Stock Price Volatility and the Equity Premium

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Abstract

A dynamic general equilibrium model of stock prices is developed which yields a stock price volatility and equity premium that are close to the historical values. Non-observability of the expected dividend growth rate introduces an element of learning which increases the volatility of stock price. Calibration to the U.S. dividend and consumption processes yield interest rate and stock price processes that conform closely to the styled facts for U.S. capital market.

Key Words: equity premium, volatility, learning

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I. Introduction

The determination of stock prices and equilibrium expected rates of return in a general equilibrium setting is still imperfectly understood. First, as Grossman and Shiller (1981) and others have argued, stock returns appear to be too volatile given the relatively smooth process for aggregate dividends and consumption. Over the period 1871-1996 the standard deviation of real annual continuously compounded stock returns in the U.S. was 17.4 percent, while the corresponding standard deviation for dividend growth was only 12.9 percent, and for consumption growth 3.44 percent. Secondly, it is argued that stock returns are too high relative to the return on short term debt to be explicable without assuming a very high level of risk aversion on the part of the representative agent: this phenomenon has been labeled the "equity premium puzzle." The apparent discrepancy between the low volatility of consumption and dividend growth rates and the high volatility associated with stock returns is a fundamental element of the equity premium puzzle, for unless we can understand why equities are risky we cannot hope to understand why they command a significant risk premium.[1]

In this paper we develop a dynamic general equilibrium model of stock prices which yields a stock price volatility and equity premium that are close to the historical values. Dividends are assumed to follow a stochastic process with a growth rate that is not observable by the representative agent but must be estimated from the realized growth rates of dividends and aggregate consumption. This non-observability of the growth rate of the dividend process introduces an element of learning into the stock valuation process which increases the volatility of the stock price and resolves the apparent discrepancy between the high volatility of stock prices and the low volatility of dividends and consumption.

Mehra and Prescott (1985) were the first to claim that the smoothness of consumption and dividend growth gives rise to an equity premium puzzle because they found that it is apparently impossible to explain the equity risk premium in a representative agent setting without an implausibly high level of risk aversion. The early work on the equity premium puzzle that followed Mehra and Prescott's classic paper, treated dividends on the equity security as either identical to aggregate consumption, or as perfectly correlated with aggregate consumption.[2] For example, Benninga and Protopapadakis (1990) and Kandel and Stambaugh (1990) were able to match the first and second moments of consumption growth and riskless and equity returns by treating equity as a levered claim on aggregate consumption and choosing the degree of leverage to match the volatility of stock returns; however, as Cecchetti, Lam and Mark (1993) point out, these authors are implicitly allowing the ratio of dividends to consumption to vary in order to match the volatility of stock returns, and the assumed variation is much greater than the data on consumption and dividends reveal. As Campbell (1996) remarks, "it is hard to produce sufficient variation in stock prices without excessive variation in expected consumption (dividend) growth and in riskless real interest rates."

A more partial approach to the equity premium puzzle is followed by authors who ignore the relation between the dividends on the equity security and its price, and instead take the volatility of the equity return as exogenous in order to consider utility specifications that will yield the equity premium. Examples include Epstein and Zin (1991) who adopt a recursive utility specification, and Constantinides (1990) who develops a model with habit formation.[3] However, the volatility of equity returns is endogenous since it reflects the volatility of stock prices, and not the volatility of some exogenously determined production process, and therefore it is important to understand whether the volatility of equity returns is consistent with a rational model.

In a paper that is related to this one, Cecchetti et. al. (1993) endogenize the price of the equity security and distinguish between the flows to equityholders which they represent by dividends, and aggregate consumption. Matching the moments of the joint dividend-consumption process by a bivariate lognormal process with a two-state Markov drift, they are able to match the risk free rate and the estimated equity premium when account is taken of the estimation error in the latter (although with a negative rate of impatience). However, they are not able to match both the first and the second moments of the equity return and risk free rate process, concluding that "the model is rejected because of its inability to match simultaneously the relatively low standard deviation of the risk free rate and the relatively high standard deviation of the equity premium." Abel (1994) considers a production economy in which there is non-traded labor income as well as the equity security, and technological uncertainty follows a Markov switching regime, but finds that the Markov switching regime exacerbates the equity premium puzzle.[4] Thus, these authors who attempt to endogenize the volatility of equity prices, are unable to calibrate their models to the high volatility of stock returns relative to the low volatility of dividend growth rates.

A common feature of almost all of the prior analyses is that they have been conducted under the assumption of perfect information; investors are assumed to know the process generating dividends or returns. Yet it seems clear that investors are uncertain about the stochastic process generating stock returns and consumption. For example, there is considerable debate today as to whether or not we have entered a new regime in which corporate profits and payouts will grow at a faster rate than in the past.[5] Indeed, even Siegel and Thaler (1997) who treat the historical level of the equity premium as a puzzle, admit to some uncertainty about the current size of the premium and therefore about the expected rate of future dividend growth.[6]

Brennan (1998) has shown that uncertainty about the size of the equity premium, and the possibility of learning about it, can have a drastic effect on investor's allocation to equities, and Timmerman (1993, 1996) demonstrates that, in a simple setting in which the stock price is equal to the sum of expected future dividends discounted at an exogenous rate, changing expectations about future dividends arising from a simple learning heuristic can increase stock return volatility relative to a setting in which the stochastic process for dividends is known.[7]

In this paper we develop a model of stock prices and dividends in a stochastic dynamic general equilibrium setting with rational learning.[8] Instead of the Markov switching regimes used by Cechetti, Lam and Mark (1990, 1993), Kandel and Stambaugh(1989, 1990) and Abel (1994), we model the dividend as a lognormal process with a time varying expected growth rate that follows an Ornstein-Uhlenbeck process; the expected growth rate is not observable by agents but must be inferred or learned from the history of the observed dividend process and aggregate endowment. In addition to allowing a continuous state-space for the current state of the economy, this framework is intended to capture the notion that agents do not face a world in which the fundamentals evolve according to a known stochastic process; rather there is risk about the parameters of the stochastic process that is generating the fundamentals. In this model the agents are unsure about the medium term expected growth rate of dividends, although for analytical tractability we assume that they know the process for the true expected growth rate and, in particular, its long run value. In addition to increased realism, the model has the advantage over the above Markov switching models of allowing for correlation between innovations in the dividend process and innovations in the (investors' estimate of the) dividend growth rate. This induces a positive correlation between the dividend innovation and the innovation in the price-dividend ratio which increases the risk of the stock return, but is missing in the models of Cechetti et. al. (1990, 1993), Kandel and Stambaugh (1989, 1990) and Abel (1994), mentioned above. Further, by endogenizing the stock price we are able to show how the volatility of stock returns depends on the level of the investor's uncertainty about the current dividend growth rate. This allows us to reconcile the volatility of stock returns with the apparent smoothness of dividends. For simplicity, we assume that the endowment process of the single consumption good follows a lognormal process with constant parameters. This implies that the riskless interest rate is constant.[9] In order to obtain a reasonable value of the riskfree interest rate with a risk aversion parameter which is consistent with the historical premium, it is necessary to assume that the representative agent's rate of impatience is negative. The need to assume a negative rate of impatience is a well-known limitation of the representative agent paradigm, which ignores life cycle considerations and makes the counterfactual assumption that the representative agent has an infinite horizon.[10]

The model is developed in Section 2. Section 3 calibrates the model to data on consumption, dividends, interest rates and stock prices. Section 4 concludes.

II. The Model

Consider an infinite horizon pure exchange economy in which the instantaneous flow of the aggregate endowment of a single, non-storable, consumption good, Y, evolves according to a lognormal diffusion with constant coefficients:

$$d \ln Y = \theta dt + \sigma_Y dw_Y, \tag{1}$$

where σ_Y is a constant and dw_Y is the increment to a standard Wiener process.

We assume that *part* of the endowment is in the form of one unit of a tradable stock that pays a continuous dividend at the rate, D; the logarithm of the dividend rate, D, is assumed to follow an arithmetic Brownian motion with constant volatility but a stochastic mean, so that:

$$d \ln D = \mu dt + \sigma_D dw_D, \tag{2}$$

and μ is unobservable and follows an Ornstein-Uhlenbeck process:

$$d\mu = \kappa(\bar{\mu} - \mu)dt + \sigma_\mu dw_\mu, \quad (3)$$

where σ_μ and σ_D are constants and dw_μ and dw_D are, possibly correlated, increments to standard Wiener processes. This structure for the dividend process, in which the instantaneous expected rate of growth of the dividend flow is stochastic and reverts to a fixed long run value, $\bar{\mu}$, allows for either positive or negative autocorrelation in the dividend growth rate. The other parameters of the joint Markov process (1)-(3) are assumed to be known[11]. In addition to the risky security, there is an instantaneously riskless security whose rate of return at time t is denoted $r(t)$. The state of the endowment and dividend process at time t can be represented by the vector of state variables (Y, D, μ).

The remainder of the endowment flow, $L \equiv Y - D$, is assumed to be non-traded and can be thought of as labor income or the return to human capital. The stochastic process for L follows from the assumed processes for Y and D.

The economy is assumed to be populated by a single representative agent who acts as a price-taker. The agent has financial wealth $W = xP + z$ where x is the agent's share of the risky security, P is the price of the risky security, and z is the amount of riskless lending. The agent is assumed to maximize the expected value of a time-additive von-Neumann-Morgenstern utility function, so that the agent's lifetime expected utility is of the form:

$$\mathrm{E}_t \left[\int_t^\infty U\left(c_\tau, \tau\right) d\tau \right] = \mathrm{E}_t \int_t^\infty e^{-\beta\tau} u\left(c_\tau d\tau\right), \quad (4)$$

where the agent's wealth dynamics are given by:

$$dW = (W - xP)rdt + x(Ddt + dP) + (L - c)dt, \tag{5}$$

and L is the non-dividend part of the endowment. To rule out unbounded solutions associated with unlimited borrowing, we impose the constraint that the investor's financial wealth be bounded from below so that $W > -kY$, for some number $k > 0$.

The dividend rate, D, is a cash flow rate and is therefore observable. However, we assume that the instantaneous expected growth rate of dividends, μ, is not observed directly by agents, but must be estimated by them from the past values of D and Y. This assumption is motivated by the consideration that agents do not in fact know the future dividend growth rate for the economy, but must use historical information to assess it. If the growth rate were a constant then agents would eventually learn it to an arbitrary degree of accuracy. By assuming a stationary stochastic process for the growth rate we eliminate the possibility of complete information about it while ensuring that the stock price is well defined. The assumption of a known asymptotic growth rate, $\bar{\mu}$, is for analytic tractability.[12] The need to estimate μ introduces learning into the agent's decision problem. As Gennotte (1986) has shown in a similar setting, the investor's decision problem may then be decomposed into two separate problems: an inference problem in which the investor updates his estimate of the current value of the unobserved state variable, μ; and an optimization problem in which he uses his current estimate of μ to choose an optimal portfolio. We consider first the inference problem:

A. The Agent's Inference Problem

At time zero the agent is assumed to have a normal prior distribution over μ, with mean m_0 and variance, ν_0. As time evolves, μ changes according to (3) and the agent updates his assessment using the information from D and Y. Following Liptser and Shiryayev (1978), it is shown in Appendix A that the mean and variance of the conditional distribution of μ_t, m_t and ν_t, evolve according to:

$$dm = \kappa(\bar{\mu} - m)dt + \left(\nu a_{s1}^T + q_{\mu s}\right) q_{ss}^{-1} d\tilde{w}, \tag{6}$$

$$d\nu = \left[-2\kappa\nu + \sigma_\mu^2 - \left(\nu a_{s1}^T + q_{\mu s}\right) q_{ss}^{-1} \left(\nu a_{s1}^T + q_{\mu s}\right)^T\right] dt, \tag{7}$$

where

$$d\tilde{w} = ds - (a_{s0} + a_{s1} m)dt. \tag{8}$$

ds is the (2×1) vector of the dividend and consumption signals $(d \ln D, d \ln Y)^T$, and $d\tilde{w}$ is a (2×1) vector of the innovations in $\ln D$ and $\ln Y$, given the agent's information set at time t, $\mathcal{F}_t = D_s, Y_s, s < t$. a_{s0} and a_{s1} are (2×1) vectors, $q_{\mu s}$ is a (1×2) vector, and q_{ss} is a (2×2) matrix, which are all defined in Appendix A. Equation (7) is the usual Riccatti equation: this shows that ν, which represents the uncertainty in the investor's assessment of μ, evolves deterministically as the agent learns. In general, the agent will never learn μ perfectly, and there will be a steady state value of ν, $\nu^* > 0$, such that for $\nu_t = \nu^*$, $d\nu = 0$. ν^* represents the investor's uncertainty about μ in the long run after he has observed the joint evolution of D and Y. It is reasonable to posit that $\nu_0 > \nu^*$, and then ν is monotonically declining towards its asymptotic value of ν^*.

B. The Agent's Optimization Problem

Given the agent's information set, $\mathcal{F}_t$, the state of the economy is completely characterized by his current assessment of the instantaneous dividend growth rate, as measured by its mean, m_t, and variance, ν_t, the current rates of endowment and dividend flow, Y and D. Thus the investor's lifetime expected utility under the optimal policy may be written as:

$$J(W, m, Y, D, t) = \max_{x,c} E_t \int_t^\infty e^{-\beta\tau} u(c_\tau) d\tau. \tag{9}$$

We conjecture, what we will show below, that the price of the risky security, P, may be written as a function of the exogenous state variables: $P(W, m, Y, D, t)$.[13] Ito's Lemma then implies that its stochastic process is of the form:

$$\frac{dP}{P} = \alpha dt + \sigma_P dw_P \tag{10}$$

where α and σ_P are to be determined. Using (10), the agent's wealth dynamics (5) may be written as:

$$dW = [rW - xrP + xD + x\alpha P + L - c]\, dt + xP\sigma_P dw_P. \tag{11}$$

Then the condition for an interior optimum in the representative agent's decision problem (9) is:[14]

$$\begin{aligned} 0 &= \max_{x,c} [U(c,t) + J_W(rW - rPx + xD + xP\alpha - c + L) + J_m\kappa(\bar{\mu} - m) \\ &+ J_D\left(m + \frac{1}{2}\sigma_D^2\right) D + J_Y\left(\theta + \frac{1}{2}\sigma_Y^2\right) Y + J_t \\ &+ \frac{1}{2}J_{WW}x^2P^2\sigma_P^2(t) + \frac{1}{2}J_{mm}\sigma_m^2(t) + \frac{1}{2}J_{DD}D^2\sigma_D^2 + \frac{1}{2}J_{YY}Y^2\sigma_Y^2 \end{aligned}$$

$$
\begin{aligned}
&+ \quad J_{WY} xPY\sigma_{PY}(t) + J_{WD} xPD\sigma_{PD}(t) + J_{Wm} xP\sigma_{mP}(t) + J_{mD} D\sigma_{mD}(t) \\
&+ \quad J_{mY} Y\sigma_{mY}(t) + J_{YD}\sigma_{YD}] \qquad (12)
\end{aligned}
$$

where σ_{mP} is the instantaneous covariance between the innovations in m and P etc. The first order conditions corresponding to the choice of optimal consumption and investment, c^* and x^* are:

$$
\begin{aligned}
U_c &= J_W \qquad (13) \\
x^* &= -\frac{J_W}{J_{WW}}(P^2\sigma_P^2(t))^{-1}(P\alpha + D - rP) - \frac{J_{Wm}}{J_{WW}}(P^2\sigma_P^2(t))^{-1}P\sigma_{Pm}(t) \\
&- \frac{J_{WD}}{J_{WW}}(P^2\sigma_P^2(t))^{-1}PD\sigma_{PD}(t) - \frac{J_{WY}}{J_{WW}}(P^2\sigma_P^2(t))^{-1}PY\sigma_{PY}(t) \qquad (14)
\end{aligned}
$$

Following Breeden (1979), these first order conditions imply that α, the expected rate of appreciation in the price of the risky security, must satisfy the condition:

$$
T(c^*)\frac{P\alpha(t) + D - rP}{P} = \mathrm{Cov}_t\left(dc^*, \frac{dP}{P}\right) \qquad (15)
$$

where $T(c) \equiv -U_c(c)/U_{cc}(c)$, is the inverse of the investor's absolute risk aversion. Equation (15) states that, at the optimum, the covariance between the changes in the agent's optimal consumption rate, c^*, and the rate of the return on the risky asset is equal to the product of the agent's risk tolerance and the expected rate of return on the risky asset.

C. Market Equilibrium

We have assumed that the aggregate endowment flow, Y, includes the flow of dividends from the risky security, which is therefore in zero net supply. Then the market equilibrium conditions for the goods and securities markets respectively, are:

$$z^* = 0 \tag{16}$$

$$x^* = 0 \tag{17}$$

$$c^* = Y. \tag{18}$$

In order to obtain simple pricing functions we assume that each agent has an instantaneous utility function of the HARA class that can be written as:

$$U(c,t) = e^{-\beta t}\frac{c^{1-\gamma}}{1-\gamma}, \quad \text{for} \quad \gamma > 0. \tag{19}$$

Since $c^* = Y$, a necessary and sufficient condition for the representative agent's expected utility of lifetime consumption to be bounded under the utility function (19) is that the expected growth rate of the instantaneous utility be less than zero. The specific condition is given in the following theorem.

Theorem 1 *Condition A:*

$$(1-\gamma)\theta + \frac{1}{2}(1-\gamma)^2\sigma_Y^2 - \beta < 0 \tag{20}$$

is the necessary and sufficient condition for the representative agent's expected utility of lifetime consumption to be bounded under the utility function (19).

Proof: See Appendix B.

The representative agent's risk tolerance at equilibrium is $T(c^*) = \frac{c^*}{\gamma}$. Then, imposing the market clearing condition (17), we have the equilibrium rate of return condition:

$$\frac{P\alpha(t) + D - rP}{P} = \gamma \mathrm{Cov}_t \left(\frac{dY}{Y}, \frac{dP}{P} \right) \tag{21}$$

Then, as shown by Cox, Ingersoll and Ross (1985), the market clearing conditions (17)-(18) imply that the riskless interest rate, r, is given by:

$$r = \beta + \gamma\theta - \frac{1}{2}\gamma^2 \sigma_Y^2. \tag{22}$$

Note that, as mentioned above, the interest rate in this model is constant, because, θ, the growth rate of the endowment process is constant. Thus, equations (21) and (22) determine the two prices, the price of the risky asset, P, and the interest rate, r.

D. The Equilibrium Price of the Risky Asset

In order to obtain an explicit expression for the price of the risky asset and the equity premium, we conjecture that the price of the risky asset depends only on D, m, and t, and that it is homogeneous of degree one in D; then it can be written as:

$$P(D, m, t) = p(m, t)D. \tag{23}$$

Since Ito's Lemma implies that:

$$\frac{dP}{P} = \frac{dp}{p} + \frac{dD}{D} + \frac{dp}{p}\frac{dD}{D}, \tag{24}$$

and the stochastic process for the price-dividend ratio is:

$$\frac{dp}{p} = \frac{p_m}{p}\kappa(\bar{\mu} - m) + \frac{1}{2}\frac{p_{mm}}{p}\sigma_m^2 + \frac{p_t}{p}dt + \frac{p_m}{p}\sigma_m dw_m. \tag{25}$$

The stochastic process for the price, P, may be written:

$$\begin{aligned} \frac{dP}{P} &= \left\{\frac{1}{2}\frac{p_{mm}}{p}\sigma_m^2 + \frac{p_m}{p}\left[\kappa(\bar{\mu} - m) + \sigma_{mD}\right] + \left(m + \frac{1}{2}\sigma_D^2\right) + \frac{p_t}{p}\right\} dt \\ &+ \frac{p_m}{p}\sigma_m dw_m + \sigma_D dw_D \equiv \alpha dt + \sigma_P dw_P. \end{aligned} \tag{26}$$

Then, using equations (1) and (26), the covariance between the change in aggregate consumption and the return on the risky security is,

$$\mathrm{Cov}\left(dY, dP/P\right) = Y\left(\sigma_{mY} p_m/p + \sigma_{DY}\right). \tag{27}$$

Finally, substituting for α and the covariance (27) in the equilibrium rate of return condition (21), the pricing conjecture is verified, and the price depends only on D, m, and t, as stated in the following theorem

Theorem 2 *The price of the risky asset at time t is given by*

$$P(D, m, t) = p(m, t)D, \tag{28}$$

where the price-dividend ratio $p(m, t)$ is the solution to the following partial differential equation:

$$A(t)p_{mm} + B(m, t)p_m + C(m)p + p_t + 1 = 0, \tag{29}$$

and:

$$A(t) = \frac{1}{2}\sigma_m^2(t), \tag{30}$$

$$B(m,t) = \kappa(\bar{\mu} - m) + \sigma_{mD}(t) - \gamma\sigma_{mY}(t), \tag{31}$$

$$C(m) = m + \frac{1}{2}\sigma_D^2 - \gamma\sigma_{YD} - r \tag{32}$$

It is shown in Appendix B that, in order for equation (29) to have a well-defined solution, it is necessary and sufficient that the expected long run growth rate in dividends satisfy the following growth condition, which requires that the expected asymptotic dividend growth rate under the equivalent martingale measure be less than the interest rate[15]:

Theorem 3 *Condition B*

$$\bar{\mu} + \frac{\sigma_{\mu D} - \gamma\sigma_{\mu Y}}{\kappa} + \frac{1}{2}\sigma_D^2 + \frac{\sigma_\mu^2}{2\kappa^2} - \gamma\sigma_{YD} < r, \tag{33}$$

is the condition that the equation (29) has a well-defined solution.

Proof: See Appendix B.

The coefficients of equation (29), the differential equation for the price-dividend ratio, depend on the coefficients of the joint stochastic process for the dividend and endowment under the investor's information set, equations (1), (2), and (6) - (8), as well as on the investor's risk aversion coefficient, γ. The coefficients depend on the coefficients of the stochastic process for μ, equation (3), only indirectly through equation (6), the process for m, which is the estimated value of μ.

The pricing function, $p(m,t)$, is time dependent because the precision of the agent's assessment of the current value of μ, ν_t^{-1}, increases over time as the agent learns. In the limit, as t becomes large, as shown in Appendix A, ν_t approaches a constant ν^*, and the coefficients of the partial differential equation (29) become time invariant; then $p_t(m,t)$ approaches zero and the equation becomes an ordinary differential equation whose solution we denote by $p(m)$. The boundary condition for equation (29) is then

$$\lim_{t\to\infty} p(m,t) = p(m), \tag{34}$$

where $p(m)$ is given by the following theorem:

Theorem 4 *(i) When $\nu_t = \nu^*$, the price-dividend ratio, $p(m)$, satisfies the ordinary differential equation:*

$$Ap_{mm} + B(m)p_m + C(m)p + 1 = 0, \tag{35}$$

where A, $B(m)$ and $C(m)$ are as given in the corresponding expressions below equation (29), with $\nu_t = \nu^*$.

(ii) Then, under Condition B, the price-dividend ratio is given by:

$$p(m) = \zeta_0 \int_0^\infty \exp\left\{\zeta_1 s + \zeta_2 e^{-\kappa s} + \zeta_3 e^{-2\kappa s}\right\} ds, \tag{36}$$

where

$$\zeta_0 = e^{-\frac{\mu^*-m}{\kappa} - 3\frac{\sigma_m^2}{4\kappa^3}}, \tag{37}$$

$$\zeta_1 = \frac{1}{2}\sigma_D^2 - \gamma\sigma_{DY} - r + \mu^* + \frac{\sigma_m^2}{2\kappa^2}, \tag{38}$$

$$
\begin{aligned}
\zeta_2 &= \frac{\mu^* - m}{\kappa} + \frac{\sigma_m^2}{\kappa^3}, && (39)\\
\zeta_3 &= -\frac{\sigma_m^2}{4\kappa^3}, && (40)\\
\mu^* &= \bar{\mu} + \frac{\sigma_{mD} - \gamma\sigma_{mY}}{\kappa}. && (41)
\end{aligned}
$$

It may be shown that expression (36) for the price-dividend ratio is equivalent to:

$$
p(m) = \mathrm{E}^Q \left\{ \int_0^\infty \frac{D(s)}{D(0)} e^{-rs} ds | \mathcal{F}_t \right\}, \tag{42}
$$

where E^Q denotes that expectations are taken with respect to the equivalent martingale measure under which:

$$
\begin{aligned}
d \ln D &= (m - \gamma\sigma_{DY})dt + \sigma_D d\tilde{w}_D^Q, && (43)\\
dm &= \kappa \left(\bar{\mu} - \gamma \frac{\sigma_{mY}}{\kappa} - m \right) dt + \sigma_m dw_m^Q, && (44)
\end{aligned}
$$

The equity premium and volatility of return are expressed in terms of the price-dividend ratio and its derivatives in the following theorem:

Theorem 5 *(i) It follows from equation (26) that the equity risk premium, $\pi(m,t)$, and volatility, σ_P, are stochastic, and depend on the investor's current assessment of the current growth rate in dividends, m, as well as calendar time:*

$$
\begin{aligned}
\pi(m,t) &= \alpha(m,t) + \frac{1}{p} - r \\
&= \frac{1}{2}\frac{p_{mm}}{p}\sigma_m^2(t) + \frac{p_m}{p}\left[\kappa(\bar{\mu} - m) + \sigma_{mD}(t)\right] \\
&+ \left(m + \frac{1}{2}\sigma_D^2 \right) + \frac{1}{p} + \frac{p_t}{p} - r, && (45)
\end{aligned}
$$

$$\sigma_P^2 \quad = \quad \left(\frac{p_m}{p}\right)^2 \sigma_m^2 + \sigma_D^2 + 2\left(\frac{p_m}{p}\right)\sigma_{mD}, \tag{46}$$

where $p(m,t)$ is the equilibrium price-dividend ratio.

(ii) Alternatively the equity risk premium, $\pi(m,t)$, may be expressed as

$$\pi(m) = \gamma\left(\frac{p_m}{p}\sigma_{mY} + \sigma_{DY}\right) \tag{47}$$

The third term in (45) is the expected growth rate in the dividend, while the fourth is the dividend yield. Thus, if m were non-stochastic ($\sigma_m = 0$) and equal to $\bar{\mu}$, then π would be a constant equal to the sum of the expected growth rate in dividends, $m + \frac{1}{2}\sigma_D^2$, and the dividend yield, $1/p$, less the riskless interest rate, r. Thus the first two terms arise from variation in m which gives rise to changes in the price-dividend ratio, p. The changes in m arise, as shown in equation (6), as the investor changes his assessment of the dividend growth rate μ, due to the mean reversion in μ and, more importantly, due to the inference he draws about μ from the innovations in the two "signals", the realized instantaneous growth rates in dividends and the aggregate endowment. Thus, the consumption risk of a security arises not only from the covariance of the innovations in its current dividend with the innovation in consumption, but also from the covariance between innovations in the investor's assessment of expected future cash flows (or, equivalently, in the growth rate) with the change in consumption. As the following lemma shows, σ_{mD} is increasing in ν, while σ_{mY} is independent of ν.

Lemma 1

$$\sigma_{mY} \quad = \quad \sigma_{\mu Y}, \tag{48}$$

$$\sigma_{mD} = \nu + \sigma_{\mu D}. \tag{49}$$

Note that the risk premium does not depend directly on σ_{mY}, or σ_{DY}, the covariances of innovations in the dividend and its estimated growth rate with the innovations in the endowment process: however, these covariances, which enter into the coefficients of the differential equation for the price-dividend ratio, do affect the risk premium indirectly through the $p(m, t)$ function and its derivatives.

E. Equilibrium Pricing in a Fully Observable Economy

It will be useful to compare the equity prices in our economy with those in a fully observable economy in which the representative investor knows not only D and the parameters of the joint stochastic process for consumption and dividends (1) - (3), but also the current dividend growth rate, μ. In the fully observable economy the price of the risky security will be of the form $P(D, \mu) \equiv q(\mu)D$; then, imposing the equilibrium return condition (15), the price dividend ratio, $q(\mu)$, satisfies

$$Aq_{\mu\mu} + B(\mu)q_{\mu} + C(\mu)q + 1 = 0, \tag{50}$$

where

$$A = \frac{1}{2}\sigma_{\mu}^2, \tag{51}$$

$$B(\mu) = \kappa(\bar{\mu} - \mu) + \sigma_{\mu D} - \gamma\sigma_{\mu Y}, \tag{52}$$

$$C(\mu) = \mu + \frac{1}{2}\sigma_D^2 - \gamma\sigma_{DY} - r. \tag{53}$$

III. Model Calibration

In order to assess the ability of the model to generate an endogenous stock price process and interest rate with properties similar to those of the empirical data, the stochastic processes for dividends and consumption, equations (1) - (3), were calibrated to the moments of the joint distribution of consumption and dividends reported in Table 1, which are based on the Shiller (1989) data. Expressions for the following six unconditional moments were calculated as shown in Appendix C: the mean and variance of *per capita* consumption growth; the mean, variance and first order autocorrelation of dividend growth; the covariance between innovations in consumption and dividends. The consumption process (1) was chosen so that the autocorrelation in consumption growth rates is zero. The values of θ and σ_Y follow directly from the first two moments of the consumption process, and $\bar{\mu}$ was set equal to the mean change in the log dividend. Then the six parameters of the stochastic process for dividends, and the correlations between innovations in μ, $\ln D$, and $\ln C$, σ_D, κ, σ_μ, $\sigma_{\mu D}$, $\sigma_{\mu C}$, σ_{DC}, were chosen to match the values of the three empirical moments: $\mathrm{Var}(\ln D_t - \ln D_{t-1})$, $\mathrm{Cov}(\ln D_t - \ln D_{t-1}, \ln C_t - \ln C_{t-1})$, and $\mathrm{Cov}(\ln D_t - \ln D_{t-1}, \ln D_{t-1} - \ln D_{t-2})$. Since this allows three degrees of freedom in the choice of parameters, different values of κ, $\sigma_{\mu D}$, and $\sigma_{\mu C}$ were prespecified and then σ_D, σ_μ, and σ_{DC} were chosen to satisfy the three moment conditions. The differential equation (35) was solved for the above-described parameters of the joint dividend/consumption stochastic process, and for values of the investor taste parameters γ equal to 15 and β equal -0.10. These values of the taste parameters were chosen to yield the historical risk premium and risk-free rate.

Table 2 reports statistics for selected combinations of the prespecified parameter values that satisfy the growth condition, *Condition B*. The selected combinations, or models, are given in Columns (a) - (c) of the table. Columns (d) - (f) show for each model, the values of σ_D, σ_μ, and σ_{DY} that are consistent with the matched moments of the joint process for dividends and consumption. Column (g) reports the steady state value of the standard error of the estimate of μ, $\sqrt{\nu^*}$. Columns (h)- (k) report the endogenously determined values of the dividend yield, equity premium, return volatility, and the correlation between the stock return and the innovation in consumption growth when the true dividend growth rate, μ, is not directly observed, the current estimate of the growth rate is $m = 2\%$, and the learning steady state has been reached so that the pricing function $p(m)$ is time-independent. In order to assess the extent to which the results are due to the fact that investors are continuously learning about the current value of μ from their observations of the dividend process, the pricing function for each model was obtained for the corresponding fully observable economy, by solving numerically the ordinary differential equation (50): the results are shown in columns (l) - (o).

Observe first that it is possible to find a wide range of parameter values that are consistent with the joint process for dividends and consumption and which, with learning in the steady state, yield stock price volatility of the order of 16-20%, an equity premium of the order of $5-6$ percent, a risk free rate of 2.5 percent, and a dividend yield of $3-5$ percent, and all with a risk aversion parameter of 15. For example, we can generate an economy (line viii) in which, for $m = 2\%$, the stock price volatility is 17.2 percent, the equity premium is 6.06 percent, the dividend yield is 5.2 percent, and the risk free rate is 2.5 percent, which compare with the corresponding historical averages of 17.4 percent, 5.75 percent, 4.8 percent, and 1.74 percent.[16] Thus, in this example the endogenous characteristics of the model stock and bond returns are very close to their empirical counterparts, while the moments of the dividend and consumption processes are equal to their empirical counterparts by construction. Moreover, the model correlations between stock returns and consumption, which range between 0.49 and 0.69, straddle the empirical value of 0.51. This example clearly demonstrates that the apparent high volatility of stock prices compared with the apparent smoothness of dividends is insufficient reason *per se* for concluding that stock prices are "too volatile" to be supported by a rational model.

The reason for the high volatility of the stock price in the model is that small changes in the slowly mean-reverting assessment of the growth rate, m, have relatively large consequences for stock prices. Figure 1 plots, for *Model viii*, the price-dividend ratios as functions of the current assessment of the instantaneous dividend growth rate both when there is learning, $p(m)$, and for the corresponding fully observable economy, $q(\mu)$. The price-dividend ratio is an increasing convex function of m, rising from 6.1 when $m = -15\%$ to 76 when $m = 15\%$. Perhaps surprisingly, the price-dividend ratio is generally lower in the fully observable economy. The reason for this is that the aggregate social risk in the two economies is the same, being determined by σ_Y, while expected future dividends are a convex function of the current growth rate, μ, so that uncertainty about that growth rate increases the value of the expectation.

$\sqrt{\nu^*}$, the steady state value of the standard error of the estimate of the current dividend growth rate, ranges from 1.9 percent to 7.1 percent for the different models. By comparing columns (i) and (m) it can be seen that the effect of learning in the economy with a non-observable dividend growth rate is to increase the *equity premium* slightly; the effect is small because $\sigma_{mY} = \sigma_{\mu Y}$ so that, as can be seen from equation (47), the only effect of the unobservability of μ on the covariance of the stock return with consumption arises from its effect on (p_m/p) which Figure 1 shows to be small. On the other hand, the effect of the unobservability of μ on the *volatility* of stock returns (columns (j) and (n)) is large in many cases; for example in *Model viii* the volatility of stock returns is only 12.5 percent when μ is observable, while in the corresponding economy with learning the volatility is 17.2 percent. This is because the volatility depends on σ_{mD} which, as shown in Lemma 1, is increasing in ν. Thus, it is plausible that the high level of stock volatility observed in the data is due to learning effects of the type that we have modeled here. In what follows we select *Model viii* for further analysis.

Figure 2 shows that for *Model viii* the equity premium is an approximately linear increasing function of the current assessment of the growth rate m. Since the dividend yield is inversely related to m, Figure 3 shows that in this model the equity premium (and since the interest rate is constant, the expected return on stocks) is decreasing in the dividend yield.

The model volatility and equity premiums reported in Table 2 are conditional on the assumed dividend growth rate, m, of 2 percent. Therefore, to gain further insight into the properties of the model, we take *Model viii* of Table 2 and use it to generate sample paths of dividends and stock prices for 1000 years. The representative investor is assumed to begin the 1000 years with a prior distribution on the dividend growth rate which is normally distributed with mean of 2 percent and standard deviation of 3.7 percent and then the consumption and dividend rates are assumed to change according to a (monthly) discrete version of the assumed stochastic process (1)-(3); then m, the estimated growth rate of dividends is updated each month using discrete versions of equations (6) and (7). The mean and standard deviation of the simulated excess returns were 5.4 percent and 17.2 percent which correspond to the historical statistics of 5.75 percent and 18.7 percent respectively. The 1000 years was then broken down into ten 100 year subperiods. In each subperiod the dividend rate was normalized to start at $\$1$ per year. The first ten panels of Figure 4 show the simulated end-of year real prices and dividends over the ten 100 year periods on a log scale. Panel (k) shows the (normalized) historical values of real log dividends and stock prices from Shiller (1989) for the period 1890-1985, and panel l shows the same historical series of log dividends, but with the prices constructed as the product of the historical dividends and the price-dividend ratio yielded by *Model viii* with Bayesian updating of the estimated real dividend growth rate from the historical path of dividends and consumption: we call these the "*Model viii* historical prices."[17] The charts are quite similar in appearance.

Since the *Model viii* historical prices reflect only the limited information set of the past dividend series, while the historical prices reflect (under the assumption of market efficiency) a possibly broader information set, we might expect the historical returns to forecast the *Model viii* historical returns. To test this, the *Model viii* historical real returns were regressed on the contemporaneous and lagged historical real market returns, and the results are reported in Panel A of Table 3. As expected, the coefficient of the lagged market return is highly significant while the coefficient of the contemporaneous market return is insignificant. Since the *Model viii* returns are driven by the dividend changes, the regression is consistent with the findings of Kothari and Shanken (1992) that market returns forecast one year ahead dividend changes. Figure 5 plots the historical annual nominal dividend and price series lagged one year and the *Model viii* nominal historical price series - the latter is obtained by applying the *Model viii* price-dividend ratio to the nominal dividend series. The two price series track each other quite well except for a period from 1955 to 1975 when the actual prices exceed the model prices for an extended period. Figure 6 plots the lagged actual return against the *Model viii* return, both expressed in nominal terms and provides visual confirmation of the strong relation between the model return and the lagged actual return.

Panel B of Table 3 reports the results of regressions, following Hodrick (1992), of log real annual returns on the dividend yield at the start of the year. Line (i) reports the results for the whole 1000 year simulation period. Line (ii) reports results using the historical data from 1890 to 1985. Line (iii) reports the results using the *Model viii* returns and dividend yields. Consistent with the flat relation between the equity premium and dividend yield shown in Figure 3, there is no significant relation between the simulated real annual return and lagged dividend yield. The historical data and the *Model viii* historical data yield similar, positive but weak, relations between returns and dividend yield.[18]

Panel C of Table 3 reports the results of regressions, following Fama and French (1988b), of real annual returns on lagged annual returns. Line (i) reports the results for the whole 1000 year simulation period. Line (ii) reports results using the historical data from 1890 to 1985. Line (iii) reports the results using the *Model viii* returns and dividend yields. The simulated data reveal no evidence of serial dependence. For both the historical data and the *Model viii* historical data there is a positive but insignificant coefficient on the lag 1 return and a negative but insignificant coefficient on the lag 2 return. In summary, the simulated model data and the historical data are qualitatively similar.[19]

IV. Conclusion

We have shown in this paper that, by distinguishing between the cash flows received by equity holders and aggregate consumption, and by allowing for learning about a stochastic but unobservable dividend growth rate, it is possible to generate a representative agent model with rational behavior which yields a stock price series whose first two moments closely match those of the historical price series and yet which is consistent with the relatively smooth behavior of the dividend and consumption series.

The high volatility of the empirical stock price series combined with the low volatility of the dividend series has suggested to some scholars that either prices are "too volatile" to be consistent with a rational model or that there are time varying components to expected returns. We have been able to generate the empirical stock price volatility from a model based on the actual dividend volatility within a rational model. That of course is not to say that there are not in fact time-varying components to expected returns, and extension of the current model to allow for time-varying interest rates and learning with respect to the consumption process seems worthwhile.

We have been able to generate a theoretical equity risk premium of around six percent, and model returns conditional on the realized path of dividends whose first and second moments are very close to those of the realized returns over the period 1890-1985.We have done this in a model real interest rate of 2.5 percent which is close to the historical average of 1.74 percent. Perhaps the most debatable element of our model is the treatment of tastes. We have assumed a coefficient of relative risk aversion for the representative agent of fifteen. Several authors, including Mehra and Prescott (1985) and Grossman and Shiller (1981), have argued that any figure for relative risk aversion above ten is unreasonable. However, Kandel and Stambaugh (1991) point out that intuitions about reasonable levels of risk aversion are typically derived either from the Friend and Blume (1975) study or from thought experiments. The Friend and Blume study implicitly assumes that consumption and stock returns are perfectly correlated and equally volatile which is clearly counterfactual, while "inferences about (risk aversion) are perhaps most elusive when pursued in the introspective context of thought experiments. It seems possible in such experiments to choose the size of the gamble so that any value of γ seems unreasonable." A further reason for distrusting the results of thought experiments is that the willingness to bear (stock market) risk can be reduced by the kind of "background risk" to which real world agents are routinely exposed.[20] As Fama (1991) argues, " ..a large equity premium is not necessarily a puzzle; high risk aversion (or low intertemporal elasticity of substitution) may be a fact."

V. Appendices

A. The Investor's Inference Problem

Each agent's prior distribution over the initial value of μ is assumed to be Gaussian. Then since μ follows an Ornstein-Uhlenbeck process (equation (3)), and the innovations in the unobservable state variable μ and the signals, $\ln Y$ and $\ln D$, follow a joint Brownian motion, the distribution of the state variable at time t, $F_t(x) = Prob(\mu_t \leq x|\mathcal{F}_t)$ is also Gaussian given the information set at time t, $\mathcal{F}_t = \{s_\tau : \tau < t\}$ where $s_t = \{\ln Y_t, \ln D_t\}$. Therefore, the agent's inference problem is a special case of the nonlinear filtering problem addressed by Theorem 12.1 in Liptser & Shiryayev (1978).

To make our setup conform to the notation of the theorem, we restate the structure of the economy as:

$$ds = (a_{s0} + a_{s1}\mu)dt + \sigma_s dw_s, \tag{A1}$$

$$d\mu = (a_{\mu 0} + a_{\mu 1}\mu)dt + \sigma_\mu dw_\mu, \tag{A2}$$

where $ds \equiv (d\ln Y, d\ln D)'$ stands for the signals that investors can use to infer the state variable μ, and $dw_s \equiv (dw_Y, dw_D)'$ is the vector of signal innovations. The parameters of this joint stochastic process are $a_{s0} = (\theta, 0)'$, $a_{s1} = (0, 1)'$, $a_{\mu 0} = \kappa\bar{\mu}$, $a_{\mu 1} = -\kappa$ and $\sigma_s = \begin{pmatrix} \sigma_Y & 0 \\ 0 & \sigma_D \end{pmatrix}$, which are all assumed to be known constants to investors. We also define $q_{\mu s} = (\sigma_{\mu Y}, \sigma_{\mu D})$ as the covariance matrix between the signals and the state variable, and $q_{ss} = \begin{pmatrix} \sigma_Y^2 & \sigma_{DY} \\ \sigma_{DY} & \sigma_D^2 \end{pmatrix}$ as the variance-covariance matrix of the signals, where σ_{ij} stands for the covariance between the innovations in variables i and j.

Let $m_t = \mathrm{E}(\mu|\mathcal{F}_t)$ and $\nu_t = \mathrm{E}((\mu - m_t)^2|\mathcal{F}_t)$ be the conditional mean and variance of the conditional distribution $F_t(x) = Prob(\mu_t \leq x|\mathcal{F}_t)$. Then it follows directly from Theorem 12.1 in Liptser & Shiryayev (1978) that m_t and ν_t satisfy equations:

$$
\begin{aligned}
dm &= (a_{\mu 0} + a_{\mu 1} m)dt + (\nu a_{s1}^T + q_{\mu s}) q_{ss}^{-1} d\tilde{w} \\
&= (\kappa\bar{\mu} - \kappa m)dt + \nu_1 d\tilde{w}_D + \nu_2 d\tilde{w}_Y \\
&= \kappa(\bar{\mu} - m)dt + \sigma_m dw_m \qquad \text{(A3)} \\
d\nu &= \left[-2\kappa\nu + \sigma_\mu^2 - (\nu a_{s1}^T + q_{\mu s}) q_{ss}^{-1} (\nu a_{s1}^T + q_{\mu s})^T\right] dt \qquad \text{(A4)}
\end{aligned}
$$

where, with ρ_{ij} denoting the correlation coefficient between the innovations in variables i and j,

$$
\begin{aligned}
\nu_1 &= \frac{\nu + \rho_{\mu D}\sigma_D\sigma_\mu - \rho_{DY}\rho_{\mu Y}\sigma_D\sigma_\mu}{(1-\rho_{DY}^2)\sigma_D}, \qquad \text{(A5)} \\
\nu_2 &= \frac{-\rho_{DY}\nu + \rho_{\mu Y}\sigma_D\sigma_\mu - \rho_{DY}\rho_{\mu D}\sigma_D\sigma_\mu}{(1-\rho_{DY}^2)\sigma_D}, \qquad \text{(A6)} \\
d\tilde{w} &= ds - (a_{s0} + a_{s1} m)dt. \qquad \text{(A7)}
\end{aligned}
$$

$d\tilde{w}$ is a (2×1) vector, and stands for the unexpected realization of the signal $ds = (d\ln Y, d\ln D)^T$, which means that

$$d\tilde{w}_D = d\ln D - mdt, \qquad \text{(A8)}$$

and

$$d\tilde{w}_Y = d\ln Y - \theta dt = dw_Y. \qquad \text{(A9)}$$

Therefore, the variance of the estimator, m, for μ, is given by $\sigma_m^2 = \nu_1^2 + \nu_2^2 + 2\rho_{DY}\nu_1\nu_2$.

When $d\nu = 0$, we can solve for $\nu = \nu^*$ in equation (A4), which is given by:

$$\nu^* = -b + \sqrt{b^2 - c}, \tag{A10}$$

where

$$\begin{aligned} b &= \rho_{\mu D}\sigma_\mu\sigma_D - \rho_{\mu Y}\rho_{DY}\sigma_D\sigma_\mu + \kappa(1 - \rho_{DY}^2)\sigma_D^2, && \text{(A11)} \\ c &= \sigma_D^2\sigma_\mu^2(\rho_{\mu D}^2 + \rho_{\mu Y}^2 + \rho_{DY}^2 - 2\rho_{\mu Y}\rho_{\mu D}\rho_{DY} - 1). && \text{(A12)} \end{aligned}$$

Note that the correlation coefficients between three inter-dependent random variables have to satisfy $c \leq 0$ to ensure that each of them is between -1 and 1 given the other two correlation coefficients. As a result, $\nu^* \geq 0$ is always guaranteed.

B. Proofs of *Conditions A and B*

B.1 Proof of *Condition A*

To prove *Condition A*, first write down the dynamics of the representative agent's utility function using Ito's Lemma:

$$dU = e^{-\beta t} c^{-\gamma} dc + (-\beta) e^{-\beta t} \frac{c^{1-\gamma}}{1-\gamma} dt + \frac{1}{2} e^{-\beta t} (-\gamma) c^{-\gamma-1} (dc)^2. \tag{B1}$$

In the equilibrium, $c^* = Y$, so that the equilibrium process for the optimal consumption is

$$d \ln c^* = \theta dt + \sigma_Y dw_Y. \tag{B2}$$

Using (B2) in (B1), we have that

$$\frac{dU^*}{U^*} = \left[(1-\gamma)\theta + \frac{1}{2}(1-\gamma)^2 \sigma_Y^2 - \beta \right] dt + (1-\gamma)\sigma_Y dw_Y, \tag{B3}$$

Or

$$d \ln U^* = \left[(1-\gamma)\theta - \beta \right] dt + (1-\gamma)\sigma_Y dw_Y. \tag{B4}$$

The representative agent's optimal expected utility of lifetime consumption is thus given by:

$$\begin{aligned}
& \max \mathrm{E}\left[\int_0^\infty e^{-\beta t} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] = \mathrm{E}\left[\int_0^\infty U^*(s) ds \right] \\
= \; & \mathrm{E}\left[\int_0^\infty U^*(0) \exp\left\{ ((1-\gamma)\theta - \beta)s + (1-\gamma)\sigma_Y \int_0^\infty dw_Y(s) \right\} ds \right] \\
= \; & U^*(0) \int_0^\infty \exp\left\{ \left[(1-\gamma)\theta + \frac{1}{2}(1-\gamma)^2 \sigma_Y^2 - \beta \right] s \right\} ds
\end{aligned} \tag{B5}$$

Equation (B5) yields a finite value if and only if $(1-\gamma)\theta+\frac{1}{2}(1-\gamma)^2\sigma_Y^2-\beta<0$ or *Condition A* holds. Under *Condition A*, the expected utility of lifetime consumption is $-\frac{U^*(0)}{(1-\gamma)\theta+\frac{1}{2}(1-\gamma)^2\sigma_Y^2-\beta}$.

B.2 Proof of *Condition B*

To obtain an expression for the price of the risky asset, we first conjecture that the price is separable in D, i.e.,

$$P(D,m,t)=p(m,t)D. \tag{B6}$$

From Ito's lemma, we can write

$$\frac{dP}{P}=\frac{dp}{p}+\frac{dD}{D}+\frac{dp}{p}\frac{dD}{D}, \tag{B7}$$

where $dp(m,t)$ is given by

$$dp=p_m dm+\frac{1}{2}p_{mm}(dm)^2+p_t \tag{B8}$$

Using equations (B6)-(B8) and (A3) in the equilibrium rate of return condition (15) yields the following partial differential equation for the price-dividend ratio:

$$A(t)p_{mm}+B(m,t)p_m+C(m)p+p_t+1=0. \tag{B9}$$

This equation has the form of a *Cauchy problem*, and we assume that the technical conditions are satisfied so that the Feynman-Kac solution exists.[21] The Feynman-Kac formula implies that the solution may be written as:

$$p(m,t) \;=\; \mathrm{E}^{m,t}\left[\int_0^\infty \exp\left(\int_t^\tau C(m)ds\right)d\tau|m_t\right]$$

$$
\begin{aligned}
&= \mathrm{E}^{m,t}\left[\int_0^\infty \exp\left(\int_t^\tau \left(m + \frac{1}{2}\sigma_D^2 - \gamma\sigma_{DY} - r\right) ds\right) d\tau | m_t\right] \\
&= \int_t^\infty \left[\exp\left(\left(\frac{1}{2}\sigma_D^2 - \gamma\sigma_{DY} - r\right)(\tau - t)\right) \mathrm{E}^{m,t}\left(e^{\int_t^\tau m_s ds} | m_t\right)\right] d\tau \\
&= \int_t^\infty \left[\exp\left(\left(\frac{1}{2}\sigma_D^2 - \gamma\sigma_{DY} - r\right)(\tau - t)\right)\right. \\
\times &\left.\exp\left(\mathrm{E}^{m,t}\left(\int_t^\tau m_s ds | m_t\right) + \frac{1}{2} Var^{m,t}\left(\int_t^\tau m_s ds | m_t\right)\right)\right] d\tau \qquad \text{(B10)}
\end{aligned}
$$

where $E^{m,t}$ indicates that m is assumed to solve the following stochastic differential equation with initial condition m_t at time t.

$$
\begin{aligned}
dm &= B(m,t)dt + \sqrt{2A(m,t)}dw_m \\
&= \kappa(\mu^* - m)dt + \nu_1 d\tilde{w}_Y + \nu_2 d\tilde{w}_D \qquad \text{(B11)}
\end{aligned}
$$

where $\mu^* = \bar{\mu} + \frac{\sigma_{mD} - \gamma\sigma_{mY}}{\kappa}$ is the adjusted long run mean of μ under the new measure, under which the expectation is taken. Note that this new measure is different from the risk neutral (or equivalent martingale) measure. The drift term of every process under the equivalent martingale measure is reduced by its risk premium[22] from its original value under the true probability measure. Let Q denotes the equivalent martingale measure. Then the log dividend follows the process:

$$
d \ln D = (m - \gamma\sigma_{DY})dt + \sigma_D d\tilde{w}_D^Q, \qquad \text{(B12)}
$$

and the process for m is given by:

$$
dm = \kappa(\bar{m} - m)dt + \sigma_m dw_m^Q \qquad \text{(B13)}
$$

and the long run mean under the equivalent martingale measure is given by

$$\bar{m} = \bar{\mu} - \frac{\gamma\sigma_{mY}}{\kappa}. \tag{B14}$$

In the steady state when $\nu = \nu^*$, $A(m,t)$ and $B(m,t)$ are time-independent. Then (B11) implies:

$$\mathrm{E}^{m,t}\left(\int_t^\tau m_s ds|m_t\right) = \frac{\mu^* - m_t}{\kappa}\left(e^{-\kappa(\tau-t)} - 1\right) + \mu^*(\tau - t), \tag{B15}$$

and

$$\mathrm{Var}^{m,t}\left(\int_t^\tau m_s ds|m_t\right) = \frac{\sigma_m^2}{2\kappa^3}\left(2\kappa(\tau-t) + 4e^{-\kappa(\tau-t)}e^{-2\kappa(\tau-t)} - 3\right). \tag{B16}$$

Substituting (B15) and (B16) into (B10), we obtain:

$$p(m) = \zeta_0\int_0^\infty \exp\left\{\zeta_1 s + \zeta_2 e^{-\kappa s} + \zeta_3 e^{-2\kappa s}\right\}ds, \tag{B17}$$

where ζ_0, ζ_1, ζ_2 and ζ_3 are defined in the text.

Since κ is positive, $\zeta_2 e^{-\kappa s}$ and $\zeta_3 e^{-2\kappa s}$ will not explode as $s \to \infty$, while ζ_1 will be negative under *Condition B* of the text:

$$\bar{\mu} + \frac{\sigma_{\mu D} - \gamma\sigma_{\mu Y}}{\kappa} + \frac{1}{2}\sigma_D^2 + \frac{\sigma_\mu^2}{2\kappa^2} - \gamma\sigma_{YD} - r < 0 \tag{B18}$$

It remains to be shown that p(m) is finite iff ζ_1 is negative.

Before we go to the proof, we first state the economic interpretation of *Condition B*. Note that

$$\begin{aligned}
\frac{\mathrm{E}^Q(\ln D_\tau - \ln D_t|m_t)}{\tau - t} &= \frac{\mathrm{E}^{m,t}\left(\int_t^\tau m_s ds|m_t\right)}{\tau - t} - \gamma\sigma_{DY} \\
&= \frac{\bar{m} - m_t}{\kappa(\tau - t)}\left(e^{-\kappa(\tau-t)} - 1\right) + \bar{m} - \gamma\sigma_{DY}, \qquad \text{(B19)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{Var}^Q(\ln D_\tau - \ln D_t|m_t)}{\tau - t} &= \sigma_D^2 + \frac{2\sigma_{mD}}{\kappa}\left(1 - \frac{1 - e^{-\kappa(\tau-t)}}{\kappa(\tau - t)}\right) \\
&+ \frac{\sigma_m^2}{\kappa^2}\left(1 - \frac{2(1 - e^{-\kappa(\tau-t)})}{\kappa(\tau - t)} + \frac{1 - e^{-2\kappa(\tau-t)}}{2\kappa(\tau - t)}\right). \qquad \text{(B20)}
\end{aligned}$$

Asymptotically,

$$\frac{\mathrm{E}^Q(\ln D_\tau - \ln D_t|m_t)}{\tau - t} \overset{\tau - t \to \infty}{\longrightarrow} \bar{m} - \gamma\sigma_{DY}, \qquad \text{(B21)}$$

$$\frac{\mathrm{Var}^Q(\ln D_\tau - \ln D_t|m_t)}{\tau - t} \overset{\tau - t \to \infty}{\longrightarrow} \sigma_D^2 + \frac{2\sigma_{mD}}{\kappa} + \frac{\sigma_m^2}{\kappa^2}. \qquad \text{(B22)}$$

The asymptotic dividend growth rate under the equivalent martingale measure, g^{asym}, is equal to its asymptotic mean plus one half its asymptotic variance, and is given by:

$$\begin{aligned}
g^{asym} &= \bar{m} - \gamma\sigma_{DY} + \frac{1}{2}\left(\sigma_D^2 + \frac{2\sigma_{mD}}{\kappa} + \frac{\sigma_m^2}{\kappa^2}\right) \\
&= \bar{\mu} - \frac{\gamma\sigma_{mY}}{\kappa} - \gamma\sigma_{DY} + \frac{1}{2}\sigma_D^2 + \frac{\sigma_{mD}}{\kappa} + \frac{\sigma_m^2}{2\kappa^2} \qquad \text{(B23)}
\end{aligned}$$

Therefore, *Condition B* is equivalent to requiring that the asymptotic dividend growth rate under the equivalent martingale measure, g^{asym}, is less than the risk free rate, r.

Now, we show that Condition B is a sufficient condition for the dividend-price ratio, $p(m)$, given in equation (B17) to be finite.

Theorem 6 *Condition B is a sufficient condition for the stock price, $p(m)$, given in equation (B17) to be finite.*

Proof: *Since $\lim_{t\to\infty} e^{\zeta_1 t} = 0$ iff $\zeta_1 < 0$, i.e., $\forall \epsilon > 0$, $\exists T_0$: $t \geq T_0 \Rightarrow e^{\zeta_1 t} < |\zeta_1|\epsilon$. This, in turn, implies that $-e^{\zeta_1 t} > \zeta_1 \epsilon$ for all $t \geq T_0$. Similarly, since $\lim \zeta_2 e^{-\kappa t} = 0$, and $\lim \zeta_3 e^{-2\kappa t} = 0$,*

$$\begin{aligned} \forall \epsilon > 0, \qquad & \exists T_1 : t \geq T_1 \Rightarrow \zeta_2 e^{-\kappa t} < \epsilon; \\ & \exists T_2 : t \geq T_2 \Rightarrow \zeta_3 e^{-2\kappa t} < \epsilon. \end{aligned}$$

Let $T_3 = \max(T_0, T_1, T_2)$. Then $\forall t \geq s \geq T_3$,

$$\begin{aligned} & \int_s^t \exp\left(\zeta_1 u + \zeta_2 e^{-\kappa u} + \zeta_3 e^{-2\kappa u}\right) du \\ \leq \;\; & \int_s^t e^{\zeta_1 u} e^{2\epsilon} du = \frac{e^{2\epsilon}}{\zeta_1}\left(e^{\zeta_1 t} - e^{\zeta_1 s}\right). \end{aligned} \tag{B24}$$

Since $T_3 \leq s \leq t$, it follows that $e^{\zeta_1 t} - e^{\zeta_1 s} \geq e^{\zeta_1 t} - e^{\zeta_1 T_3} \geq -e^{\zeta_1 T_3} \geq \zeta_1 \epsilon$, and this implies that $\frac{e^{\zeta_1 t} - e^{\zeta_1 s}}{\zeta_1} \leq \epsilon$. Therefore,

$$\int_s^t \exp\left(\zeta_1 u + \zeta_2 e^{-\kappa u} + \zeta_3 e^{-2\kappa u}\right) du \leq e^{2\epsilon}\epsilon. \tag{B25}$$

Thus, the function $f_t \equiv \int_0^t \exp\left(\zeta_1 u + \zeta_2 e^{-\kappa u} + \zeta_3 e^{-2\kappa u}\right) du$ is Cauchy, and the completeness of the real line implies that $\lim f_t$ exists and is given by:

$$\lim f = lim_{t\to\infty} f_t = \int_0^\infty \exp\left(\zeta_1 u + \zeta_2 e^{-\kappa u} + \zeta_3 e^{-2\kappa u}\right) du \tag{B26}$$

which is finite.

It is immediate that when $\zeta_1 \leq 0$ *the integral (B17) does not converge; therefore* $p(m)$ *is finite and given by expression (B17) under Condition B.*

C. Moments of the stochastic Processes

Under steady state when we assume $t \to \infty$, we can derive some theoretical moment conditions for our joint Markov process, and use them to calibrate our model to derive parameter values consistent with historical stylized empirical moments. The following are the moment conditions we use for this purpose.

1. Unconditional mean of the dividend innovation process:

$$\begin{aligned} \mathrm{E}[\ln D_t - \ln D_{t-1}] &= \frac{\bar{\mu} - m_0}{\kappa} e^{-\kappa(t-1)} \left(e^{-\kappa} - 1\right) + \bar{\mu} \\ &\xrightarrow{t\to\infty} \bar{\mu} \end{aligned} \tag{C1}$$

2. Unconditional variance of the dividend innovation process:

$$\begin{aligned} &\mathrm{Var}[\ln D_t - \ln D_{t-1}] \\ = \quad & \frac{\sigma_m^2}{2\kappa^3} \left(2\kappa - e^{-2\kappa t} - e^{-2\kappa(t-1)} + 2e^{-2\kappa t + \kappa} + 2e^{-\kappa} - 2\right) \\ + \quad & \sigma_D^2 + \frac{2\sigma_{mD}}{\kappa^2} \left(\kappa - 1 + e^{-\kappa}\right) \\ \xrightarrow{t\to\infty} \quad & \sigma_D^2 + 2\left(\frac{\sigma_m^2}{2\kappa^3} + \frac{\sigma_{mD}}{\kappa^2}\right) \left(\kappa - 1 + e^{-\kappa}\right) \end{aligned} \tag{C2}$$

3. Unconditional first order autocovariance of the innovation dividend process:

$$\begin{aligned} &\mathrm{Cov}(\ln D_t - \ln D_{t-1}, \ln D_{t-1} - \ln D_{t-2}) \\ = \quad & \frac{\sigma_{mD}}{\kappa^2} \left(e^{-2\kappa} - 2e^{-\kappa} + 1\right) \\ + \quad & \frac{\sigma_m^2}{2\kappa^3} \left[\left(e^{-2\kappa} - 2e^{-\kappa} + 1\right) + 2e^{-2\kappa(t-1)} - e^{-2\kappa t + \kappa} - e^{-2\kappa t + 3\kappa}\right] \\ \xrightarrow{t\to\infty} \quad & \left(\frac{\sigma_m^2}{2\kappa^3} + \frac{\sigma_{mD}}{\kappa^2}\right) \left(1 - e^{-\kappa}\right)^2 \end{aligned} \tag{C3}$$

4. Unconditional mean of innovations of the endowment process:

$$\mathrm{E}[\ln Y_t - \ln Y_{t-1}] = \theta \tag{C4}$$

5. Unconditional variance of innovations of the endowment process:

$$\mathrm{Var}[\ln Y_t - \ln Y_{t-1}] = \sigma_Y^2 \tag{C5}$$

6. Covariance between the innovations in the dividend and endowment processes:

$$\mathrm{Cov}(\ln D_t - \ln D_{t-1}, \ln C_t - \ln C_{t-1}) = \sigma_{DY} + \frac{\sigma_{mY}}{\kappa^2}\left(\kappa - 1 + e^{-\kappa}\right) \tag{C6}$$

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Footnotes

[1]Campbell (1996) claims that “the deeper puzzle is the high volatility of stock prices, which seems to be associated with predictable time-variation in excess stock returns.” This “excess volatility” is sometimes taken as evidence against market efficiency. Hagiwara and Herce (1997) state that “Empirical research carried out during the past 15 years has repeatedly found that news about future real dividends cannot account for the variability of stock prices observed in annual US data.”

[2]In contrast, Hagiwara and Herce (1997) measure the consumption of the representative agent by the dividend flow on the equity security.

[3]Constantinides(1990) actually assumes a production economy and identifies the return on equity with the return to a levered equity claim on a constant returns to scale production process. Ferson and Constantinides (1991) find that habit formation cannot explain the mean equity premium.

[4]Abel (1994) (as well as Kandel and Stambaugh (1989,1990)) makes the strong assumption that the innovation in the dividend is independent of the innovation in the dividend growth rate.

[5]“We must stress that our analysis has all been on historical data which suggest that the equity premium has been too large in the past...there is reason to believe that it is lower than it has been in the past..”. Siegel and Thaler (1997).

[6]A typical quote is: “Copious efforts have been made to explain the (high) valuation ratio away. It is suggested, for example, that economic growth - and so returns on capital - will be

higher than previously, or that intangible assets are more important than before. Neither of these arguments is compelling". Martin Wolf, *Financial Times*, October 7, 1997, p14.

[7]Barsky and de Long (1993) also develop a heuristic stock valuation model with stochastic dividend growth in which the discount rate is exogenous and investors ignore the fact that their current estimate of the dividend growth rate will be revised in the future.

[8]Kandel and Stambaugh (1996) analyze the implications of the predictability of stock returns for a risk-averse Bayesian investor who allocates his wealth between stocks and cash and uses available data to update his prior beliefs about returns. The one period investment horizon assumption eliminates the dynamic effects induced by learning. In a more closely related setup, Gennotte (1986) models an economy in which mean returns are unobservable and must be estimated, and shows how this affects the asset allocation. Brennan (1998) also treats the mean return of stock as an unobservable state variable. Everything else the same, investors tend to hold a much smaller percentage of their wealth in the risky asset.

[9]Campbell (1996) finds that it is easier to explain the equity risk premium when consumption growth rates are negatively autocorrelated, but concludes (p23) "It is troubling that the standard model (for which $C = D$) must rely so heavily on negative autocorrelation of consumption growth for which there is no strong evidence."

[10]Constantinides, Donaldson and Mehra (1997) model life-cycle considerations in an overlapping generations model. For a general discussion of the risk free rate "puzzle" see Kocherlakota (1996). Kocherlakota (1990) points out that a negative rate of impatience is consistent with the existence of equilibrium in an infinite horizon growth model.

[11]This simplifying assumption means that our model fails to capture significant aspects of the learning process.

[12]A more realistic assumption would be that agents have a prior distribution over $\bar{\mu}$ which they gradually update as more information becomes available; however this would add considerable modeling complexity. For small t the current assumption captures the notion that there may be considerable uncertainty about the growth rate in the medium term if not in the very long run.

[13]In writing the agent's lifetime expected utility in (9) we are assuming that the investor knows the form of the equilibrium pricing function $P(m, Y, D, t)$.

[14]A necessary and sufficient condition for the expected utility of lifetime consumption for the representative agent to be finite is given in *Condition A* below. Since the representative agent holds the whole of the risky asset and consumes the whole endowment, the constraint $W > -kY$ is not binding under his optimal policy, which therefore corresponds to an interior solution.

[15]This is the stochastic economy equivalent of the condition in the Gordon constant growth rate model that the discount rate exceed the growth rate.

[16]However, the current yield on the newly issued US indexed bonds is around 3.5 percent.

[17]Shiller reports dividends for the calendar year and prices for January of each year. We treat the (closing) price for a given year as the price in January of the following year.

[18]This is consistent with Fama and French (1988a) and Hodrick (1992). Goetzmann and Jorion (1993), and Kothari and Shanken (1997) offer a more circumspect interpretation of the evidence.

[19]It is important to recall that there are significant small sample biases when returns are regressed on their lagged values or on lagged dividend yields using ordinary least squares.

[20]Gollier and Pratt (1996).

[21]Please refer to Duffie (1988, 1996) for details.

[22]The risk premium in this economy is determined by equation (15) in the text.

Table 1: Empirical Moments of Consumption, Dividends, and Interest Rates

Source: Shiller (http://www.econ.yale.edu/ shiller)

Dividends (1871-1996)	
Log real dividend growth rate:	
Mean	1.55%
Standard deviation	12.9%
First order autocorrelation	0.07
Dividend Yield (1871-1996)	
Mean	4.8%
Standard deviation	1.2%
Returns (1871-1996)	
Log Real Returns	
Mean	6.99%
Standard deviation	17.43%
Equity Premium	
Mean	5.75%
Standard deviation	18.72%
Consumption (1889-1985)	
Log real consumption per capita growth rate	
Mean	1.69%
Standard deviation	3.44% (M-P 3.56%)*
Consumption-Correlations (1889-1985)	
Correlation between growth rates of $\ln D$ and $\ln C$	0.34
Correlation between growth rate of $\ln C$ and $\ln$(Stock Return)	0.51
Real Interest Rate(1889-1985)	
Mean	1.74%
Standard Deviation	5.57%

* Mehra and Prescott (1985)

Table 2: Equity Premium, Stock Volatility, Dividend Yield and Interest Rate for different Exogenous Parameter Values

The table shows values of the equity premia, stock volatility, dividend yield and interest rate for various parameter combinations for the joint process of dividends and consumption which are consistent with the empirical moments of dividend and consumption shown in Table 1. For all the examples shown, the risk aversion coefficient, γ, is equal to 15, the risk free interest rates is equal to 2.5%, the impatience parameter, β, is -0.10, and m, the current estimate of the dividend growth rate, is 2%. $\sqrt{\nu^*}$: the standard error of estimate of dividend growth rate, μ, in steady state. D/P: dividend yield; $\pi - r$: equity risk premium; σ_P: stock return volatility; ρ_{RC}: correlation between equity return and innovation in consumption.)

	Selected Parameters			Calculated Parameters			Model Outcomes								
							With Unobservable μ					With Observable μ			
	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)	(l)	(m)	(n)	(o)
	κ	$\sigma_{\mu Y}$	$\sigma_{\mu D}$	σ_D	σ_μ	σ_{DY}	ν^*	D/P	$\pi - r$	σ_P	ρ_{RC}	D/P	$\pi - r$	σ_P	ρ_{RC}
(i)	0.01	0.6	-0.5	12.4%	0.6%	0.34	2.7%	3.9%	5.12%	18.4%	0.54	4.5%	4.71%	11.0%	0.83
(ii)	0.01	0.6	-0.1	12.4	0.5	0.34	2.2	3.7	5.02	19.0	0.51	4.1	4.70	14.2	0.64
(iii)	0.02	0.4	-0.7	12.4	0.9	0.34	3.7	3.6	5.11	20.1	0.49	4.5	4.65	9.5	0.95
(iv)	0.02	0.7	-0.2	12.4	0.7	0.33	2.5	4.5	5.63	17.9	0.61	4.9	5.42	13.9	0.76
(v)	0.03	0.5	-0.5	12.4	1.1	0.33	3.5	4.3	5.48	18.8	0.56	4.8	5.22	12.2	0.83
(vi)	0.03	0.7	0.2	12.4	0.8	0.33	1.9	4.3	5.5	18.7	0.57	4.5	5.45	16.9	0.62
(vii)	0.04	0.7	-0.3	12.4	1.1	0.32	2.9	5.2	6.07	17.1	0.69	5.4	5.94	13.8	0.84
(viii)	**0.05**	**0.6**	**-0.5**	**12.4**	**1.1**	**0.32**	**3.7**	**5.2**	**6.06**	**17.2**	**0.68**	**5.6**	**5.91**	**12.5**	**0.92**
(ix)	0.05	0.8	0.0	12.4	1.1	0.32	2.1	5.2	6.09	17.1	0.69	5.3	6.04	15.8	0.74
(x)	0.13	0.4	-0.7	12.4	3.3	0.30	5.9	5.0	5.60	16.6	0.65	5.3	5.55	12.6	0.86
(xi)	0.16	0.3	-0.8	12.4	4.2	0.31	7.1	4.5	5.06	16.5	0.59	4.8	5.02	12.5	0.78
(xii)	0.18	0.8	0.2	12.4	1.8	0.30	1.8	4.7	5.18	16.2	0.62	4.7	5.18	16.0	0.63
(xiii)	0.20	0.6	-0.1	12.4	2.6	0.29	3.3	4.7	5.13	16.0	0.62	4.7	5.13	15.4	0.64

Table 3: Historical and Simulated Return Regressions

Panel A: Regression of log *Model viii* historical returns (R_{mviii}) on current and lagged log historical real market returns (R_M). (1890-1985)

$$\ln R_{mviii,t} = a_0 + a_1 \ln R_{M,t} + a_2 \ln R_{M,t-1}$$

	a_0	a_1	a_2	R^2	F-stat	Nobs
	0.023	0.045	0.703	0.50	46.77	95
	(1.56)	(0.62)	(9.62)			

Panel B: Regressions of log Real Annual Returns on Dividend Yields*

$$\ln R_t = a_0 + a_1 (D/P)_{t-1}$$

	a_0	a_1	R^2	F-stat	Nobs
(i) 1000 year simulation	0.002	0.059	0.000	0.31	999
	(0.28)	(0.55)			
(ii) Historical Data	-0.030	1.873	0.035	3.34	95
(1890-1985)	(0.55)	(1.83)			
(iii) *Model viii* Historical Data	-0.083	2.732	0.017	1.60	95
(1890-1985)	(0.69)	(1.26)			

Panel C: Regression of log Real Market Returns on Lagged Values

$$\ln R_{M,t} = a_0 + a_1 \ln R_{M,t-1} + a_2 \ln R_{M,t-2}$$

	a_0	a_1	a_2	R^2	F-stat	Nobs
(i) 1000 year simulation	0.005	0.048	0.002	0.0003	1.17	998
	(3.10)	(1.48)	(0.06)			
(ii) Historical Data	0.067	0.058	-0.167	0.031	1.44	94
(1890-1985)	(3.27)	(0.56)	(1.62)			
(iii) *Model viii* Historical Data	0.074	0.194	-0.178	0.037	2.81	94
(1890-1985)	(3.54)	(1.21)	(1.62)			

* The dividend yield is defined as the dividend for the previous year divided by the price for January of the current year. ** $t-$statistics in the parenthesis.

Legends

Figure 1: Price-Dividend Ratio and Current Assessed Dividend Growth Rate.

The figure shows the relation between the price-dividend ratio, $p(m)$, and m, the estimated dividend growth rate for *Model viii*, and the price-dividend ratio, $q(\mu)$, and μ for the corresponding fully observable economy.

Figure 2:. Equity Premium and Current Assessed Dividend Growth Rate.

The figure shows the relation between the steady state equity premium, $\pi(m)$, and m, the estimated dividend growth rate for *Model viii*.

Figure 3: Equity Premium and Dividend Yield.

The figure shows the relation between the steady state equity premium, $\pi(m)$, and $p(m)^{-1}$, the instantaneous dividend yield for *Model viii*.

Figure 4: Simulated and Historical Real Annual Stock Prices and Dividends.

Figures 4a - 4l show time series plots of 100 year samples of log real annual dividends (dotted) and stock prices (continuous) generated from Model vii starting with $D = 1$ and prior on μ normally distributed with mean 2% and variance $(3.7\%)^2$. For the plots the dividend is renormalized to $D = 1$ for each 100 year subsample. Figure 4k shows the historical log real annual dividends and stock prices for the period 1890-1995. Figure 4l shows the *Model viii* historical log real annual dividends and stock prices for the period 1890-1995, starting in 1890 with a prior on μ normally distributed with mean 2% and variance $(3.7\%)^2$.

Figure 5: Normalized Lagged Historical and *Model viii* **Historical Nominal Annual Stock Prices and Dividends (1890-1985).**

The figure shows the nominal annual dividends (crosses), the historical nominal annual stock prices (lagged one year) (continuous) and the *Model viii* (dotted) historical nominal annual stock prices starting in 1890 with a prior on μ with mean 2% and variance $(3.7\%)^2$.

Figure 6: Lagged Actual and *Model viii* Historical Nominal Annual Stock Returns (1890-1985).

The figure plots the lagged historical nominal annual returns (vertical axis) against the *Model viii* historical nominal annual returns (horizontal axis) which are constructed from the *Model viii* historical prices and the historical dividends.

Figure 1: Price-Dividend Ratio and Current Assessed Dividend Growth Rate

The figure shows the relation between the price-dividend ratio, $p(m)$, and m, the estimated dividend growth rate for *Model viii* (solid line), and the price-dividend ratio, $q(\mu)$, and μ for the corresponding fully observable economy (dashed line).

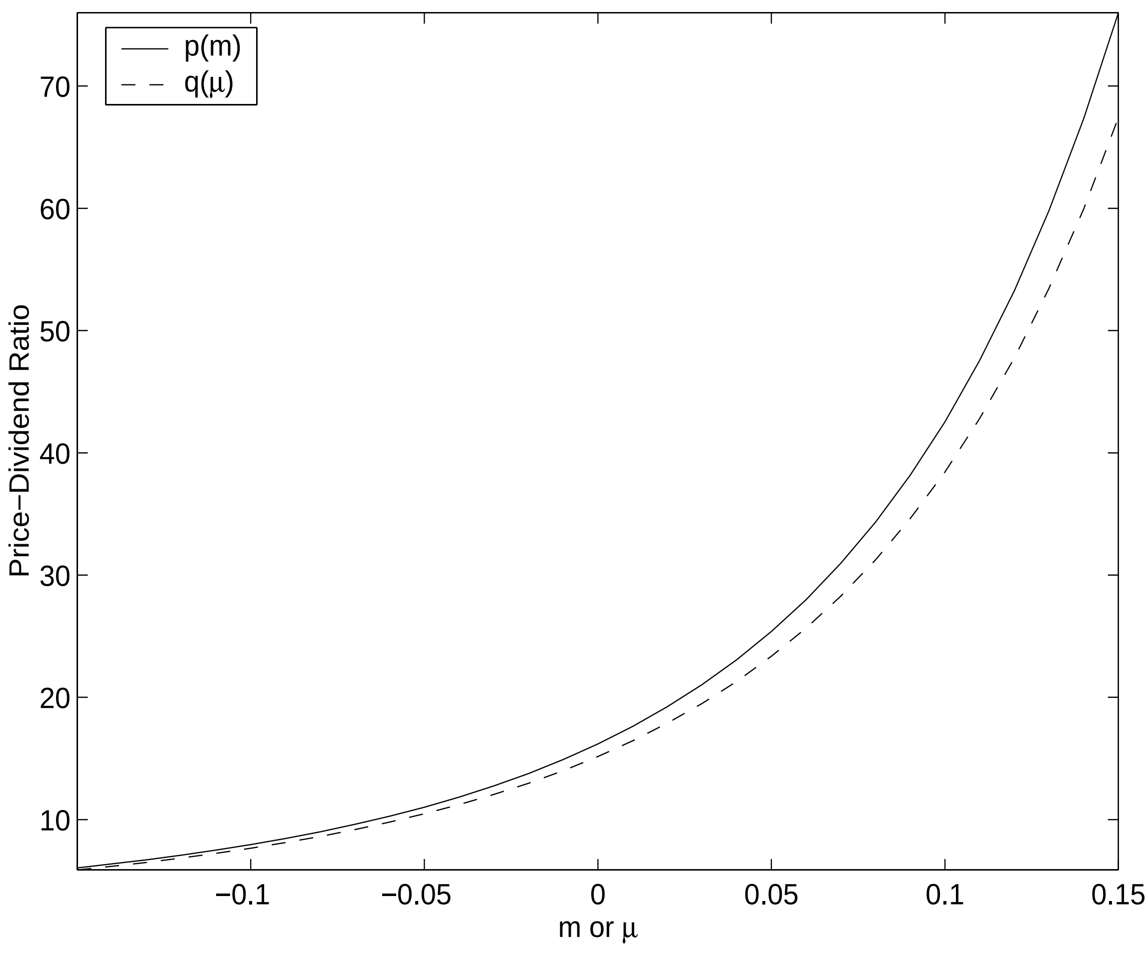


Figure 2: Equity Premium and Current Assessed Dividend Growth Rate

The figure shows the relation between the steady state equity premium, $\pi(m)$, and m, the estimated dividend growth rate for *Model viii*.

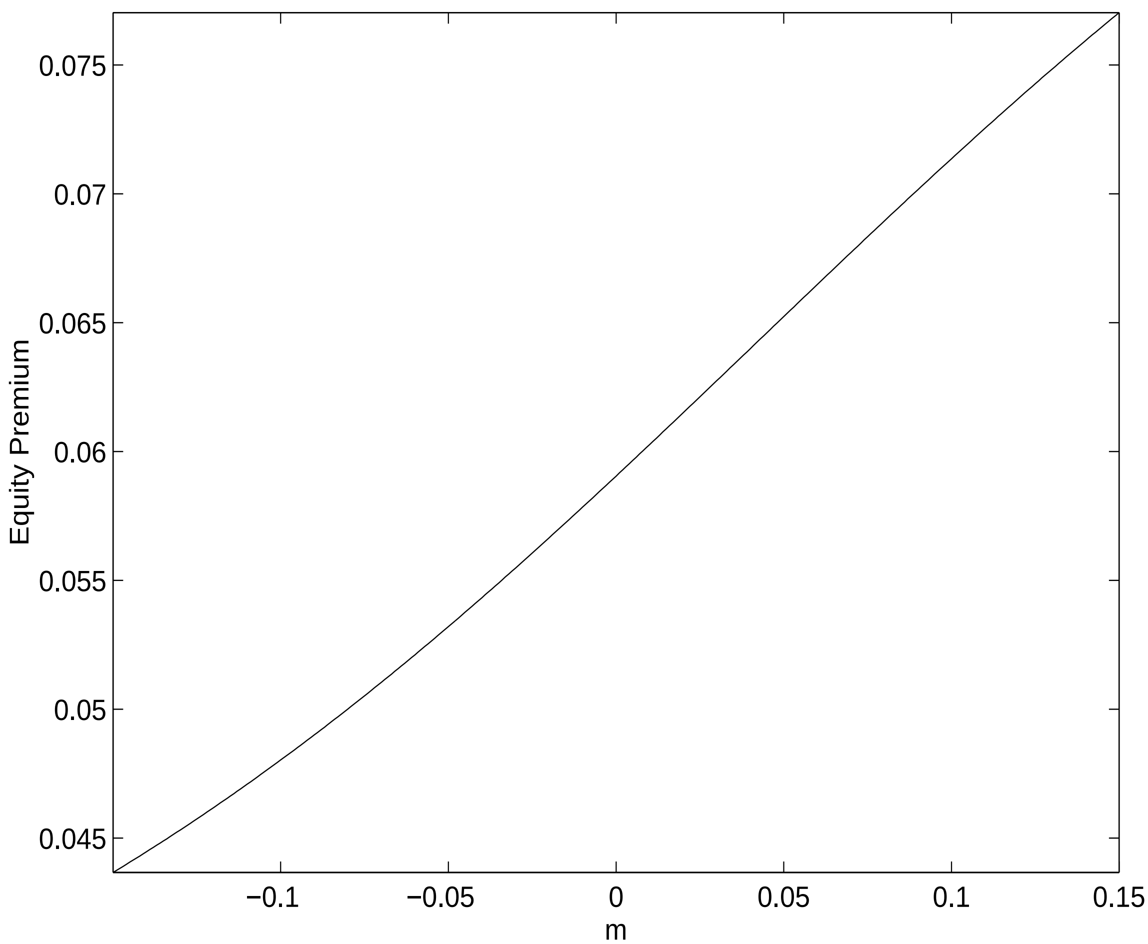


Figure 3: Equity Premium and Dividend Yield

The figure shows the relation between the steady state equity premium, $\pi(m)$, and $p(m)^{-1}$, the instantaneous dividend yield for *Model viii*.

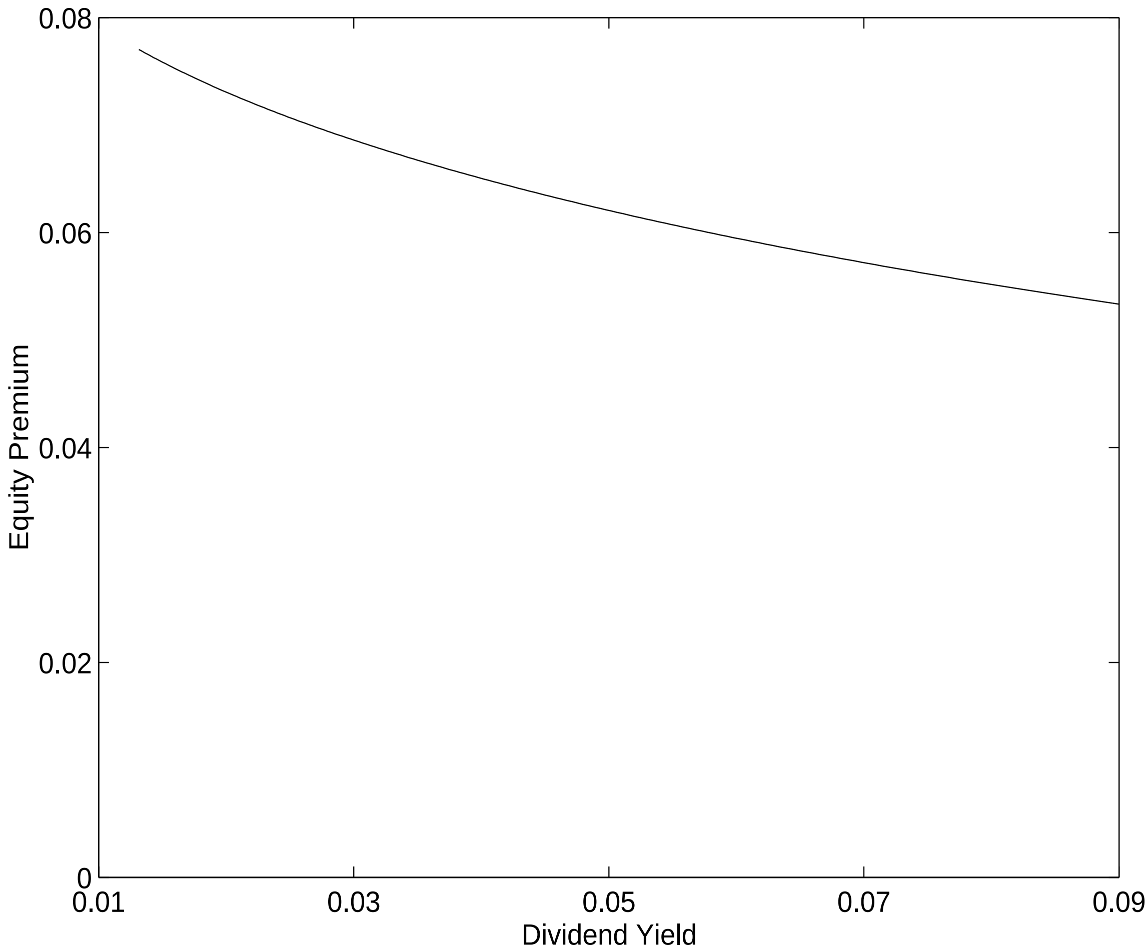
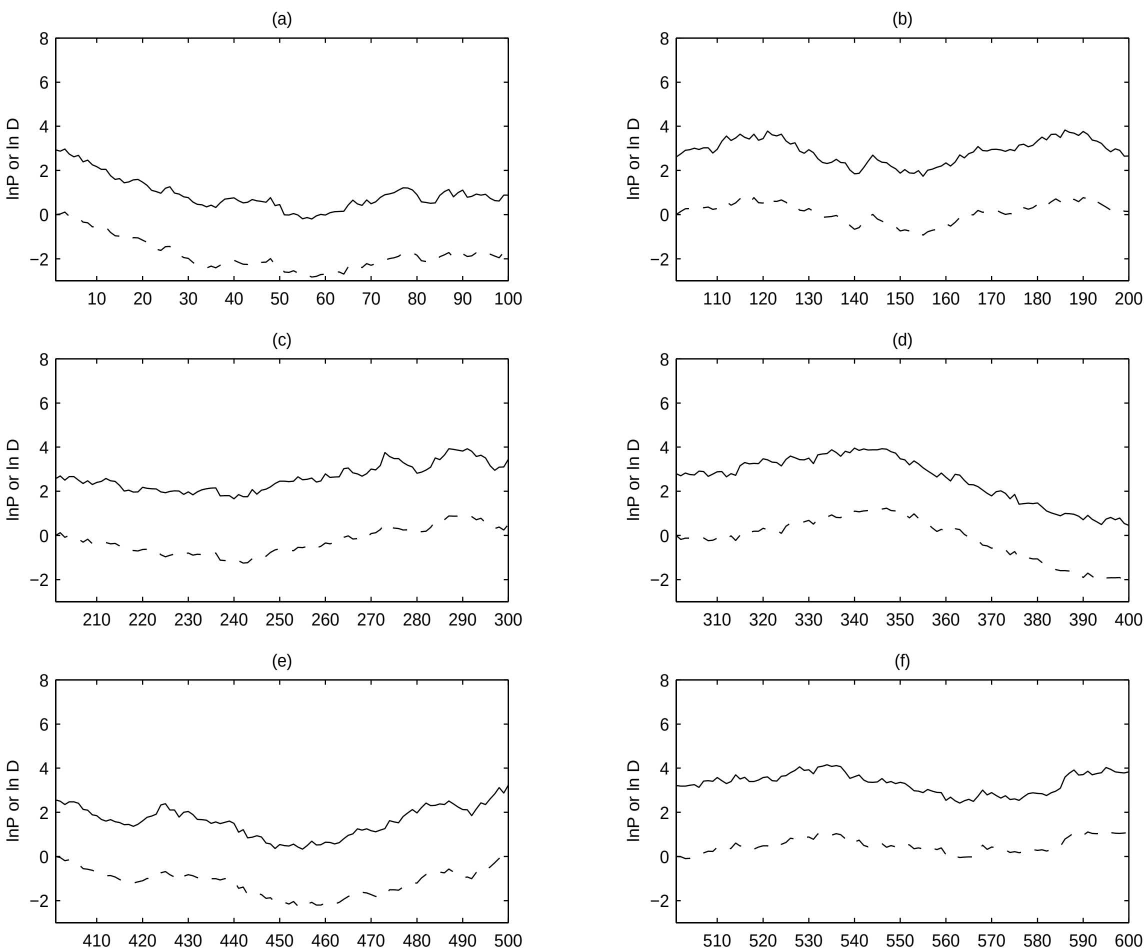


Figure 4: Simulated and Historical Normalized Log Real Annual Stock Prices and Dividends

Figures 4a - 4l show time series plots of 100 year samples of log real annual dividends (dotted) and stock prices (solid) generated from *Model viii* starting with $D = 1$ and prior on μ normally distributed with mean 2% and variance $(3.7\%)^2$. The dividend is renormalized to $D = 1$ for each 100 year subsample. Figure 4k shows the *Model viii* historical log real annual dividends and stock prices for the period 1890-1995, starting in 1890 with a prior on μ normally distributed with mean 2% and variance $(3.7\%)^2$. Figure 4l shows the historical log real annual dividends and stock prices for the period 1890-1995.



Simulated and Historical Normalized Log Real Annual Stock Prices and Dividends (continued)

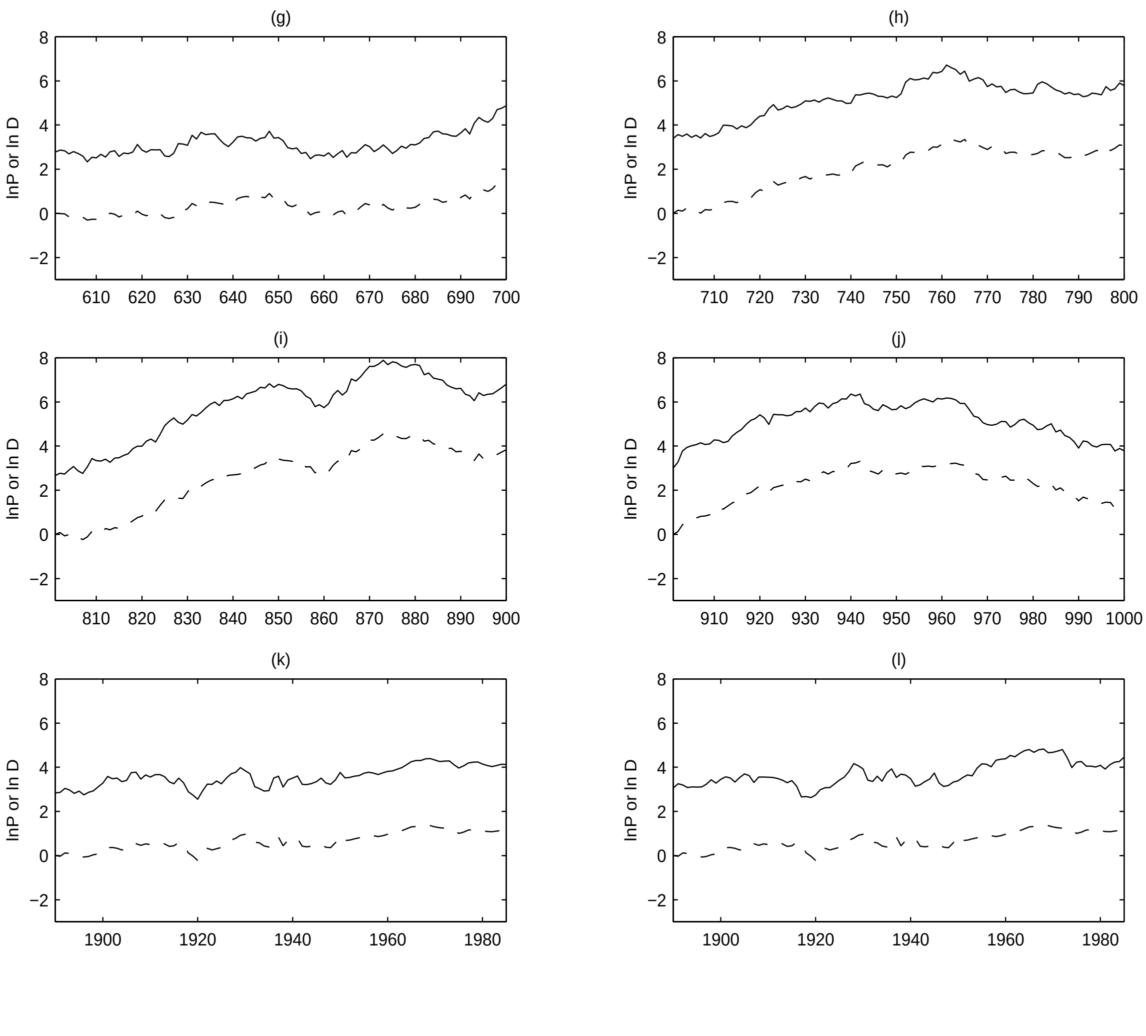


Figure 5: Normalized Lagged Actual and *Model viii* Historical Annual Stock Prices and Dividends (1890-1985)

The figure shows the log nominal annual dividends, the log historical nominal annual stock prices (lagged one year) and the log *Model viii* historical nominal annual stock prices starting in 1890 with a prior on μ with mean 2% and variance $(3.7\%)^2$

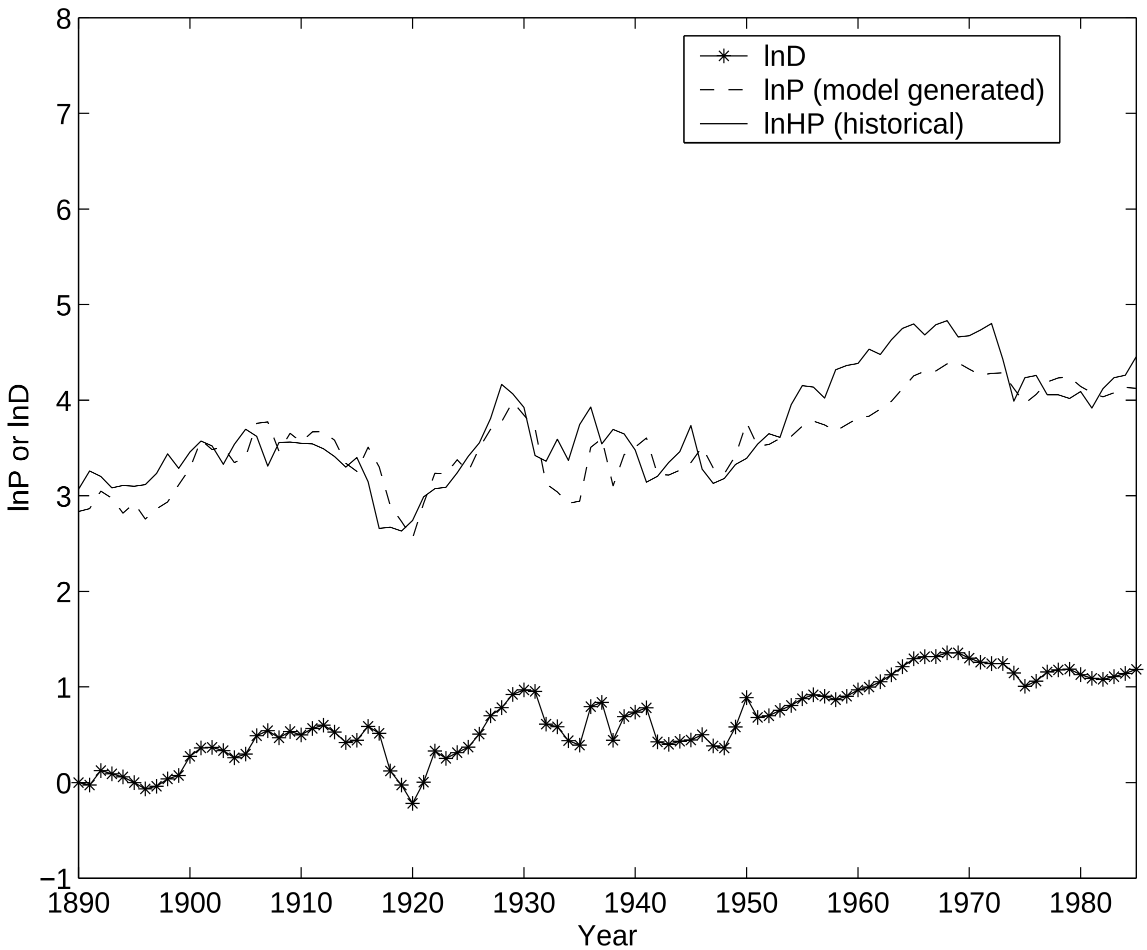


Figure 6: Lagged Actual and *Model viii* Historical Real Stock Returns (1890-1985)

The figure plots the lagged actual annual stock returns against the *Model viii* stock returns constructed from the *Model viii* prices and the historical dividends.