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MEREOLOGICAL SETS OF DISTRIBUTIVE CLASSES

Abstract. We will present an elementary theory in which we can speak of mereological sets composed of distributive classes. Besides the concept of a *distributive class* and the *membership relation*, it will possess the notion of a *mereological set* and the relation of *being a mereological part.* In this theory we will interpret Morse's elementary set theory (cf. Morse [11]). We will show that our theory has a model, if only Morse's theory has one.

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I. Motivations

In von Neumann-Gödel type set theories (NG for short) we distinguish between *sets* and *classes*. Every set is a class, yet not every class is a set. Sets are those and only those classes which are members of some classes. Classes that are not sets are called *proper classes*; for instance: the class of all sets, the class of all singleton sets, the class of all groups, etc.

Within set theory we cannot, however, deal with «objects» («collections», «complexes», «multitudes», «assemblies») whose elements are proper classes. Such «objects», nevertheless, are quite handy in some cases. We will mention three examples: from mathematics, from meta-mathematics, and from philosophy of science.

1. A definition of category frequently begins somewhat like: "We say that a category $\mathfrak{A}$ is defined if the following are specified: ...". Then, three objects are mentioned, all of which can be proper classes, for instance when we deal with the category of all sets, all groups, or all metric spaces (cf. [2]).[1] In such cases, obviously, the category $\mathfrak{A}$ cannot be an object in an NG-type set theory. This barrier can be gotten around in several ways, which are presented, for instance, in [14]. A «partial» solution is to define categories in set theories tailored especially to suit this purpose, e.g., in Grothendieck's, or MacLane's systems. A drawback of this solution is that we are not able to consider «the whole» category of all sets, «the whole» category of all groups, etc. A «full» overcoming of the difficulties has been proposed by Lawvere, who, instead of construing category theory within a set theory, has built up an axiomatic category theory and construed his set theory within it.

2. To deal with models of Zermelo-Fraenkel set-theory in Morse's class theory the following approach is often adopted: Take a class M and a relation $e \subseteq M \times M$, which is to be the interpretation of the predicate 'ϵ' (cf. [4, p. 25], [3, p. 21]).[2] Then, introduce, inductively, the notion of a formula being *true* in the class M, even though it is clear that this notion depends on the relation e as well. Consequently, the class M alone is called a model of the theory, despite the fact that, with different interpretations of the

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[1] Sometimes it is straight forwardly said that a category is a triplet, consisting of these objects.

[2] The counterpart of 'ϵ' in the metalanguage, i.e., the language of the paper, will be the symbol '$\in$'.

predicate 'ϵ' ($e_1 \neq e_2$) the very notion of model becomes equivocal.[3] Within Morse's class-theory we cannot, obviously enough, define model as a pair $\langle M, e\rangle$ (such as in [4]).[4]

3. In [12], with theories there are associated certain "resources", or "systems", of their concepts. The author uses the term "resources" because the very concepts are proper classes already. To quote: "This, of course, brings about certain difficulties in formulating theorems and leads to employing, 'unofficially', terms like 'resources', or 'systems' (but only in such a way that they could be eliminated altogether)" [12, p. 105].

The above examples evidence the «natural» need for a simple and consistent way of construing objects composed of proper classes. Any such construction must, of course, go beyond the framework of the theory of distributive classes.

In search of a solution we will look into mereology, namely, we will construct mereological sets (or: collective sets, aggregates, fusions, conglomerates) composed of distributive classes.[5]

II. Outline of a set-theoretical ontology

The ontology proposed below is, obviously, of a metaphoric character. It is a draft project of a set-theoretical ontology. Its precise formulation must be left to a formal, relatively consistent, theory.

1. (a) Among other ontological assumptions of set-theory, we could adopt the principle that — apart from *distributive classes* — there exist some ob-

[3] Equivocality disappears only when the standard interpretation of 'ϵ' is adopted, under which e is the «natural» membership relation in M, i.e., $e = \{\langle m_1, m_2\rangle \in M^2 : m_1 \in m_2\}$.

[4] In Morse's class-theory, we could define a model of Zermelo-Fraenkel set-theory as a function from the class $M \times M$ into the set $\{0, 1\}$ (such a function is a proper class whenever M is). Thus, to define a model we would need only one object, unequivocally determining both its universe (the class M) and the interpretation of the predicate 'ϵ' in it (the above function is the characteristic function of a binary relation).

[5] Basic intuitions connected with the notion of *collective set* can be found, for instance in [5], [13] or [16]. It is sometimes said that mereological sets — as opposed to distributive classes — are so-called concrete objects. This standpoint precludes collective sets with abstract elements. However, S. Leśniewski in [7] allowed such objects. He considered a geometrical interval as a collective set composed of other intervals (there seems to be no reason to believe that geometrical intervals, which he analyzed in a paper on the «foundations of mathematics», were for Leśniewski concrete objects). The present author is of the opinion that a mereological set is concrete if and only if all its elements are.

jects, which (together with the *empty class*) are so-called *ur-elements* of distributive classes. These have been named differently: 'non-classes' (cf. [10, p. 160]), 'individuals' (cf. [12, p. 48], [21, p. 101), 'atoms' (cf. [4, § 26]). We will adopt this latter term[6] prefixed with the adjective 'distributive'.[7] No ur-element has distributive elements.

Distributive atoms can, however, be mereological sets (i.e., collective sets).[8] Therefore, they can possess components, fragments, chunks, or pieces, which are not dealt with in set-theory. These will be called proper mereological parts (collective parts), or mereological elements (collective elements). We assume that the following principle is one of the ontological assumptions of set-theory:

Principle. *No distributive class is a mereological part of any distributive atom.*[9]

Let us underline again that the principle above is never adopted *explicitly* in set-theory. And quite naturally so: in set-theory we never speak about mereological parts of anything. However, it seems that another solution would contradict the intuitive meaning of the notion of 'ur-element'.

(b) The most frequently encountered version of set-theory assumes that its universe[10] consists entirely of so-called *pure classes*. These comprise: the empty class, and the classes which have the former as their only ur-element.

(c) It can be assumed that the universe of mereology (resp. of the calculus of individuals) consists of the above distributive atoms. If the adopted version of mereology is atomic, then the objects under consideration are mereological sets uniquely defined by their *mereological atoms*, i.e., objects that have no proper mereological parts.

To expand our ontology *ad infinitum*:

2. The relation of *being a mereological part* we extend onto distributive classes, assuming that each of these is a mereological atom.[11]

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[6] The first has to broad a meaning, the second is used in philosophy in a technical sense; both are inappropriate for our purpose.

[7] Apart from distributive atoms, we will be considering so-called mereological atoms.

[8] These can, for example, be forests, herds, solar systems, or the like; hence our reluctance to call them 'individuals'.

[9] This principle is also — in some version — in Lewis [8] as the Priority Thesis: "No class is part of any individual." (p. 7) and the Second Thesis: "No class has any part that is not a class." (p. 6)

[10] The universe of a theory is the «range» of objects considered in the theory. The universe itself cannot, obviously, be one of the objects investigated within the theory.

[11] As D. Lewis in [8], I mean that not only ur-elements have mereological parts. But the First Thesis of Lewis says: "One class is a part of another iff the first is a subclass

3. Then, we form mereological sets of all distributive atoms and distributive classes. Thus, we assume that there exists mereological sets that have at least two mereological parts, of which at least one is a distributive class. The objects we have construed this way are neither distributive classes (by 2), nor distributive atoms (by our Principle).

4. To the objects postulated in 3, we add distributive atoms and distributive classes.

5. «Forming» distributive classes from the ur-elements can be metaphorically described as a certain construction. Looking for its precise, metaphor-free formulation we turn to the formal set-theory. Such a «construction», starting, as it does, from ur-elements (described in 1), cannot lead outside the universe of distributive classes (also considered in 1), since this is precisely how the latter has been formed.

Applying the «construction» to the universe of 4, we obtain new distributive classes. These will have at least one (distributive) element that is neither a distributive atom, nor a «standard» distributive class.

6. The universe of 4 we further extend, by adding the non-standard distributive classes described in 5.

7. Into the universe of 6 we extend the relation of being a mereological part, assuming that each distributive class (be it standard, or not) is a mereological atom.

8. We form mereological sets of all objects mentioned in 6. In other words: we repeat step 3 for the objects of 6. We assume, thus, that there exist mereological objects that have at least two mereological parts, of which at least one is a «non-standard» distributive class.

9. We extend the universe of 6 to include the objects postulated in 8.

And so on, *ad infinitum*.

The realization of the above project — embodied in a formal system — we will postpone to another occasion. Here, we will focus on a small fragment of it. Namely, starting from pure classes (as in 1c), we will extend the universe to include the objects postulated in 3 and stop at this stage. The

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of the second." (p. 4) "The conjunction of the First and Second Thesis [see footnote 9] is our Main Thesis: The parts of a class are all and only its subclasses." (p. 6–7)

Lewis adds: "To explain what the First Thesis means, I must hasten to tell you that my usage is a little idiosyncratic. By 'classes' I mean things that have members. By 'individuals' I mean things that are members, but do not themselves have members. Therefore there is no such class as the null class. I don't mind calling some memberless thing — some individual — the null *set*. But that doesn't make it a memberless class. Rather, that makes it a 'set' that is not a class. Standardly, all sets are classes and none are individuals." (p. 4) extended universe will comprise solely «standard» distributive classes, and mereological sets construed from them.

We will present a formal theory, suitable to deal with such a universe. We will show that the set-theoretical universe will remain the same. Our mereology will be atomic. We will also show its relative consistency.

III. Elementary theory of distributive classes

All theories that will be considered here are formulated in first-order language with identity over a countable set of variables:

$$x,\ y,\ z,\ u,\ v,\ x_0,\ x_1,\ x_2,\ \ldots$$

(ordered alphabetically, which will be tacitly assumed in the sequel). The Greek letters 'α', 'β', 'γ' (with or without indices) are meta-variables ranging over variables.

Elementary theory of distributive classes **DC** (originating with Morse) will be expressed in first-order language with identity $\mathscr{L}_{\mathbf{DC}}$. Its sole specific symbol is the binary predicate 'ϵ', read "is an element of".

Axioms of the theory, as presented here, are modelled on [3, p. 9–11]. First comes the extensionality axiom:

(DC 1) $$\forall_{x,y}(\forall_z(z \mathrel{\epsilon} x \leftrightarrow z \mathrel{\epsilon} y) \rightarrow x = y)$$

Then, we assume an infinite set of axioms of class existence, in the form of the following scheme: for any mutually different variables α, β and γ, and a formula φ, in which α is and γ is not free

(DC 2) $$\exists_\gamma\forall_\alpha(\alpha \mathrel{\epsilon} \gamma \leftrightarrow (\exists_\beta\, \alpha \mathrel{\epsilon} \beta \wedge \varphi(\alpha)))$$

With the same assumptions, by the above axiom and the axiom (DC 1), we obtain (for each formula φ)

(†) $$\exists!_\gamma\forall_\alpha(\alpha \mathrel{\epsilon} \gamma \leftrightarrow \exists_\beta\, \alpha \mathrel{\epsilon} \beta \wedge \varphi(\alpha))$$

in other words: for a given φ the class postulated by (DC 2) is unique.

To be able to shorten the formulas in the axioms to come, we introduce the convention that for any variable α different from 'v':

$$\mathrm{S}\alpha \qquad \text{abbreviates} \qquad \exists_v \alpha \mathrel{\epsilon} v$$

i.e., $\mathrm{S}\alpha$ can be read as "α is a set". Moreover, for any variables α, β, different from 'v', and for any γ different from the first two:

$$\gamma\mathrm{p}\alpha\beta \qquad \text{abbreviates} \qquad \forall_u(u \mathrel{\epsilon} \gamma \leftrightarrow u = \alpha \vee u = \beta)$$

i.e., $\gamma \mathrm{p} \alpha \beta$ can be read as "γ is a pair consisting of α and β". Furthermore, for any $\alpha \neq \beta \neq \gamma$, and different from '$v$' and '$u$':

$\gamma \mathsf{U} \alpha$	abbreviates	$\forall_u (u \mathbin{\epsilon} \gamma \leftrightarrow \exists_v (v \mathbin{\epsilon} u \wedge u \mathbin{\epsilon} \alpha))$
$\gamma \mathsf{P} \alpha$	abbreviates	$\forall_u (u \mathbin{\epsilon} \gamma \leftrightarrow \mathsf{S} u \wedge \forall_v (v \mathbin{\epsilon} u \to u \mathbin{\epsilon} \alpha))$
$\gamma \mathrm{I} \alpha \beta$	abbreviates	$\forall_u (u \mathbin{\epsilon} \gamma \leftrightarrow u \mathbin{\epsilon} \alpha \wedge u \mathbin{\epsilon} \beta)$

i.e., $\gamma \mathsf{U} \alpha$ can be read as "γ is a generalized sum of α"; $\gamma \mathsf{P} \alpha$ — "γ is a power--set of α; $\gamma \mathrm{I} \alpha \beta$ — "γ is the intersection of α and β".

Employing the above abbreviations we formulate the axioms of empty set, pair, generalized sum, power-set, infinity, and foundation (regularity):

(DC 3) $$\forall_x (\neg \exists_y y \mathbin{\epsilon} x \to \mathsf{S} x)$$

(DC 4) $$\forall_{x,y,z} (\mathsf{S} x \wedge \mathsf{S} y \wedge z \mathrm{p} x y \ \to \ \mathsf{S} z)$$

(DC 5) $$\forall_{x,y} (\mathsf{S} x \wedge y \mathsf{U} x \ \to \ \mathsf{S} y)$$

(DC 6) $$\forall_{x,y} (\mathsf{S} x \wedge y \mathsf{P} x \ \to \ \mathsf{S} y)$$

(DC 7) $$\exists_x (\forall_u (\neg \exists_v v \mathbin{\epsilon} u \to u \mathbin{\epsilon} x) \wedge \\ \wedge \forall_{y,z} (y \mathbin{\epsilon} x \wedge \forall_u (u \mathbin{\epsilon} z \leftrightarrow u \mathbin{\epsilon} y \vee u = y)) \ \to \ z \mathbin{\epsilon} x)$$

(DC 8) $$\forall_x (\exists_u u \mathbin{\epsilon} x \ \to \ \exists_y (y \mathbin{\epsilon} x \wedge \forall_z (z \mathrm{I} x y \to \neg \exists_u u \mathbin{\epsilon} z)))$$

For example: (DC 3) can be read as "if a class has no elements, it is a set". Usually, the axiom is formulated in terms of a previously defined — by (†) — individual constant '$\varnothing$' (empty set), and has the form '$\mathsf{S}\varnothing$'. After eliminating (in the standard way) the constant '$\varnothing$', we obtain: '$\exists_x (\neg \exists_y y \mathbin{\epsilon} x \wedge \mathsf{S} x)$'. It is easy to notice that, by (†), this is equivalent to (DC 3). Sometimes, the same axioms are formulated in terms of another (previously defined by (†)) constant 'Υ' (universal class), and has the form '$\varnothing \mathbin{\epsilon} \Upsilon$'. After eliminating both the constants, we obtain: '$\exists_{x,z} (\neg \exists_y y \mathbin{\epsilon} x \wedge \forall_u (u \mathbin{\epsilon} z \leftrightarrow \mathsf{S} u) \wedge x \mathbin{\epsilon} z)$'. This is again equivalent, by (†), to (DC 3).

Other axioms can be similarly commented upon.

In order to «legibly» formulate the last axiom of replacement, we will need some more abbreviations. Let

$z \mathsf{O} x y$ abbreviate
$$\forall_{x_2} (x_2 \mathbin{\epsilon} z \ \leftrightarrow \ \mathsf{S} x_2 \ \wedge \ \exists_{x_1} (x_1 \mathbin{\epsilon} y \wedge \forall_{x_0} (x_0 \vec{\mathrm{p}} x_1 x_2 \to x_0 \mathbin{\epsilon} x)))$$

where for any variables $\alpha \neq \beta \neq \gamma$ different from 'v' and 'u':

$\gamma \vec{\mathrm{p}} \alpha \beta$ abbreviates $\forall_v (v \mathbin{\epsilon} \gamma \ \leftrightarrow \ v \mathrm{p} \alpha \beta \vee v \mathrm{p} \alpha \alpha)$

'$z \mathsf{O} x y$' can be read as "z is the image of x, restricted to y", and '$z \vec{\mathrm{p}} x y$' as "z is the ordered pair of x and y". The last abbreviation employed will be:

$$\mathsf{F}x \quad \text{abbreviates} \quad \forall_y(y \mathbin{\epsilon} x \rightarrow \exists_{x_1,x_2}(\mathsf{S}x_1 \wedge \mathsf{S}x_2 \wedge y\vec{\mathsf{p}}x_1x_2)) \wedge \\ \wedge \forall_{y,z,x_0,x_1x_2}(y\vec{\mathsf{p}}x_0x_1 \wedge y \mathbin{\epsilon} x \wedge z\vec{\mathsf{p}}x_0x_2 \wedge z \mathbin{\epsilon} x \rightarrow x_1 = x_2)$$

i.e., '$\mathsf{F}x$' can be read as "x is a function". Employing the above abbreviations we state the axiom of foundation:

(DC 9) $$\forall_x(\mathsf{F}x \;\rightarrow\; \forall_{y,z}(\mathsf{S}y \wedge z\mathsf{O}xy \;\rightarrow\; \mathsf{S}z))$$

Apart from the above, we can also assume the axiom of choice.

IV. Elementary mereology

Leśniewski's mereology (cf. [7], [5]) was formulated in a way specific for his logic. For instance, it made use of «creative definitions», and its language contained variables of different syntactic categories bound by quantifiers. It was built up on the basis of another system of his: "ontology" (theory of the copula 'is').

Leśniewski does not introduce the unanalyzable predicate 'is a part of', or 'is a proper part of'. These are for him compositions of the copula 'is' with a name-forming functor 'a part of', or 'a proper part of'. The schema 'a proper part of N' produces either an empty name (if 'N' represents an empty name, a general name, or a singular name of a mereological atom), or a general name (if 'N' represents a singular name of something other than a mereological atom). The schema 'a part of N' produces either an empty name (if 'N' represents an empty name, or a general name), or a singular name (if 'N' represents a singular name of a mereological atom), or a general name (if 'N' represents a singular name of something other than a mereological atom).

Mereology has also been formulated as a second-order theory (e.g., in [18], [19], and — under the name "calculus of individuals" — in [6]). In this guise, mereology can be viewed as a theory of certain algebraic structures that cannot be axiomatized in elementary language (one of the axioms uses a variable ranging over the power-set of the universe). Mereology formulated as an elementary theory can be found in [17] and [9].

Elementary mereology is formulated in first-order language $\mathscr{L}_{\mathbf{M}}$ (with identity). Its sole specific symbol is the binary predicate '$\sqsubseteq$', read "is a mereological part of". Based on the primitive notion we can define two predicates. The first is a binary predicate 'O' (read "overlaps with"):

$$x \,\mathsf{O}\, y \;\leftrightarrow\; \exists_z(z \sqsubseteq x \;\wedge\; z \sqsubseteq y)$$

i.e., two objects overlap iff they have a cammon part. The second defined notion is a unary predicate ‘A’ (read “is a mereological atom”):

$$\mathsf{A}\, x \;\leftrightarrow\; \forall_y (y \sqsubseteq x \;\rightarrow\; y = x)$$

i.e., the object is an atom iff it has no parts except itself (no proper parts).

We assume that the relation of being a mereological part (i.e., the predicate ‘$\sqsubseteq$’) is a partial order:

(M 1) $$\forall_x \; x \sqsubseteq x$$

(M 2) $$\forall_{x,y} (x \sqsubseteq y \wedge y \sqsubseteq z \;\rightarrow\; x \sqsubseteq z)$$

(M 3) $$\forall_{x,y} (x \sqsubseteq y \wedge y \sqsubseteq x \;\rightarrow\; x = y)$$

satisfying moreover:

(M 4) $$\forall_{x,y} (\neg\, x \sqsubseteq y \;\rightarrow\; \exists_z (z \sqsubseteq x \wedge \neg\, z \,\mathsf{O}\, y))$$

i.e., if one object is not a part of another, then the first has a part discrete from the second.

Finally, and characteristically for elementary mereology, we adopt an infinite number of axioms that postulate, for each satisfiable formula $\varphi(\alpha)$ (in which the variable α is free), that there exists the *mereological set* (*mereological sum*) of all objects satisfying $\varphi(\alpha)$.

To be able to shorten the axioms we introduce the following abbreviation. For any variables $\alpha \neq \beta$ different from the variable ‘u’, and for any formula $\varphi(\alpha)$ from $\mathscr{L}_{\mathbf{M}}$, in which α is free and ‘u’ does not occur:

$\beta \,\mathsf{Sum}_\alpha\, \varphi(\alpha)$ abbreviates

$$\forall_\alpha (\varphi(\alpha) \rightarrow \alpha \sqsubseteq \beta) \wedge \forall_u (u \sqsubseteq \beta \rightarrow \exists_\alpha (\varphi(\alpha) \wedge \alpha \,\mathsf{O}\, u))$$

The abbreviation $\beta \,\mathsf{Sum}_\alpha\, \varphi(\alpha)$ can be read as “β is a mereological set of all objects α satisfying $\varphi(\alpha)$”. The given object is a mereological set of all φers iff every φer is part of this object and every part of it overlaps with some φer.

From (M 1) we obtain:

$$\exists_\beta\, \beta \,\mathsf{Sum}_\alpha\, \varphi(\alpha) \;\rightarrow\; \exists_\alpha\, \varphi(\alpha)$$
$$\forall_y\, y \,\mathsf{Sum}_x \text{‘} x = y \text{’}$$
$$\forall_y\, y \,\mathsf{Sum}_x \text{‘} x \sqsubseteq y \text{’}$$

i.e., if there exists a mereological set of all φers, then there exists at least one φer; every object is a mereological set of itself; and every object is a mereological set of all parts of it.

From (M 1)–(M 4), it follows (under the assumptions of β, β_1 and β_2 analogous to the above):

$$\beta_1 \,\mathsf{Sum}_\alpha\, \varphi(\alpha) \wedge \beta_2 \,\mathsf{Sum}_\alpha\, \varphi(\alpha) \;\rightarrow\; \beta_1 = \beta_2$$

i.e., there exists at most one mereological set of all φers. Moreover:

$$\beta \,\mathsf{Sum}_\alpha\, \varphi(\alpha) \;\leftrightarrow\; \forall_u (u \,\mathsf{O}\, \beta \leftrightarrow \exists_\alpha(\varphi(\alpha) \wedge \alpha \,\mathsf{O}\, u))$$

i.e., the object is a mereological set of all φers iff it overlaps with exactly those objects which overlap with some φers.

Under the above assumptions, the axioms of mereological sets have the form:

(M 5) $$\exists_\alpha \varphi(\alpha) \;\rightarrow\; \exists_\beta\, \beta \,\mathsf{Sum}_\alpha\, \varphi(\alpha)$$

Remark 1. There is a connection between (M 1) and (M 5). This will be commented on in Remark 4, in the last section. Now, let us only state the following:

Assume, that we have adopted (M5) and — instead of reflexivity (M1) — only quasi-reflexivity (i) '$\forall_x(\exists_y y \sqsubseteq x \rightarrow x \sqsubseteq x)$' and non-reflexivity (ii) '$\exists_x \neg\, x \sqsubseteq x$'. Then, we get a contradiction taking $\varphi(x) =$ '$\neg\, x \sqsubseteq x$'.

Indeed, by (ii) and (M 5) we obtain:

$$\exists_z(\forall_x(\neg x \sqsubseteq x \rightarrow x \sqsubseteq z) \;\wedge\; \forall_y(y \sqsubseteq z \rightarrow \exists_x(\neg x \sqsubseteq x \wedge \exists_u(u \sqsubseteq x \wedge u \sqsubseteq y))))$$

Hence, by first-order logic:

$$\exists_z((\exists_x \neg x \sqsubseteq x \rightarrow \exists_x x \sqsubseteq z) \;\wedge\; \forall_y(y \sqsubseteq z \rightarrow \exists_x(\neg x \sqsubseteq x \wedge \exists_u(u \sqsubseteq x \wedge u \sqsubseteq y))))$$

Thus, by (ii) and detachment, we get:

$$\exists_z(\exists_x x \sqsubseteq z \;\wedge\; \forall_y(y \sqsubseteq z \rightarrow \exists_x(\neg x \sqsubseteq x \wedge \exists_u(u \sqsubseteq x \wedge u \sqsubseteq y))))$$

Now, by (i), we obtain:

$$\exists_z(z \sqsubseteq z \;\wedge\; \forall_y(y \sqsubseteq z \rightarrow \exists_x(\neg x \sqsubseteq x \wedge \exists_u(u \sqsubseteq x \wedge u \sqsubseteq y))))$$

Dropping the universal quantifier, we get:

$$\exists_z(z \sqsubseteq z \;\wedge\; (z \sqsubseteq z \rightarrow \exists_x(\neg x \sqsubseteq x \wedge \exists_u(u \sqsubseteq x \wedge u \sqsubseteq z))))$$

And, by detachment, we have:

$$\exists_{z,x}(\neg\, x \sqsubseteq x \;\wedge\; \exists_u(u \sqsubseteq x \wedge u \sqsubseteq z))$$

Putting this in prenex form, we obtain:

$$\exists_{z,x,u}(\neg\, x \sqsubseteq x \,\wedge\, u \sqsubseteq x \,\wedge\, u \sqsubseteq z)$$

From which logically follows:

$$\exists_{x,u}(\neg\, x \sqsubseteq x \,\wedge\, u \sqsubseteq x)$$

Which contradicts the assumption (i). □

Elementary atomic mereology is an elementary theory in which, apart from (M 1)–(M 5), we adopt as an axiom:

(M 6) $$\forall_x \exists_y (\mathsf{A} y \wedge y \sqsubseteq x)$$

For elementary atomic mereology defined as above, we will present a proof of the following theorem:

Theorem 1. *Let A be a non-empty set. Let $\mathfrak{A} = \langle |\mathfrak{A}|, \sqsubseteq_{\mathfrak{A}} \rangle$ be a structure for $\mathscr{L}_{\mathbf{M}}$, in which:*

$$|\mathfrak{A}| \;:=\; \mathcal{P}(A) \setminus \{\emptyset\}\,,$$
$$\sqsubseteq_{\mathfrak{A}} \;:=\; \{\langle X, Y\rangle \in |\mathfrak{A}|^2 \,:\, X \subseteq Y\}\,.$$

Then $\mathfrak{A}$ is a model of elementary atomic mereology and for all $X, Y \in |\mathfrak{A}|$ we have:

$$\bigvee_{Z \in |\mathfrak{A}|} \Big(Z \subseteq X \;\&\; Z \subseteq Y\Big) \;\overset{\text{df}}{\iff}\; \langle X, Y\rangle \in \mathsf{O}_{\mathfrak{A}} \iff X \cap Y \neq \emptyset\,,$$

$$\bigwedge_{Z \in |\mathfrak{A}|} \Big(Z \subseteq X \Rightarrow Z = X\Big) \;\overset{\text{df}}{\iff}\; X \in \mathsf{A}_{\mathfrak{A}} \iff \bigvee_{a \in A} X = \{a\}\,.$$

Proof. It is clear that (M 1)–(M 4) and (M 6) are true in $\mathfrak{A}$. We will show that (M 6) is also true in $\mathfrak{A}$.

Let $\varphi(\alpha)$ be a formula from the statement of (M5) with $n{+}1$ free variables $\alpha, \alpha_1, \ldots, \alpha_n$ (for $n \geqslant 0$). Let $\mathfrak{A} \models \exists_\alpha\, \varphi(\alpha)\, [\alpha_1/X_1, \ldots \alpha_n/X_n]$, i.e., the valuation $[\alpha_1/X_1, \ldots \alpha_n/X_n]$ in the family $|\mathfrak{A}|$ satisfies the antecedent of the implication (M5). Then the family $\{X \in |\mathfrak{A}| \,:\, \mathfrak{A} \models \varphi(\alpha)\, [\alpha/X, \alpha_1/X_1, \ldots \alpha_n/X_n]$ is non-empty. Thus, the fact that $[\alpha_1/X_1, \ldots \alpha_n/X_n]$ satisfies the consequent of (M 5), follows from the general set-theoretical lemma below:

Lemma. *Let $\mathcal{R}$ be a non-empty family of sets from $\mathcal{P}(A) \setminus \{\emptyset\}$. Then:*

(a) *for any $X \in \mathcal{R}$, we have $X \subseteq \bigcup \mathcal{R}$;*

(b) $\bigcup R \neq \emptyset$;

(c) *for any $Y \in \mathcal{P}(A) \setminus \{\emptyset\}$, if $Y \subseteq \bigcup \mathcal{R}$, then there are $Z \in \mathcal{P}(A) \setminus \{\emptyset\}$* and $X \in \mathcal{R}$, such that $Z \subseteq X \cap Y$.

Proof of the lemma. Condition (a) follows from the implication: if $a \in X \in \mathcal{R}$, then there is a $Y \in \mathcal{R}$ such that $a \in Y$, i.e., $a \in \bigcup \mathcal{R}$. Condition (b) is a consequence of (a) and the assumption $\mathcal{R} \neq \emptyset$. For (c), let us take any $Y \in \mathcal{P}(A) \setminus \{\emptyset\}$ contained in $\bigcup \mathcal{R}$. By the definition of a generalized sum, for every $a \in Y$ there is an $X \in \mathcal{R}$ such that $a \in X$. Let us choose a b_0 in Y. We have, thus, an $X_0 \in \mathcal{R}$, such that $b_0 \in X_0$. Since $\mathcal{P}(A) \setminus \emptyset$ contains all the singletons from A, we obtain for $Z_0 := \{b_0\}$ that $Z_0 \in \mathcal{P}(A) \setminus \{\emptyset\}$ and $Z_0 \subseteq X_0 \cap Y$.

Coming back to the main proof, put $\mathcal{R}_0 := \{X \in |\mathfrak{A}| \ : \ \mathfrak{A} \vDash \varphi(\alpha)\,[\alpha/X, \alpha_1/X_1, \dots \alpha_n/X_n]$. In the structure $\mathfrak{A}$, the consequent of the implication (M 5) says that in $|\mathfrak{A}|$ we can find a Y_0 such that (i) for any $X \in \mathcal{R}_0$ we have $X \subseteq Y_0$, and (ii) for any $Y \in |\mathfrak{A}|$ if $Y \subseteq Y_0$ then there are $X \in \mathcal{R}_0$ and $Z \in |\mathfrak{A}|$ such that $Z \subseteq X$ and $Z \subseteq Y$. By the lemma, we can take $\bigcup \mathcal{R}_0$ to be Y_0. □

Remark 2. (a) For a non-empty set A, it is impossible to construe a model in the (whole of) family $\mathcal{P}(A)$. From the axioms (M 1), (M 3) and (M 4) it follows that if the universe contains at least two elements, then there is no smallest element with respect to the parthood relation, i.e., we have:

$$\exists_{x,y}\, x \neq y \;\rightarrow\; \neg \exists_x \forall_y\, x \sqsubseteq y$$

1.	$\exists_{x,y}\, x \neq y$	assumption
2.	$\exists_x \forall_y\, x \sqsubseteq y$	assumpt. indir.
3.	$\forall_y\, x_0 \sqsubseteq y$	2
4.	$\neg\, x_1 \sqsubseteq x_0$	1, 3, (M 3)
5.	$\neg\, x_2 \,\mathsf{O}\, x_0$	4, (M 4)
6.	$\forall_y (y \sqsubseteq x_0 \;\rightarrow\; \neg\, y \sqsubseteq x_2)$	5, def. O
7.	$x_0 \sqsubseteq x_0$	(M 1)
8.	$\neg\, x_0 \sqsubseteq x_2$	6, 7
	contr. 3 and 8	

Now, for a non-empty set A we have at least two elements in the family $\mathcal{P}(A)$, namely A and $\emptyset$, and $\emptyset$ is the smallest element with respect to inclusion.

Of course, it can also be checked directly, that the interpretation in $\mathcal{P}(A)$ falsifies (M 4), as $A \nsubseteq \emptyset$ and: $\{\emptyset\} \in \mathcal{P}(A)$, $\emptyset \subseteq \emptyset$ and $\{\emptyset\} \subseteq Z$ for any $Z \subseteq A$.

(b) If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$ and the interpretation in $\mathcal{P}(A)$ is a model of our theory. The same holds for any "degenerate" interpretation in a singleton set $\{a\}$, if only $a \sqsubseteq_{\mathfrak{A}} a$. The validity of (M 1)–(M 3) is evident, and the antecedent of (M 4) is always false. Moreover, if the antecedent of (M 5) is true, then the only object that can satisfy $\varphi(\alpha)$ must be a, so we have $\varphi(a)$. Thus, a satisfies the condition $\beta\, \mathsf{Sum}_\alpha\, \varphi(\alpha)$. □

V. Class theory plus mereology

Let $\mathscr{L}_{\mathbf{MDC}}$ be the first-order language with identity, in which, apart from the predicates 'ϵ' and '$\sqsubseteq$' — as a primitive notion — we have unary predicate 'C', read "is a distributive class".[12]

The elementary theory with identity **MDC** (*Mereology of Distributive Classes*) will be expressed in $\mathscr{L}_{\mathbf{MDC}}$. Its first specific axiom is:

$$\text{(MDC 1)} \qquad \forall_{x,y}(x \mathrel{\epsilon} y \to \mathsf{C}x \wedge \mathsf{C}y)$$

In **MDC** we will have the following extensionality axiom:

$$\text{(MDC 2)} \qquad \forall_{x,y}(\mathsf{C}x \wedge \mathsf{C}y \wedge \forall_z(z \mathrel{\epsilon} x \leftrightarrow z \mathrel{\epsilon} y) \;\to\; x = y)$$

We say that in a formula φ all quantifiers (that occur in it) are restricted by the predicate 'C', if they occur only in sub-formulas of the form $\forall_\alpha(\mathsf{C}\alpha \to \psi)$ and $\exists_\beta(\mathsf{C}\beta \wedge \chi)$.

Similarly as in **DC**, we will have a set of axioms stating that distributive classes exist: for any pairwise different variables α, β, γ, and any formula $\varphi(\alpha)$ of $\mathscr{L}_{\mathbf{MDC}}$ with $n+1$ ($n \geqslant 0$) free variables $\alpha, , \alpha_1, \ldots, \alpha_n$, such that γ is not free in $\varphi(\alpha)$ and all quantifiers in $\varphi(\alpha)$ are restricted by 'C'

$$\text{(MDC 3)} \quad \mathsf{C}\alpha_1 \wedge \cdots \wedge \mathsf{C}\alpha_n \to \exists_\gamma(\mathsf{C}\gamma \wedge \forall_\alpha(\alpha \mathrel{\epsilon} \gamma \leftrightarrow \exists_\beta(\mathsf{C}\beta \wedge \alpha \mathrel{\epsilon} \beta) \wedge \varphi(\alpha)))$$

With the above assumptions, by extensionality, we obtain:

$$(\ddagger) \quad \mathsf{C}\alpha_1 \wedge \cdots \wedge \mathsf{C}\alpha_n \to \exists!_\gamma(\mathsf{C}\gamma \wedge \forall_\alpha(\alpha \mathrel{\epsilon} \gamma \leftrightarrow \exists_\beta(\mathsf{C}\beta \wedge \alpha \mathrel{\epsilon} \beta) \wedge \varphi(\alpha)))$$

Let $\mathbf{I}$ be a syntactic interpretation of $\mathscr{L}_{\mathbf{DC}}$ in $\mathscr{L}_{\mathbf{MDC}}$, such that $\mathbf{I}(\text{'}\epsilon\text{'}) = \text{'}\epsilon\text{'}$ and the "universe" of $\mathbf{I}$ is the predicate 'C' (cf. [15, § 4.7]). For any formula φ of $\mathscr{L}_{\mathbf{DC}}$ we define a formula $\varphi^{(\mathbf{I})}$ of $\mathscr{L}_{\mathbf{MDC}}$, called the interpretation of φ in $\mathbf{I}$. Let φ_{C} be a formula of $\mathscr{L}_{\mathbf{MDC}}$ produced by relativisation of φ to 'C' (i.e., replacing all sub-formulas $\forall_\alpha\psi(\alpha)$ by $\forall_\alpha(\mathsf{C}\alpha \to \psi(\alpha))$ and $\exists_\alpha\psi(\alpha)$ by $\exists_\alpha(\mathsf{C}\alpha \wedge \psi(\alpha))$). Now, if $\alpha_1, \ldots, \alpha_n$ are all the variables that occur free in φ (hence, in φ_{C}), enumerated in alphabetic order, then $\varphi^{(\mathbf{I})}$ is $\mathsf{C}\alpha_1 \wedge \cdots \wedge \mathsf{C}\alpha_n \to \varphi_{\mathsf{C}}$.

Taking, for example, φ to be '$x = x$', we obtain, by (MDC3) that $\exists_x \mathsf{C}x$ is a theorem of **MDC**. Thus, $\mathbf{I}$ is an interpretation of $\mathscr{L}_{\mathbf{DC}}$ in the theory **MDC** (cf. [15, § 4.7]).

[12] This was redundant in **DC**, as all objects from its universe were distributive classes (i.e., we would have an axiom '$\forall_x \mathsf{C}x$').

The next seven axioms of **MDC** will be the interpretations, by **I**, of (DC 3)–(DC 9) from **DC**, i.e., the formulas (DC 3)$^{(\mathbf{I})}$–(DC 9)$^{(\mathbf{I})}$. In a few cases, due to the presence of (MDC 1), we could dispense with restricting some of the quantifiers. For instance, instead of (DC 3)$^{(\mathbf{I})}$ we could take:

$$\forall_x(\mathsf{C}x \to (\neg\exists_y\, y \mathbin{\epsilon} x \to \exists_y\, x \mathbin{\epsilon} y))$$

since '$\exists_y(y \mathbin{\epsilon} x)$ and '$\exists_y(x \mathbin{\epsilon} y)$' are, by (MDC 1), equivalent to '$\exists_y(\mathsf{C}y \wedge y \mathbin{\epsilon} x)$' and '$\exists_y(\mathsf{C}x \wedge x \mathbin{\epsilon} y)$' respectively.

Similarly, employing (MDC1) and (MDC2), we can prove in **MDC** the formula (DC 1)$^{(\mathbf{I})}$.

Since the set of formulas of $\mathscr{L}_{\mathbf{DC}}$ is contained in the set of formulas of $\mathscr{L}_{\mathbf{MDC}}$, we have — by (MDC 3) and (MDC 1) — that, for any formula $\varphi(\alpha)$ in $\mathscr{L}_{\mathbf{DC}}$, (DC 2)$^{(\mathbf{I})}$ holds in **MDC**.

It follows from the above that **I** is an interpretation of **DC** in **MDC** (cf. [15, § 4.7]). Therefore, by a well-known interpretation theorem (cf. [15, § 4.7]), we obtain that for any theorem φ of **DC**, $\varphi^{(\mathbf{I})}$ is a theorem of **MDC**.

We proceed with construing **MDC** in such a way that will ensure the validity of the converse as well. Namely, we will have, for any formula φ in $\mathscr{L}_{\mathbf{DC}}$: if $\varphi^{(\mathbf{I})}$ is a theorem of **MDC**, then φ is a theorem of **DC**.

The predicate '$\sqsubseteq$' is characterized in **MDC** by the axioms (M 1)–(M 6), and the predicates 'O' and 'A, by the definitions from section IV.

In **MDC**, we assume that distributive classes are mereological atoms:

(MDC 4) $$\forall_x(\mathsf{C}x \to \mathsf{A}x)$$

By (M 1), (MDC 4) and the definition of 'At', we obtain:

$$\forall_x(\mathsf{C}x \to \forall_y(y \sqsubseteq x \leftrightarrow x = y))$$

Thus, if in the formula $\varphi(\alpha)$, from (MDC 3), the predicate '$\sqsubseteq$' occurs, we can, equivalently, replace it by the identity predicate.

The following relative consistency theorem holds for **MDC**:

Theorem 2. *Let* $\mathfrak{m} = \langle|\mathfrak{m}|, \epsilon_{\mathfrak{m}}\rangle$ *be a normal*[13] *model of* **DC**. *Let* $\mathfrak{M} = \langle|\mathfrak{M}|, \sqsubseteq_{\mathfrak{A}}, \mathsf{C}_{\mathfrak{M}}, \epsilon_{\mathfrak{M}}\rangle$ *be a structure for* $\mathscr{L}_{\mathbf{MDC}}$, *in which:*

$$\begin{aligned}
|\mathfrak{M}| &:= \mathcal{P}(|\mathfrak{m}|) \setminus \{\emptyset\}\,,\\
\sqsubseteq_{\mathfrak{M}} &:= \{\langle X, Y\rangle \in |\mathfrak{A}|^2 : X \subseteq Y\}\,,\\
\mathsf{C}_{\mathfrak{M}} &:= \{\{a\} : a \in |\mathfrak{m}|\}\,,\\
\epsilon_{\mathfrak{M}} &:= \{\langle X, Y\rangle \in |\mathfrak{A}|^2 : \bigvee_{a,b \in |\mathfrak{m}|} (X = \{a\} \;\&\; Y = \{b\} \;\&\; a \mathbin{\epsilon_{\mathfrak{m}}} b)\}\,.
\end{aligned}$$

[13] In the sense that the interpretation of '=' is the «true» identity.

Then $\mathfrak{A}$ *is a normal model of* **MDC**, *in which:*

$$\begin{aligned} \mathsf{A}_{\mathfrak{M}} &= \mathsf{C}_{\mathfrak{M}}\,, \\ \mathsf{O}_{\mathfrak{A}} &= \{\langle X, Y\rangle \in |\mathfrak{M}|^2 \,:\, X \cap Y \neq \emptyset\}\,. \end{aligned}$$

Proof. (i) By Theorem 1, the axioms (M 1)–(M 6), and the definitions of 'O' and 'A', are true in $\mathfrak{M}$.

That the axioms «coming from» **DC** are true as well follows from the fact that in $\mathfrak{M}$ they «act» only on singletons, whose elements belong to $|\mathfrak{m}|$.

(ii) The axiom (MDC 1) is also true, as a consequence of the fact that, if $X \,\epsilon_{\mathfrak{m}}\, Y$, then by the interpretation of 'ϵ' in $\mathfrak{M}$', X and Y are singleton sets, hence they belong to $\mathsf{C}_{\mathfrak{M}}$.

(iii) To show that (MDC 2) is true in $\mathfrak{M}$ let us assume that $X, Y \in \mathsf{C}_{\mathfrak{M}}$ and that for any Z in $|\mathfrak{M}|$ we have: $Z \,\epsilon_{\mathfrak{M}}\, X$ iff $Z \,\epsilon_{\mathfrak{M}}\, Y$. By the first assumption, there exist a and b in $|\mathfrak{m}|$ such that $X = \{a\}$ and $Y = \{b\}$. For any element c of $\mathfrak{m}|$ we then get: $c \,\epsilon_{\mathfrak{m}}\, a$ iff (by the interpretation) $\{c\} \,\epsilon_{\mathfrak{M}}\, \{a\}$ iff (by the second assumption) $\{c\} \,\epsilon_{\mathfrak{M}}\, \{b\}$ iff $c \,\epsilon_{\mathfrak{m}}\, b$. Thus, by the fact that $\mathfrak{m}$ is a normal model of **DC**, we obtain $a = b$. Therefore, $X = Y$.

(iv) To show that (MDC 3) is true in $\mathfrak{M}$ let us assume that $\varphi(\alpha)$ is the relevant formula with $n + 1$ free variables α, $\alpha_1, \ldots, \alpha_n$.

Let the valuation $[\alpha_1/X_1, \ldots, \alpha_n/X_n]$ satisfy the antecedent of (MDC 3) in $\mathfrak{M}$. Thus, by the definition of $\mathsf{C}_{\mathfrak{M}}$, there exist $a_1, \ldots, a_n$ in $|\mathfrak{m}|$ such that $X_1 = \{a_1\}, \ldots, X_n = \{a_n\}$.

Let us now replace $\varphi(\alpha)$ by $\psi(\alpha)$ in such a way that the predicate '$\sqsubseteq$' is replaced by '$=$', and the predicate 'A', by 'C'. Consider the first-order language $\mathscr{L}'_{\mathbf{DC}}$ with identity and two specific symbols 'ϵ' and 'C'. Define the elementary theory $\mathbf{DC}'$, in this language, with the axioms (DC 1), (DC 2) extended onto the formulas of $\mathscr{L}'_{\mathbf{DC}}$, the axioms (DC 3)–(DC 9), and '$\forall_x \mathsf{C}x$'. It is evident that a structure $\mathfrak{m}' = \langle |\mathfrak{m}|, \epsilon_{\mathfrak{m}}, \mathsf{C}_{\mathfrak{m}}\rangle$ with $\mathsf{C}_{\mathfrak{m}} = |\mathfrak{m}|$ is a model of $\mathbf{DC}'$. Therefore, by the suitably extended (DC 2), there exists a c_0 in $|\mathfrak{m}|$ such that the valuation $[\gamma/c_0, \alpha_1/a_1, \ldots, \alpha_n/a_n]$ satisfies in $\mathfrak{m}'$ the formula $\forall_\alpha(\alpha \,\epsilon\, \gamma \;\leftrightarrow\; \exists_\beta \alpha \,\epsilon\, \beta \wedge \psi(\alpha))$. Hence, for any a in $|\mathfrak{m}|$, we have: $a \,\epsilon_{\mathfrak{m}}\, c_0$ iff there exists a b in $|\mathfrak{m}|$, such that $a \,\epsilon_{\mathfrak{m}}\, b$ and the valuation $[\gamma/c_0, \alpha/a, \alpha_1/a_1, \ldots, \alpha_n/a_n]$ satisfies $\psi(\alpha)$ in $\mathfrak{m}'$ iff there exists a Y in $\mathsf{C}_{\mathfrak{M}}$, such that $\{a\} \,\epsilon_{\mathfrak{M}}\, Y$ and the valuation $[\gamma/\{c_0\}, \alpha/\{a\}, \alpha_1/X_1, \ldots, \alpha_n/X_n]$ satisfies $\psi(\alpha)$ in $\mathfrak{M}$.

Take now any X from $|\mathfrak{M}|$ and assume that $X \,\epsilon_{\mathfrak{M}}\, \{c\}$. Then, for some a_0 in $|\mathfrak{m}|$ we have $X = \{a_0\}$ and $a_0 \,\epsilon_{\mathfrak{m}}\, c_0$. Therefore, by the fact proved above, there is a $Y \in \mathsf{C}_{\mathfrak{M}}$ such that $X \,\epsilon_{\mathfrak{M}}\, Y$ and $[\gamma/\{c_0\}, \alpha/X, \alpha_1/X_1, \ldots, \alpha_n/X_n]$ satisfies $\psi(\alpha)$ in $\mathfrak{M}$. Thus, we have shown that the valuation $[\gamma/\{c_0\}, \alpha/X, \alpha_1/X_1, \ldots, \alpha_n/X_n]$ satisfies in $\mathfrak{M}$ the formula $\alpha \,\epsilon\, \gamma \;\rightarrow\; (\exists_\beta \alpha \,\epsilon\, \beta \wedge \psi(\alpha))$.

Let us now assume that for an arbitrary X in $|\mathfrak{M}|$ the valuation $[\alpha/X, \gamma/\{c_0\}, \alpha_1/X_1, \ldots, \alpha_n/X_n]$ satisfies in $\mathfrak{M}$ the formula $\exists_\beta \alpha \epsilon \beta \wedge \psi(\alpha)$, i.e., (a) there is a Y in $\mathsf{C}_\mathfrak{M}$ such that $X \epsilon_\mathfrak{M} Y$ and (b) the valuation $[\alpha/X, \gamma/\{c_0\}, \alpha_1/X_1, \ldots, \alpha_n/X_n]$ satisfies in $\mathfrak{M}$ the formula $\psi(\alpha)$. From (a) and the definition of $\epsilon_\mathfrak{M}$ it follows that for some $a_0 \in |\mathfrak{m}|$ we have $X = \{a_0\}$ and $X \epsilon_\mathfrak{M} Y$. Hence, by (b) and the fact proved above, we have $X \epsilon_\mathfrak{M} \{c_0\}$. Thus, we have shown that the valuation $[\alpha/X, \gamma/\{c_0\}, \alpha_1/X_1, \ldots, \alpha_n/X_n]$ satisfies in $\mathfrak{M}$ the formula $(\exists_\beta \alpha \epsilon \beta \wedge \psi(\alpha)) \to \alpha \epsilon \gamma$.

From the two previous paragraphs, it follows that for any X in $|\mathfrak{M}|$ the valuation $[\alpha/X, \gamma/\{c_0\}, \alpha_1/X_1, \ldots, \alpha_n/X_n]$ satisfies in $\mathfrak{M}$ the formula $(\alpha \epsilon \gamma \leftrightarrow \exists_\beta \alpha \epsilon \beta \wedge \psi(\alpha))$. Thus, the valuation $[\gamma/\{c_0\}, \alpha_1/X_1, \ldots, \alpha_n/X_n]$ satisfies in $\mathfrak{M}$ the formula $\forall_\alpha(\alpha \epsilon \gamma \leftrightarrow \exists_\beta \alpha \epsilon \beta \wedge \psi(\alpha))$.

From the definitions of the predicate 'A', the set $\mathsf{C}_\mathfrak{M}$, and the relation $\sqsubseteq_\mathfrak{M}$, it follows that $\mathsf{C}_\mathfrak{M} = \mathsf{A}_\mathfrak{M}$ and for any $X, Y \in \mathsf{C}_\mathfrak{M}$: $X \sqsubseteq_\mathfrak{M} Y$ iff $X = Y$. Since all the quantifiers in $\psi(\alpha)$ are restricted by 'C', and no variable other than α, $\alpha_1, \ldots, \alpha_n$ occurs free in $\psi(\alpha)$, the fact proved above is true for $\varphi(\alpha)$ as well. Thus, the valuation $[\alpha_1/X_1, \ldots, \alpha_n/X_n]$ satisfies in $\mathfrak{M}$ the formula $\exists_\gamma \forall_\alpha(\alpha \epsilon \gamma \leftrightarrow \exists_\beta \alpha \epsilon \beta \wedge \varphi(\alpha))$.

Therefore, we have proved that any valuation satisfying in $\mathfrak{M}$ the antecedent of (MDC 3) satisfies its consequent as well.

(v) To show that $(\mathrm{DC}\,3)^{(\mathbf{I})}$ is true, let us assume that the valuation $[x/X]$ satisfies in $\mathfrak{M}$ the formula '$\mathsf{C}x \wedge \neg \exists_y(\mathsf{C}y \wedge x \epsilon y)$'. Then, there exists such an $a \in |\mathfrak{m}|$ that $X = \{a\}$ and for no Y in $|\mathfrak{M}|$ do we have $X \epsilon_\mathfrak{M} Y$. Therefore, neither is there such a b in $\mathfrak{m}|$, that $a \epsilon_\mathfrak{m} b$. Thus, as (DC 3) is true in $\mathfrak{m}$, we have a $c \in |\mathfrak{m}|$, such that $a \epsilon_\mathfrak{m} c$. Thus, the valuation $[x/X]$ satisfies the formula '$\exists_y(\mathsf{C}x \wedge x \epsilon y)$' in $\mathfrak{M}$ as well.

Other axioms are proved to be true in $\mathfrak{M}$ in a similar way. □

Corollary. *If* **DC** *is consistent, so is* **MDC**.

We have also the «conservative interpretation» theorem for **MDC**[14]:

Theorem 3. *Let φ be a formula of $\mathscr{L}_{\mathbf{DC}}$. Then, for $\varphi^{(\mathbf{I})}$ to be a theorem of* **MDC** *it is necessary and sufficient that φ be a theorem of* **DC**.

Proof. "$\Rightarrow$" Let φ be an arbitrary formula of $\mathscr{L}_{\mathbf{DC}}$.

In the proof we will make use of a well-known fact (cf. e.g., [1, p. 79]), implying that, for any structure $\mathfrak{A}$ of $\mathscr{L}_{\mathbf{MDC}}$: if $\mathfrak{A}^{\mathsf{C}}$ is a structure for $\mathscr{L}_{\mathbf{MDC}}$

[14] The interpretation **I** of **DC** in **MDC** can be termed «conservative». Notice that **MDC** is not a conservative extension of **DC**, as it is not even its extension (for instance (DC 1) is not a theorem of **MDC**).

with the universe $\mathsf{C}_{\mathfrak{M}}$ ($\mathsf{C}_{\mathfrak{M}} \neq \emptyset$) and with relations from $\mathfrak{A}$ restricted to $\mathsf{C}_{\mathfrak{M}}$, then $\mathfrak{A} \models \varphi^{(\mathbf{I})}$ iff $\mathfrak{A}^{\mathsf{C}} \models \varphi$.

Let us assume that $\varphi^{(\mathbf{I})}$ is a theorem of **MDC**. Take any model $\mathfrak{m}$ of **DC**. With no loss of generality we can assume that $\mathfrak{m}$ is a normal model. We will show that $\mathfrak{m} \models \varphi$. Thus, by Gödel's completeness theorem (the model was chosen arbitrarily), it will follow that φ is a theorem of **DC**.

In Theorem 2, starting from a model $\mathfrak{m}$, we have built a structure $\mathfrak{M}$, such that the former is a model of **MDC**. Thus, by the assumption and the completeness theorem, we obtain $\mathfrak{M} \models \varphi^{(\mathbf{I})}$. Hence, by the fact mentioned above, we have $\mathfrak{M}^{\mathsf{C}} \models \varphi$.

By the construction of $\mathfrak{M}$, it follows that $\epsilon_{\mathfrak{M}}|_{\mathsf{C}_{\mathfrak{M}}} := \epsilon_{\mathfrak{M}} \cap (\mathsf{C}_{\mathfrak{M}} \times \mathsf{C}_{\mathfrak{M}}) = \epsilon_{\mathfrak{M}}$. Since the predicates '$\mathsf{C}$', '$\mathsf{A}$', and '$\sqsubseteq$' do not occur in φ, we have that for the structure $\langle \mathsf{C}_{\mathfrak{M}}, \epsilon_{\mathfrak{M}} \rangle$ (i.e., $\mathfrak{M}^{\mathsf{C}}$ without the three above relations) $\langle \mathsf{C}_{\mathfrak{M}}, \epsilon_{\mathfrak{M}} \rangle \models \varphi$ as well. This, by the definitions of $\mathsf{C}_{\mathfrak{M}}$ and $\epsilon_{\mathfrak{M}}$ in Theorem 2, implies that $\mathfrak{m} \models \varphi$.

"$\Leftarrow$" Follows from the interpretation theorem (cf. [15, § 4.7]). □

Remark 3. Assume that instead of (MDC 4) we adopted as axiom the formula '$\mathsf{C}y \rightarrow (x \sqsubseteq y \leftrightarrow (\mathsf{C}x \wedge x \neq \varnothing \wedge \forall_z (z \,\epsilon\, x \rightarrow z \,\epsilon\, y)))$', i.e., "The parts of a class are all and only its [non-empty] subclasses."[15] Then for the formula $\varphi(x)$ = '$\mathsf{C}y \wedge \mathsf{C}z \wedge (x = y \vee x = z)$' we get the thesis:

$$u \,\mathsf{Sum}_x\, \varphi(x) \;\leftrightarrow\; \mathsf{C}u \wedge \forall_v (v \,\epsilon\, u \leftrightarrow v \,\epsilon\, y \vee v \,\epsilon\, z)$$

i.e., the mereological set (sum) of two distributive classes is the distributive sum of theirs.[16] □

Remark 4. Assume that instead of (M 1) and (MDC 4) we adopted as axioms the formulas '$\neg \mathsf{C}x \rightarrow x \sqsubseteq x$' and '$\exists_y\, y \sqsubseteq x \rightarrow \neg \mathsf{C}x$', stating that, except for distributive classes, the relation $\sqsubseteq$ is reflexive, and that distributive classes have no mereological parts. Then, we get a contradiction (a counterpart of the Russell paradox) by taking φ = '$\neg x \sqsubseteq x$'.

To show this, notice that the axioms adopted above imply quasi-transitivity '$\forall_x (\exists_y\, y \sqsubseteq x \rightarrow x \sqsubseteq x)$' and '$\forall_x (\mathsf{C}x \rightarrow \neg x \sqsubseteq x)$'. Further, as we have proved '$\exists_x \mathsf{C}x$' without recourse to (M 1) and (MDC 4), we obtain non-reflexivity '$\exists_x \neg x \sqsubseteq x$'. Hence, by Remark 1, a contradiction follows. □

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15 See [8, p. 6–7] and my footnote 11.

16 Informal, for two distibutive classes c_1 and c_2: the mereological sum $[\![c_1, c_2]\!]$ is equal to the class $c_1 \cup c_2$; i.e. we have no pair. If $c_1 \subseteq c_2$ then $[\![c_1, c_2]\!] = c_2$.

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