to the surprising conclusion that as the mass of a white dwarf star increases its radius decreases ($M \propto 1/R^3$) until the Fermi energy becomes relativistic, when

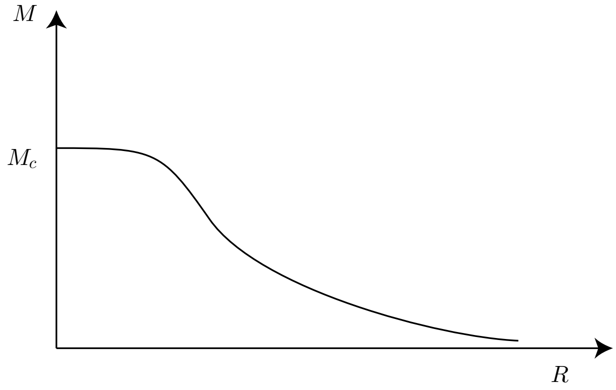

Figure 12.3: The variation of the mass of a white dwarf star is illustrated schematically as a function of radius.

the mass approaches a constant, cf. Figure 12.3. This limiting mass is called the Chandrasekhar mass. Its value is

$$M_c = \frac{3.1}{\zeta_e^2 m_H^2} \left(\frac{\hbar c}{G} \right)^{3/2} = \frac{1.45}{(\zeta_e/2)^2} M_\odot, \tag{12.25}$$

where $M_\odot$ is the mass of the Sun, and where $\zeta_e = 2$ corresponds to fully ionized He^4, C^{12} or O^{16}, which are the likely constituents of white dwarf stars. No white dwarf star can exist with a mass greater than this limiting value. Most observed white dwarf stars have a mass between $\sim 0.5 M_\odot$ and $M_\odot$.

Exercise Set 12

12.1). Evaluate the integral in equations (12.6) and (12.7) by performing the integrals in terms of p in the nonrelativistic ($p_F \ll mc$) and ultrarelativistic ($p_F \gg mc$) separately. Show that your answers reproduces the expressions obtained from (12.6) and (12.7) by assuming $\sinh\xi \approx \xi \ll 1$ and $\sinh\xi \approx \frac{1}{2}e^{\xi} \gg 1$ in these two limits, respectively.

12.2) Fill in the steps between (12.11) and (12.12) by using the integral

$$\int_0^\infty dz\, \frac{z^{x-1}}{e^z+1} = \Gamma(x)\zeta(x), \qquad (E12.1)$$

where the gamma function satisfies

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(1/2) = \sqrt{\pi}, \qquad (E12.2)$$

and hence $\Gamma(3/2) = \sqrt{\pi}/2$ and $\Gamma(5/2) = 3\sqrt{\pi}/4$, and where the Riemann zeta function

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \qquad (E12.3)$$

for n an integer has the value

$$\zeta(2n) = \frac{2^{2n}-1}{2n}\,\pi^{2n} B_n, \qquad (E12.4)$$

where B_n are the Bernoulli numbers,

$$B_1 = \frac{1}{6}, \qquad B_2 = \frac{1}{30}, \qquad B_3 = \frac{1}{42}, \qquad \cdots. \qquad (E12.5)$$

12.3) Show that the chemical potential for an almost degenerate, nonrelativistic Fermi gas is related to the Fermi energy by

$$\mu = \varepsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2 + \cdots \right]. \qquad (E12.6)$$