

INTRODUCTION

Alonzo Church: Life and Work

Life is not the same as *work*, but in the case of Alonzo Church the connection is tight. His name is preserved in “Church’s Thesis,” referring to the identification of effective calculability with a precisely defined notion, and in “Church’s Theorem,” referring to the undecidability of the the concepts of validity and of satisfiability in first-order logic. And ω_1^{CK} is an ordinal number (the Church–Kleene ordinal) associated with his name.

Church’s work in symbolic logic spans a wide range, both in time and in subject matter. His first published paper, *Uniqueness of the Lorentz Transformation*, appeared in 1924. The following year *On Irredundant Sets of Postulates* appeared. Seven decades later, his paper *A Theory of the Meaning of Names* was published in 1995. This amazing span of seventy-two years embraces a remarkable collection of publications.

The nature of Church’s contributions to logic will be discussed below, arranged under the three headings (i) calculability, (ii) set theory and foundations, and (iii) intensional logic and philosophy. The subjects Church chose to work on were not selected at random. (Indeed, one suspects that very few of Church’s decisions involved any elements of randomness.) He was guided by a sense of what the field needed. Being a person with great determination—even ambition—he then resolved to get at the heart of the important problems. For example, his extensive work (especially from 1950 on) in intensional logic seems to have stemmed from a feeling that following symbolic logic’s success in explicating extensional logic, the field of philosophy stood in need of an analogous analysis of intensional logic. As he wrote in 1951, “intensional logic also must ultimately receive treatment by the logistic method.”

Church’s commitment to the needs of the young field of symbolic logic is also evident in his deep involvement with *The Journal of Symbolic Logic*. Church was influential in the founding in 1935 of the Association for Symbolic Logic, the publisher of JSL. From the first volume (1936), he was an editor and also edited the Reviews Section of JSL.

The field needed a precise and and sufficiently comprehensive textbook for undergraduate students. By 1936, mimeographed notes, *Mathematical Logic*, had been prepared by Church and his students. In 1944, *Introduction to Mathematical Logic*, Part I, was published in the Annals of Mathematical Studies series, based in part on notes by C. A. Truesdell on Church’s 1943 lectures. Finally in 1956, the greatly revised and enlarged textbook, *Introduction to Mathematical Logic*, Volume I, was published by Princeton University Press. This was the book that defined the subject for a generation of logicians. It remains in print after more than fifty years, and has set a high standard of rigor and precision.

Page x of this book give a “tentative table of contents for volume two,” listing Chapter VI–XII (to follow the Chapters I–V in Volume I). While Volume II was never published, material for Chapter VII (*Second Order Arithmetic*)

was mimeographed for class use, and is printed in these *Collected Works* (see *The Logistic System A²*). Material for Chapter VIII (*Gödel's Incompleteness Theorems*) is also included in these *Collected Works*, based in part on notes written by Church and in part on student notes.

The topics for some of the other chapters were addressed in subsequent research papers. In connection with Chapter VI (*Functional Calculi of Higher Order*) see Church's 1972 paper, *Axioms for Functional Calculi of Higher Order*. In connection with Chapter IX (*Recursive Arithmetic*) see his 1957 paper, *Binary Recursive Arithmetic* (and his 1965 paper, *An Independence Question in Recursive Arithmetic*).

Chapter X was to be *An Alternative Formulation of the Simple Theory of Types*. See his 1974 paper, *Russellian Simple Type Theory*. Chapter XI was to be *Axiomatic Set Theory*. While it is uncertain what was to be included in this chapter, certainly the topic was represented in such papers as his 1974 article *Set Theory with a Universal Set*. Finally, Chapter XII was to be *Mathematical Intuitionism*. Here it is even less certain what was to be included.

Church's influence on the field of symbolic field remains strong today. Thirty-one students received doctorates under his supervision, from 1931 to 1985. In addition, he taught many more students, either in person or through his textbook. And his papers continue to form the basis for further research. The purpose of these *Collected Works* is to facilitate the continued study of Church's writings.

Life. Alonzo Church was born on June 14 (Flag Day), 1903, in Washington, D.C. His great-grandfather (also named Alonzo Church) had moved from Vermont to Georgia, where he was a professor of mathematics and astronomy—and then president—of Franklin College, which later became the University of Georgia. His grandfather Alonzo Webster Church was at one time Librarian of the U.S. Senate. His father, Samuel Robbins Church, was a Justice of the Municipal Court of the District of Columbia, until failing vision and hearing compelled him to give up that post. The family then moved to rural Virginia, where Alonzo Church and his younger brother grew up.

Alonzo Church had an uncle (also named Alonzo Church) living in Newark, New Jersey, who was financially helpful to the family, taking an interest in the children's education. An airgun incident in high school left Church blind in one eye. He attended the Ridgefield School, a preparatory school in Connecticut, graduating in 1920.

After graduating from Ridgefield School, Church enrolled at Princeton, where his uncles had attended college. For a while he worked part-time in the dining hall to help pay his way. He was an exceptional student; his first published paper, *Uniqueness of the Lorentz Transformation*, was written while he was an undergraduate. Its object was to obtain a set of logically independent postulates that uniquely determine the Lorentz transformation in one dimension. He graduated with an A.B. in mathematics, 1924. He then continued graduate work at Princeton, completing a Ph.D. in three years (in 1927) under Oswald Veblen. The title of his dissertation was *Alternatives to Zermelo's Assumption*.

While a graduate student, he married (1925) Mary Julia Kuczinski, who was training to be a nurse. (This in spite of the fact that his senior class had voted him the “most likely to remain a bachelor”—his handsome features notwithstanding. In the summer of 1924, Church stepped off the curb and was hit by a trolley car coming from his blind side; Mary was a nurse-trainee at the hospital.) They were inseparable for the next 51 years, until Mary’s death in 1976. Mary was an excellent cook; over the years many a mathematician enjoyed dining at the Church home.

On receiving his degree, he was awarded a two-year National Research Fellowship. So after serving briefly as an Instructor at the University of Chicago in the summer of 1927, he spent two years visiting first Harvard (1927–28) and then Göttingen and Amsterdam (1928–29). In particular, he met with Bernays and with Heyting on this trip.

At the end of his term as a National Research Fellow, Church was invited to return to the Princeton Mathematics Department, to begin his academic career there. He was an Assistant Professor 1929–39, an Associate Professor 1939–47, and a Professor 1947–67 (of Mathematics and Philosophy, 1961–67).

Meanwhile, he and Mary had three children. Alonzo Church, Jr., was born in 1929 (in Amsterdam), Mary Ann in 1933, and Mildred in 1938. (Mary Ann later married the logician John Addison.) While Church was developing what came to be known as Church’s Thesis, for example, there were two small children in the house. Somehow he balanced Sunday afternoon family outings with the demands of his work.

Early in his career, Church put together *A Bibliography of Symbolic Logic*—nothing less than a complete annotated bibliography of every publication in symbolic logic up to that time. Thus it would be an understatement to say that he had a complete familiarity with the literature: he had read, organized, and indexed (by authors and subject) that literature. Of course, the literature was in many languages. Church not only had a wide knowledge of modern languages; he also had studied Latin and Greek.

Princeton in the 1930’s was an exciting place for logic. There was Church together with his students Stephen Kleene and Barkley Rosser. John von Neumann was there. Alan Turing, who had been thinking about the concept of effective calculability, came as a visiting graduate student in 1936. He ended up writing his dissertation under Church. Kurt Gödel visited the Institute for Advanced Study in 1933–34 (when he lectured on his then-recent incompleteness theorem) and 1935, before moving there permanently. Since Church had recently been to Göttingen and Amsterdam, he knew personally almost every logician.

As mentioned above, Church was one of the principal founders of the Association for Symbolic Logic, in 1935. He made certain that *The Journal of Symbolic Logic* got off to a good start; he was an editor for contributed papers for its first 15 volumes (1936–50). (The other editor of the early volumes was C. H. Langford.) More importantly, for most of his career, Church was its editor for reviews, a task he performed for for its first 44 volumes (1936–79).

Church had two goals for the Reviews Section. First, it continued his *A*

Bibliography of Symbolic Logic (which was published in the first volume of JSL) in providing a complete and accurate bibliographical record of publications in symbolic logic. And secondly, it provided critical analytical commentary on these publications. In a 1950 letter to Rudolf Carnap, Church explained the importance he attached to this commentary: “The situation in mathematical logic has always been much worse than in other branches of mathematics, in the matter of the frequency of publication of matter containing errors and even absurdities, with the effect that there is serious danger of the field itself falling into disrepute. . . . I believe that the Journal has made already considerable progress, not only in reducing somewhat the proportion of erroneous or purposeless publications, but more important, in making it possible for the general body of mathematicians and philosophers to have at least some approximate idea as to how the publications in the field are to be sorted out and which ones are worth giving attention to.” The reviews were needed to defend the field and to separate wheat from chaff.

And a motivation for Church’s 1967 departure from Princeton after so many years was that after his retirement (which would have been mandatory in 1971 at age 68), Princeton would be unwilling to accommodate the small staff working on the JSL reviews, while UCLA promised to support the reviews office as long as Church was its editor.¹

Thus Church left Princeton in 1967, and moved to UCLA, where he was Flint Professor of Philosophy and Mathematics, 1967–90.

Finally in 1979, Church retired from editing JSL reviews, after 44 years. He retired from teaching at UCLA in 1990, at age 87, ending a teaching career that had started 63 years before. But his research continued until he died in 1995, at the age of 92.

Throughout his career, Church supervised research of graduate students. Altogether, he had 31 doctoral students. The first 28 were at Princeton, 1931–67; the last three were at UCLA, 1976–85. They form a distinguished list (see *Doctoral Dissertation Students of Alonzo Church* in these *Collected Works*).

Alonzo Church had the polite manners of a gentleman who had grown up in Virginia. He was never known to be rude, even with people with whom he had strong disagreements. A deeply religious person, he was a lifelong member of the Presbyterian church.

In his habits he was careful and deliberate—very careful and very deliberate. The students in his classes would discover this on the first day, when they saw how he would erase a blackboard. The material he wrote out on paper (he did not type) was often done in several colors of ink—sometimes colors made by mixing bottles together—and always done in his distinctive unslanted handwriting. He was a master at using white-out fluid to eliminate imperfections. Finally, an important piece of writing could be made permanent by coating it with a thin layer of Duco cement. (Duco was the one brand of household cement that did not shrink the paper.)

¹See H. B. Enderton, *Alonzo Church and the Reviews*, *The Bulletin of Symbolic Logic*, vol. 4 (1998), pp. 172–180.

Church enjoyed reading science fiction magazines. He did not like to see writers get their facts wrong; he was known to have written several letters to the editors.

He preferred a nocturnal schedule, working late at night, when it was quiet and he would not be disturbed. His staff at the JSL reviews office would leave material on his desk in the afternoon. On arrival in the morning, they would find the replies from Church.

He never drove a car. But he would walk substantial distances, in varying weather and at all hours. Many a student crossing the campus at night would see a well-dressed portly gentleman with white hair, carrying a briefcase and humming softly to himself.

While living in Princeton in the 1960's, Church and his family liked to visit the Bahamas. Eventually Church bought property there and built two duplexes. Even after moving to Los Angeles, Church would spend summers at his place in the Bahamas. He did not spend his time lying on the beach, but it was both a place to escape to, and a place where the children and grandchildren could gather.

Although Church had solitary work habits, that is not to say that he lived in isolation from his colleagues. He attended and spoke at professional meetings. As time allowed, he answered his mail. He maintained an active correspondence both with those whose viewpoints were close to his (e.g. Kleene and Rosser) and with those whose viewpoints were not (e.g. Carnap and Quine).²

Church belonged to a large number of professional organizations, including the American Academy of Arts and Sciences, the American Association for the Advancement of Science, the American Mathematical Society, the American Philosophical Association, the Association for Symbolic Logic, Circolo Matematico di Palermo, and Académie Internationale de Philosophie des Sciences.

In 1966 he was elected a Corresponding Fellow of the British Academy. And in 1978 he was elected to the National Academy of Sciences. He received honorary D.Sc. degrees from Case Western Reserve (1969), Princeton (1985), and SUNY at Buffalo (1990).

Church died in Hudson, Ohio (where his son lived), on August 11, 1995. He was buried in the Princeton Cemetery, where his wife and his parents had been buried. The papers he left behind were donated to the Princeton University Library.

Work in calculability. Church focused on the concept of effective calculability at least by 1934, the year in which both Stephen Kleene and Barkley Rosser received their Ph.D.'s under Church's direction. The concept of effective calculability arises in problems throughout mathematics (e.g. given two simplicial complexes, can we effectively decide whether or not they are homeomorphic?). But it is central to the logical notion of an *acceptable proof*. A proof must be *verifiable*, that is, there must in principle be some effective procedure

²See also María Manzano, *Alonzo Church: His Life, His Work and Some of His Miracles, History and Philosophy of Logic*, vol. 18 (1997), pp. 211–232.

that, given an alleged proof, will verify its syntactic correctness. But exactly what is an effective procedure?

In 1936 a pair of papers by Church changed the course of logic. *An Unsolv-able Problem of Elementary Number Theory* presents a definition and a theorem: “The purpose of the present paper is to propose a definition of effective calculability which is thought to correspond satisfactorily to the somewhat vague intuitive notion . . . , and to show, by means of an example, that not every problem of this class is solvable.” The “definition” now goes by the name Church’s Thesis: “We now define the notion . . . of an *effectively calculable* function of positive integers by identifying it with the notion of a recursive function of positive integers (or of a λ -definable function of positive integers).” (The name, “Church’s Thesis,” was introduced by Kleene.)

The theorem in the paper is that there is a set that can be defined in the language of elementary number theory (viz. the set of Gödel numbers of formulas of the λ -calculus having a normal form) that is not recursive—although it is recursively enumerable. Thus truth in elementary number theory is an effectively unsolvable problem.

A sentence at the end of the paper adds the consequence that if the system of *Principia Mathematica* is ω -consistent, then its decision problem is unsolvable. It also follows that the system (if ω -consistent) is incomplete, but of course Gödel had shown that in 1931.

As indicated above, the paper identifies the effective calculability of a function with two equivalent precisely defined concepts: One is *recursiveness*, which here means that a set of recursion equations exists from which can be derived exactly the correct values of the function (a concept Gödel formulated in his 1934 lectures at Princeton, crediting in part a suggestion by Jacques Herbrand). The other is *λ -definability*, meaning that for a suitable formula of the λ -calculus, exactly the correct values of the function are derivable. (In a footnote, Church writes, “The question of the relationship between effective calculability and recursiveness . . . was raised by Gödel in conversation with the author. The corresponding question of the relationship between effective calculability and λ -definability had previously been proposed by the author independently.”)³

Church had been working on the λ -calculus in connection with his two-part *A Set of Postulates for the Foundations of Logic* (1932 and 1933), where the intent was to develop axioms that “would lead to a system of mathematical logic free of some of the complications entailed by Bertrand Russell’s theory of types, and would at the same time avoid the well known paradoxes, in particular the Russell paradox.” As it turned out, the future of the λ -calculus lay not in that direction (it turned out that contradictions had not been avoided), but in computer science. Church’s 1941 monograph *The Calculi of Lambda-conversion* was later useful to others in the development of semantics for programming languages.⁴ Today the λ -calculus is a major research topic in

³See Wilfried Sieg, *Step by Recursive Step: Church’s Analysis of Effective Calculability*, *The Bulletin of Symbolic Logic*, vol. 3 (1997), pp. 154–180.

⁴See Henk Barendregt, *The Impact of the Lambda Calculus in Logic and Computer Science*, *The Bulletin of Symbolic Logic*, vol. 3 (1997), pp. 181–215.

theoretical computer science.

One person who read Church's paper with great interest—not to say dismay—was Alan Turing. Turing at that time was a student at Cambridge, working with Max Newman. Turing had independently formulated an exact mathematical concept to make rigorous the informal concept of effective calculability. Turing's concept was equivalent to Church's, as Turing showed in an appendix to his paper. But it was formulated in very different terms, involving a step-by-step simulation—by an imaginary machine—of a calculational procedure a person might carry out. (The phrase “Turing machine” first appeared in Church's review of Turing's paper in *The Journal of Symbolic Logic*.) Thus Turing's definition stood in contrast to the formulations in terms of formal logical calculi.

Turing wrote to Church, and he came to Princeton as a visiting graduate student. At the encouragement of John von Neumann, Turing stayed a second year, eventually completing his Ph.D. degree under Church. Turing's dissertation involved transfinite extensions of logical systems, a topic taken up much later by Solomon Feferman.

It is an interesting fact that the idea of formalizing the concept of effective calculability occurred at roughly the same time to several people: Gödel (who mentions the problem in his 1934 Princeton lectures), Church, Turing, and Emil Post (who independently developed a formulation somewhat along the lines Turing used).

The name for the class of effectively computable function (as defined by any one of these equivalent definitions) came to be “recursive functions.” The name made sense from the viewpoint of the Gödel–Herbrand formulation. (Gödel had used the term “rekursiv” in his 1931 paper for what are now called the primitive recursive functions. Kleene applied the name “general recursive functions” to the functions obtained from the primitive recursive functions by the least-zero search operator.) But the other equivalent formulations did not involve *recursion* in an essential way. So the name of the class of functions (and of the subject of “recursive function theory” or “recursion theory”) was something of a misnomer. The adjective Church used was “calculable”; the adjective Turing used was “computable.” Recently there has been a movement to change the name of the subject to “computability theory.”

Another 1936 paper that changed the course of logic was *A Note on the Entscheidungsproblem*. This short paper (two pages, followed by a two-page correction) presents what is now called Church's theorem: The problem of deciding validity of formulas in first-order logic is unsolvable. The method of proof is first to make a finitely axiomatizable theory (in an expanded language) in which a certain function enumerating a recursively enumerable, but non-recursive, set can be represented. And then one can eliminate the added function symbols.

Moreover, Church published two other papers in 1936, together with his former students. *Some Properties of Conversion*, with Barkley Rosser, dealt with the λ -calculus. *Formal Definitions in the Theory of Ordinal Number*, with Stephen Kleene, was followed by Church's 1938 paper, *The Constructive Second Number Class*. It is here that the least non-constructive ordinal, now called ω_1^{CK} , arises.

Work in set theory and foundations. Church's dissertation, *Alternatives to Zermelo's Assumption* (published in 1927) already displayed a broad-minded (and even skeptical) attitude toward set theory. "Zermelo's assumption" is of course the axiom of choice, and Church's attitude was not to regard the axiom of choice as received doctrine, but instead was to examine what array of other set theories might serve instead for the foundations of mathematics.

A similar spirit can be seen in his much later paper *Set Theory with a Universal Set* (presented at a symposium honoring Alfred Tarski in 1971 and published in 1974). In this instance it was the so-called principle of limitation of size that was the issue. This paper presents a system of set theory in which every set has a complement, and proves its consistency relative to ZF. As the paper says, "there is room for exploration of the axiomatic possibilities." And it concludes with the intriguing statement: "Indeed an interesting possibility which must not at this stage be excluded is a synthesis or partial synthesis of ZF set theory and Quine set theory."

It is not surprising that Church found Quine's system New Foundations (NF) interesting. While many mathematicians—through fondness for the principle of limitation of size or for universal choice—found NF unappealing, Church was more than willing to take it seriously. The key idea of avoiding the paradoxes by restricting comprehension to stratified formulas fit well into the Frege–Russell tradition. What was lacking was a relative consistency proof.

Church devoted a substantial effort, especially in the 1970's, toward supplying a consistency result for NF. As he said, his "purpose is not specially to advocate the Quine set theory but to explore the possibilities in regard to non-Neumannian set theories." In the end, the effort was unsuccessful. He left in his *Nachlass* several three-ring binders presenting his work—describing what he had tried, what worked, what did not work. The work was left in well-organized form, written out in several colors of ink. These notebooks, with his other papers, went to the Princeton University Library. Future graduate students may yet mine this material for dissertations.

In connection with set theory, it might also be mentioned that in his 1953 paper, *Non-normal Truth-tables for the Propositional Calculus*, where Church considers replacing the set $\{0, 1\}$ of two truth values by larger Boolean algebras, he wrote: "Another motive is the suggestion, which was made to me by Paco Lagerström ten years or more ago, that use may be made [of the method] in order to extend to the functional calculi of first and higher order, and other related systems, the method of proving independence of axioms." A decade later, Boolean-valued models of set theory were used to present Paul Cohen's results on the relative independence of the axiom of choice in Zermelo–Fraenkel set theory.

—H.B.E.

Work in intensional logic and philosophy.