TEACHING NOTE 97-13:
OPTION PRICES AND STATE PRICES[1]

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A contingent claim is a security that provides a payoff that is dependent (contingent) on something specific happening. An option is one form of a contingent claim in that it provides a positive payoff under the condition that the option expires in-the-money. If the option does not expire in-the-money, the payoff is obviously zero. Another form of a contingent claim is a security that pays $1 in a given outcome and zero otherwise. These outcomes are referred to as *states* or *states of nature* and the security is often called a *state-contingent claim*. Other common names for this type of security are *pure security*, the term we shall use, and *Arrow-Debreu security*, the latter arising out of the work of Nobel Laureates Kenneth Arrow and Gerard Debreu. In this Teaching Note, we examine some properties of pure securities and demonstrate how they relate to options.

Suppose we are facing a risky situation, which could be something as simple as the next day in the stock market. Let us define the possible outcomes in terms of three states, which might be as simple as the market goes down 2% (state 1), the market is unchanged (state 2), and the market goes up 2% (state 3). Naturally the possible outcomes are infinite and cannot be reduced to such simple statements, but the framework provided by this simplification is, nonetheless, useful and generalizes to the case of a continuous spectrum of states.

Consider a stock that will be worth $110 in state 1, $100 in state 2 and $90 in state 3. Another security might be worth $105 in state 1, $101 in state 2 and $98 in state 3. Suppose the risk-free rate is 2%. Then a risk-free security worth $100 today would have a value of $102 in each state.

Now consider a state-contingent claim that pays $1 in state 1 and zero in the other states. Another state-contingent claim pays $1 in state 2 and zero in the other states. A third state-contingent claim pays $1 in state 3 and zero in the other states. Our first stock, whose three possible future values are $110, $100 and $90, can be viewed as a portfolio of 110 units of the first-state

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[1]These notes have benefitted from helpful conversations with and similar notes of Professor Richard Rendleman of the University of North Carolina at Chapel Hill. The notes also appear in published form in *Financial Engineering News*, March/April 2003, pp. 8-10.

contingent claim, 100 units of the second state-contingent claim and 90 units of the third state-contingent claim. Our second security can be viewed as a portfolio of 105 units of the first state-contingent claim, 101 units of the second and 98 units of the third. A risk-free security worth $100 today can be viewed as 102 units of all three state-contingent claims. The price of a state-contingent claim is called a *state price*. It follows that the price of each security must be the value today of the equivalent portfolio of state-contingent claims. In other words, if we know the state prices, we can determine the security prices. Alternatively, if we know the security prices, we can determine the state prices.

The state-contingent claims are the fundamental securities in the market. We cannot literally see them or trade them, but they are there, the financial atoms of the marketplace. Ordinary securities, being combinations of these pure securities, are sometimes called *complex securities*, though there is nothing particularly complex about them. They are just portfolios of pure securities. Let us now develop a formal framework for understanding these concepts.

First let us establish the fact that there must always be at least as many securities as there are states. This is referred to as the *spanning property*, which means that the pure securities will be sufficient to reproduce any complex securities. Here we shall make the number of securities equal to the number of states. Specifically let there be n states, where each state is identified as state i, i = 1, 2, ...n and n complex securities, with each security defined as security j, j = 1, 2, ...,n with price S_j. Let X_{ij} be the payoff of complex security j in state i. A complex security can be defined in terms of the number of units of each pure security required to replicate the payoffs of the complex security. We can alternatively define each pure security in terms of the number of units of each complex security required to replicate its outcomes. Define pure security i as a security that pays $1 in state i and zero in all other states. Then α_{ij} is the number of units of complex security j that should be held to reproduce the payoff of pure security i. Alternatively we can view the payoff X_{ij} as the number of units of pure security i that are implicit in complex security j. Let us now organize this information in a more meaningful way. We shall use both matrix and scalar notation, though the matrix notation is somewhat more useful in facilitating the solution of simultaneous equations.

As stated, a pure security is a combination of complex securities. The payoffs of pure security 1 in each of the possible states are as follows:

$$\alpha_{11}X_{11} + \alpha_{12}X_{12} + ... + \alpha_{1n}X_{1n} = 1 \text{ (outcome in state 1)}$$

$$\alpha_{11}X_{21} + \alpha_{12}X_{22} + ... + \alpha_{1n}X_{2n} = 0 \text{ (outcome in state 2)}$$

. . .

$$\alpha_{11}X_{n1} + \alpha_{12}X_{n2} + ... + \alpha_{1n}X_{nn} = 0 \text{ (outcome in state n).}$$

In other words, pure security 1 is a combination of α_{11} units of complex security 1, α_{12} units of complex security 2, ..., and α_{1n} units of complex security n. Similarly the payoffs of pure security 2 in each of the possible states are as follows:

$$\alpha_{21}X_{11} + \alpha_{22}X_{12} + ... + \alpha_{2n}X_{1n} = 0 \text{ (outcome in state 1)}$$
$$\alpha_{21}X_{21} + \alpha_{22}X_{22} + ... + \alpha_{2n}X_{2n} = 1 \text{ (outcome in state 2)}$$

. . .

$$\alpha_{21}X_{n1} + \alpha_{22}X_{n2} + ... + \alpha_{2n}X_{nn} = 0 \text{ (outcome in state n).}$$

Pure security 2 is, thus, a combination of α_{21} units of complex security 1, α_{22} units of complex security 2 and α_{2n} units of complex security n. The payoffs of pure security n in each of the possible states are as follows:

$$\alpha_{n1}X_{11} + \alpha_{n2}X_{12} + ... + \alpha_{nn}X_{1n} = 0 \text{ (outcome in state 1)}$$
$$\alpha_{n1}X_{21} + \alpha_{n2}X_{22} + ... + \alpha_{nn}X_{2n} = 0 \text{ (outcome in state 2)}$$

. . .

$$\alpha_{n1}X_{n1} + \alpha_{n2}X_{n2} + ... + \alpha_{nn}X_{nn} = 1 \text{ (outcome in state n).}$$

Pure security n is, thus, a combination of α_{n1} units of complex security 1, α_{n2} units of complex security 2, and α_{nn} units of complex security n.

These conditions can be easily expressed in matrix notation as

$$X = \begin{vmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{vmatrix},$$

$$A = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix},$$

$$I = \begin{vmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \ldots & 1 \end{vmatrix}.$$

The relationship is expressed as $\mathbf{XA'} = \mathbf{I}$. We can then solve for the weights using the expression[2]

$$A = (X^{-1}I)'.$$

We can also obtain this result using scalar notation. Letting I_{ij} be the ij^{th} element of vector $\mathbf{I}$, then

$$I_{ij} = \sum_{k=1}^{n} X_{ik}\alpha_{jk}.$$
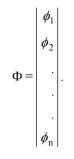
In other words the rows and columns of matrix $\mathbf{I}$ are

$$I_{11} = X_{11}\alpha_{11} + X_{12}\alpha_{12} + \ldots + X_{1n}\alpha_{1n} = 1$$
$$I_{12} = X_{11}\alpha_{21} + X_{12}\alpha_{22} + \ldots + X_{1n}\alpha_{2n} = 0$$

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$$I_{1n} = X_{11}\alpha_{n1} + X_{12}\alpha_{n2} + \ldots + X_{1n}\alpha_{nn} = 0$$
$$I_{21} = X_{21}\alpha_{11} + X_{22}\alpha_{12} + \ldots + X_{2n}\alpha_{1n} = 0$$
$$I_{22} = X_{21}\alpha_{21} + X_{22}\alpha_{22} + \ldots + X_{2n}\alpha_{2n} = 1$$

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[2]To obtain the inverse of X, we require the condition that no row or column of X is a linear function of any other row or column. This will always be the case if no complex security is a linear function of any other combination of complex securities. Otherwise, that security would be redundant.

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$$I_{2n} = X_{21}\alpha_{n1} + X_{22}\alpha_{n2} + \ldots + X_{2n}\alpha_{nn} = 0$$

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$$I_{n1} = X_{n1}\alpha_{11} + X_{n2}\alpha_{12} + \ldots + X_{nn}\alpha_{1n} = 0$$

$$I_{n2} = X_{n1}\alpha_{21} + X_{n2}\alpha_{22} + \ldots + X_{nn}\alpha_{2n} = 0$$

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$$I_{nn} = X_{n1}\alpha_{n1} + X_{n2}\alpha_{n2} + \ldots + X_{nn}\alpha_{nn} = 1$$

Now let us introduce a vector **Φ** where the element φ_i is the price today of pure security i:

$$\Phi = \begin{vmatrix} \phi_1 \\ \phi_2 \\ . \\ . \\ . \\ \phi_n \end{vmatrix}.$$

The price of a complex security can be obtained, in matrix notation, as

$$S = X'\Phi,$$

or in scalar notation as

$$S_j = \sum_{k=1}^{n} X_{kj}\, \phi_k.$$

The above equations written out are as follows:

$$S_1 = X_{11}\varphi_1 + X_{21}\varphi_2 + \ldots + X_{n1}\varphi_n$$

$$S_2 = X_{12}\varphi_1 + X_{22}\varphi_2 + \ldots + X_{n2}\varphi_n$$

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$$.$$

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$$S_n = X_{1n}\varphi_1 + X_{2n}\varphi_2 + \ldots + X_{nn}\varphi_n.$$

Here we see how the complex securities are combinations of the pure securities. Alternatively we can obtain the state prices from the prices of the complex securities. This would be found as

$$\Phi = (X')^{-1}S.$$

Alternatively one could solve for the φ_i values in the scalar equations for S_j shown above.

In other words, a complex security j can be priced by multiplying its payoff in each state by the price of the pure security that pays off in that given state. In this way we see that the payoffs of complex securities, or what we ordinarily just call "securities," can be expressed in terms of the payoffs of more fundamental securities, those whose payoffs are contingent on the given states.

The risk-free security is extremely easy to see in this context. Its payoffs are the same in all states. Denoting the risk-free security as security r and its payoff as R, we have $S_r = \sum_{k=1}^{n} R\phi_k = R\sum_{k=1}^{n} \phi_k$. One plus the risk-free rate is, by definition, R/S_r. Consequently, the risk-free rate, which we write as r_f is given as[3]

$$r_f = \left(\frac{1}{\sum_{k=1}^{n} \phi_k} \right) - 1$$

We see that the risk-free rate is just the inverse of the sum of the state prices minus 1. This should make sense. A risk-free asset is one that pays $1 in each state. Thus, a portfolio of 1 unit of each pure security will replicate the payoff of the risk-free asset. It follows that the sums of the values of 1 unit of each pure security will give the present value of $1, which will be the price of the risk-free asset. Inverting the price gives one plus the rate.

Numerical Example

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[3]To solve for r_f in matrix notation we would introduce an nx1 row vector, ι, which contains 1 as each element. Then $r_f = (1/\iota\Phi) - 1$.

Let there be four states and four complex securities. The payoffs of these securities are shown in the four columns of the matrix **X** below, where the rows are the states and the columns are the securities:

$$X = \begin{vmatrix} 100 & 75 & 20 & 85 \\ 100 & 100 & 65 & 72 \\ 100 & 125 & 90 & 135 \\ 100 & 150 & 118 & 110 \end{vmatrix}.$$

The **A** matrix will, of course, be

$$A = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{vmatrix}.$$

We can solve for **A** by obtaining $A = (X^{-1}I)'$.[4] The solution is

$$A = \begin{vmatrix} -0.0236 & 0.0624 & -0.0574 & -0.0020 \\ 0.0564 & -0.0822 & 0.0653 & -0.0091 \\ 0.0181 & -0.0626 & 0.0418 & 0.0242 \\ -0.0409 & 0.0825 & -0.0496 & -0.0131 \end{vmatrix}.$$

Let us assume we have the prices of the complex securities:

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[4]The matrix operations of transposing, multiplying and taking the inverse can be easily done using Excel's array formulas =transpose(), =mmult(), and =minverse().

$$S = \begin{vmatrix} 92.00 \\ 118.00 \\ 85.86 \\ 99.36 \end{vmatrix}.$$

Then we can find the prices of the pure securities as $\Phi = (X')^{-1}S$. The solution is

$$\Phi = \begin{vmatrix} 0.06 \\ 0.18 \\ 0.26 \\ 0.42 \end{vmatrix}.$$

In scalar notation, this would be found by solving the equations,

$$92.00 = 100\varphi_1 + 100\varphi_2 + 100\varphi_3 + 100\varphi_4$$

$$118.00 = 75\varphi_1 + 100\varphi_2 + 125\varphi_3 + 150\varphi_4$$

$$85.86 = 20\varphi_1 + 65\varphi_2 + 90\varphi_3 + 118\varphi_4$$

$$99.36 = 85\varphi_1 + 72\varphi_2 + 135\varphi_3 + 110\varphi_4$$

Alternatively, if we had the prices of the pure securities, we could obtain the prices of the complex securities as $S = X'\Phi$ or in the above scalar equations, inserting values for each φ and leaving the left-hand sides as the unknowns.

The risk-free rate is then $(1/\iota\Phi) - 1$ or simply

$$r_f = \frac{1}{0.06 + 0.18 + 0.26 + 0.42} - 1 = .086957.$$

State Pricing and Options

Consider a one-period binomial option pricing world. Let an asset worth V today be worth either Vu or Vd one period later, where u and d are one plus the return on the stock in each of the two outcomes. From what we have previously learned about state pricing, we know that the V must be $Vu\varphi_1 + Vd\varphi_2$. Let us divide through by V and also specify the formula for the risk-free rate in terms of the state prices,

$$1 = u\phi_1 + d\phi_2$$

$$r_f = \left(\frac{1}{\phi_1 + \phi_2}\right) - 1.$$

Solving these simultaneously for ϕ_1 and ϕ_2, we obtain

$$\phi_1 = \frac{1 + r_f - d}{(1 + r_f)(u - d)}$$

$$\phi_2 = \frac{u - (1 + r_f)}{(1 + r_f)(u - d)}.$$

We know that we can obtain the price of a call option on this asset with an exercise price of K by using the standard binomial pricing formula,

$$c = \frac{pc_u + (1 - p)c_d}{1 + r_f},$$

where $p = (1 + r_f - d)/(u - d)$, $c_u = Max(0, Vu - K)$ and $c_d = Max(0, Vd - K)$. We know that p and $1 - p$ are the risk neutral or equivalent martingale probabilities of the two states. Given what we learned about state pricing, we can also obtain the option price by weighting its payoffs by the state prices. Note also that comparing the formula for p to that for ϕ_1, we see that

$$p = (1 + r_f)\phi_1$$

and $1 - p = (1 + r_f)\phi_2$. Thus, the state prices and risk neutral probabilities differ only by the risk-free rate. In fact if we are given the risk-free rate, we can easily solve for the state prices as $\phi_1 = p/(1 + r_f)$ and $\phi_2 = (1 - p)/(1 + r_f)$. Let us demonstrate these results with an example.

Consider the following case. We have a stock priced at $100 that can go up to either $125 or down to $80 in the next period. Thus, $u = 125/100 = 1.25$ and $d = 80/100 = 0.80$. The risk-free rate is 7%. Using the standard binomial pricing approach, we first calculate the risk neutral probability as

$$p = \frac{1.07 - 0.80}{1.25 - 0.80} = 0.6,$$

and 1- p is, therefore, 0.4. The payoffs of the call are obviously 25 and 0. The call price today is found as

$$c = \frac{0.6(25) - 0.4(0)}{1.07} = 14.02.$$

Now let us look at how this problem is consistent with state pricing. Given the risk neutral probabilities of .6 and .4, we can find the state prices as

$$\phi_1 = \frac{0.6}{1.07} = 0.5607$$

$$\phi_2 = \frac{0.4}{1.07} = 0.3738.$$

We have three financial instruments, a stock, an option and a riskless bond. Let us take two of these at a time and, using the formulas we previously developed, re-derive the state prices, which should be 0.5607 and 0.3738. Let us first use the stock and risk-free bond. We can set the price of the risk-free bond to anything as long as its payoff is 7% higher than its price. Let us just set it at $100, the same as the stock price. Then our **X** matrix is

$$X = \begin{vmatrix} 107 & 125 \\ 107 & 80 \end{vmatrix}.$$

Our **S** matrix is

$$S = \begin{vmatrix} 100 \\ 100 \end{vmatrix}.$$

Then performing the matrix operations $(\mathbf{X'})^{-1}\mathbf{S}$, we obtain

$$\Phi = \begin{vmatrix} 0.5607 \\ 0.3738 \end{vmatrix},$$

which are the correct values for the state prices.

Alternatively, we could use the option and the stock. Then our **X** matrix would be

$$X = \begin{vmatrix} 25 & 125 \\ 0 & 80 \end{vmatrix}.$$

Our **S** matrix would be

$$S = \begin{vmatrix} 14.02 \\ 100 \end{vmatrix}.$$

Performing the necessary matrix operations would again gives us the correct values in **Φ**.

Using the option and the risk-free bond, our **X** matrix is

$$X = \begin{vmatrix} 25 & 107 \\ 0 & 107 \end{vmatrix}.$$

The **S** matrix is

$$S = \begin{vmatrix} 14.02 \\ 100 \end{vmatrix}.$$

Again, performing the necessary matrix operations gives us the correct values in **Φ**. Obviously it does not matter which assets we use.

State Prices in a Continuous-State World

In the real world there are an infinite number of possible states. This makes it difficult, if not impossible, to identify the specific states and obtain their prices. It is possible, however, to make some rough approximations of state prices from the prices of traded options.

A standard European call option on a stock can be decomposed into two components. One is a long position in an *asset-or-nothing option*, which pays the value of the asset if its price at expiration exceeds the exercise price and nothing otherwise. The other component is a short position in a certain number of *cash-or-nothing option*, which obligates the seller to pay a certain amount of money if the asset price at expiration exceeds the exercise price and nothing otherwise. The amount of money owed is the exercise price.

Letting c be the call price, V be the underlying asset price, K be the exercise price, r_f be the risk-free rate, σ be the volatility of the return on the asset price and T - t be the time to expiration, the value of the European call is well-known as the Black-Scholes formula of

$$c = VN(d_1) - Ke^{-r_f(T-t)}N(d_2), \text{ where}$$
$$d_1 = \frac{\ln(V/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2 = d_1 - \sigma\sqrt{T - t}.$$

The value of the asset-or-nothing component is known to be $VN(d_1)$ while the value of the cash-or-nothing component is known to be $Ke^{-r_f(T-t)}N(d_2)$. For our purposes here, we need the value of a more general cash-or-nothing option, one that pays off $1 if it expires with the asset value above the exercise price and zero otherwise. Such an option is sometimes called a *digital option*.[5] Let us denote the price of that call option as dc, and we see that its formula is

$$dc = e^{-r_f(T-t)}N(d_2).$$

Given the Black-Scholes put option pricing formula,

$$p = Ke^{-r_f(T-t)}[1 - N(d_2)] - V[1 - N(d_1)],$$

we can find the price of a digital put, which is an option that pays $1 if the asset value at expiration is less than the exercise price. Its formula is

$$dp = e^{-r_f(T-t)}[1 - N(d_2)].$$

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[5]Cash-or-nothing options are also sometimes called digital options. Another name for these types of options is *binary options*.

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[5]Cash-or-nothing options are also sometimes called digital options. Another name for these types of options is *binary options*.

These digital option formulas are also the partial derivatives of the Black-Scholes call and put formulas with respect to the exercise price.[6]

Now let us divide the uncertain outcomes into three possibilities. Let V_T be the value of the asset at a specific future date, K_1 be one possible level of the asset and K_2 be another possible level of the asset where $K_2 > K_1$. Now let us define three states: $V_T \leq K_1$, $K_1 < V_T \leq K_2$, and $V_T > K_2$. Although this specification oversimplifies the real world, it does allow us to define three easily identifiable states from which we can determine the three state prices.

A digital put with an exercise price of K_1 is a security that pays $1 if the first state, $V_T < K_1$, occurs and zero otherwise. Thus, its price is the price of the first pure security. The second pure security is identical to a long position in a digital call with an exercise price of K_1 and a short position in a digital call with an exercise price of K_2. To see this note that if $V_T \leq K_1$, both options expire out-of-the-money so there is no payoff. If $K_1 < V_T \leq K_2$, the long digital call struck at K_1 pays $1 and the short digital call struck at K_2 pays nothing for a total payoff of $1. If $V_T > K_2$ the long digital call struck at K_1 pays $1 and the short digital call struck at K_2 will require a payment of $1, thereby offsetting and leaving a zero payoff. The third pure security, which pays $1 if the state $V_T > K_2$ occurs, can be replicated with a long digital call with an exercise price of K_2.

Let us illustrate these results by estimating the prices of certain pure securities from the prices of options on the Dow Jones Industrial Average (DJIA). The DJIA is, of course, the most widely followed market indicator. Its options, which trade at the Chicago Board Options Exchange, are European-style, which means that the Black-Scholes model is appropriate, at least for our purposes. Consider the following information: The index is at 9234.47, which per contract specifications is converted to 92.34 for option trading purposes. We shall look at some options expiring in 37 days with exercise prices of 90 and 94. The continuously compounded risk-free rate is 4.95%. An estimate of the dividend yield of the Dow Jones Industrials is 2% and an estimate of the volatility is 18%. Given the exercise prices of 90 and 94, we can define three states: DJIA $\leq$

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[6]The derivative of the call formula with respect to the exercise price has a minus sign, which would have to be ignored if one were using the derivative as the price of a digital option.

9000, 9000 < DJIA ≤ 9400, and DJIA > 9400. Obviously these three states are more general than exist in practice, but they are useful for understanding state pricing.

Using the Black-Scholes dividend-adjusted option pricing model,[7] we obtain the following values: $e^{-r_f(T-t)} = 0.9950$, $N(d_2|K=90) = 0.6813$ and $N(d_2|K=94) = 0.3869$. The following prices are obtained for the digital options:

digital call struck at 90:	0.6779 (0.9950*0.6813)
digital put struck at 90:	0.3171 (0.9950*(1 − 0.6813))
digital call struck at 94:	0.3850 (0.9950*0.3869).

Thus, our state prices are

pure security 1:	0.3171
pure security 2:	0.6779 - 0.3850 = 0.2929
pure security 3:	0.3850.

We can then obtain the risk-free rate over that period as 1/(0.3171 + 0.2929 + 0.3850) - 1 = .0050. This is consistent with the 4.95% continuously compounded rate since the equivalent discrete compounded rate for 37 days would be $e^{.0495(37/365)}$ - 1 = .0050.

Final Comments

State-pricing theory, sometimes known as *state-preference theory*, provides a framework for the valuation of financial assets. It can be shown to provide a general equilibrium theory of asset pricing, consistent with a market in which assets are risky, and investors have homogeneous beliefs and aversion to risk. State-preference theory was developed around the same time as the Capital Asset Pricing Model, but has not received as much attention as the CAPM. This is probably because state-preference theory is a more abstract theoretical framework, relying as it does on the existence of pure securities, whose prices cannot be observed in financial newspapers, from the Bloomberg, or on the Internet. It is more appropriately viewed as what one would see if one took a microscopic look at the financial markets.

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[7]Recall that the dividend adjustment is simply to change the asset price from V to $Ve^{-y(T-t)}$ where y is the dividend yield. This reduces the asset price by the present value of the dividends.

With the development of option pricing theory, state-preference theory has stepped to the back in the family of valuation models. While, as we have seen here, state-preference theory is clearly consistent with option pricing theory, the implications of the latter are much easier to observe in the real world, and hence, it has become more widely used in practice as well as in scholarly work. Keep in mind that just as a biologist cannot simply observe a specimen with the naked eye and expect to learn much about it, so must a serious student of finance observe the internal structure of the financial pricing process. State-preference theory provides the framework to accomplish that task.

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