A History of Factor Tables with Notes on the Birth of Number Theory 1657–1817

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Abstract

The history of the construction, organisation and publication of factor tables from 1660 to 1817, in itself a fascinating story, also touches upon many topics of general interest for the history of mathematics. The considerable labour involved in constructing and correcting these tables has pushed mathematicians and calculators to organise themselves in some network. Around 1660 J. Pell was the first to motivate others to calculate a large factor table, for which he saw many applications, not only in Diophantine analysis but also in arithmetic and even philosophy. Some hundred years later (1770), J.H. Lambert launched a table project that would engage many computers and mathematicians to (re)produce Pell's table and extend it. Importantly, Lambert also pointed out that a theory of numbers, of divisors and factoring methods was still lacking. Lambert's ideas were taken up by his colleagues at the Berlin Academy, and indirectly by L. Euler in St Petersburg. Finally, the many number-theoretical essays that were written in the context of Lambert's table project contributed importantly to the birth of higher arithmetic around 1800, starting with A.-M. Legendre's and C.F. Gauss's work.

Une histoire des Tables des Diviseurs, avec des Notes sur la Naissance de la Théorie des Nombres 1657–1817

L'histoire de la fabrication, l'organisation et la publication des tables de diviseurs, de 1600 à 1817, offre en et pour soi une histoire

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fascinante, mais elle enveloppe aussi beaucoup de thèmes d'importance plus générale pour l'histoire des mathématiques. Le travail qu'implique la fabrication et la correction de ces tables était considérable et a poussé les mathématiciens et calculateurs à s'organiser dans un réseau scientifique. Autour de 1660 J. Pell était le premier à motiver d'autres mathématiciens à produire une large table de diviseurs, pour laquelle il voyait une grande utilité, non seulement dans l'analyse Diophantienne, mais aussi dans l'arithmétique et même dans la philosophie. Quelque cent ans plus tard (1770), J.H. Lambert lança un projet de tables qui engagerait beaucoup de calculateurs et de mathématiciens afin de (re)construire la table de Pell et de l'étendre. Lambert indiquait aussi qu'il manquait une théorie des nombres, des diviseurs et des méthodes de factorisation. Ces idées étaient reprises par les collègues de Lambert à l'Académie de Berlin, et, indirectément, par L. Euler à St.-Petersbourg. Finalement, ces textes sur la théorie des nombres, écrits dans le contexte du projet tabulaire de Lambert, contribuaient de manière importante à la naissance de l'arithmétique transcendante autour de 1800, dans les travaux de A.-M. Legendre et C.F. Gauss.

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Contents

1 Introduction

Like other scientific experts, the makers of tables which list prime numbers or factors of positive integers have cultivated the memory of the work of their forerunners. In the case of factor tables, there is also a more specific reason to consult existing tables, because for tables where the interpolation of entries is impossible, comparison is the only way to test a newly produced table. The most complete list of prime and/or factor tables compiled before the twentieth century can be found in Dickson [1919–1927, I, pp. 347–356]; the most extensive comment and analysis was provided by Glaisher [1878], in a companion essay to his *Factor Table for the Fourth Million*.

A topic closely connected with the construction of factor tables is the development of primality tests and factoring algorithms. Since the advent of the digital computer and especially since the invention of RSA-encryption, primality tests and factoring algorithms are considered as an important research field for mathematics and its applications. Before 1945, however, the topic mainly figured in research on and construction of tables in number theory. The history of primality tests and factoring algorithms has already been well documented by Dickson [1919–1927, I, pp. 357–374] and more recently been reappraised by Williams and Shallit [1994] and Mollin [2002].

The aim of the present paper is to show how a proper historiographical appraisal of the construction of factor tables may yield insights which go considerably beyond a mere chronology of computational techniques and of the production of tables. We will first show that the circumstances of production of the earliest prime and factor tables provide insight into the way in which mathematicians and calculators organised themselves in communities or networks in the 17th and 18th centuries. Further, the use and production of factor tables were gradually embedded in an emerging theoretical framework of its own. It will be shown how this theoretical framework played an essential role, next to Pythagorean, Diophantine and Fermatian problems, for the emergence of number theory, culminating in A.-M. Legendre's *Essai* (1798) and C.F. Gauss's seminal *Disquisitiones Arithmeticae* (1801). Indeed, the birth of higher arithmetic or number theory is in fact partly constituted by its differentiation from Diophantine analysis. The theories and methods that provide the theoretical background for factor tables are one of the sources that complement the classical tradition in the process of forming number theory.[1]

2 Anatomia Numerorum

The oldest and smallest factor and/or prime tables stem from the early 17th century and were constructed by Cataldi, Guldin and van Schooten. Cataldi [1603] gave a list of the factors of all numbers to 750 (in a supplement to 800). As the title of Cataldi's treatise indicates, the table was used in connection with perfect numbers, numbers that are equal to the sum of their divisors (including 1 but excluding the number itself). Cataldi proved with the help of his table that $2^{17} - 1$ and $2^{19} - 1$ (known as Mersenne numbers) were prime, hence $2^{16}(2^{17} - 1)$ and $2^{18}(2^{19} - 1)$ were perfect numbers. Cataldi's claim that the exponents 23, 29, 31, 37 also generated perfect numbers, could not be checked against his small table and was later proven false, except for 31, by Fermat and Euler [Dickson 1919–1927, I, pp. 10–19]. Paul Guldin [1641, pp. 383–401] made a factor table to 10000 in the fourth book of his series *De Centro Gravitatis*.[2]

Finally, Frans van Schooten [1657, pp. 393–403] published a table of primes to 9979.[3] Van Schooten's *Exercitationes Mathematicae* consisted of five books. The first four books contained geometrical problems, problems from Euclid's *Elements*, from Apollonius's works and showed how Descartes's *calculus geometricus* might be applied to these. The fifth and last book contained "Miscellaneous Problems" and may be situated in the then newly

[1]On the role of the Pythagorean, Diophantine and Fermatian problems for the formation of number theory, see [Weil 1984, Chap. IV] and [Shanks 1993, Chap. 1]. On the shaping of this discipline after the publications of Gauss and Legendre we would like to refer to the rewarding collection of essays in [Goldstein et al. 2007].

[2]Guldin's table is missing in [Dickson 1919–1927]. The context of Guldin's table is unclear, his own explanation for his *Tabula ultima* is not really satisfying. In his second book of *De Centro Gravitatis* the table is announced while defining what a non-divisible number is. The name *Tabula ultima* only refers to the fact that it is the last table in the last book of Guldin's four volume work.

[3][van Schooten 1657] was published two years later in a Dutch version, viz. [van Schooten 1659]. The *syllabus numerorum primorum* is there reprinted as *Tafel der eerste getallen*, with a correction. In the table of the latin version 809 was dropped by accident in the process of printing, this error is corrected in the Dutch version. Jacques Ozanam [1697, pp. 30–32] put van Schooten's table in another format and reprinted it (with the 809 error) in his *Récreations mathématiques et physiques*. In discussing van Schooten's *Exercitationes mathematicae* we will refer of the Dutch version.

emerging tradition of books on recreational mathematics. Quite a lot of these miscellaneous problems were arithmetic problems. They were mainly derived from Michael Stifel's edition of Christoph Rudolff's *Coss* [1553] and Claude-Gaspard Bachet de Méziriac's edition of Diophantus's arithmetic books [1621]. Although the problems were seemingly disconnected, van Schooten often inserted programmatic remarks on the excellence of algebra and especially Descartes's analysis for solving these problems.[4] The *Exercitationes Mathematicae* may therefore be seen as an early attempt in developing a (general) method to attack problems with numbers. In this (Cartesian) context we find the list of the primes under 10000. Van Schooten recommends it as a useful for solving problems of parts and divisors, for avoiding fractions, for finding the roots of equations, for calculating logarithms, actually, "helpful for nearly all kind of calculation".[5]

2.1 John Pell's Table of Incomposits

The first factor table that took more effort and time than a day well spent on calculation was published by J.H. Rahn (1622–1676) in his *Teutsche Algebra* (1659). It gives only the smallest factors of the numbers less than 24.000 which are not divisible by 2 and 5.[6] The subsequent English translation of Rahn's book, *An Introduction to Algebra* published 1668, was begun by Thomas Brancker (1633–1676) in 1665. Through the mediation of John Collins, John Pell (1611–1685) was involved in reading, correcting and supplementing the translation, in the end nearly replacing half of Rahn's text with his own [Malcolm 2004, pp. 250–252].[7] For this translation, Brancker

[4]E.g., van Schooten remarked that contrary to Stifel's opinion algebra is useful for finding amicable numbers (numbers that are equal to the sum of the divisors of a friend-number and vice versa) [van Schooten 1659, pp. 390–391]. Similarly, van Schooten showed how algebra could solve a rather disconnected collection of 15 problems with one and the same trick [van Schooten 1659, pp. 459–462].

[5]Original full quote:"Hier by komt, dat dese getallen mede niet weynig tottet minderen der gebroocke getallen, in 't delen der AEquatien of Vergelijckingen, en in hare wortelen te soecken, gelijck oock in het vinden der Logarithmi of Reden-tallen, en eyndelijck by-na in alle reeckeningen behulpsaem zijn." [van Schooten 1659, p. 365].

[6]According to [Malcolm and Stedall 2005, p. 200] it was a student of Pell, Balthasar Keller, who calculated the table; Rahn only copied and published it.

[7]As H. M. Pycior [1997] showed, this English translation was one of the attempts, on the insistence of John Collins, to provide the English with a current English Algebra book. Pell added pages 79–82 and 100–192.

had calculated the factor table afresh up to 100.000, following Pell's directions. The authorship of this book has been a matter of debate, but it is by now certain that Rahn was a student of Pell in Zürich and mainly used Pell's lectures to write his book.[8] Already in 1668 the book has therefore been known as Pell's *Algebra*, and the *Table of Incomposits* has likewise been known as Pell's Table, though Keller and Brancker, independently, calculated it. Both the *Algebra* and the *Table* are products of Pell's particular mind set and the details of their execution should be set in the context of Pell's general philosophy and should also be interpreted as part of a dialogue that Pell (indirectly) took up with van Schooten's *Exercitationes*.

Pell[9] entertained particular ideas on mathematics and of the organisation and transmission of knowledge. As his *Idea on Mathematicks* (1638) shows, Pell was supportive of a reorganisation of knowledge, as professed in Hartlib's circle and inspired by or close to Comenius's ideas.[10] In this compression and presentation of mathematical knowledge, due place was given to "the usefullest Tables and the Precepts for their use, in solving all Problems" [Pell 1638/1650, p. 40], amongst them a table of sines/logarithms to solve higher equations that Pell often mentioned, but never published. Samuel Hartlib had characterised Pell as a man who "vrges mainly a perfect Enumeration of all things" (1639)[11] and this emphasis on complete enumeration is indeed a recurring theme in Pell's work. In this context, Pell happened upon the idea (or rather powerful metaphor) that knowledge could be organised through combining "prime truths" [Malcolm and Stedall 2005, pp. 263–5]. This idea of tables, prime and factor tables especially, in connection with a (re)organisation of knowledge will prove to be quite persistent into the 18th century, though altering its modes.

Some general characteristics of Pell's *Algebra* fit within this ideology. Pell had developed a "method" to present his algebra which an anonymous

[8] Already Kästner [1786, pp. 555-56] found this translation of a German algebra book most curious and suspected Rahn might have been a student of Pell. More recently Scriba [1974] and Malcolm [2000; 2004], on the basis of new unpublished correspondence, have removed nearly any doubt that Rahn's original text owes much, certainly its innovations, to Pell's teachings.

[9] A extensive biographic and scientific portrait of Pell, detailing his life phases, was recently published by Malcolm and Stedall [2005].

[10] See [Malcolm and Stedall 2005, Parts I and II]. Yates [1972, p. 233] also suspects a strong influence of John Dee, and of course, the project is close in time and spirit to Bacon's proposal for the advancement of science, and the foundation of the Royal Society.

[11] Quoted after [Malcolm and Stedall 2005, p. 63].

reviewer in the *Philosophical Transactions*[12] characterised as follows:

> the Method is such, that most of the Book, if not all, may be understood by those not vers'd in the English tongue, that are vers'd in Specious Algebra, most of the Questions being propounded in Symbols, and the progress of the work so described by the Marginal quotations, that for those exercised in Algebra, that would transcribe a Problem in this Method, it were sufficient, only to take the Margent, omitting the work it self, till farther leisure is afforded to perform it.[13] [Phil. Trans. 3 (1668), 689]

This is the essence of Pell's tri-column method. From right to left one had per line three columns: 1) the text and its algebraic transliteration; 2) a numbering of the line; 3) a summary of the operations that led to that line using the numberings, e.g., 3*7 means: line 3 times line 7.[14] This near tabular arrangement in Pell's *Algebra* not only depended on the complete and consecutive numbering of the lines, but it also opened up the book for a variety of readers. It became accessible to non-English-speakers, and had moreover the pedagogic advantage that one could choose one's style of reading (a quick glance or detailed).

If one takes a closer look at the specific parts that Pell added to Brancker's translation (pp. 79–82 and 100–192), one sees that Pell had added an additional layer to the book. All additions belong to indeterminate or Diophantine analysis. Pell had been interested in Diophantine problems since the 1640s. He had corresponded with Father Mersenne (1639–40) and had given lectures on Diophantine problems in Amsterdam (1644–46), with Vossius in the audience, and later in Zürich to Rahn (1654–58).[15] Pell had often announced or promised an edition of Diophantus (in his own tri-column style) but as so many of Pell's projects, nothing was ever published [Malcolm and Stedall 2005, p. 289–290 et passim]. It seems that Pell took advantage of the

[12]Actually Collins according to Malcolm and Stedall [2005, p. 205].

[13]Cfr. "a sequence of marginal annotations summarizing the working out of the problem in the text" [Malcolm 2000, p. 287].

[14]More on the tri-column method in [Stedall 2002, pp. 137–138] and in general on Pell's mathematical style, that is close to "a mechanisation of unit steps", [Malcolm and Stedall 2005, pp. 235–319].

[15]See [Malcolm 2000, p. 276] on Mersenne plus the references there; [Kästner 1786, p. 555] on Vossius; and [Malcolm 2004, p. 253] on Rahn, see also Malcolm and Stedall [2005], keyword Diophantus.

occasion and used his involvement in the English translation of Rahn's book to make up, at least in part, for the missed edition or commentary of Diophantus. Instead of going for a general treatment, however, Pell suspended his considerations from a set of particular problems. Problems XV and XVI dealt with Pythagorean triangles and allowed Pell to give a small exposition on what exactly constitutes an indeterminate problem. All further problems (XXVII–XXXI) were problems which

> Bachet [...] left obscure; and [...] the celebrated DesCartes and Van Schooten have left doubtful, as not being by them throughly understood. [Phil. Trans. 3 (1668), 689]

Indeed, Pell's problem XXVII–XXVIII corresponds to Diophantus V.19 and van Schooten's Problem XIII; Pell's XXIX–XXXI to van Schooten's XII.[16] Likewise, the *Table of Incomposits* corresponds to van Schooten's V (the *syllabus numerorum primorum*) and pages 194 to 195 ("XXIX different examples of a Composit") to van Schooten's III and IV. In the process of writing his own solutions to these indeterminate problems, Pell used up 83 pages, van Schooten 4 pages.

It may be clear from this list that Pell seems to engage into a discussion with van Schooten's *Exercitationes mathematicae*, if not with its underlying Cartesianism in things mathematical. Pell's main criticism on the solutions presented in van Schooten (solutions by Bachet, van Ceulen and Descartes) was that these mathematicians had failed to appreciate the indeterminate nature of these problems. Namely, Van Schooten's sources had given only one (or two) answer(s) to the problems, whereas they are "capable of innumerable answers" [Rahn 1668, pp. 80; 116; 138]. Pell adapted the methods so as to produce innumerable answers and then added a "review" of the list of solutions.[17] This "review" is actually an analysis of the order of the solutions. For Problem XXVII this "review" states:

[16] Unfortunately, Stedall's essay on Pell's mathematics [Malcolm and Stedall 2005, pp. 247–328] somewhat neglects Pell's work on Diophantine problems. In the case of problem XXIX the comment is restricted to "A further problem [...] fills most of the remaining sixty pages" (p. 311). The sources and solutions to problem XXIX are discussed in [Costabel 1950, Hofmann and Costabel 1952], an analysis of Pell's treatment is unfortunately lacking as Costabel [1986, p. 324] later acknowledged.

[17] This "review" can be found pp. 121–128 for Problem XXVII, pp. 142–174 for Problem XXIX. The solutions are actually integer triangles and thus consist of three numbers (a, b, c), Pell's "review" focuses mainly on the hypotenuse of the triangle.

> this Pattern shews you a disorderly mixture of Answers in Great Numbers amongst Smaller Numbers. [...] So that here is need of another Rule for the orderly selecting of values of b and c, apt to lead us, in order, to Answers falling under any prescribed limit [as for example 100,000] that so we may not be cumbted with huge Numbers, when there are many smaller ones fit to answer the Question. [Rahn 1668, p. 142]

This disorder displays "inverted repetitions" (i.e. (a, b, c) and (c, b, a)) and "confused Anticipations" (smaller hypotenuses before larger ones). Near the end of the "review" Pell can say that

> In the two preceding Pages you have some Solutions of Probl. XXIX proposed p. 131, which was declared capable of innumerable Answers. And therefore I prescribed a Limit [No side greater than 100,000] Pag. 152, I required that the Enumeration of them should be orderly, pag. 159. I declared that I would have that Enumeration Complete, giving All the answers that do not exceed 100,000 in their greatest side. [Rahn 1668, p. 168]

This conclusion sums up the aspects that are particular to Pell's take on Diophantine problems: Methods to generate an orderly and complete enumeration of answers.[18] Finally, after the "review", Pell gives another way of solution where the disentanglement of the disorder in the answers is easier.[19]

The crucial tools in the "review" of van Schooten's methods and the list of solutions were tables.[20] For both Problems XXVII and XXIX Pell used

[18] Two remarks pertaining more to the philosophy of mathematics may be added here. First, one can wonder if Pell's "orderly and complete Enumeration" is connected in any way to Regula VII of Descartes's *Regulae ad directionem ingenii* where a "sufficienti et ordinata enumeratione complecti" [Descartes 1701, p. 18] is demanded. Although this requirement is absent in Descartes's *Discours de la Méthode* (1637) and the *Regulae* were only published posthumously, Pell may have had knowledge of Descartes's manuscript during his years in the Low Countries. Second, in the analysis of the sources of Problem XXIX Pierre Costabel and J. E. Hofmann both remarked that there is "une impuissance [...] à pousser les questions de théorie des nombres jusqu'aux considérations exhaustives et au point de vue existentiel" [Hofmann and Costabel 1952, p. 326] in the 17th century, with the notable exception of Pierre de Fermat's work. One should add Pell as an exception.

[19] These are Problems XXVIII for XXVII and XXX for XXIX (pp. 131 and 174–188).

[20] It is informative to compare Pell's approach to number problems with Frenicle de Bessy's methods as analysed in [Goldstein 2001].

a table of squares[21] and the *Table of Incomposits*.[22] More specifically, the *Table of Incomposits* was pivotal in establishing Pell's orderly enumeration. It was used to find the greatest common divisor of three numbers (which is easier with the help of a table than with a repeated Euclidean algorithm). Now, to avoid the "inverted repetition" it was mandatory to divide by the greatest common divisor at a certain point in the solution method (p. 147) and to reorder the "confused anticipation" one had to restructure the list of solutions according to their greatest common divisors (p. 152 ff.).

Mathematically, it seems that Diophantine problems and specifically the generation of solutions in the right order indeed stimulated Pell to calculate auxiliary tables. Both Brancker's *Table of Incomposits* and Pell's own *Table of Squares* are instances of these, directly linked with specific problems. Both tables also have the same upper limit (100,000) and fulfill the condition that all numbers in the table should be in natural order without gaps. This substratum clarifies the philosophical importance Pell ascribed to a table of "prime truths", since such a table would not only combinatorially generate all composed truths, but would also be able to generate them without gaps, in the right order, and could be used to show the non-existence (i.e. falsehood) of a composed statement.

Of course, as already van Schooten had pointed out, the *Table of Incomposits* has many possible applications. The announcement of the book in the *Philosophical Transactions* listed some of these:

> Thirdly, as to the Table of Incomposits, no Book but this extends it to above Ten thousands, some of the uses whereof are declared in the Title [i.e. "factors or coefficients"], others in the Book; and even in Common Arithmetick, it is of excellent Use for the Abbreviation of Fractions, and for giving of all the aliquot parts of a Number proposed, useful for the Depression and Resulotion of AEquations, as is taught by Albert Gerard [sic], and van Schooten. [Phil. Trans. 3 (1668), p. 689]

The reduction of a fraction to its least denominator was indeed mentioned p. 34 and taken up again in the explanation of the Table.

Pell's Table is arranged in 21 columns indicating the hundreds, and 40

[21]See p. 130; 148. In 1668 Pell used Paul Guldin's table of squares [Guldin 1635, post p. 228], later he calculated a square table of his own [Pell 1672].

[22]See pp. 129; 147; 152–153.

rows containing the unities. Numbers divisible by 2 and 5 (though not 3) are excluded and only the least factor is given. The result is:

> a complete and orderly enumeration of all incomposits between 0 and 100000 [Rahn 1668, p. 193]

Little is said of the actual calculation of this *Table of Incomposits*. Brancker wrote that Pell had taught him methods of calculating and extending the table, but no theoretical details are given.[23]

> He [Pell] shewed me the way of making the *Table of Incomposits*, of examining it, and of continuing as far as I would. He encouraged me to extend it to 100 thousand. [Rahn 1668, non-pag. preface]

More even, Brancker himself nearly despairs at the correctness of the table:

> I was very sensible of the bad effects of perfunctoriness in Supputating, Transcribing or Printing of it [the Table]. My care therefore was not small, yet pag. 198 is almost filled with Errata, and I dare not warrant that non have escaped unseen. [Rahn 1668, non-pag. preface]

Page 198 contains a list of 96 errors, some of them printing errors. During the publication process of the *Algebra* (Jan.-Feb. 1667), organiser John Collins had written to John Wallis (1616–1703) regarding the translation. In a letter to Collins that arrived after the publication, Wallis gave a list of 145 errata in the Table, at least 10 of which Brancker had not spotted. Collins communicated the list to Pell, who in his turn communicated it to Brancker, who in the meanwhile had found 19 errors more.[24] This slow process of control led Brancker to conclude: "I yet doubt its exactness".

Wallis published his "Catalogue of Errors" some years later in *A Discourse on Combinations, Alternations and Aliquot Parts* [Wallis 1685a, pp. 135–136] appended to his *Treatise of Algebra* [Wallis 1685b].[25] Wallis claimed to have examined the whole table "in the same method and with the same pains as if I were to Compute it anew", and listed 30 additional errors or misprints. More convinced than Brancker, Wallis added:

[23] But compare to earlier work of Pell on factoring, by way of summing up the digits in [Malcolm and Stedall 2005, pp. 256–7].

[24] All letters in July 1668, [Beeley and Scriba 2005, II, pp. 469–470; 525–528; 533–535].

[25] This text, including the error list, is also reprinted in the Latin translation of the *Treatise* in [Wallis 1693, p. 483ff.].

Tho. Branker's Table of Incompoſit numbers,leſs than 100 000. 1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
01	1. p	p	3	7	p	3	p	p	3	17	7	3	p	p	3	19	p	3	p	p
03	p	p	7	3	13	p	3	19	11	3	17	p	3	p	23	3	7	13	3	11
07	p	p	3	p	11	3	p	7	3	p	19	3	17	p	3	11	p	3	13	p
09	3	p	11	3	p	p	3	p	p	3	p	p	3	7	p	3	p	p	3	23
11	p	3	p	p	3	7	13	3	p	p	3	11	7	3	17	p	3	29	p	3
13	p	p	3	p	7	3	p	23	3	11	p	3	p	13	3	17	p	3	7	p
17	p	3	7	p	3	11	p	3	19	7	3	p	p	3	13	37	3	17	23	3
19	p	7	3	11	p	3	p	p	3	p	p	3	23	p	3	7	p	3	17	19
21	3	11	13	3	p	p	3	7	p	3	p	19	3	p	7	3	p	p	3	17
23	p	3	p	17	3	p	7	3	p	13	3	p	p	3	p	p	3	p	p	3
27	3	p	p	3	7	17	3	p	p	3	13	7	3	p	p	3	p	11	3	41
29	p	3	p	7	3	23	17	3	p	p	3	p	p	3	p	11	3	7	31	3
31	p	p	3	p	p	3	p	17	3	7	p	3	p	11	3	p	7	3	p	p
33	3	7	p	3	p	13	3	p	7	3	p	11	3	31	p	3	23	p	3	p
37	p	p	3	p	19	3	7	11	3	p	17	3	p	7	3	29	p	3	11	13
39	3	p	p	3	p	7	3	p	p	3	p	17	3	13	p	3	11	37	3	7
41	p	3	p	11	3	p	p	3	29	p	3	7	17	3	11	23	3	p	7	3
43	p	11	3	7	p	3	p	p	3	23	7	3	11	17	3	p	31	3	19	29
47	p	3	13	p	3	p	p	3	7	p	3	31	29	3	p	7	3	p	p	3
49	7	p	3	p	p	3	11	7	3	13	p	3	p	19	3	p	17	3	43	p
51	3	p	p	3	11	19	3	p	23	3	p	p	3	7	p	3	13	17	3	p
53	p	3	11	p	3	7	p	3	p	p	3	p	7	3	p	p	3	p	17	3
57	3	p	p	3	p	p	3	p	p	3	7	13	3	23	31	3	p	7	3	19
59	p	3	7	p	3	13	p	3	p	7	3	19	p	3	p	p	3	p	11	3
61	p	7	3	19	p	3	p	p	3	31	p	3	13	p	3	7	11	3	p	37
63	3	p	p	3	p	p	3	7	p	3	p	p	3	29	7	3	p	41	3	13
67	p	p	3	p	p	3	23	13	3	p	11	3	7	p	3	p	p	3	p	7
69	3	13	p	3	7	p	3	p	11	3	p	7	3	37	13	3	p	29	3	11
71	p	3	p	7	3	p	11	3	13	p	3	p	31	3	p	p	3	7	p	3
73	p	p	3	p	11	3	p	p	3	7	29	3	19	p	3	11	7	3	p	p
77	7	3	p	13	3	p	p	3	p	p	3	11	p	3	7	19	3	p	p	3
79	p	p	3	p	p	3	7	19	3	11	13	3	p	7	3	p	23	3	p	p
81	3	p	p	3	13	7	3	11	p	3	23	p	3	p	p	3	41	13	3	7
83	p	3	p	p	3	11	p	3	p	p	3	7	p	3	p	p	3	p	7	3
87	3	11	7	3	p	p	3	p	p	3	p	p	3	19	p	3	7	p	3	p
89	p	3	17	p	3	19	13	3	7	23	3	29	p	3	p	7	3	p	p	3
91	7	p	3	17	p	3	p	7	3	p	p	3	p	13	3	37	19	3	31	11
93	3	p	p	3	17	p	3	13	19	3	p	p	3	7	p	3	p	11	3	p
97	p	p	3	p	7	3	17	p	3	p	p	3	p	11	3	p	p	3	7	p
99	3	p	13	3	p	p	3	17	29	3	7	11	3	p	p	3	p	7	3	p

Figure 1: A specimen of Brancker's *Table of Incomposits*

> the Table will then be very accurate; and (I think) without any Error [Wallis 1685a, p. 135]

Wallis mentioned the table in connection with the problem of decomposing compound numbers in their aliquot parts, presenting the problem both in a numeric (1123) and a coefficient/literal (*aabc*) notation. This was in line with van Schooten's Problem III and IV. Historically, Wallis's *Treatise* and *Discourse* both helped to strengthen an "arithmetic approach to algebra" [Pycior 1997, p. 125]. This approach had already been announced by Oughtred and Pell[26], but Wallis explictly advocated the use of the more powerful and universal methods of algebra and arithmetic in demonstrations over the geometric approach [Pycior 1997, pp. 118–134].

The *Table of Incomposits* was often recycled. It was first reprinted by John Harris in his *Lexicon Technicum*, following the entry *Incomposite Numbers* [Harris 1707 & 1710, II, Incomposite]. Harris's entry was a near word for word repetition of Brancker's description of the Table (i.e., [Rahn 1668, pp. 193 and 196]) and gave the table, correcting the errors given by Brancker but not those given by Wallis, exactly as printed in the *Algebra*.[27]

Pell's Table was also reprinted as an appendix to volume XIII (1765) of the famous *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*.[28] The table is described in the lemma *premiers, nombres*.

> A l'occasion des nombres premiers, nous insérons, à la fin de ce volume, une table qui nous paroît assez bien étendue, & qui est tirée d'un livre anglois d'algebre assez ancien & assez peu connu [Diderot and D'Alembert 1751–1765, XIII, p. 289]

As in Harris's *Lexicon*, the table was reprinted with Brancker's corrections, but not Wallis's.[29]

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[26]On Pell's influence on Wallis's work see [Stedall 2002, pp. 141–153] and [Malcolm and Stedall 2005, pp. 313–20].

[27]The second edition of Harris's *Lexicon* dropped the table but kept the description.

[28]Dickson [1919–1927, I, p. 349] consulted a later edition of the *Encyclopédie* (1780, vol. II) and failed to note this is a reprint of Pell's table.

[29]A superficial inspecting revealed that, e.g., the entry for 99443 is changed to 77 in Harris's table according to Brancker's instructions. In the *Encyclopédie*, it has been correctly changed to 277. However, it seems unlikely that a new recalculation was done, since the errors Wallis indicated went unnoticed and in 1770 Lambert found 60 more errors (see 3.1). It is quite probable that Rallier des Ourmes contributed the entry to the *Encyclopdíe*, since he published a small article on factor tables later that year [Rallier des

2.2 German Reception in the early 18th Century

The endeavours of Brancker would remain unbeaten for over a century, but his table to 100,000 was also hard to get hold of. Whereas, e.g., Wallis's *Treatise on Algebra* and its Latin translation was rather accessible in continental Europe, the book by Rahn, Brancker and Pell was not.[30] Only the inclusion of the table in the *Encyclopédie* in 1765 made it accessible for a larger public. However, many knew of Pell's table, mainly through the mediation of Wallis. The most ironic part then of the history of factor tables in the 18th century is that many interested in mathematics knew that Pell's table existed, but were unable to obtain a copy, and thus had to calculate the table again. This was the case of Poetius and Lambert.

The reception of 17th century English algebra on the continent, and especially in the German-speaking, protestant countries, was a complex and multi-faceted process in which many personalities and media figured, but that has so far not been adequately described and/or studied. Our treatment will therefore be rather short and will focus mainly on some developments that are important for the history of factor tables. It is, however, important to keep in mind that this history has to be situated in the more general context of the transmission of ideas between England and protestant Germany.

G.W. Leibniz (1646–1717) was one of the persons who were rather well informed of the doings of the British algebraists through his correspondence with the secretary of the Royal Society, Henry Oldenburg, who was of German origin. Oldenburg, and indirectly Collins who made up drafts for Oldenburg's letters, had started the correspondence with Leibniz in 1670, a correspondence that lasted until 1679 [Gerhardt 1849–1863, I, pp. 11–168]. In 1673 Leibniz also visited London and met with Hooke, Boyle, but also Pell.[31] When Wallis's *Treatise on Algebra* appeared in 1785, Leibniz wrote a re-

Ourmes 1768]. Perhaps in this process did Rallier des Ourmes correct some entries while adapting the *Lexicon*-reprint of Pell's table for inclusion in the *Encyclopédie*.

[30]E.g., Poetius, Lambert, Kästner and Lagrange knew the book by hearsay, but none of them was ever able to inspect a copy. Rahn's German version seems to have been even rarer. All four, however, know Wallis's work, Poetius and Kästner often quoting it.

[31]In this context, consider Leibniz's letter to Abbé Galloys from December 1678. Leibniz claimed to have a method to resolve all Diophantine problems, giving all solutions in proper order or showing its impossibility, after a discourse on the use of tables in literal algebra [Gerhardt 1849–1863, I, p. 185]. The words of this letter very nearly match up with Pell's own statements though Leibniz's letter does not mention Pell.

view for the *Acta Eruditorum* [Leibniz 1686].[32] In 1695 Leibniz started a correspondence with Wallis that lasted until 1700 [Gerhardt 1849–1863, IV, pp. 1–82].

Whereas in England Wallis had to defend his algebraic/arithmetic approach in mathematics against the attacks of Hobbes and Barrow, who, in line with the new empiricist philosophy, preferred geometric, sense-based demonstration[33], Wallis's reception in Germany was in this respect less biased.[34] Through Leibniz's mediation there was a considerable reception of English algebra in the German states. When Augustinus Vagetius sollicited Leibniz's opinion on how to write a textbook on algebra in 1696, Leibniz [1923–2006, III, 6, pp. 780-81] replied that algebra and arithmetic (letters and numbers) should best be explained at the same time, an idea rather close to Wallis's. Johann Michael Poetius is also one of the writers who drew largely on Wallis in writing a textbook, and Christian August Hausen (1693–1743), mathematics professor in Leipzig, was one of the first to introduce the English style of algebra in Germany, using Newton's *Universal Arithmetick* in his courses [ADB, 15, pp. 440–441]. One of Hausen's students was Abraham Gotthelf Kästner (1719–1800), who would later become mathematics professor in Göttingen and write influential textbooks that stressed the need to base arithmetic on the concept of number [Kästner 1758, non-pag. Vorrede]. Another irony of history then is that Wallis's arithmetic approach to mathematics got more or less lost in England, but was developed in a different variety in 18th century Germany.

As mentioned, one of Wallis's readers was J.M. Poetius[35], who followed Wallis's work in writing his *Anleitung zu[r] arithmetischen Wissenschaft, vermittelst einer parallelen Algebra* (1728).[36] Poetius wrote this arithmetic and algebra textbook following a hint by Leibniz.[37]

> I will in this book follow Mr. Leibniz's proposition, and conju-

[32] Later, Leibniz also reviewed I. Newton's *Arithmetica Universalis* [Leibniz 1708].

[33] This debate had a political-religious undertone, Wallis and Pell were connected to Cromwell's puritan movement, whereas Hobbes and Barrow were royalists, see [Pycior 1997, pp. 135ff.]. (The Restauration is in 1660)

[34] In France, the Cartesian heritage probably blocked an unbiased reception of Wallis.

[35] Unfortunately, we have been unable to find any biographical information on Poetius so far.

[36] The dependence is clear by the content, but also by the many references. Poetius, however, quotes many authors, e.g., also Oughtred, French textbooks and of course many German writers.

[37] The reference is to Leibniz's letter to Vagetius.

> gate common arithmetic as much as possible with literal calculus [i.e. algebra], but I will set this latter part apart through a smaller print, so that the beginners may skip it by a first reading if they want. In this way, one will have on the one hand the main rules and examples of the operations, on the other hand one will comprehend the reasons, from which these rules spring, and consequently learn to understand the modus procedendi through proofs. [Poetius 1728 & 1738, p. 54][38]

Indeed, Poetius subscribed to a programme Christian Wolff (1679–1754) had started. In the introduction to *Auszug aus den Anfangs=Gründen aller mathematischen Wissenschaften* Wolff [1717] had argued for the introduction of arithmetic into general education, on the one hand because of its practical use, on the other, because of its logical order and structure, both aspects working together for the Enlightenment of common sense.[39] This implied two changes to the in Germany prevalent tradition of *Rechenbücher*: A multi-layered text was given, appealing to both beginning and advanced students, and all rules were proven to emphasize the order and structure. Poetius implemented both requirements, but contrary to Wolff, did not rely on Euclidean proofs with definitions and logical deductions, but endorsed the development started by the British algebra writers and used algebraic demonstrations. This is the meaning of arithmetic in parallel with algebra: Poetius gives the proofs of the arithmetic rules through algebraic translation.

The construction and structure of numbers was a major topic in Poetius's book, in line not only with the English tradition, but also with Leibniz's and Wolff's philosophy of *cognitia symbolica*, knowing through signs. Knowledge of how (numerical) signs are structured enhances the use of these signs to know and investigate the world.[40] In an introductory chapter Poetius [1728 &

[38]Original: "Ich werde hierinne dem Vorschlag des Hernn von Leibniz folgen, und die Buchstaben=Rechnung mit der gemeinen Rechnung so viel mglich conjungiren, jedoch jene bey dieser vermittelst des kleinern Drucks a parte setzen, damit die Anfnger, oder auch diejenige, so sich vor der Algebra scheuen, sie nach Gefallen im ersten Druchlesen auch bergehen knnen. denn also wird man dennoch eines Theils die Haupt=Reguln und Exempel zu denen Operationen haben, andern Theils aber wird man die Grnde, woraus dieselben Reguln entsprungen, verstehen, und einfolglich den Modum procedendi per Demonstrationes begreiffen lernen."

[39]We should add that Erhard Weigel (1625–1699), professor in Jena and one of Leibniz's influences, was the first to launch these ideas, see also [Vleeschauwer 1932].

[40]For a good introduction into 18th century semiotics see [Meier-Oeser 1998], and espe-

1738, p. 13] stressed the Indian provenance of the Hindu-arabic numerals[41], and devoted 4 pages to the explication of number systems to a different base.[42] The conclusion of this introductory chapter was:

> In this way, one needs but few signs and names to designate expressions of both the largest and smallest numbers. [Poetius 1728 & 1738, p. 13][43]

In this context the first German factor table appeared.

Appended to Poetius's book was an *Anatomia Numerorum, Oder Zergliederung der Zahlen Von 1 bis 10000.* It lists all factors of all numbers to 1000, of all odd numbers not divisible by 5 and 3 between 1000 and 10,000. In the introduction Poetius referred to Pell's table to 100,000, mentioned in Wallis. Since Poetius had been unable to find a copy, he had calculated a (smaller) table by himself to 10,000 [Poetius 1728 & 1738, pp. 39–40]. This anatomy of numbers was followed by a *Practicam*, a section on the advantages and uses of this table: The manipulation of fractions, arithmetic and geometric series. Diophantine problems are not mentioned. As far as the construction of the table is concerned, Poetius referred back to the main text on arithmetic, where he had explained the "Kenn-Zeichen" of the prime numbers, or the "Symbolum primi geneticum" [Poetius 1728 & 1738, p. 141]. A first class of these characteristics were under the heading "Division", where the rules are given to determine whether a number is divisible by 2, 3, 4, 5, 7, 9 and 11. These rules were standard in nearly all textbooks on arithmetic, and were generally called the "Kenn-Zeichen" of the numbers, so, e.g., in the classic *Demonstrative Rechenkunst* by Clausberg [1732], who devoted some

cially for the semiotics within mathematics [Knobloch 1998].

41This was a matter of dispute between the *Treatise* by Wallis [1685b, p. 8] and John Gerard Vossius's *De scientiis mathematicis* (Amsterdam 1650).

42The references mentioned by Poetius are Erhard Weigel's Tetractys [Weigel 1672]; [Leibniz 1703/1720] on dyadic; Dangicoure in the *Misc. Berol.* (1710) on quaternic; Wiedeburg's Dissertation (1718); Pelecanus (1712) on triadic; and Weidler's *Computo Duodecadico* on duodecimal number systems. We quote the references to show the popularity of the topic in Germany. Leibniz [1703/1720, p. 226]. thought that the dyadic/binary numeration system might provide a way to find a law behind the progression of prime numbers, see [Mahnke 1912/13, Zacher 1973].

43Original: "Auf solche Art hat man zu den Expressionen so wohl der größten, als auch der kleinsten Zahlen nicht viele Zeichen und Nahmen von nöthen." Authorities quoted at this point are Wolff (Elem. Math. Univ. Arithm.) and Christian August Hausen's *De mathesi semiotica* (1716).

60 pages to these "Kenn-Zeichen". Poetius turned these characteristics into tools that were useful for the construction of factor tables. A second class of factoring auxiliaries could be found in the section on powers:

> To find whether a number with large aliquot parts has factors, or if it is a prime number?
> We treated of this problem in §397, and this problem also belongs to the use of square tables.
> One subtracts the given number (if it is not a square) from the next greater square, until the remainder is a perfect square, then the greater root + the smaller root give the larger factor, and the greater root – the smaller root the smaller factor. [Poetius 1728 & 1738, p. 299][44]

With a table of squares at hand, this elementary method can indeed help to determine the factors. Poetius was thus the first factor table maker who had explicitly indicated the methods used in the construction. Curiously enough, Eratosthenes's sieve procedure was not mentioned at all[45], but only methods to factor single numbers, surely a more tedious method.

Poetius's factor table was later reprinted in the *Vollständiges mathematisches Lexicon* [Wolff and Richter 1734/1742, II, pp. 530ff.], originally edited by Christian Wolff (1716), later reworked and extended against Wolff's will by G.F. Richter, who inserted the table in Volume II (1742).[46] Some years later, a Nürnberger military, Peter Jäger, calculated a list of primes to the full 100,000 (actually to 100,999) and offered his complete table for sale at the expensive price of 2000 Thalers. Halle's professor of medecine, J.G. Krüger

[44]Original: "Zu finden, ob eine ungrade schwehrtheilige Zahl einige Factores hat, oder ob sie eine Prim-Zahl ist?
Hiervon ist schon oben im 397 §gedacht worden, und gehört dieses Problema auch mit zur Nutzung der Quadrat-Taffeln.
Man subtrahiere die vorgegebene Zahl, (so sie nicht selbst eine Quadratzahl ist,) von denen nechstfolgenden grössern Quadraten, so lange biß der Rest ein vollkommen Quadrat, so giebt des grösseren Wurtzel + des kleinern Wurtzel den grössern Factorem, und des grössern Wurtzel – des kleinern wurtzel, des kleinern Factorem."

[45]As Verdonk [1966, p. 167] has shown, the original descriptions of the *cribrum Erasthostenis* in Nichomachus and Boethius seem to have been rather badly known in the 16th century. This seems to have persisted in the 17th century. Only in the 18th century did Rallier des Ourmes [1768] publish a description and did Horsley [1772] edit both text excerpts.

[46]See [Kästner 1786, pp. 556-57] for the publication history.

[1746], however, published the table without paying. Apparently without knowing of Pell or Poetius, H. Anjema of Franeker (Netherlands) also undertook the calculation of all factors of numbers under 100,000, but died when he was at 10,000.[47] After his death, the editors Sam. and Joh. Luchtmanns published the extant part. Anjema's table consisted of 302 pages, because he not only indicated all factors for all numbers, but also included 1 and the number itself in the list [Kästner 1786, pp. 558-59].

3 Johann Heinrich Lambert's Table Project

"It seems that calculation and the construction of tables have become Mr Lambert's second nature, as it seems to cost him no more time and effort than plain writing."[48]

This was the state of the art regarding factor tables anno 1770, the year in which J.H. Lambert published several appeals for tables in general, factor tables in particular, an event that will dramatically change the history of factor tables and leave a mark deep into the 19th century. Johann Heinrich Lambert (1728–1777) was born in Mulhouse (Alsace) from poor parents, but through private study and determination he became a philosopher, physicist, linguist and mathematician of importance. In 1748 Lambert became tutor of the children of the Swiss confederation president von Salis in Chur. In this way, Lambert did not only became acquainted with many works in the rich library of the Salis familiy, but 1756-58 also visited Göttingen, Hannover, Utrecht, Leiden, Turin with his pupils. This voyage connected him with various centers and persons of learning in Europe. Among the mathematical books Lambert read quite early are Wolff's *Anfangsgründe* and Poetius's *Anleitung*.

As recorded in the *Monatsbuch* (his scientific diary) Lambert started thinking about the divisors of integers in June 1756. An essay by G.W. Krafft

[47] Some biographic detail can be found in [*Mathematical Tables and other Aids to Computation* 3 (24), (Oct. 1948), pp. 331–332].

[48] Original: "Wenigstens muß das Rechnen und das Tabellenmachen dem H. Lambert schon so zur andern Natur worden seyn, daß es ihm nicht mehr Zeit und Mühe kostet, als gemeine Schrift." (review of Lambert's *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, Band II (1770), in: *Allgemeine Deutsche Bibliothek* 14 (2), p. 322, 1771.)

(1701–1754) in the St Petersburg *Novii Commentarii* seems to have triggered Lambert's interest [Bopp 1916, p. 17, 40]. The physicist, astronomer and mathematician Krafft was Euler's colleague in St Petersburg and had already published on perfect and amicable numbers. His [Krafft 1751/1753] was a survey of known facts and techniques to factor large integers. Krafft referred to van Schooten's list of primes and Poetius's factor table and indicated some errors in the latter table. Then he went on to discuss the "prime-formula" $6n \pm 1$ (due to Jacob Bernoulli and Leibniz [1678]) and explained how versions of Fermat's little theorem ($a^p - a$ is divisible by p, p odd) may be used for factoring, referring to Euler [1732/1738].[49]

3.1 Simple Ideas, Prime Numbers and Tables

From 1760 to 1765 Lambert worked on his major philosophical works [Bopp 1916, p. 47] in which he merged Wolff's *cognitia symbolica* with Locke's anatomy of concepts [Locke 1690]. Lambert wanted a reformation of philosophy where the basic principles would not be rather arbitrary definitions (Wolff) but would be acquired through an anatomy of the concepts available (Locke). A kind of philosophy that had a scientific method, though not Wolff's Euclidean method, but rather an algebraic method. This reform plan should enable philosophy to make progress, just as science did (Bacon). The scientific method[50] is explained in the *Neues Organon* [Lambert 1764], not by accident referring to Bacon's project, the anatomy of some basic concepts in *Anlage zur Architectonic* [Lambert 1771].[51]

For carrying out this anatomy of concepts and for re-combining simple concepts ("das Chaos auseinander lesen"), tables were of great value to Lambert. Lambert often used a topical table as a heuristic tool in his investigations, be they philosophical or scientific:

> A topical system, which would be the abstraction of what can be

[49] It must be remarked that Euler's paper only contained a statement of Fermat's little theorem, not a proof. Euler proved the theorem in 1736 (published 1741) and showed in 1747 (publ. 1750) how to use it for factoring. Krafft did not refer to those later papers, but must have known their contents as he gave a proof and application of Fermat's little theorem.

[50] Amongst the plurality of the methods proposed by Lambert are an algebraic calculus for logical deduction (part 1); an algebraic view on language (part 3) and an algebraic calculus to determine the degree of verisimilitude of historical accounts (part 4).

[51] This work was published 1771, but already finished in 1765.

> thought, determined, researched for any object [...] an inventory, a form of all things [...] that one can use when one wants to know something, both the thing in itself as in its relationships to other things. [Lambert Briefe 1781–1787, I, pp. 284–85][52]

Lambert saw the anatomy of a number system as an instance of such a topical table:

> The architecture of numbers is the abstraction of all things, where one calculates with numbers or discrete quantities. It is a general type, a form, and the relationships and transformations of numbers have arithmetic as their own theory. [Ibid.][53]

In this analogy, it need not surprise that a table of factors becomes the metaphor for a philosophy having a table of simple concepts at its disposition.[54]

> [In this anatomy of concepts] one takes a concept and looks up its inner determinations, which are more or less like its factors and prime numbers. [Lambert Briefe 1781–1787, I, p. 24][55]

The *Anatomia Numerorum* or Arithmetic, not only the title of Poetius's table but also in the title of [Lambert 1769], was therefore a theory of utmost importance to Lambert.

Reform of philosophy meant for Lambert also reform of the organisation of knowledge. In 1765 Lambert became a member of the Berlin Academy, a position he kept until his death in 1777. During this Berlin period he

[52]Original: "Ein topisches System, welches ein Abstractum wäre, von allem was sich bey einem jeden Objecte gedenken, betrachten, bestimmen, untersuchen läßt [...] ein Inventarium, ein Formular etc. von allem [...], was bey jeder Sache, wenn sie an sich und nach ihren Verhältnissen erschöpft werden sollte, zu suchen ist." (Lambert an Holland, 15.8.1768) Lambert's topical table was published in *Nova Acta Eruditorum* [Lambert 1768].

[53]Original: "Nun ist das Zahlengebäude gleichsam das Abstractum alles dessen, wo man mit Zahlen rechnet oder aller Discreten-Quantitäten. Es ist ein allgemeiner Typus, ein Formular davon, und die Verhältnisse und Verwandlungen der Zahlen haben die Arithmetik als ihre eigene Theorie."

[54]To our knowledge Lambert must have come upon this metaphor independently of John Pell (see 2.1).

[55]Original: "[Bey dieser Anatomie der Begriffe] hält man sich schlechthin an den Begriff selbst, und sucht seine inneren Bestimmungen auf, welche gleichsam seine Factores und numeri primi sind." (Lambert an Holland, 21.04.1765)

devoted himself to the dissemination and advancement of sciences in general, to the editing of tables in particular. From 1770 onwards, two extensive table projects absorbed nearly all of Lambert's time. The first one, supported by the Academy and in collaboration with his colleagues Bernoulli, Schulze, Lagrange and Bode, was a collection of astronomical tables.[56] The aim was to bring together all useful and necessary astronomical tables in one collection so that they would be accessible to the individual astronomer and would contain (through comparison and recalculation) no more printing and calculation errors. In this way, Lambert claimed, it was possible:

> if all the best astronomical tables [...] would get lost, they could be reconstructed from our collection [Lambert Briefe 1781–1787, V, p. 154][57]

In 1770 Lambert started up a similar kind of compilation.

> There are numbers, proportions, formulae and calculations that deserve to be done and written down once and for all, because they occur very often, so as to avoid the trouble to find or calculate them over and over again. This is the reason why in all parts of mathematics one has tried to put everything in tables that can possibly be put in tables. [Lambert 1770, p. 1][58]

The publication of the *Zusätze zu den logarithmischen und trigonometrischen Tafeln* [Lambert 1770] was a personal project of Lambert (without the support of the Academy). Therefore, at the same time, Lambert launched an urgent appeal to the general public for helping him to extend this collection of tables.

Prominently featuring in Lambert's table collection is a table of factors and primes, filling pages 2–117 of the 210 pages with tables. As Lambert recounted in the introduction to the *Zusätze*, Poetius's table to 10,000 was the first factor table he saw and although Poetius referred to Pell's table,

[56] These are the *Sammlung astronomischer Tafeln* in 3 volumes, published 1776 in Berlin.

[57] Original: "Wenn die besten astronomischen Tafeln [...] sollten verlohren gehen, so würden sie aus unserer Sammlung wieder hergestellt werden können."

[58] Original: "[es gibt] Zahlen, Verhältnisse, Formeln und Rechnungen, die eben daher, daß sie öfters vorkommen, ein für allemahl gemacht und aufgezeichnet zu werden verdienen, damit man der Mühe, sie immer von neuen zu finden oder zu berechnen, überhoben seyn könne. Dieses ist der Grund, warum man in allen Theilen der Mathematick, was sich in Tabellen bringen liesse, in Tabellen zu bringen gesucht hat."

NUMERORUM. 69

	992	995	998	1001	1004	1007	1010	1013	1016	1019
3	13	19	11	—	—	—	7	17	—	181
9	11	151	—	—	31	7	—	13	17	101
11	7	191	151	11	—	13	83	7	—	223
17	47	11	—	53	—	23	7	71	307	—
21	313	23	173	7	137	47	—	11	13	—
23	—	—	—	59	233	~~7~~	13	—	151	227
27	67	—	7	223	29	11	—	19	—	7
29	13	—	—	—	7	263	31	107	11	—
33	—	7	—	11	67	—	71	—	7	13
39	7	11	—	13	47	131	23	7	37	—
41	—	13	7	239	11	—	79	—	—	7
47	61	7	11	17	—	—	37	—	7	97
51	—	—	31	—	13	7	—	43	11	269
53	7	113	13	—	17	53	139	7	—	43
57	—	29	61	47	7	19	11	79	59	—
59	—	—	—	37	—	17	7	—	277	11
63	17	—	37	7	11	13	—	—	—	—
69	53	17	7	—	—	—	211	167	19	7
71	37	—	—	109	7	11	~~53~~	17	293	107
77	—	—	—	7	13	179	61	—	17	—
81	7	—	—	17	89	31	—	7	—	11
83	101	11	7	—	—	97	271	—	23	7
87	43	53	59	19	17	—	7	11	61	—
89	—	7	23	—	317	13	—	53	7	79
93	31	13	191	—	—	7	43	41	—	~~29~~
99	109	137	283	11	7	—	17	—	13	—

	991	994	997	1000	1003	1006	1009	1012	1015	1018
51	13	11	23	7	17	251	157	19	173	~~179~~
57	229	2 ~~4~~71	7	—	—	17	—	13	41	7
61	17	79	—	13	—	11	7	109	—	37
63	53	7	67	47	—	43	17	131	7	—
67	131	17	—	11	167	7	31	—	47	23
69	7	—	19	—	29	—	11	7	13	—
73	—	11	17	19	7	—	37	—	—	—
79	41	31	113	7	—	83	241	[illegible]	~~157~~	—
81	—	53	11	41	37	7	—	—	—	13
87	11	—	—	13	7	107	—	—	29	139
91	—	7	73	101	—	17	~~11~~	199	7	—
93	281	37	—	7	—	—	23	—	19	11
97	7	—	23	199	11	101	13	7	283	19
99	19	29	7	31	13	—	—	11	—	7

E 4 102000.

101279 est num. prim.

Figure 2: A specimen of Lambert's *Table of Divisors*, with C.F. Gauss's hand written correction

Lambert had been unable to find a copy. At first, Lambert wanted to use Poetius's table for his collection:

> I satisfied myself with the table calculated by Poetius, and just brought it in a more flexible order.[59] I occasionally showed my table, before its printing, to Mr de la Grange. He did not know of any other tables similar to it, and thus he wished to have copies of the table once it would be printed, to send them to his correspondents. As the printing was delayed, Mr de la Grange looked if he could not find more of these tables. He did not search in vain. Pell's table, indeed to 100,000 und thus 10 times further than Poetius or Anjema, could be found in the *Dictionnaire encyclopédique* and in Harris's Lexicon of Arts and Sciences. As I thereupon looked into Wallis's Opera, I found the 30 printing errors Wallis had indicated in Pell's tables and Pell himself had missed, all just as Poetius mentions. [Lambert 1770, pp. 4–5][60]

Lambert used Poetius's table, Pell's table in the two reprint versions and Wallis's corrections to check the table before publication. After Pell's table, Lambert excluded numbers divisible by 2 and 5 and only noted the smallest factor, but he also excluded the numbers divisible by 3 and likewise changed the arrangement of the table. As Lambert had largely expounded in [Lambert 1765-1772, II, pp. 42–53] the regularities of the decimal system could be used for checking the table and he advised the following arrangement. The hundreds still figure over the columns, but not in counting progression but

[59]This specimen, a factor table to 10,200 was printed in [Lambert 1765-1772, II, pp. 52–53].

[60]Original: "[ich ließ] es bey der von Poetius berechneten Tafel bewenden, und begnügte mich sie in eine geschmeidigere Ordnung zu bringen. Ich zeigte hierauf meine Tafel, ehe sie abgedrückt wurde, gelegentlich dem Herrn de la Grange. Es war ihm ebenfalls weiter nichts davon bekannt, und so bezeugte er ein Verlangen, die Tafel, wenn sie einmal abgedrückt wäre, zu haben, und selbst an seine Correspondenten Exemplarien davon zu verschicken. Da es sich inzwischen mit dem Abdrucke verzögerte, so suchte der Herr de la Grange, ob er nichts weiteres von solchen Tafeln finden könne. Er suchte auch nicht vergebens. Pells Tafel, die in der That biß auf 100000, und demnach 10mal weiter als die von Poetius und Anjema geht, findet sich sowohl in dem Dictionaire encyclopedique als in des Harris Lexicon der Künste und Wissenschaften. Und da ich daraufhin noch in den operibus Wallisi nachsuchte, so fand ich darinn auch die von Poetius erwähnte 30 Druckfehler, die Wallis in Pells Tafel angemerkt hat, und die sich unter den von Pell selbst angemerkten nicht fanden."

in three separate progressions ($3n$, $3n+1$, $3n+2$), the unities are still in the rows but excluded are those divisible by 2, 3 and 5. In this way, there are 3000 numbers per page, and some regularities are easier noted – Lambert noted over 60 errors more than Wallis [Lambert 1770, p. 5].

3.2 Lambert's Contributions to a Theory of Composite and Incomposite Numbers

Lambert did not only deliver the factor table, but also noted the absence of any coherent theory on prime numbers and divisors. This was of use in finding primality criteria and factoring tests, but for Lambert it was also an instance of a fragmentary theory which needed philosophical and mathematical efforts to become mature.

> To this aim [prime recognition] and others I have looked into the theory of prime numbers, and I did find some pieces, but singular and incomplete, without the apparence that these could be assembled quickly and could build a formal system. Euclid has few, Fermat singular but mostly unproven theorems, Euler single fragments, that anyway are far removed from the elementary level and between them and the elementary results are many gaps. [Lambert 1770, p. 20][61]

This lack of theory was much regretted by Lambert. In a comprehensive discussion on the structure and acquirement of scientific knowledge in his *Neues Organon* [Lambert 1764, I, pp. 386–450], Lambert had given many tools and strategies for finding and repairing the gaps in a theory, for building a theory and making it more complete. The theory of numbers seemed to Lambert a domain that badly needed such a treatment.

In an essay on decimal periodic fractions, Lambert had already constructed some primality tests and had found a new criterium for primality.[62]

[61]Original: "Ich habe mich zu diesem Ende [der Primzahlerkennung] so wie auch zu andern Absichten um die Theorie der Primzahlen näher umgesehen, und da fand ich freylich nur einzelne abgebrochne Stücke, ohne sonderlich Anschein, daß dieselbe so bald sollten zusammengehängt und zum förmlichen System gemacht werden können. Euclid hat wenig, Fermat einzelne meistens unbewiesene Sätze, Euler einzelne Fragmente, die ohnehin von den ersten Anfängen weiter entfernet sind und zwischen sich und den Anfängen Lücken lassen."

[62]On decimal fractions and Lambert's essay, see [Bullynck 2008].

> In this way, there are for any given prime a progressions
> $1, m, m^2, m^3, m^4$, etc.
> for which a period of $a-1$ members is produced, which never happens in the case of composite numbers. It is obvious, that this criterium for prime numbers can be checked. [Lambert 1769, pp. 127–128][63]

This criterium, however, is hardly efficient in practice, since its worst case amounts to a divisions of length a in a numerical positional systems.

In 1770 Lambert presented two sketches of what would be needed in such a theory of numbers. The first one mainly dealt with factoring methods [Lambert 1765-1772, II, pp. 1–41], the second one gave a more axiomatic treatment [Lambert 1770, pp. 20–48]. In the first essay, Lambert explained how Eratosthenes's sieve worked, and how it could be optimised through exclusions of test divisors if one factors numbers with small factors. For larger factors, appromixation from above, starting from the square root of the tested number p, was more advantageous.[64] For both methods, Lambert advised the use of tables: Tables of primes for the sieve, a table of the last digits of a square and a table of repeated division for the large factor method. The second essay had more theoretical bearings. Lamberts rephrased Euclid's theorems for use in factoring, included the greatest common divisor algorithm, and put the idea of relatively prime numbers to good use. He also noted that binary notation, because of the often occurring symmetries, could be helpful. Finally, Lambert also discovered Fermat's little theorem as a good though not infallible criterium for primality, "but the negative example is very rare" [Lambert 1770, p. 43].

Through Lambert's efforts, the topic of factoring got discussed in the Berlin Academy. J.-L. Lagrange (1736–1813) showed firm interest in factor

[63] Original: "Sic et pro quovis numero primo a dantur progressiones
$1, m, m^2, m^3, m^4$, etc.
quae periodum producant $a-1$ membrorum, quod cum de numeris compositis nunquam locum habeat, patet, et hinc peti posse numerorum primorum criterium." This criterium comes down to the statement that every prime number has a primitive root, or alternately, that there exists a number system to a base m for which $\frac{1}{p}$ will have a period of length $p-1$.

[64] One should bear in mind that Lambert's range of tested numbers is between 1 and some millions, say, up to 10-digit-numbers. Since this upper limit is nowadays much higher, the methods fall into 3 categories: small, middle and large.

tables[65], and Johann III Bernoulli (1744–1807), pursuing Lambert's method of factoring with decimal periods, also regretted the absence of a theory [Bernoulli 1771/1773, p. 318]. In this atmosphere, Academy member Nikolaus von Beguelin (1716–1789), the former tutor of the Prussian crown prince, wrote some essays on factoring. To this end, Beguelin devised a new number system that combined the conveniences of algebraic and numeric notation.

> Although the science of numbers is of geometric necessity, based on the principle of contradiction, we know that the number signs & and the methods of expressing the various combinations are not of absolute necessity. [...] It is evident that the more we diminish the numbers of primitive elements the more the arithmetical operations will be simplified, & the sooner also we can hope to see the nature of numbers & their mutual relationships in their expressions. [von Beguelin 1772, p. 296][66]

Though the 18th century had proven Leibniz wrong to expect that the binary notation would reveal the mysteries of prime numbers, other notations could be helpful in finding factors, as Lambert had pointed out, providing examples of binary numbers that can be factorised at sight. Beguelin applied his principle of sufficient reason (”raison suffisante”) to this problem of finding an optimal notation. The binary system would be the best, if it would not, as Beguelin remarked, have two disadvantages: The numbers become too long and one cannot transpose the digits at will as one can in literal algebra. His solution was a mixture of systems: He notated the powers of a binary number, thus 0.2.4. (or 4.0.2. etc.) means 19 or 10101 in binary notation [von Beguelin 1772, p. 297].

Using the advantages of his notation, Beguelin constructed formulae for numbers with 2, 3 and 4 factors that show the skeleton of the notation to be expected. In this way, Beguelin had an approach that mechanised Lambert's

[65] See the introduction to Lambert's *Zusätze* [Lambert 1770, p. 4], Lambert's correspondence [Lambert Briefe 1781–1787, V, pp. 51–52; 120–121; 194] and Lagrange's correspondence [Lagrange 1867-1892, XIII, p. 193].

[66] Original: “Quoique la science des nombres soit de nécessité géométrique, fondée sur le princip de contradiction, on fait que les signes des nombres, & les méthodes d'en exprimer les diverses combinaisions ne sont pas d'une nécessité absolue. C'est une affaire de choix, ou de convention. [...] Il est évident que plus le nombre des élémens primitifs, plus les operations arithmétiques seront simplifiées, & plutôt aussi on pourra se promettre d'appercevoir la nature des nombres & leurs rapports mutuels dans leur expression.”

remarks on binary numbers. Instead of manipulating the binary number to find symmetries, Beguelin could look them up in a table of formulae. Unfortunately, Beguelin soon found out that the problem was thus reduced to the combinatory problem of excerption, i.e., writing a number as a sum of certain other numbers having a specific form. For most forms, Beguelin's approach reduced factoring to an at least as difficult problem in additive number theory. For numbers that in Beguelin's notation had no gaps (i.e. contain all exponents starting from 0), numbers of the form $2^n - 1$ (Mersenne numbers), or have many gaps, $2^n + 1$, von Beguelin [1777] tediously deduced formulae. Beguelin's essays again show the connection between a rational philosophy of *cognitia symbolica* and the research done on number systems.

Finally, it should be remarked that in 1773 and 1775 Lagrange published his *Recherches d'arithmétique* in the *Mémoires* of the Berlin Academy, bringing Euler's and his own results on quadratic forms together, giving the first general treatment of reduction and equivalence (in modern terms) of binary quadratic forms. In his preface, he explicitly referred to the use of this theory for factoring numbers:

> These studies are devoted to numbers that can be represented by the formula $Bt^2 + Ctu + Du^2$ [...] First I will give a way to find the different forms the divisors of these numbers can have; then I will give a method to reduce these forms to a minimal set; I will show how to set up tables for practice, and I will show how to use these tables in the search for factors of numbers.[Lagrange 1773 and 1775, p. 265][67]

Lagrange had tabulated the linear divisors of a certain family of quadratic forms. These were useful as a tool in finding particular forms of prime numbers, or forms that exclude them. Lagrange closed his paper with an application, showing how to factor 10001, 10003 and 100003 with the help of these tables, and with list of divisibility criteria.[68]

[67]Original: "Ces Recherches ont pour objet les nombres qui peuvent être représentées par la formule $Bt^2 + Ctu + Du^2$ [...] Je donnerai d'abord la manière de trouver toutes les différentes formes dont les diviseurs de ces sortes de nombres sont susceptibles; je donnerai ensuite une méthode pour réduire ces formes au plus petit nombre possible; je montrerai comment on en peut dresser des Tables pour la pratique, et je ferai voir l'usage de ces Tables dans la recherche des diviseurs des nombres."

[68]The tables are in pp. 311–312 (1773); pp. 329–330; 332–333 (1775), or also in [Lagrange 1867-1892, III, p. 695] on pages 757–758, 766–767 & 769–770. The theorems on divisibility

3.3 Lambert's Appeal for Factor Tables

Lambert's first public appeal for the production of mathematical tables was advertised in the 2nd part of his *Beyträge zum Gebrauch der Mathematik* (1770). In the description of his first specimen of a factor table (to 10,200), Lambert had encouraged other mathematical practitioners to extend the table. To all calculators that wanted to join his project, Lambert had promised honour if not scientific immortality just like Napier, Briggs, etc. had earned:

> I would like to remark that I specifically publish this [factor] table, so that the flexible arrangement of the table would motivate someone to add 9 more, or if he wants real immortal fame, 99 more [i.e. to 1,020,000] [Lambert 1765-1772, II, p. 49][69]

In the introduction to the table collection, the *Zusätze* (1770), Lambert elaborated his argument. He deplored that the factor table had at least been calculated four times already (Pell/Brancker, Poetius, Anjema, Krüger/Jäger), but that those duplicate efforts had not advanced the table in any way.

> I said, that in the future these tables will be calculated from scratch again. [...] Because it is a tedious labour, to calculate the table of all divisors of the numbers from 1 to 102000 again, I have a plea for the journalists and other writers who will see this work. Viz., they will act out of humanity and serve the mathematical sciences if they contribute as much as possible to the advertisement of this work. Because if someone in the future feels like calculating such tables, he will better spend his time [...] on extending the table, instead of recalculating it again.[Lambert 1765-1772, II, pp. 8–9][70]

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on pp. 345–349 (1775) resp. pp. 783–788. The divisibility criteria had been announced two years before by Johann III Bernoulli [1771/1773, p. 321].

[69]Original: "Vielmehr werde ich anmerken, daß ich die Tabelle vorzüglich deswegen durch den Druck bekannt mache, daß etwann jemand durch die so geschmeidige Einrichtung derselben sich bewegen lasse, noch 9 andere, oder wenn er sich einen recht unsterblichen Namen machen will, noch 99 andere beyzufügen. [d.h. bis 1020000]"

[70]Original: "Ich sagte erst, daß solche [Faktoren- und Prim-]Tafeln auch künftig noch von neuem werden berechnet werden. [...] Da es indessen eine langwierige Arbeit ist, die Tafeln der Theiler der Zahlen von 1 biß 102000 von neuem zu berechnen, so werde ich an die Herren Journalisten und an jede andere Schriftsteller, denen dieses Werckchen vorkommen wird, eine Bitte thun. Sie werden nemlich aus Menschenliebe handeln, und den

Here again, we see Lambert's vision on science at work: Science should advance, not stabilise, and its results should be known and accessible to everyone.

In 1771 and 1772 two updates on the table project are given. Since a few people joined in Lambert's table project, some table desiderata had become obliterate, therefore, it was of importance to announce the progress, so that duplicate efforts would be avoided. Lambert inserted a note in the widely read review journal *Allgemeine Deutsche Bibliothek* (1771, vol. 14, nr. 1, pp. 305–306), communicating that one person (Oberreit) had calculated the factor table to 150,000 and planned to proceed to 200,000, and that another (Wolfram) had calculated hyperbolic logarithms. He also repeated that everyone was invited to complete his system of tables. Lambert concluded with:

> My address is:
> A Monsieur Monsieur [sic!] Lambert, Professeur Royal, Membre de l'Acad. R. de Berlin et diverses Academies et Sociétés des Sciences, à Berlin.

Parallel to this public appeal, Lambert had also inserted a standard appeal at the end of all letters he sent during the years 1770–1771.

> Every now and then there are lovers of mathematics who like to calculate. And I have reason to hope that my invitation [...] will not be in vain. If you would, dear sir, find someone in your surroundings, who would like to undertake such calculations, it would be very agreeable to me. [Lambert Briefe 1781–1787, I, pp. 367–368][71]

mathematischen Wissenschaften einen guten Dienst thun, wenn sie zur Bekanntmachung dieses Werkchens so viel möglich beytragen. Denn wer nur auch künftig Lust hat, solche Tafeln zu berechnen, der wird dann immer besser seine Zeit darauf verwenden [...] [die Tabelle] weiter [zu führen], als das bereits berechnete nochmals [zu berechnen]"

[71] Original: "Es gibt hin und wieder Liebhaber der Mathematik, die gerne rechnen. Und ich habe Ursache zu hoffen, daß die Einladung [...] nicht ohne Frucht seyn werde. Sollten Sie, mein Herr, in dortigen Gegenden jemand finden, der zu solchen Berechnungen Lust hätte, so würde es mir sehr angenehm seyn." (Lambert to Kant). Cfr. Bernoulli's footnote on this page. In this context, it is also interesting to note that Lambert proposed to Röhl (Sept. 1771) that the calculation of tables might be good topics for PhD students! Lambert Briefe [1781–1787, II, p. 391–392]

A last publicly printed update of the table project appeared in the last volume of Lambert's *Beyträge* (1772). It included a four page list of errors in the factor table which the officer Wolfram had found and a list of tables that Lambert had already received or calculated himself, amongst them a factor table to 339000 [Lambert 1765-1772, III, non-pag. *Vorrede*]. Most of the organisation of the table project had been through letters.

> everything else can be easily adjusted in a written correspondence [*A.D.B.* **14** (1), p. 305]

This correspondence with about a dozen of people comprises most of parts IV and V in Lambert's *Deutscher Gelehrter Briefwechsel*, posthumously edited by Johann III Bernoulli from 1782 to 1787.

4 Calculators in Correspondence

4.1 Early Answers to Lambert's Appeal

One year after the publication of the *Zusätze* the first contributions arrived in Lambert's hands. All in all, between 1770 and 1777, the year of his death, Lambert entertained a correspondence with 11 persons on tabular topics, 6 of whom were working on factor tables.[72] Restricting ourselves to factor tables, the contributions occurred in two quite different phases: A first one 1770–1776 (Wolfram, Oberreit, von Stamford, Rosenthal) and a second one 1776–1777 (Felkel, Hindenburg). It is remarkable how differenciated the social provenance of Lambert's collaborators was: Wolfram was an artillery officer, Oberreit an accountant, von Stamford an engineer, Rosenthal a baker, Felkel a teacher and Hindenburg a student and later professor. This list already gives a hint why the first phase is different from the second; Felkel and Hindenburg were near professionals, thinking in career moves, whereas the others were amateurs.

As already mentioned, Isaac Wolfram had contributed a list of errors to the factor table in Lambert's *Zusätze*, his other main contributions to the

[72]These are Wolfram, Oberreit, von Stamford, Rosenthal, Felkel, Hindenburg on factor tables; Schönberg and Röhl on square and cube tables; Baum on sine tables; Schulze and Eißenhardt on issues of publication. The correspondences are in [Lambert Briefe 1781–1787, II, IV & V].

table project were logarithms.[73] Oberreit had devoted himself solely to factor tables. He had calculated to 150,000 in 1770, to 260,000 in the Summer of 1771 and to 339,000 in 1772 [Lambert 1765-1772, II, non-pag. *Vorrede*]. In 1774–5 Lambert had received the factor table to 500,000 from Oberreit, but also the announcement that professional problems made further calculations quite impossible. Acting on this problem, Lambert found von Stamford willing to calculate the part 504,000 to 1 million, but seeing the work did not advance quickly enough, he divided the work between von Stamford (504,000 to 750,000) and Rosenthal (750,000 to 1 million).[74] In the end, two tables were finished, the one by Oberreit to 504,000 and the one by Rosenthal 750,000 to 1 million, unfortunately neither of them were published. J.K. Schulze, who was designated to continue Lambert's table project after 1777, could dispose of Lambert's table-related *Nachlass*.[75] He published Wolfram's logarithms and Röhl's squares and cubes in [Schulze 1778], but he never got to publish Oberreit's table.[76] After Lambert's death, Rosenthal sent his table to the professor J.G. Kästner of the Göttingen University, but the gap between 500,000 and 750,000 hindered a publication [Kästner 1786, pp. 564–565].[77]

Being practically oriented, Lambert had a clear plan for publishing these tables. He repeatedly promised his correspondents they would get their "paper and ink" paid back when it came to a publication, and that he would take care of finding a publisher [Lambert Briefe 1781–1787, V, p. 61]. Of course, publishers were mostly rather unwilling to invest in a hard-to-print and hard-to-sell volume of tables. Intimately knowing the mechanisms of the printing trade of his days, Lambert even devised strategies to find a publisher:

> I have to find various ways, to gradually publish [these tables]. I thought the *Leipziger Buchmesse* would offer the best opportunity to find a publisher, in particular those publishers who cannot find enough manuscripts because they live in remote places.

[73] Some biographic details and a thorough survey of Wolfram's contributions in [Archibald 1950].

[74] The respective correspondence with Oberreit, von Stamford and Rosenthal in [Lambert Briefe 1781–1787, II, pp. 366–382 & V, pp. 10–23 & V, 24–33], a summary of the content of these letters is given by Glaisher [1878, pp. 111–113].

[75] The rest, letters and unpublished essays, was at the disposition of Johann III Bernoulli.

[76] See Bernoulli's footnote [Lambert Briefe 1781–1787, I, p. 368].

[77] In a letter to Z. Dase [1856, pp. 76–77] (dated 1850), C.F. Gauss wrongly described this table as a factor table for numbers between 500,000 and 750,000, probably a slightly confused recollection of Kästner's lectures and books, which Gauss had read some 50 years before writing this letter. Cfr. [Glaisher 1878, p. 113].

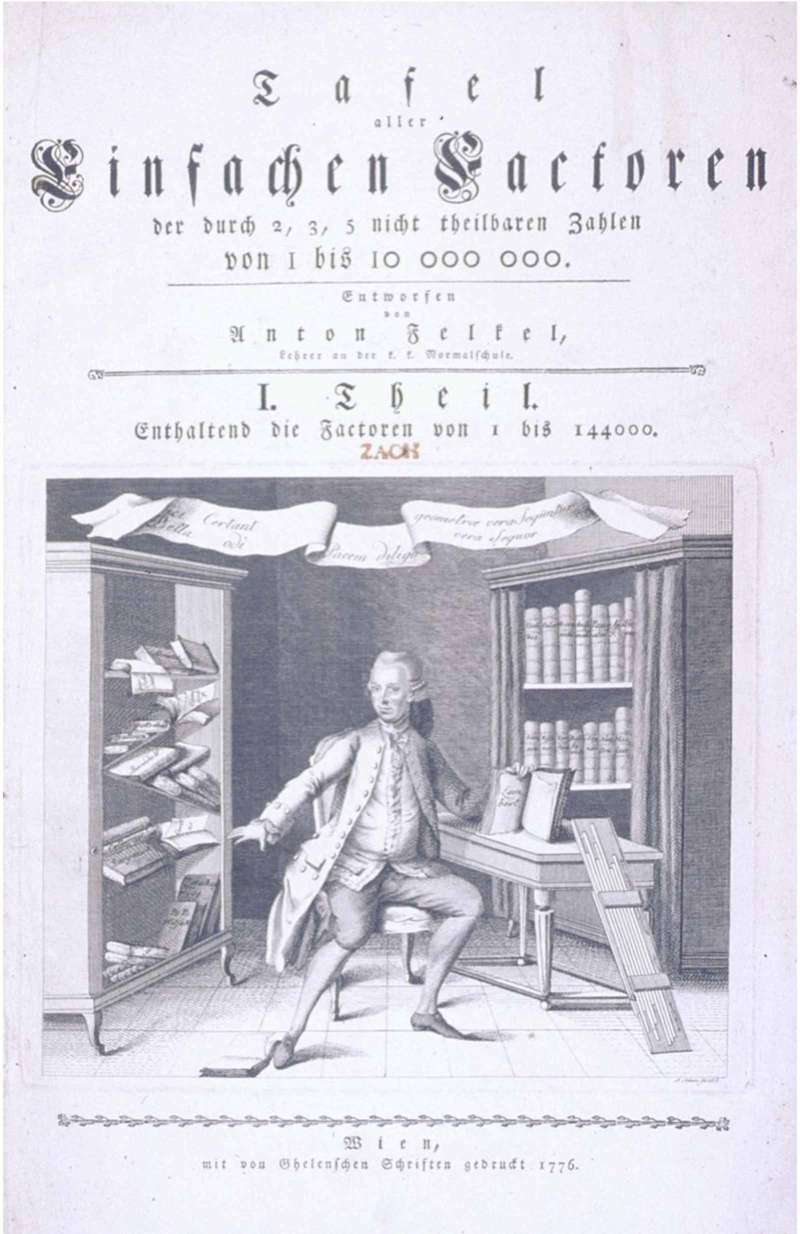
> From time to time one can easily convince a publisher, who had a bad experience with a supposedly trendy book, to speculate more on the durability of sales than on the often failing rapidity. That has indeed always been my best argument. [Lambert Briefe 1781–1787, V, p. 313][78]

A more unfortunate result of Lambert's appeal was the case of J. Neumann, who, upon reading the 2nd part of the *Beyträge*, had decided to extend and correct Lambert's small table (to 10,200) to 100,100. Working on his own without contact with Lambert, Neumann [1785] finished and published his table in Dessau. His table lists all factors, not only the smallest one. At the time of publication, his table was unfortunately already superseded by Lambert's own table to 102,000 in the *Zusätze*, although, as Kästner [1786, pp. 562–3] remarked, it could be used for control. Incidentally, Vega [1797, I, pp. 1–86] used exactly this factor table for his collection of mathematical tables.

4.2 Mechanising the Production of Factor Tables: The Felkel-Hindenburg Priority Battle

January 1776 the Vienna-based teacher Anton Felkel (1740–1800?) announced to Lambert that he had found an apparatus, consisting of rods, that could mechanically find the divisors of all integers. He had come upon this device by reading Lambert's *Zusätze*, especially the remarks on the best arrangement of a table. Felkel's device was a mechanisation of factor table making, based on Eratosthenes's sieve. In his letter, Felkel had inserted an short announcement of his method, of a soon-to-be-published table to 144,000, and of a promise to extend it to 1,000,000. Felkel's idea was that Lambert would publish the announcement in a journal to help him find financial support (and to deter other competitors) [Lambert Briefe 1781–1787, V, pp. 41–44]. Lambert inserted Felkel's circular in the *Leipziger Neue Zeitungen von gelehrten*

[78]Original: "[ich] muß auf verschiedene Mittel bedacht seyn, sie nach und nach herauszugeben. Ich dächte inzwischen, daß sich auf der Leipziger Messe die beste Gelegenheit anbieten sollte, Verleger zu finden, zumal solche die, weil sie an abgelegenen Orten wohnen, in ihrer Gegend nicht immer genug Manuscripte aufbringen können. Zuweilen läßt sich ein Buchhändler, dem ein Mode=Buch fehlgeschlagen, leicht bereden, mehr auf die Dauerhaftigkeit als auf die meistens sehr mißliche Schnelligkeit des Verkaufs zu setzen. Dieses war auch in der That immer mein bester Beweggrund." (Lambert to von Schönberg)



Tafel
aller
Einfachen Factoren
der durch 2, 3, 5 nicht theilbaren Zahlen
von 1 bis 10 000 000.

Entworfen
von
Anton Felkel,
Lehrer an der k. k. Normalschule.

I. Theil.
Enthaltend die Factoren von 1 bis 144000.

Wien,
mit von Ghelenschen Schriften gedruckt 1776.

Figure 3: The frontispice of Felkel's 1776 Factor Table, depicting him with a book by Lambert in his hand, his machine at his table

Sachen (L.G.Z., nr. 63, 5 August 1776, pp. 507–510), but advised Felkel to collaborate with the other calculators and start the second million. End of March 1776, however, Felkel wrote back to Lambert with a change of plan. Constrained by the design of his machine, processing 240 numbers per hour, on the one hand, and encouraged by his patrons in Vienna, on the other hand, Felkel had decided to play *cavalier seul.* He now announced a table from 1 to 2,016,000, the numbers arranged in a way differing from Lambert's set-up [Lambert Briefe 1781–1787, V, pp. 62–70]. Felkel held back with details on his method and machine, to avoid the danger of "seeing my own work thwarted" [Lambert Briefe 1781–1787, V, p. 67]. Felkel's ensuing letters of June and July kept insisting that Lambert would convince the other calculators to stop their work [Lambert Briefe 1781–1787, V, pp. 70–80].

Lambert's reaction was irritation and silence *vis à vis* Felkel, though he aired his disappointment to Rosenthal [Lambert Briefe 1781–1787, V, p. 30]. In the meantime, triggered by Felkel's announcement in the *L.G.Z.*, the publisher S.L. Crusius had a note inserted in the same *Zeitung* (L.G.Z., nr. 64, 8 August 1776, pp. 515–522). The note said that also the Magister Carl Friedrich Hindenburg (1741–1808) from Leipzig had since considerable time (i.e. 1774) found a mechanism for producing factor and other tables, and now planned to publish a description of the mechanism and a factor table to the 5th million [Hindenburg 1776b]. Also Hindenburg's mechanism was a continuation of Lambert's ideas:

> This advantageous artifice is for the greater part the result of a careful study of the decimal number system structure, and is in itself so considerable that it surpasses everything what one could hope and wish for, because it changes the tedious looking up of divisors into a near immediate finding, and produces the prime numbers in their natural order without searching and without loss of time. The method is, as would be suspected, totally mechanic and so secure that that it becomes impossible to make errors that would not immediately be betrayed by a contradiction: a circumstance that takes away the danger of the usual, unavoidable miscalculations with so many numbers. [Hindenburg 1776b, pp. 144–45][79]

[79]Original: "Dieser Vortheil ist größtentheils das Resultat einer sehr sorgfältigen Untersuchung des Decimalzahlengebäudes, und ist in seiner Art so beträchtlich, daß es alles übertrifft, was man nur wünschen und hoffen konnte, indem er das mühsame Aufsuchen

Upon the publication of the announcement, Hindenburg opened his correspondence with Lambert, forwarding him this text from the *L.G.Z.* (August 1776). Trying to save the collaborative spirit, Lambert immediately wrote to all his factor table correspondents August 13th. He informed all his collaborators of Felkel's and Hindenburg's plans, adding complaints when writing to his older correspondents, and adding an insistent request in the letters to Felkel and Hindenburg to divide amongst them the 2nd and 3rd million [Lambert Briefe 1781–1787, V, pp. 81–82 & 151–154]. To Rosenthal, Lambert wrote:

> These Gentlemen apparently want to be ahead one of the other, instead of the better option, to start where the other stops. The first one praises his machine, the second his method. Time will show what they are worth. [Lambert Briefe 1781–1787, V, p. 30][80]

Instead of the scientific collaboration that Lambert so vividly promoted, a series of disputes, (unfulfilled) promises and discussions ensued.[81]

Felkel's reaction on Hindenburg's announcement was immediate. In September he published a more extensive, though not more informative, announcement of his plans, promising now a table to 10 million, to assert his priority in mechanising the production of factor tables [Felkel 1776a]. As Crusius had antedated Hindenburg's announcement to May 1776, Felkel did the same, antedating to June 1776. Hindenburg meanwhile had prepared a manuscript for Lambert and Kästner that described his method, and although nearly all presses were busy during the *Leipziger Messe*, he succeeded in publishing his *Beschreibung* end of 1776 [Hindenburg 1776a]. By that time, Felkel, in his turn, had printed a first specimen of his table to 144,000 [Felkel 1776b]. Both sent their work to Lambert, Felkel included an error list for his table [Lambert Briefe 1781–1787, V, pp. 112–113].

der Theiler, in ein fast augenblickliches Finden verwandelt, und selbst die Primzahlen, in ihrer natürlichen Ordnung nach einander, ohne sie zu suchen, und also ohne allen Zeitverlust, giebt. Das Verfahren hierbey ist, wie man leicht vermuthen wird, ganz mechanisch, und so zuverläßig, daß es unmöglich wird einen Fehler zu begehen, der sich nicht sogleich auf der Stelle durch einen Widerspruch verrathen sollte: ein Umstand, der das gewöhnliche, bey einer so großen Menge Zahlen, ganz unvermeidliche Verrechnen nicht befürchten läßt."

[80]Original: "Diese Herren wollen, wie es scheint, einander zuvorkommen, anstatt dass unstreitig besser wäre, wenn der eine da anfienge, wo der andere aufhöret. Der eine rühmt seine Maschine, der andere seine Methode. Die Zeit muss lehren, was an beiden ist."

[81]See also [Glaisher 1878, pp. 113–118] for an account of the priority discussion.

Following this series of announcements and publications, Lambert presented the plans of Felkel and Hindenburg to the Berlin Academy. In this matter, he closely communicated with Lagrange, who acted as a kind of second referee on Felkel's and Hindenburg's pretentions and productions.[82] The ordeal was in favour of Hindenburg, because his procedure was so to say 'open source', whereas Felkel spent 2 long pages describing his mechanism without being all too clear. Lambert made up for this obscure description and explained it himself in a posthumously published review of both Felkel's and Hindenburg's books [Lambert 1778]. In this review, Lambert also mentioned a serious drawback of Felkel's table: It grouped numbers not in groups of $3n + 0, 1, 2$ as Lambert had done, but in groups of $30n + \ldots$[83] and it used letters as abbreviations for the factors, which made any connection with previous (and later) work quite difficult.

After Lambert's death in 1777, the dispute did not stop, but Felkel travelled in the Autumn of 1783 to Leipzig to discuss the case with Hindenburg [Lambert Briefe 1781–1787, V, pp. 487-88]. The conflict was, however, not resolved, and both issued a new announcement, Felkel a circular in Halle [Felkel 1784], Hindenburg an announcement of his table to the million in the *Leipziger Messkatalog*. The tables to the n-th million they promised were never printed. Felkel though had calculated a table to the second million during the years 1775–1776. The first part to 144,000 was printed (i.e. [Felkel 1776b]) and two additions up to 408,000 were issued. These would later be used by Vega [1797, I, pp. 87–128] for his list of primes.[84] Most copies of these additions did not get sold and were unfortunately destroyed and/or lost during the Austro-Turkish War (1787–1791) – the paper of Felkel's table was recycled for gunpowder cartridges. From 1793 to 1794 Felkel occupied himself with finding methods to reduce the large extent of the tables to 5 or 10 millions. He finally took refuge to other bases of numeration than the decimal one, a topic Felkel had pursued while studying periodic decimal fractions

[82]The subcorrespondence with Lagrange over Felkel's first circular, Felkel's table and Hindenburg's *Beschreibung* can be found in [Lambert Briefe 1781–1787, V, pp. 51–52 & 120–121 & 194].

[83]This ordering is similar to Euler's plan, see 5.1.

[84]A description of Felkel's 1776 tables to 408,000 and their arrangement is given by Glaisher [1878, pp. 106–108]. Remark that in [Glaisher 1873/1874] there still reigns some confusion regarding the original 1770 and the latin 1798 edition of Lambert's *Zusätze*, Glaisher seems to have seen only the 1798 edition, thus getting the data slightly meddled up sometimes.

[Felkel 1785]. Felkel considered 15 bases for the tables to 24 million, 65 were needed for a table to 100 million. In 1798 Felkel reappears with a latin translation of Lambert's *Zusätze*, commissioned and published by the Academy of Sciences in Lisbon [Felkel 1798]. The introduction recounted his story, the book contained a table to 102,000 though not in Lambert's but in Felkel's arrangement [Glaisher 1878, pp. 119–122]. Hindenburg, as J. Bernoulli confirmed [Lambert Briefe 1781–1787, I, pp. 386–87 & V, 242], had prepared the manuscript for the first two millions, but just as his *Primtariffe*[85] they never appeared in print.

4.3 A Short Description of the Sieve Mechanisms

Both Felkel's and Hindenburg's contrivances[86], as well as a third one proposed by Lambert, are mechanisations of multipication, based on the simple idea that every multiple of n in natural order is n units removed from the next and from the preceding multiple. The most expensive step then in Eratosthenes's sieve, checking the multiples of already known prime numbers, is simplified by these devices. They all use Lambert's remark in the *Beyträge*, that a well chosen arrangement of a table displays certains patterns. E.g., the multiples of 7 can be discerned by the eye in Lambert's arrangement. Such patterns allows to check certain n-tuples and the consistency of the factor table.

Felkel's mechanism, depicted on the frontispice of his *Tafel* (Figure 3), is a variation on multiplication rods. Because Felkel based his device on a step-30 procedure, there are 8 of these rods corresponding to resp. $30n + 1, 7, 11, 13, 17, 19, 23, 29$ (the only forms $30n + a$ that are not divisible by 2, 3 or 5). On each rod are inscribed all integers $30n + a$ with $n = 0$ to 99, but the digits of the thousands are dropped. If one now wants to find the multiples of 47, one first has to calculate the multiples of 47 under 1000 by hand, and then look these numbers up on the 8 rods. Then, one has to align the rods so, that the multiples are all on a horizontal line. Now, one has to calculate the first multiple of 47 over 1000, drop the 1 of the thousands (but take it down on a sheet of paper), and look up the remaining digits on the first rod. The numbers (on the other rods) that are horizontally aligned with that number are the further multiples of 47. This procedure

[85] See [Bullynck 2008].

[86] Bischoff [1804/1990, pp. 73–83] contains detailed drawings and descriptions of Felkel's and Hindenburg's devices.

(with some slight improvements) can be repeated to find further multiples [Lambert 1778, pp. 494–95].

As Lambert remarked, this mechanism has some disadvantages. First, one has to take down the thousands manually, a possible source of errors; second, the mechanism is limited by the limits of the rods, though one can produce extra rods [Lambert 1778, p. 494] [Lambert Briefe 1781–1787, V, pp. 120–121]. Hindenburg noticed one more inconvenience in a letter to Lambert, viz., that the numbers that are found are on the rods and have to be taken down (by dictation or by sight) on paper, "machine and journal are separated" [Lambert Briefe 1781–1787, V, p. 204]. Lambert also noticed in his review that this same mechanism can be brought in another form, viz., a cylinder with circular discs on it. On one disc the numbers 0 to 99 are inscribed, on the other a segment is indicated, corresponding to a number p whose multiples one wants to know. By turning this second disk over the first one, one can find the last 2 digits of the desired multiples in natural order; by counting how many complete circles have already been turned, one can manually take down the other digits [Lambert 1778, p. 495]. This system has, of course, the same disadvantages as Felkel's system.

Hindenburg's solution avoided the first and third disadvantages, and adapted the procedure to printing practice, since it was an ink-and-paper implementation of Erastosthenes's sieve procedure. As the title of his work, *Beschreibung einer ganz neuen Art, nach einem bekannten Gesetze fortgehende Zahlen, durch Abzählen oder Abmessen bequem und sicher zu finden*, told, Hindenburg relied (as Felkel and Lambert) on the property of the decimal positional system that every multiple of n in its order is n removed from the next one. To use this property, he designed perforated cartridges that fitted on a folio page with 10 times 30 cells on which were imprinted all odd numbers from left to right, and were continued left to right in the next row. The perforations were, relative to the grid on the folio, at a distance n from each other. This cartridge method worked for 7, 11, 13, 17 and 19[87] but for larger numbers, Hindenburg conceived of a sizeable cartridge (see Figure 4). This sizeable cartridge is a frame of wood with various sliders in it that can be adjusted and/or changed. The sliders cover exactly one row of 10 cells on the folio, except for one cell at position n ($1 < n < 10$). Thus, the number 23 can be sieved using two sliders without holes and one with a hole

[87] 3 and 5 can be discerned by the eye because of the arrangement of the folio, viz., all 5-tuples were contained in two horizontal columns, all triples could be found be a diagonal.

at position 3, followed by one slider without a hole and one with a hole at position 6, etc. It is clear this arranging of sliders takes quite some time, but the pattern repeats after every 23 folio pages (in our example that is), and it is also possible to re-use one installment with a slight reshuffling after one or two folios.

Using the cartridges for all prime numbers in their natural order, Hindenburg wrote the factor m and its "distance" n (which he called "Ordnungszahl", index) in the measured cell, $m.n$ making up the number of the cell. The end product were many folios that only differed in one respect to the to-be-printed tables, viz., they still contained the multiples of 3 and 5.[88]

Remarkably, Hindenburg from the beginning insisted on the fact that this procedure could be used for constructing other tables. In his *Beschreibung*, he explained how to use this device for making tables of squares, of trigonal numbers, of remainders after division, and even how to use it for solving linear Diophantine equations [Hindenburg 1776a, pp. 39–92 and 106–116].[89] The foundations of the later Combinatorial Analysis, as laid down in [Hindenburg 1781], sprang from this study of the positional number system [Hindenburg 1776a, pp. 92–104], as Hindenburg [1795b, p. 247] recounted.[90] Hindenburg recurred on the idea that it is remarkable how so many numbers can be represented with such a small set of signs. By showing that it does not matter if one uses hindu-arabic ciphers or letters, Hindenburg [1776a, pp. 96–100] rediscovered the combinatory nature of positional number systems to an arbitrary base.[91] He also pointed out that a positional number system essentially implicates the possibility of mechanisation. Referring to Leupold's encyclopaedic work on arithmetic instruments [Leupold 1727], Hindenburg indicated how his system was affiliate to the abacus, to multiplication rods and to many other devices of calculation [Hindenburg 1776a, pp. 101–104].

[88]Hindenburg later realised that his folios could be used directly for printing if he deleted the multiples in his standard folios [Lambert Briefe 1781–1787, V, p. 178 note]. This method of working has the advantage of minimising copying errors, but the disadvantage that the folios cannot be used for other purposes such as tables of squares, of remainders after division, etc., tables which figured in Hindenburg's original plan.

[89]This last issue is theoretically pursued in [Hindenburg 1786].

[90]Cfr. Bernoulli's remark in the *Vorrede* of [Lambert Briefe 1781–1787, V].

[91]Hindenburg referred in this context to Beguelin's paper, p. 100.

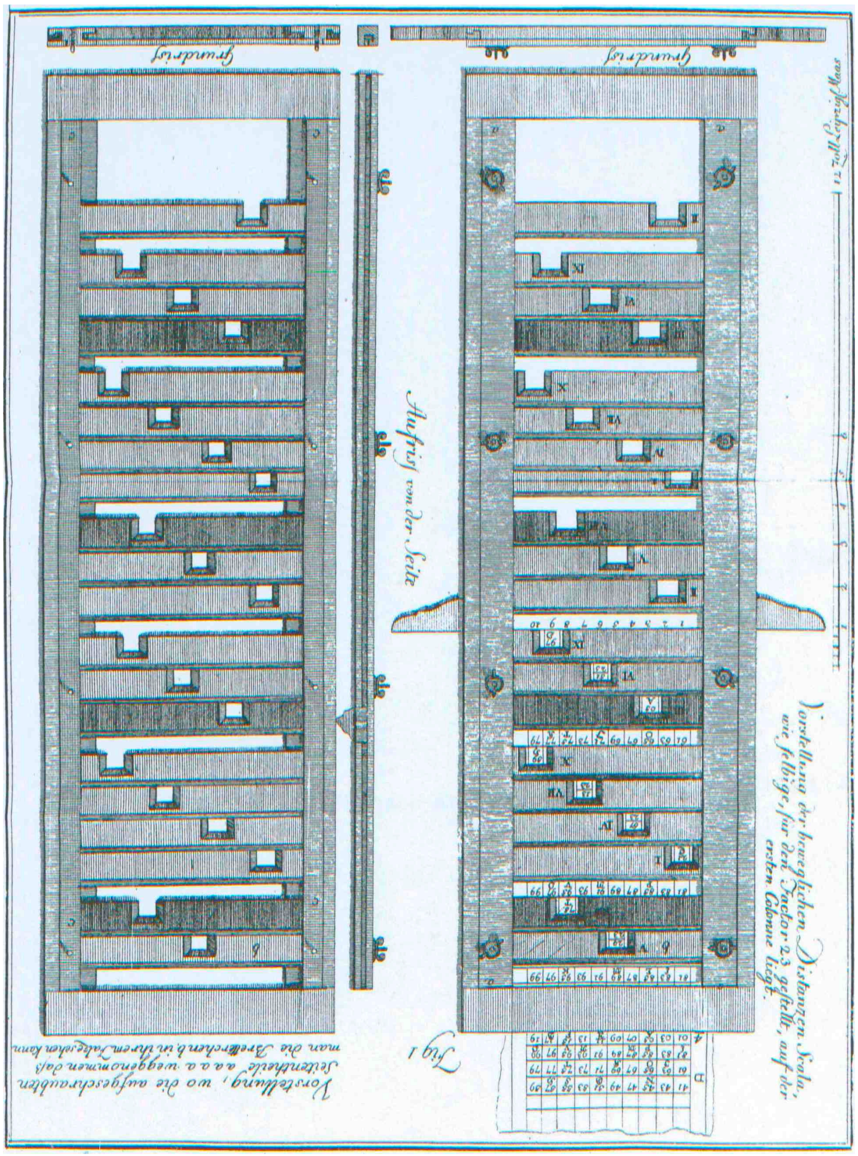


Figure 4: C.F. Hindenburg's sizeable paper-and-ink prime sieve set up for the number 23

5 Factor Tables and the Birth of Number Theory

"Most of the important classical theorems in number theory were discovered as a by-product of the production and inspection of tables."[92]

As Lambert had predicted, "this thing [i.e. the factor tables] may in the future be an important part of the history of mathematics" [Lambert Briefe 1781–1787, II, p. 30]. Indeed, Johan III Bernoulli copiously documented the whole history while editing the scientific correspondence of Lambert during the years 1781–1787. Bernoulli addressed the involved calculators, asking not only for their part of the correspondence, but also for notes and additions to the letters. Also A.G. Kästner [1786, pp. 549–564], mathematics professor in Göttingen, wrote extensively on the topic, but afterwards, partly through external circumstances (the Napoleonic Wars), partly through internal circumstances (a new generation of mathematicians after 1800), the history of Lambert's project was largely forgotten.[93] Indirectly, however, Lambert's project would stimulate the minds of some of the most important mathematicians of the turn of the century, contributing importantly to the birth of number theory as an independent discipline.

5.1 Euler commenting

During the time of the project (1770–1777), its reverberation within the at that time small research community was considerable. As we noticed earlier (p. 28), tables and factoring were a much discussed issue at the Berlin Academy. Lagrange also made sure that a specimen of Lambert's Tables were sent to his correspondent d'Alembert [Lagrange 1867-1892, XIII, pp. 202–203]. Next to the project of astronomical tables and the factor tables, there was in this period a third collaborative project running at the Academy. Following a wish of d'Alembert, Johann III Bernoulli began a French translation of Leonhard Euler's *Algebra*, published 1770 in St Petersburg. He was aided in this venture by J.L. Lagrangre who wrote his famous Additions on indeterminate analysis that complemented Euler's second volume. When the French *Élémens d'Algèbre* appeared 1774, Bernoulli wrote in the foreword:

[92] [Lehmer 1969, p. 118].

[93] Though Glaisher [1878] rediscovered it in his account of producing factor tables.

> I will not say much of the notes I added to the first part [...] they could clear up some points in the history of mathematics & make lot of rather unknown subsidiary tables known. [Euler 1774, p. xvj][94]

Indeed, in the subtext of Bernoulli's translation one finds all tables listed which Lambert had mentioned in the preface to his *Zusätze* and one also finds a eulogy of Lambert's own table work [Euler 1774, pp. 26–28].

Leonhard Euler (1707–1783) had left the Berlin Academy for the St Petersburg Academy in 1766, after a conflict with Frederick of Prussia over the presidency of the Academy which was offered to d'Alembert (and declined) and perhaps also after a conflict with Lambert on running the Academy [Biermann 1985]. Euler has only a role in the margin of this history of tables, be it a curious one. Though he corresponded on the topic with neither Lambert nor Lagrange[95], he did so with Johann III Bernoulli and Nikolaus von Beguelin. Quite some of Euler's contributions concerning factoring methods were actually letters, presentations and papers that were initially conceived as a response to their work, with the exception of one essay. This essay, published 1774 in the *Novi Commentarii* of the Petersburg Academy, discussed how to arrange a factor table to the first million in the best possible way [Euler 1774/1775]. Without any reference to Lambert's work, Euler argued for the importance of factor tables and described a method of arrangement that was based in the grouping into $30n + 1, 7, 11, 13, 17, 19, 23, 29$. Thinking in quarto pages (not folio as most did), Euler constructed subsidiary tables to facilitate the application of Eratosthenes's sieve over the succession of distinct pages. At the end of his paper Euler delivered some (quite error-ridden) samples of a factor table.[96]

Euler's plan was known (i.e., its abstract in the *Journal Encyclopédique* (1776) was read), but its influence was not considerable. Euler's arrangement

[94] Original: "Je ne dirai rien non plus des notes que j'ai ajoutées à la premiere Partie; [...] elles peuvent d'aillieurs répandre du jour sur différens points d'histoire des Mathématiques, & faire connoitre un grand nombre de tables subsidiaires peu connues."

[95] Euler's last letter to Lagrange in 1775, that ends with a reference to [Euler 1774/1775], is the sole exception [Euler 1911-..., IV, 5, pp. 244-45]. A letter from Lambert to Euler with a specimen of the *Zusätze* from October 1771 received only a polite and short response of Euler's secretary [Juskevic et al. 1975, OO1419].

[96] Remark that the editors of [Euler 1907, X–XIII] give a short overview of the history of factor tables from the perspective of Euler's paper, only noticing in a footnote that Lambert's work occurred earlier.

was essentially the same as Felkel's, as Hindenburg remarked in a letter to Lambert, differing only in the quarto and folio format, but Felkel claimed he did not know of Euler's paper [Lambert Briefe 1781–1787, V, pp. 204 & 501]. After 1800, Euler was often referred to in the introduction of factor tables, but no one used his arrangement, all stuck to the one Lambert described. There seems to have been only one exception, an unpublished table of primes by the Swede Schenmark, which was presented to Lexell in Lund. Regarding this table, Nikolas Fuss, Euler's assistant, wrote to Joh. III Bernoulli in December of 1781:

> Mr. Lexell [...] has brought from Lund a table of prime numbers in manuscipt, ready to 1 million after the plan Euler gave and constructed by Mr. Schenmark and some other calculators under his direction. We talked about publishing at the expense of the Academy, but we doubt, because since I communicated to Mr. Euler what you have remarked to me considering Mr. Hindenburg, who has promised two millions before Easter. [Bernoulli 1781/1783, p. 31][97]

Bernoulli advised not to publish Schenmark's tables, defending at large his friend Hindenburg's work. Schenmark's work was, however, conserved and later used by Burckhardt for checking his tables.

Euler's responses to the factoring methods developed by Bernoulli and Beguelin were more momentous for the history of mathematics. These texts announced important methods that soon would be secured theoretically by C.F. Gauss. In a letter to Bernoulli regarding his publication on factoring numbers of the form $10^n \pm 1$, Euler presented a simple, necessary and sufficient criterium to decide whether p divides either $10^{2p+1} - 1$ or $10^{2p+1} - 1$ (p prime) instead of the fragmentary collection of rules that Bernoulli had assembled from Euler's earlier papers [Euler 1772/1774b]. The result rested upon Fermat's theorem and on a theorem Euler did not enunciate, but described as unproven, i.e., the law of quadratic reciprocity. Simultaneously to this letter, Euler read at the St Petersburg Academy on the theory on which

[97]Original: "M. Lexell [...] a apporté de Lund, en manuscript, une Table des nombres premiers, éxecutée jusqu'à un million d'après le plan que M. Euler a donné [...] par M. Schenmark & quelques autres Calculateurs sous sa direction. On avoit parlé d'abord de la faire imprimer aux dépens de l'Académie; mais on hésite depuis que j'ai communiqué à M. Euler ce que Vous m'avez marqué touchant M. Hindenbourg, qui doit avoir promis 2 millions pour Pâques."

Berlin Academy	Euler in St Petersburg	Topic
J.H. Lambert, *Zusätze zu den logarithmischen und trigonometrischen Tafeln* (1770) Lambert to Euler, 18 October 1771	De tabula numerorum primorum usque ad millionem et ultra continuanda (E467, 1774/1775)	manufacture of factor tables
Jean III Bernoulli: Sur les fractions décimales périodiques. Suivi de: Recherches sur les diviseurs de quelques nombres très grands (1771/1773)	Extrait de la correspondance de M. Bernoulli (E461, 1772/1774) Demonstrationes circa residua ex divisione potestatum (E449, 1772/1774) Observationes circa divisionem quadratorum; Disquitio accuratior circa residua etc. (E552 & E554, 1772/1783)	periodic fractions, quadratic residues, instances of quadratic reciprocity
	Earlier work: E54, E134, E271 (Fermat's little theorem); E242, E262, (quadratic residues) Later work: E557, E792	
Nikolaus von Beguelin: Solution particulière du Problème sur les nombres premiers (1775)	Extrait d'une lettre de M. Euler le père à M. Beguelin en mai 1778 (E498) Extrait d'une lettre de M. Fuss à M. Beguelin écrite de Pétersbourg le 1920 juin 1778	factorisation using quadratic forms with idoneal numbers
	Earlier work: E29, E164, E228, E241, E255, E256, E272, E283, E369 Later work: E699, E708, E715, E718, E719, E725	
J.L. de Lagrange: Recherches arithmétiques (1773 and 1775)	De insigni promotione scientiae numerorum (E598, 1775/1785)	theory of quadratic forms
	Earlier work: E29, E279, E323, E452, E454, E559 Later work: E610, E683, E744	

Table 1: A tabular overview of the interactions between research at the Berlin Academy and Euler's research. Indicated are Euler's previous work on the same topics and Euler's later (often posthumously published) work, using the numbers of his papers in the Eneström-index.

Bernoulli's paper was built, i.e., on power residues and now introduced the term "primitive root" [Euler 1772]. Later in the same year, Euler also read on quadratic residues of powers, especially on criteria to decide whether ± 1 is or is not a quadratic residue of a prime p. This way, Euler provided proofs for specific instances of quadratic reciprocity [Euler 1772/1774a].

A similar pattern of interactions ensued some years later (1778) when Euler reacted on an essay by von Beguelin [1775]. This essay had sprung from Beguelin's investigations on his new number system and proposed a new method for factoring numbers. The idea was to isolate a particular sequence of numbers, trigonal or square numbers for instance, and gradually exclude elements until one or more were left.

> it is only in determining the compound elements that one finds the primitive elements[98], through the gaps that result of this determination, in the manner of Eratosthenes.[von Beguelin 1775, p. 301][99]

Beguelin relied for his exclusions on a result of Euler, viz., that the formula $ppxx + 1$ contains all odd numbers, but only compound numbers can be decomposed in more than one sum of squares $a^2 + b^2$. Beguelin's procedure had changed Euler's method a little bit; through a certain arrangement of lists, only one series of possible squares had to be checked instead of two [von Beguelin 1775, p. 308].

Euler's criterium that a prime number can only be expressed in one way as the sum of two squares (or a square and certain multiples of a square, $a^2 + nb^2$) was of course a special case of a more general law. In slightly more modern terms, prime numbers can be expressed in only one way by a binary quadratic form that has only one principal genus. Without this theoretical luggage, Euler wrote exactly that to Beguelin, a list of all n for which the form $x^2 + ny^2$ had only one principal genus [Euler 1776/1779]. These numbers were later coined idoneal numbers by Euler, but the letter to Beguelin contained the first mention (without the name) of these numbers. As Beguelin and Lagrange desired to know more details of this novel method, Euler's assistant N. Fuss compiled a resumé which was published somewhat later in the *Mémoires* of the Berlin Academy [Fuss 1776/1779]. As was

[98] Primitive elements are here p's that are prime, and generative to the elements of the sequence, e.g., of the form $ppxx + 1$.

[99] Original: "ce n'est qu'en déterminant les élémens composés qu'on trouve les primitifs, par les lacunes qui résultent de cette détermination, á la manière d'Erastosthene."

the case with the Bernoulli-paper, Euler used the occasion to write up some papers about the topic of factoring large numbers using the idoneal numbers. They only got published long after Euler's death, between 1801 and 1806.[100]

Lastly, Euler also replied to Lagrange's *Recherches d'arithmétique* (the 1773-part), though not in a letter but with a presentation to the St Petersburg Academy October 26, 1775 (published posthumously in the *Opuscula Analytica*, 1785). Euler's objective in that presentation was to make Lagrange's overview more elegant and more complete (thus partly "anticipating" Lagrange's 1775-part). Euler used the opportunity to point out that he had found many of the results much earlier by induction, but that only the joint efforts of Lagrange and himself had advanced this part of mathematics. According to Euler, Lagrange "ha[d] brought light into the science of numbers, that ha[d] hitherto been enveloped in darkness" [Euler 1775/1785, p. 163].[101] Although Euler had occasionnally used the term "science of numbers" before[102], this paper explicitly addressed this new science in its title[103] and thus indirectly connected Lagrange's and his own endeavours with Lambert's programme – announcing the official birth of number theory around 1800.

Euler's contributions dealt mainly with theoretical aspects of producing factor tables, more specifically, were contributions to factoring. Though these contributions often re-used (partial, often unproven) results Euler had obtained earlier[104], they now appeared in a different context, as reactions to publications on factoring, publications that in their turn were instigated by Lambert's project. Although Euler never referred to this context, the chronology of these papers is telling. So is Euler's adaptation of results from Diophantine problems to the problem of factoring (large numbers), a problem that almost never appeared in his work before 1770.[105] One of Euler's most important results in number theory, the idoneal numbers, were mentioned for

[100]These are E708, E715, E718, E719, E725 [Euler 1907, II, pp. 249–260 & 198–214 & 215–219 & 220–242 & 261–262].

[101]Original: "Eximia omnino sunt, quae La Grange [...] demonstravit, et maximam lucem in scientia numerorum, quae etiamnunc tantis tenebris est involuta, accendunt.".

[102]So in [Euler 1911-..., I, 2, p. 611].

[103]Viz., *De insigni promotione scientiae numerorum.*

[104]Some were mentioned in his correspondence with Christian Goldbach, some were published in the Petersburg *Commentarii*, see [Dickson 1919–1927, I, pp. 360–1] and Table I.

[105]Though the topic of finding large prime numbers did appear, factoring itself was not a major topic.

the first time exactly in this context of tables and factoring, current at the Berlin Academy. Although Euler elaborated further on these results at the St Petersburg Academy, they remained fragmentary and were often based upon induction alone. Such was the case for the law of quadratic reciprocity, for the genera of quadratic forms, for the idoneal numbers. Consequently, a framework for these results, the theory Lambert had called for, was still but fragmentary and lacking. However, just as the table of (in)composits had emancipated itself from the context of Diophantine problems, so were the papers written at the Berlin and Petersburg Academy stepping stones towards a further emancipation of number-related problems, such as factoring, from the Diophantine context. Not only a framework for these results, but a new theory, the theory of numbers, was soon to be wrought by C.F. Gauss in his *Disquisitiones Arithmeticae* (1801), and A.M. Legendre in his *Essai sur la théorie des nombres* (1798) and its consequent editions.

5.2 Tables and the Distribution of Primes

The empirical law, 1st Version Though C.F. Hindenburg never returned to the topic of factor tables later, one issue (vol. 2, nr. 2) of the *Leipziger Magazin für reine und angewandte Mathematik* (1787), of which Hindenburg was the editor, contained quite some essays on factoring. None of these essays, however, went beyond what Hindenburg himself had described much earlier (as he did not fail to note in his editorial comments). With the exception of Karl Christian Friedrich Krause, nobody affiliated to the Combinatorial School seems to have pursued the topic of prime numbers.[106] Krause, better known as a philosopher, elaborated on one remark in Hindenburg's introduction to his *Beschreibung*:

> I have neither time nor motivation, to tediously and strictly prove or disprove a theorem that for my purpose is not useful at all, since it rather satisfies a displaced curiosity than real use. [Hindenburg 1776a, p. 15][107]

[106] Johann Christian Burckhardt, originally a student of Hindenburg, and Carl Friedrich Gauss, quite familiar with Hindenburg's work, should be treated seperately (see infra).

[107] Original: "Ich habe weder Zeit noch Lust, einen für meine Absicht völlig unbrauchbaren Satz [wie sich die Primzahlen unter den Zahlen verteilen], der eher eine unzeitige Neugierde, als einen reellen Nutzen zu befördern scheint, durch einen weitläufigen strengen Beweis *a priori* zu unterstützen oder zu verwerfen."

The theorem in question is a determination of the distribution of primes. Krause had written a dissertation at the university of Jena on this topic, *De inventione numerorum primorum* (1801), probably under the supervision of the Hindenburg-influenced professor D.M.C. Stahl. The manuscript is now unfortunately lost [Riedel 1941].

In 1804 Krause published a factor table to 100,000 and returned to the question of the distribution. Considering the successive application of the sieve procedure, eliminating by and by all composites, Krause arrived at a description of the "Primzahlgesetz".

> We have found the law [that gouverns the distribution of primes], it is a law that is continuously changing with every series of prime numbers; it is a infinite and multi-sided law, that specifies through progression. That is why we will not trouble us any further to find a finite algebraic law, where an infinite one rules. [Krause 1804, p. 12][108]

Krause considered the distribution of primes as an infinite process of eliminating the doubles, triples, 5-tuples etc. and since the series of prime numbers was infinite (with reference to Euclid) the obtained fractions formed an infinite decreasing series, but since the rate of decrease was neither regular nor rapidly decreasing itself, a finite algebraic formula was impossible [Krause 1804, p. 11].[109] Of course, Krause had been anticipated by Euler. In a letter from 1752 [Fuss 1843, I, p. 595], Goldbach had claimed he could prove that no closed algebraic formula could generate only primes, the theorem was later published and proven by Euler [Euler 1762/1763, p. 99].

Also in 1752 [Fuss 1843, I, p. 587] Euler had remarked to Goldbach that the number of prime numbers relative to the number of integral numbers ($= x$) converges to $\ln x$ without proof. Euler repeated this observation in [Euler 1762/1763, p. 101]. Since Lambert's project, however, various tables of factors and primes had been printed, creating the opportunity to make empirical observations about the distribution of primes. On being received

[108] Original: "Wir haben das Gesetz gefunden, es ist nehmlich ein beständig durch jede Reihe der Primzahlen gesetzmäßig verändertes; es ist ein unendlich vielseitiges, bei immer weiterem Fortschreiten weiter bestimmtes. Daher werden wir uns nicht weiter bemühen, ein endliches, algebraisches Gesetz aufzufinden, wo ein unendliches waltet."

[109] Most famous in connection with this finite algebraic formula for primes is Euler's $41 - x + xx$, where the first 40 terms (for $x = 0$ to 39) are all prime numbers. This formula was mentioned in his letter to Bernoulli [Euler 1772/1774b, p. 36].

by the Duke of Braunschweig (1791), the young C.F. Gauss (1777–1856) got Schulze's *Sammlung*, somewhat later (1793), Lambert's *Zusätze* and Hindenburg's *Beschreibung*. In the margin of his copy of Hindenburg's work, Gauss had entered his objections to Hindenburg's opinion on the law of prime distribution. Indeed, Gauss was among the first to make counts and draw up a formula, though he never published it. According to a letter to Encke (1849), Gauss had begun counting in Lambert's and Schulze's tables as early as 1792–3, "even before I had occupied myself with subtler investigations within higher arithmetic" [Gauss 1863–1929, II, p. 444]. In his scientific diary, he had noted:

> Comparationes infinitorum in numeris primis et factoribus cont[entorum] [1796] 31. Mai G[ottingae]
> Leges distributionis [1796] 19. Iun. G[ottingae] [Gauss 1863–1929, X/I, pp. 493 & 495]

The formula in question was that the primes under a converge to [Gauss 1863–1929, X/1, pp. 11–16]

$$\frac{a}{\ln \quad a}$$

a result already conjectured by Euler. Using Vega's list of primes (taken from Felkel's tables), Adrien-Marie Legendre (1753–1833) came up with a similar, though slightly more accurate formula in the second edition of his *Essai sur la Théorie des nombres* (1808):

$$\frac{x}{log.x - 1.08566}$$

With $log.x$ being the hyperbolic logarithm (ln).

New Tables In 1811, Ladislas Chernac [1811], of Hungarian origin but professor of mathematics in Deventer (Netherlands), published the *Cribrum Arithmeticum* on his own account. His table, over 1000 pages, gave all factors for the numbers not divisible by 2, 3 and 5. His introduction contained a very complete lists of all factor tables until 1811, even mentioning a table by Adolph Marci (Amsterdam, 1772) that was calculated in response to Lambert's appeal but that has hitherto been lost [Chernac 1811, pp. V–X; IX]. Chernac did not tell how he constructed his table, except for a reference to Nicomachus's *Arithmetic* and the title *Cribrum Arithmeticum*, a title that

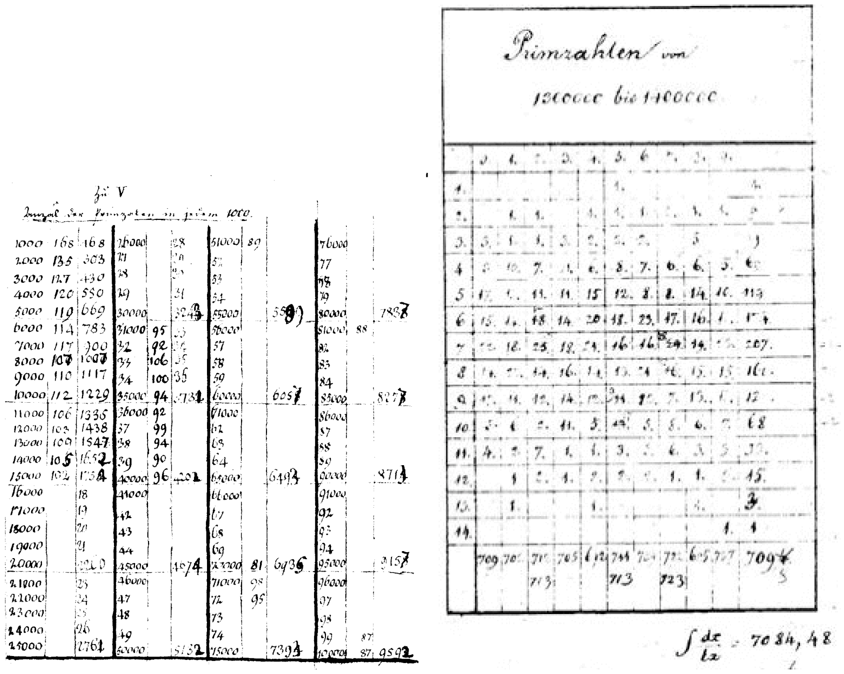


Figure 5: Gauss's Table counting the primes in Lambert's *Zusätze* & a specimen of C.F. Gauss's specially printed paper slips for counting primes and composites for the first chiliad in Chernac's Table

seemed to denote both the method and the content of the table. Chernac's introduction closed with applications and examples: Logarithms, divisions and several tricks where the factor table can be put to use, but no reference at all to more abstract number problems.

Three years later, Johann Karl Burckhardt (1773–1825) published the second million, in 1816 the third million (actually to 3,036,000) and in 1817 the first million, corrected and in the same format as the other two. For this last edition, Burckhardt had compared Chernac's table with Schenmark's manuscript, had found some but few errors in Chernac's table, many in Schenmark's and had checked each inconsistency through re-calculation [Burckhardt 1814, p. i]. In his own words, Burckhardt undertook these calculations while "one was occupied with comparing my moon tables with those of Mr. Burg, a circumstance that made it impossible for me to begin other astronomical studies." [Burckhardt 1814, p. vij][110]

Burckhardt's career was a curious one. Burckhardt was born in Leipzig and he had acquired an extensive knowledge of mathematics and astronomy as a 15-year-old through private study before going to the Leipzig university in 1792. There he had studied under C.F. Hindenburg and had become a *Magister* in 1794 with a dissertation on the expression of continued fractions through combinatoric signs.[111] Through a scholarship and with Hindenburg's recommendation, Burckhardt had gone to the observatory in Gotha to work and study under the astronomer Franz-Xaver von Zach, who was a central figure in the internationalisation of astronomy [Brosche 2001]. With Zach's recommandation, Burckhardt then had gone to Paris in 1798 where the astronomer Lalande had him hired as his assistent in the *Bureau des Longitudes*. In the years 1800–1802, Burckhardt and Gauss had rivaled each other for the best calculations of the courses of Ceres and Pallas. Burckhardt had also translated Laplace's *Mécanique céleste* in German (1800–1802, with lengthy comments and added examples) and was known as a skilled and precise calculator. After Lalande's death in 1807, Burckhardt became the director of the observatory of the Ecole Militaire. He died 1825 in Paris.[112]

The very accurate tables of Burckhardt were calculated with the stencil

[110]Original: "j'ai entrepris et fort avancé ce travail dans le tems qu'on s'occupait de comparer mes Tables de la lune à celles de M. Burg, circonstance qui m'empêchait de commencer d'autres recherches astronomiques."

[111]Hindenburg [1795a, pp. 174–178] described Burckhardt's procedure.

[112]His biography in [ADB, 3, pp. 571–72], where "Hindenburg" is consistently misspelled as "Hindenberg"!

method that would become a standard in the 19th and early 20th century for the production of factor tables. Burckhardt described his method as an improvement on Hindenburg's method in the *Beschreibung*.[113] First, Burckhardt immediately eliminated all numbers divisible by 3 and 5:

> Half of the work [using Hindenburg's method] is done in vain, because in the printed tables, one omits all numbers divisible by 3 or 5, so one is obliged to copy that part of the work that one wants to conserve in print. I have avoided these problems and I have obtained the factors in the same way and time as in my printed tables [Burckhardt 1814, p. v][114]

To this aim, Burckhardt had let a copper plate be engraved in 81 horizontal lines and 78 vertical ones, obtaining 80 times 77 little squares. Next to the first horizontal line were engraved all numbers under 300 not divisible by 2, 3 and 5, thus repeating the arrangement Lambert had advised and Hindenburg had followed. Due to the 77 columns, the multiples of 7 and 11 could immediately be engraved on the plate.

With this plate, the individual sheets were printed, immediately reducing the work to divisors above 11. For these larger divisors, e.g. 13, Burckhardt took an empty, squared sheet, started cutting out the squares that were multiples of 13 and stopped after the 13th column, since "this factor will return in the same order [...] because of the distance" [Burckhardt 1814, p. v].[115] By putting together two sheets, three sheets etc. this procedure could be expanded for larger divisors.

> for divisors over 500, I have preferred to find the multiples by successive additions [...] I have checked the last multiple by a direct multiplication [Burckhardt 1814, p. vi][116]

[113]As mentioned earlier, Hindenburg had made some of these improvements earlier, [Lambert Briefe 1781–1787, V, p. 178 note].

[114]Original: "la moitié de l'ouvrage [est fait] en pure perte; car dans les Tables imprimées on rejette les nombres divisibles par 3 ou par 5, ce qui oblige de copier au net la partie de l'ouvrage qu'on conserve. J'ai évité ces deux inconvéniens et j'ai obtenu en même tems que dans mes Tableaux imprimés les facteurs"

[115]Original: "ce facteur retournera dans le même ordre, puisque la distance d'une colonne à l'autre est toujours de 300."

[116]Original: "quant aux facteurs qui surpassent 500, j'ai préféré de trouver les multiples par des additions successives. [...] le dernier multiple [...] a été vérifié par une multiplication directe."

This procedure of cutting out squares in a sheet of squared paper is essentially the stencil method, not very much different from Hindenburg's sieve, but more practical.[117]

The empirical law, 2nd Version Using these new tables, Legendre had checked his empirical formula again in 1830, for the third edition of his book that was now simply entitled *Théorie des Nombres*. His comparison was quite favorable for the formula [Legendre 1830, II, p. 65]. Gauss had pursued his countings as well, at times together with Goldschmidt. Gauss had even a kind of standard paper slips made to custom for counting the chiliads in Chernac's and Burckhardt's tables. In a letter to Encke, Gauss corrected and/or specified his formula for the distribution of primes. Gauss saw it in inverse proportion to the integral

$$\int \frac{dn}{ln.n}$$

Gauss compared his integral with Legendre's formula and found that his integral approximated the distribution of primes better [Johnson 1884].

5.3 Introducing Number Theory

The birth of a new discipline is always accompanied by legitimation practices, especially by a chronological series that presents a prehistory that leads up to this present new science. The two works that were to found number theory as a discipline in its own right, Gauss's *Disquisitiones Arithmeticae* (1801) and Legendre's *Essai sur la Théorie des nombres* (1798), used their prefaces exactly to do this, construct a historical lineage that culminated in the present efforts.[118] The lineage in both works was constructed in a similar

[117] C.F. Gauss reviewed both Chernac's and Burckhardt's tables [Gauss 1863–1929, II, pp. 181–186], referring in Chernac's review to Lambert's project and describing at length Burckhardt's procedure, strangely enough with referring to Hindenburg. Also, Gauss summarized the history of factor tables around 1800, most probably using Kästner's account [Kästner 1786, pp. 549–564] in Chernac's review. Gauss repeated this same summary some 30 years later in a letter to Zacharias Dase [Dase 1856], stimulating Dase to undertake the calculation of the missing millions. The 6th, 7th and 9th million were eventually calculated by Dase, the 4th and 5th were calculated (with lots of errors) by Leopold Crelle and preserved at the Berlin Academy, cfr. [Crelle 1853].

[118] To complement our observations on the birth of number theory, [Goldstein and Schappacher 2007] is essential reading. In general, see [Goldstein et al. 2007] for the development of number theory after Gauss (and Legendre).

way: Starting with Euclid's Books VII and VIII, passing through Diophantus and Fermat, and finally arriving at Euler and Lagrange.

The most immediate predecessor were Lagrange's *Recherches Arithmétiques.* Legendre explicitly referred to this text as the first general theory on indeterminate questions ("la Théorie de Lagrange", [Legendre 1798, p. X]) and Gauss may be said to have translated Lagrange's title into his *Disquisitiones Arithmeticae* [Weil 1984, pp. 319–320]. A second important impetus, however, had come from Euler's *De insigni promotione scientiae numerorum*, posthumously published in 1785. Legendre's first publication on number theory, that was to be an important part in his later book, was written exactly to prove some of the theorems Euler had put forward in that paper [Legendre 1785, pp. 523–524]. Equally, it figured as one of the most frequently quoted papers in Gauss's work.

Legendre's legitimation was a more tentative one, putting himself in the tradition of Diophantus:

> I do not make a disctinction between number theory and indeterminate analysis, and I regard these two parts as one and the same branch of algebraic analysis. [Legendre 1798, p. xj][119]

Gauss, on the contrary, explicitly addressed a new discipline, probably taking up Lambert's suggestion to fill in the gaps between elementary and advanced arithmetic, and separated its content from mere Diophantine analysis.

> The inquiries which this volume will investigate pertain to that part of Mathematics which concerns itself with integers. [...] The Analysis which is called indeterminate or Diophantine [...] is not the discipline to which I refer but rather a special part of it, just as the art of reducing and solving equations (Algebra) is a special part of universal Analysis. [Gauss 1801, Preface]

For Gauss, arithmetic comprised "all investigations on the general properties and relations between numerical quantities", so that "integers are the sole object of arithmetic", containing both elementary arithmetic (reckoning) and higher arithmetic (now called number theory).

Whereas Legendre proposed to present and expand Lagrange's theory, focussing on quadratic Diophantine problems, Gauss's reference framework

[119]Original: "Je ne sépare point la Théorie des Nombres de l'Analyse indéterminée, et je regarde ces deux parties comme ne faisant qu'une seule et même branche de l'Analyse algébrique"

was broader from the start. Euclid's books VII & VIII, on composite and incomposite numbers, constituted the basis of this new discipline according to Gauss's preface. In this respect, Gauss seemed to continue John Wallis's programme, of re-writing Euclid in arithmetical terms, thus founding mathematics on numbers [Wallis 1657, pp. 14–20].[120] Most probably, C. S. Remer's *Demonstrativische Rechenkunst* (1739), which the 8-year-old Gauss received as gift and which he called his "liebes Büchlein" [Maennchen 1928, p. 17], acquainted Gauss early on with such ideas. Remer, who often quoted Poetius as one of his sources, dealt extensively with the topics of divisors, odd and even numbers, prime and composite numbers and factoring methods, including Eratosthenes's sieve procedure [Remer 1739, pp. 232–321]. The book also contained a large section on "the properties of numbers in relation to each other", discussing properties of the greatest common divisor process [Remer 1739, pp. 324–356].[121]

In Gauss's original set-up (dating from 1796–7) for the *Disquisitiones*, section V on quadratic forms was still rather modest in volume, and Gauss had planned to conclude the book with his construction of the 17-sided polygon and a treatise on the general solution of higher order congruences, the section VIII which only got published posthumously [Merzbach 1981, Bachmann 1911, pp. 6–8]. During the later re-workings of his text, Gauss expanded section V, building on his lecture of Euler's, Lagrange's and Legendre's work, which he had greedily read upon his arrival at Göttingen university from Decembre 1795 to May 1796.[122] As a consequence, the original focus and section VIII disappeared in the ultimate publication.

However, turning to the older parts of the *Disquitiones Arithmeticae*, sections II, III & VI, one notices their connection with the developments from the period 1770–1800. Section VI was mainly concerned with factoring methods, taking up Lagrange's linear divisors of quadratic forms and Euler's idoneal numbers (for which the theorems are proven in section V). Section

[120]There was a copy of Wallis's work in the library of the Collegium Carolinum where Gauss studied [Küssner 1979, p. 37]. The idea of arithmetisation was discussed by e.g. Poetius, Lambert and Kästner in protestant Germany. Of course, Lambert's tentative presentation of a theory of numbers in [Lambert 1770] is also along these lines and was read by Gauss as a 16-year-old. Gauss's Sections I and II may be called the more mature equivalent of Lambert's essay.

[121]Dr. Christian Siebeneicher (Bielefeld) most kindly drew my attention to Remer's book and pointed out its relevance for this topic.

[122]See his letters to Zimmermann, [Poser 1987, pp. 20 and 24].

III (the theoretical part) and part of section VI (the application) treated power residues (using Euler's term "primitive root") and decimal periods (referring to Lambert's, Bernoulli's and Euler's work).[123] This rather hidden sub-focus of the *Disquitiones* on factoring only becomes apparent in the light of its prehistory, and more so, the *Disquisitiones* themselves, often described as a book of wonders falling from the sky, becomes a phenomenon that can be accounted for. It turns out to be the brilliant culmination of half a century of research, started by Euler's various texts and actively stimulated by Lambert's project on factor tables and his appeal for a coherent and complete theory of numbers in general, of factoring specifically.

6 Conclusions

19th century mathematics is marked by a rapidly growing professionalisation, due to the growing importance (and sometimes introduction) of mathematics in the university curriculum and the foundation of mathematical seminars.[124] As a consequence, in harmony with the industrial idea of labour division, table-making became a rather mechanical job, for reckoners and less talented mathematicians, and often got separated from academic mathematics. A prime example of this was de Prony's logarithmic and trigonometric table project in Paris, where the job was divided between Legendre, who set up the formulae, and jobless haircutters, who calculated additions and subtractions [Grattan-Guinness 1990b]. The factor tables of Burckhardt, calculated in 'spare time', and of the reckoning wonder Zacharias Dase were also typical exponents of this tendency. The reports of the British Committee on Mathematical Tables [Glaisher 1873/1874, Cayley 1875/1876] constituted the eventual outcomes of this evolution. They listed all existing tables so as to produce the missing tables as efficiently as possible. Renowned mathematicians such as Cayley and Stokes pointed out the most urgent tasks, J.W.L. Glaisher executed and/or commissioned the missing tables.[125]

Before 1800 the picture is different. Important mathematicians as Pell, Wallis, Lambert and Euler spent quite some time on factor tables, either producing, correcting or promoting them. The main medium for the orga-

[123]For more detail on this part, see [Bullynck 2008].

[124]See e.g. [Jahnke 1990] for Germany, [Grattan-Guinness 1990a] for France.

[125]For the history of the Committee and its dissolution in the 1930ies that announced the computer era, see [Thompson 1949].

nization and promotion of these factor tables was private correspondence, though scientific societies (Collins at the Royal Society and Lambert at the Berlin Academy) acted as catalyzers in this scientific communication. Due the expansion and vulgarization of written communication in the 18th century, Lambert could also use popular journals and book publications for the dissemination of his appeal, and could hope for non-professional, but well educated amateurs to enter in on his plans.

Apart from its importance for the production and lay-out of factor tables, Lambert's appeal also had a considerable impact on the birth of number theory. His scientific essays on tables and numbers put factoring on the academic agenda and pointed out that a gapless and coherent theory of numbers was a scientific desideratum. Through Lambert's active propagation these questions spread not only in academic circles, inspiring contributions by Bernoulli, Béguelin, Lagrange and Euler, but acquired an even wider public in the German states. The popular professors Kästner (Göttingen), Karsten, Klügel (Halle) and Hindenburg (Leipzig) often referred to Lambert's project in their textbooks and lectures, introducing a generation of university students to the problem of tables, factoring and some kind of "theory of numbers". This scientific project is one of the influences on Gauss's *Disquisitiones Arithmeticae* that would ultimately found that missing discipline, the theory of numbers.

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