PROPERTIES OF DICE

by Klaus Æ. Mogensen

Proper Dice

Gamers use all sorts of dice. Dice are usually some sort of regular polyhedra. Best known are the Pythagorean solids, the classical d4, d6, d8, d12, and d20, but other shapes are also used, primarily the d10. In addition to these, a d30 and a d100 are available in well-stocked game stores. Considerations of symmetry tells us that all of these except the d100 are "proper" dice since all of their faces are equivalent. The d100 is less certain, since its surfaces aren't clearly defined. The questions I'm going to answer is, can this d100 be a proper dice (with clearly equal chances of landing on all faces)? Are there more proper dice than those we know? And, can you make dice with curved surfaces that can't be made with flat surfaces?

The requirements for a die with a certifiably equal chance of landing on each face, are that all the faces must be identical (though mirror symmetry is allowed), and must be placed identically in relation to each other. In other words, the faces on the die must be indistinguishable barring mirror symmetry. The die must be also be convex (it can't curve in on itself), or it would be unable to land on some faces.

We now turn to the math. This isn't too complex, but a bit time-consuming, so you may want to jump directly to the results.

Euler's equation, which is good for any convex polyhedron of three or more faces (curved or otherwise), states that $V + N = E + 2$, where V is the number of vertices in the polyhedron, N the number of faces, and E the number of edges. Since the faces on a proper die are identical, they must all have the same number of sides (and corners), a number we'll call M. We have $E = M*N/2$, which tells us that N only can be odd if M is even (or E won't be a whole number). Using this substitution, Euler's equation becomes:

(1) $V = N(M/2 - 1) + 2$

This equation does not cover the situations where M=0 or M=1. These, however, only have one solution each, namely:

(A) N=1, M=0, V=0 (a sphere)
(B) N=2, M=1, V=0 (a lens)

In the case M=2, (1) simplifies to V = 2. This has solutions for all N greater than 2, but requires curved surfaces. The shape of these "dice" are prisms that taper in both ends, with N-sided cross sections:

(C) N>=3, M=2, V=2 (edged cigar shapes)

As will be shown later, this is the only way to make dice with an odd number of faces (not counting the sphere).

We turn to the cases M > 2. Since a vertex must contain at least three face corners, we have that 3V =< N*M, which with (1) tells us that M =< 6 - 12/N, which means that M <6. The faces must thus be triangles, quadrangles, or pentagons.

While we require that all the faces on the dice are equivalent, the same is not true for the vertices. There can theoretically be as many types of vertices as there is types of corners in the faces. Since a given type of corner must always be part of the same type of vertex (or the faces wouldn't be equivalent), there can't be more types of vertices than there are corners, but there can be less (everal types of corners may meet in one type of vertex).

Let's label the different types of vertices with a number i, i=1,2,... up to the number of different vertices. Let's call the number of corners that meet in a vertex the *rank* of the vertex, and label that rank **Ri**. Similarly, we label the number a vertex type occurs **Vi**, and the number of different corners a vertex is made out of **Mi**. We get:

(2) V = SUM(Vi) (i=1,2,...)
(3) M = SUM(Mi) (i=1,2,...)
(4) Vi = N*Mi/Ri

The two first are obvious: The total number of vertices is the sum of the numbers of the individual vertices, and the total number of corners on a face is the sum of the corners that are part of different vertices. (4) must be true since all the corners in vertices must be provided by the faces, and vice versa.

Combining (2), (3), and (4) with (1) gives us:

(5) SUM(N*Mi/Ri) = N(M/2 - 1) + 2

Which reduces to:

(6) N = 2/(SUM(Mi/Ri) - M/2 + 1)

In the cases where there's only one type of vertex we have Mi=M, and (6) becomes N = 2/(M/R - M/2 + 1), which has these solutions:

(D) N=4, M=3, R=3, V=4 (tetrahedron, standard d4)
(E) N=8, M=3, R=4, V=6 (octahedron, standard d8)
(F) N=20, M=3, R=5, V=12 (icosahedron, standard d20)
(G) N=6, M=4, R=3, V=8 (cube, standard d6)
(H) N=12, M=5, R=3, V=20 (dodecahedron, standard d12)

Some or all the corners on the die faces may be equivalent. By this I mean that they can't be told apart when looking at the face (barring mirroring or rotation) and are part of the same type of vertex. An example of this is the cube: Each face has four corners, but the vertices only have three corners meet. If the four corners of the faces all were different, some of the vertices would obviously also be different, and the faces would end up being aligned differently.

Equivalent faces must all belong to the same type of vertex (or else they could be told apart by their placement, and thus wouldn't be equivalent). Thus, if all the corners are equivalent, we can only have one type of vertex, and get the solutions shown above.

In the cases $2 < M < 6$, unless all the corners are equivalent, the only rotational symmetry is for a 180 degree rotation: There isn't room in a quadrangle for three-way symmetry, or in a pentagon for three-way or four-way symmetry. Thus, at most two corners can be equivalent (through 180 degree rotation or mirroring), though quadrangles and pentagons may have two such pairs. In other words, unless all the corners are equivalent, the corners must either be *singlets* or *doublets*.

For triangles, we find two possible combinations: 1 doublet plus 1 singlet, and 3 singlets.

For quadrangles, we find three possible combinations: 2 doublets (either side-by-side or opposite corners), 1 doublet plus 2 singlets (with the doublet being two opposite corners), and 4 singlets.

For pentagons, we find two possible combinations: 2 doublets plus 1 singlet (with one doublet being the corners on either side of the singlet, and the other doublet being the remaining two corners), and 5 singlets. (You can't have 1 doublet plus 3 singlets: the doublet corners could not be placed symmetrically if the other three corners are different.)

Let's examine a vertex consisting of only one type of corner. If this corner has two different adjacent corner types, then if the vertex has odd rank, those two different corners will be forced together in one of the adjacent vertices. This observation can be expressed as follows:

(7) If a type of corner belongs to a vertex that contains no other types of corners, and the two corners adjacent to this corner type aren't a doublet pair, then the vertex must have even rank, or the adjacent vertices must contain both adjacent corners types.

A vertex can consist of more than one type of corners. Since a specific type of corner only can be part of one type of vertex, the following must be true (since all corners must occur an equal number of times in the polyhedron):

(8) If a vertex consists of more than one type of corner, they must occur in proportion to their singlet/doublet status (different singlets equally, different doublets equally, a doublet twice as often as a singlet).

E.g., if a vertex is made up of a singlet and a doublet types of corners, the vertex must contain 1 singlet corner plus 2 doublet corners, or multiples thereof. The rank of such a vertex must be a multiple of three. However, if a vertex only consists of one type of doublet corner, there is no requirement that the rank is even (see the example with the cube above). Similarly, if it consists of two types of doublet corners, (8) would be satisfied if it contained three corners of each type.

With the restrictions provided by (7) and (8), we can solve (6) for all the shapes described above. We only examine the cases where there's more than one type of vertex, since if there's only one, we just get the solutions (D) to (H) above.

Let's start with the triangular case of 1 doublet, 1 singlet. If the singlet and doublet corners are part of different types of vertices, (7) applies to the doublet vertices, which must have even rank. We get N = 2/(2/R1 + 1/R2 - 1/2), R1 even, which has the solutions:

(I) N even (N>=6), M=3, R1=4, V1=N/2, R2=N/2, V2=2 (double pyramid)
(J) N=12, M=3, R1=6, V1=4, R2=3, V2=4 (pyramids on tetrahedron faces)
(K) N=24, M=3, R1=6, V1=8, R2=4, V2=6 (pyramids on octahedron faces)
(L) N=60, M=3, R1=6, V1=20, R2=5, V2=12 (pyramids on dodecahedron faces)
(M) N=24, M=3, R1=8, V1=6, R2=3, V2=8 (pyramids on cube faces)
(N) N=60, M=3, R1=10, V1=12, R2=3, V2=20 (pyramids on icosahedron faces)

By "pyramids" in the above I mean any number of identical triangles meeting in a vertex. The standard d8 is a special case of (I) above, which only if N/2 is even has opposing faces (and thus an "up" face when lying on a table).

In the triangular case of 3 singlets, let's first consider the case where two types of corners are part of the same type of vertex. We find that the equation becomes the same as for the case above, with R1 forced to be even by (8) rather than (7). Same equation = same solutions. We get nothing new.

If all three corners are part of different types of vertices, (7) requires that all the vertices have even rank. We get N = 2/(1/R1 + 1/R2 + 1/R3 - 1/2), all Ri even. We get the solutions:

(O) N=8,12,16,.., M=3, R1=R2=4, V1=V2=N/4, R3=N/2, V3=2 (variant of (I) above)
(P) N=24, M=3, R1=4, V1=4, R2=R3=6, V2=V3=4 (variant of (K) above)
(Q) N=48, M=3, R1=4, V1=12, R2=6, V2=8, R3=8, V3=6 (shape seen in crystals)
(R) N=120, M=3, R1=4, V1=30, R2=6, V2=20, R3=10, V3=12 (largest non-bipolar die)

By "variant" in the above, I mean that the shapes are topologically equivalent, the way that a cube elongated by pulling out opposite vertices is equivalent to an undeformed cube. Essentially, (O) and (P) are deform versions of (I) and (K). You can build (Q) and (R) by putting squeezed pyramids on the rhombic faces of (T) and (U) below, respectively.

We now turn to quadrangles. In the case 2 doublets,we get N = 2/(2/R1 + 2/R2 - 1). We see that either R1 or R2 must be smaller than 4. Of the two possible ways of having 2 doublets, only the one where the doublets are opposite corners (i.e., rhombic faces) allow non-even ranks according to (7). We get the solutions:

(S) N=6, M=4, R1=3, V1=4, R2=3, V2=4 (elongated cube, a variant of (G))
(T) N=12, M=4, R1=3, V1=8, R2=4, V2=6 (rhombic dodecahedron)
(U) N=30, M=4, R1=3, V1=20, R2=5, V2=12 (the d30 sold in game shops)

The next case is 1 doublet, 2 singlets (kites). We first consider the case where the two singlets are part of the same type of vertex. We get the same equation as above (with the added limitation by (8) that R2 must be even), thus no new solutions. Next we consider the case where the doublet and one of the singlets are part of the same type of vertex. We get the equation N = 2/(3/R1 + 1/R2 - 1), but we know from (8) that R1 must be a multiple of 3. We get this solution only:

(V) N even (N>=6), M=4, R1=3, V1=N, R2=N/2, V2=2 (double cone made from kites)

The cube and the d10 are both special cases of (V). Only if N/2 is odd do these shapes have opposing faces (and thus an "up" face when lying on a flat surface).

If the doublet and both singlets all form their own vertices, we get the equation N = 2/(2/R1 + 1/R2 + 1/R3 - 1), and we know from (7) that R2 must be even. We get the solutions:

(W) N=12, M=4, R1=4, V1=6, R2=R3=3, V2=V3=4 (a variant of (T))
(X) N=24, M=4, R1=4, V1=12, R2=3, V2=8, R3=4, V3=6 (shape seen in crystals)
(Y) N=60, M=4, R1=4, V1=30, R2=3, V2=20, R3=5, V3=12

You can make (X) by replacing each face of a cube with four kites, and (Y) in a similar manner from a dodecahedron.

The last case with quadrangles is where all the corners are singlets. If all belong to different corners, we get the equation N = 2/(1/R1 + 1/R2 + 1/R3 + 1/R4 - 1). We know from (7) that all Ri must be even, and thus 4 or greater. No solutions can satisfy that. If two corner types are part of the same type of vertex, this vertex must have even rank according to (8). You thus can't have two double-singlet vertices (this would require that all Ri are even, which we have shown has no solution). If the two corner types belonging to the same vertex are adjacent, both singlet vertices must also have even rank (= no solutions), so the two corner types must be opposite. This gives us the same equation that gave us (W), (X), and (Y), and the same solutions. Similarly, in the case where three singlets are part of the same type of vertex, we reproduce solution (V). Thus, we find no new solutions by only having singlet corners.

We now turn to pentagons. We start with the case of 2 doublets and 1 singlet. If both doublets are part of the same type of vertex, we get N = 2/(4/R1 + 1/R2 - 3/2). R1 must be even according to (8), and thus 4 or greater. No solutions can satisfy that. If one doublet and the singlet are part of the same type of vertex, we get the equation N = 2/(3/R1 + 2/R2 - 3/2), R1 a multiple of 3. We get one solution:

(Z) N=12, M=5, R1=3, V1=12, R2=3, V2=8 (a deform dodecahedron)

Next we have the case where both doublets and the singlet are part of different types of vertices. We get N = 2/(2/R1 + 2/R2 + 1/R1 - 3/2). From (7) we know that R1 and R2 both must be even, and thus 4 or greater, so we get no solutions.

We turn to the last case, namely all singlet corner types. If all but one are part of one type of vertex, we get N = 2/(4/R1 + 1/R2 - 3/2), and R1 even. We have already shown that this has no solutions. Next we consider the case where 3 corner types are part of one type of vertex, and the remaining two both are part of another type. We get N = 2/(3/R1 + 2/R2 - 3/2), with R2 even and R1 a multiple of 3 (from (8)). This has no solutions. The next case has one type of vertex with three corner types, and two vertices with one corner type. We get N = 2/(3/R1 + 1/R2 + 1/R3 - 3/2), R1 a multiple of 3. This has these solutions:

(Æ) N=12, M=5, R1=3, V1=12, R2=R3=3, V2=V3=4 (another deform dodecahedron)
(Ø) N=24, M=5, R1=3, V1=24, R2=3, V2=8, R3=4, V3=6
(Å) N=60, M=5, R1=3, V1=60, R2=3, V=20, R3=5, V3=12

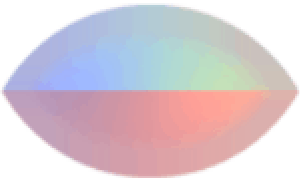
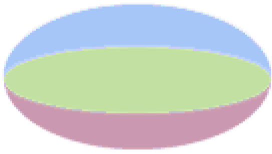
We can make (Ø) by placing sets of four pentagons on each face of a cube, turned a bit to make the corners interlace. (Å) can be made in a similar manner from a dodecahedron. Even though the faces connect asymmetrically, these polyhedrons can be built with faces that have bilateral symmetry (if you don't consider how they connect).

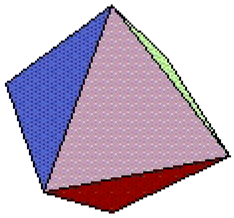
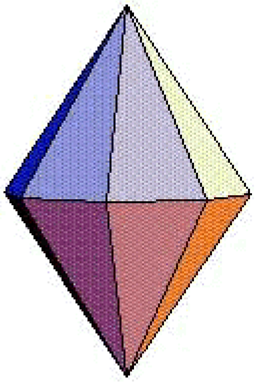
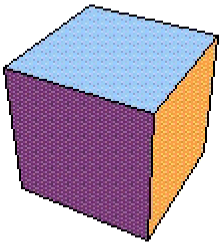
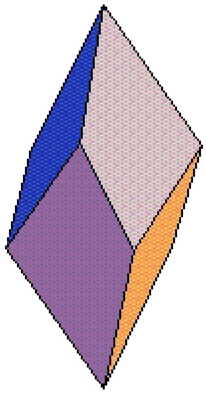
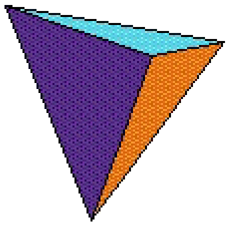
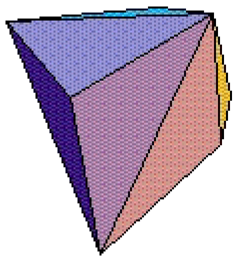
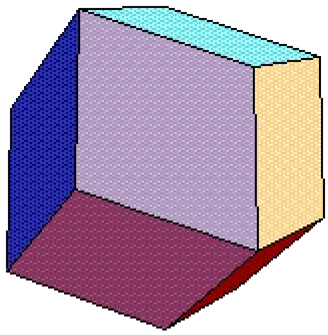
The next case has two vertex types each with two corner types, and one vertex type with only one. We get N = 2/(2/R1 + 2/R2 + 1/R3 - 3/2), R1 and R2 both even (from (8)). This has no solutions. The next case has one vertex type with two corner types, and three vertex types with only one corner type. We get N = 2/(2/R1 + 1/R2 + 1/R3 + 1/R4 - 3/2), R1 even, and this also has no solutions. The last case has five different vertexes, each made from one corner type. (7) requires all of the vertices to have even rank, which cannot have a solution.

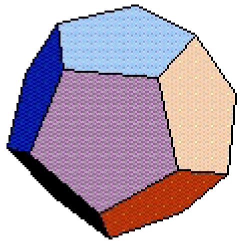
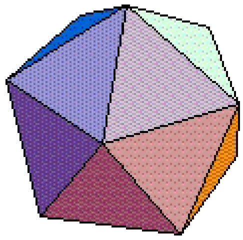
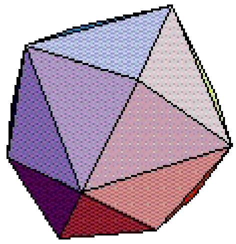
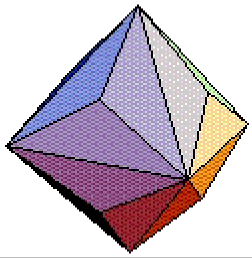
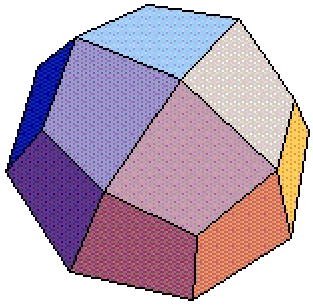
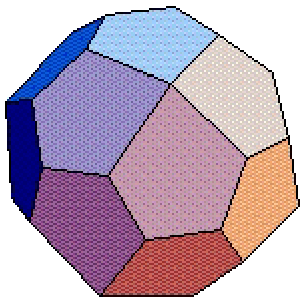
We now have gone through all the possibilities. There can be no proper dice except those mentioned in the solutions (A) to (Å) above. We have made no requirements that the faces or edges must be non-curving, but it can be shown (by example) that all the solutions for M = 3, 4, or 5 can be made with non-curving faces and edges, as I have suggested in the text (and will show for some in illustrations below). In fact, I'm pretty sure (but offer no proof) that all of these can be made so all vertex points touch an enclosing sphere.

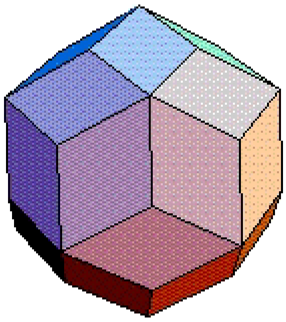
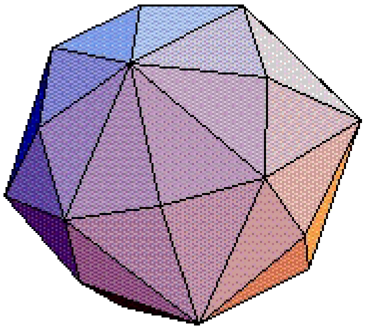
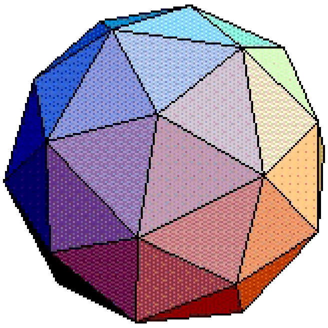
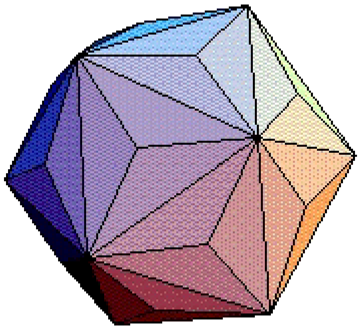
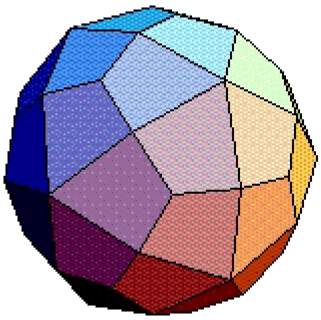
While it may be considered interesting that some of the solutions are more-or-less deform versions of each other, we are only interested in topologically distinct shapes for the purposes of using them as dice. With this restriction, (E) and (O) are subsumed by (I), (G) and (S) are subsumed by (V), (Z) and (Æ) are subsumed by (H), (P) is subsumed by (K), and (W) is subsumed by (T). We can put the remaining, topologically different solutions into a table:

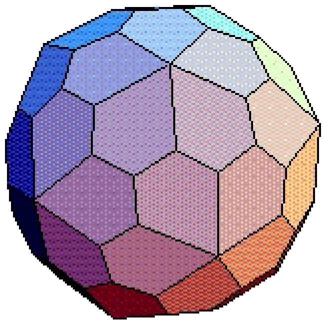
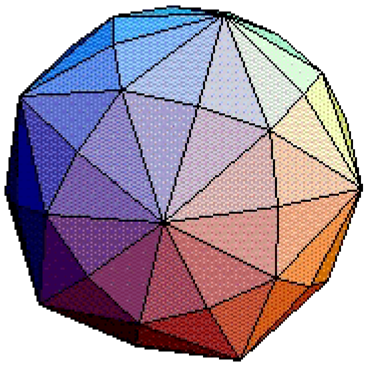
TABLE OF PROPER DICE
The top three illustrations are done by me, the rest are done by Ed Pegg, Jr.

N	M	R1	V1	R2	V2	R3	V3	Notes	Illustration
1	0	-	-	-	-	-	-	Sphere	
2	1	-	-	-	-	-	-	Lens	
>2	2	-	-	-	-	-	-	Has curved faces	

Even, >4	3	4	N/2	N/2	2	-	-	Triangular Dihedron (opposing faces if N/2 even)		
Even, >4	4	3	N	N/2	2	-	-	Trapezoidal Dihedron (opposing faces if N/2 odd)		
4	3	3	4	-	-	-	-	Regular Tetrahedron		
12	3	6	4	3	4	-	-	Triakis Tetrahedron		
12	4	3	8	4	6	-	-	Rhombic Dodecahedron		

12	5	3	20	-	-	-	-	Regular Dodecahedron	
20	3	5	12	-	-	-	-	Regular Icosahedron	
24	3	6	8	4	6	-	-	Tetrakis Hexahedron	
24	3	8	6	3	8	-	-	Triakis Octahedron	
24	4	4	12	3	8	4	6	Trapezoidal Icositetrahedron	
24	5	3	24	3	8	4	6	Pentagonal Icositetrahedron	

30	4	3	20	5	12	-	-	Rhombic Triacontahedron	
48	3	4	12	6	8	8	6	Hexakis Octahedron	
60	3	6	20	5	12	-	-	Pentakis Dodecahedron	
60	3	10	12	3	20	-	-	Triakis Icosahedron	
60	4	4	30	3	20	5	12	Trapezoidal Hexecontahedron	

60	5	3	60	3	20	5	12	Pentagonal Hexecontahedron	
120	3	4	30	6	20	10	12	Hexakis Icosahedron (largest non-bipolar die)	

The regular octahedron (d8) and the cube (d6) are special cases of the infinite series having M=3 and M=4, respectively. The d10 also belongs to the latter of these.

The d100 you can buy in some gaming stores cannot be found on the list (it clearly doesn't belong to any of the infinite series). It can't be trusted absolutely. The d30 you can buy in gaming stores is on this list and thus can be trusted.

Dice Probabilities

The following formulae can be used to find the probability of rolling a sum S using N dice of M faces. The formulae provide the number of *permutations* that give the desired result; to obtain the *probability* as an absolute number, divide by M^N.

In the formulae, {A:B} is the binomial coefficient A!/B!/(A-B)!, 0!=1, and SUM[i=0,j](Xi) is the sum of Xi for all values of i from 0 to j.

Number of permutations that give sum = S:
P(S,N,M) = SUM[i=0,j]((-1)^i * {N:i} * {(S-i*M-1):(N-1)})
Where j = Int((S-N)/M).

For large S the identity P(S) = P(Z), where Z = N*(M+1)-S, can be used to simplify the calculation.

Number of permutations that give sum =< T:
P(=<T,N,M) = SUM[i=0,j]((-1)^i * {N:i} * {(T-i*M):N})
Where j = Int((T-N)/M).

For large T the identity P(=<T) = 1-P(=<Y), where Y = N*(M+1)-T-1, can be used to simplify the calculation.

In some games you roll a pool of dice and have to count the number of dice that have a certain value or higher. For that and similar problems, the following formulae can be used to find the probability of various results. The notation is the same as above.

Number of permutations that have exactly I dice coming up with a value of exactly J:
P(I,J,N,M) = (M-1)^(N-I) * {N:I}

Number of permutations that have exactly I dice coming up with a value of J or higher:
P(I,>=J,N,M) = (J-1)^(N-I) * (M-J+1)^I * {N:I}

Number of permutations that have exactly I dice coming up with a value of J or less:
P(I,<=J,N,M) = (M-J)^(N-1) * J^I * {N:I}