Cognitive Foundations of Arithmetic: Evolution and Ontogenisis*

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Abstract: Dehaene (this volume) articulates a naturalistic approach to the cognitive foundations of mathematics. Further, he argues that the 'number line' (analog magnitude) system of representation is the evolutionary and ontogenetic foundation of numerical concepts. Here I endorse Dehaene's naturalistic stance and also his characterization of analog magnitude number representations. Although analog magnitude representations are part of the *evolutionary* foundations of numerical concepts, I argue that they are unlikely to be part of the *ontogenetic* foundations of the capacity to represent natural number. Rather, the developmental source of explicit integer list representations of number are more likely to be systems such as the object–file representations that articulate mid–level object based attention, systems that build parallel representations of small sets of individuals.

1. Introduction

In his précis of *The Number Sense* (*TNS*, Dehaene, 1997) Dehaene argues that the ultimate cognitive foundation[1] of mathematics rests on core representations that have been internalized in our brains through evolution. At the end of his précis, he mentions several distinct core representations that may play a foundational role: 'number line' representations of number, representations of space (which may ground geometrical understanding), representations of continuous quantities such as length, distance, and time, iterative capacities, logical capacities—and I would add—the capacity to represent ordered relations, the syntactic/semantic representation of number in natural language, and the system of parallel indexing of small sets of individuals in mid–level attentional

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[1] Cognitive foundations are representational primitives out of which more complex representations are built.

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systems. Dehaene suggests that language and the human capacity to build explicit symbolic representational systems allows for the extension of these core representational systems, and for the drawing of links between them, thus creating mathematical knowledge.

However, in most of his précis, Dehaene describes *one* core representational system, the number line system he calls *the* number sense, as the evolutionary and ontogenetic foundation of the human capacity to represent number and to create arithmetical understanding. I endorse Dehaene's naturalistic approach to the cognitive foundations of arithmetical knowledge, as well as his characterization of the number line representation of number as evolutionarily ancient, available to prelinguistic infants, automatically activated in adult numerical reasoning, and encoded in the human brain by a dedicated neural circuit. These are signature properties of any core representational system (see Carey and Spelke, 1994, 1996; Leslie, 1994; Spelke, Breinlinger, Macomber and Jacobson, 1992, for characterizations of core knowledge). Dehaene's wonderful book, *TNS,* as well as his précis of the book, pulls together massive evidence, much of it collected by Dehaene and his collaborators, for these properties of number line representations. I shall not review this evidence here. In my commentary, I refer to what Dehaene calls 'number line representations' as 'analog magnitude representations.'

In analog magnitude number representations, each number is represented by a physical magnitude that is proportional to the number of individuals in the set being enumerated. The neural underpinnings of analog magnitude representations are unknown, but the idea can be conveyed by examination of an external analog magnitude representational system for number. Such a system might represent 1 as '—', 2 as '——', 3 as '———', 4 as '————', 5 as '—————', …7 as '———————', 8 as '———————', etc. In such systems, numerical comparisons are made by processes that operate over these analog magnitudes, in the same way that length or time comparisons are made by processes that operate over underlying analog magnitude representations of these continuous dimensions of experience. Importantly, there is a psychophysical Weber–fraction signature of analog magnitude representations: the discriminability of two numbers is a function of their ratio. Examining the external analogs above, it is easy to see that it is easier to discriminate 1 from 2 than 7 from 8, (what Dehaene calls the magnitude effect), and it is easier to discriminate 1 from 3 than 2 from 3 (what Dehaene calls the distance effect). This Weber–fraction signature applies to discrimination of continuous quantities as well, such as representations of lengths (as can be experienced directly by examining the above lengths), distances, time, and so forth, and is the primary evidence that number is being represented by a quantity that is linearly related to the number of individuals in the set.

In spite of my agreement with Dehaene on all the above points, I consider it extremely unlikely that analog magnitude models of number are *the* ontogenetic foundation of human arithmetical abilities. First, human arithmetical abili-

ties derive from the integer list representation of number 'one, two, three, four, five…', for this representational system, including the counting routine, is built on the successor function in a way that analog magnitude representations are not. Second, the integer list representation of number is not itself a core representational system; it is a cultural construction that is most likely built not from analog magnitude representations but from a different core system of representation—the individual indexing mechanisms of mid-level vision. It is still possible that Dehaene will turn out to be right that analog magnitude representations underpin, developmentally, explicit integer list representations. I agree that the analog magnitude system of number representation is one of the evolutionarily given representational systems that ground numerical understanding, but I suggest that it is integrated with the integer list representation only after the latter has been constructed as a representation of number. If this is so, the integer list representation itself must be constructed from other building blocks.

Though I disagree with Dehaene on this issue of detail, I share his basic approach to the subject. Thanks to *TNS*, and other work by such pioneers as Gelman and Gallistel (1978), Gallistel (1990), Butterworth (1999), and Meck and Church (1983), we are embarked on a program of research that holds the promise of characterizing the representational primitives from which mathematical concepts are built, as well as the processes through which these primitives are enriched and extended.

2. The Natural Numbers

One natural position concerning the cognitive foundations of arithmetic is inspired by Leopold Kronecker's famous remark: 'The integers were created by God; all else is man-made' (quoted in Weyl, 1949, p. 33). I don't know exactly what Kronecker meant, but I am concerned with conceptual primitives, not ontological ones. If we replace 'God' with 'evolution,' the position would be that evolution provided us with the capacity to represent the positive integers, the *natural* numbers, and that the capacity to represent the rest of arithmetic concepts, including the rest of the number concepts (rational, negative, 0, real, imaginary, etc.) was culturally constructed by human beings. I assume that the rest of arithmetic is built upon a representation of the natural numbers; I shall not argue for this here. Rather, my goal is to convince you that God did not give man the positive integers either. Rather, the capacity to represent the positive integers is also a cultural construction that transcends core knowledge.

The extent of my disagreement with Dehaene depends upon what he considers the relation to be between the analog magnitude representations of number and the natural numbers. He does not explicitly consider this question in his précis or in *TNS*. Sometimes he writes as if he thinks they are identical, as when he says that the verbal system merely provides external lexical symbols

that map onto states of the analog magnitude system. If this were true, the analog magnitude system would be the cognitive foundation of the capacity to represent the natural numbers in the strongest of senses (see Gallistel and Gelman, 1992, for a defence of this position). More weakly, he could hold that even though historically the capacity to represent the natural numbers came into being only when cultures constructed an explicit integer list system, and ontogenetically only when the child masters this cultural construction, nonetheless, the analog magnitude system could be the system of core knowledge of which this cultural construction is an extension, and thus its evolutionary foundation.

The burden of my commentary is that analog magnitude representations are unlikely to be the cognitive foundation of the capacity to represent the natural numbers in either the strong sense or the weaker sense.

The historically and ontogenetically earliest explicit representational system with the potential to represent natural number are integer list systems. Most, but not all, cultures have explicit ordered lists of words for successive integers ('one, two, three, four, five, six…' in English; body parts in some languages, see Butterworth, 1999, and Dehaene, 1997, for examples of body part integer lists). Integer lists are used in conjunction with counting routines to establish the number of individuals in any given set. In a very important work, Gelman and Gallistel (1978) argued that if young toddlers understand what they are doing when they count (i.e., establishing the number of individuals there are in a given set), then, contra Piaget (1952), they have the capacity to represent number. Gelman and Gallistel (1978) analyzed how integer list representations work: there must be a stably ordered list of symbols (the stable order principle). In counting, the symbols must be applied in order, in 1–1 correspondence to the individuals in the set being enumerated (1–1 correspondence principle). The cardinal value of the set is determined by the ordinal position of the last symbol reached in the count (cardinality principle). While these principles indeed characterize counting, they fail to make explicit another central feature of integer list representations, namely that they embody the successor function: For any symbol in an integer list, if it represents cardinal value n, the next symbol on the list represents cardinal value n + 1. It is the successor function (together with some productive capacity to generate new symbols on the list) that makes the integer list a representation of natural number.

3. Why Analog Magnitude Representations are not Representations of Positive Integers

Analog magnitude representational systems do not have the power to represent natural number. This fact alone defeats the proposal that the analog magnitude system of numerical representation is the ontogenetic foundation of explicit numerical representations on the strong reading of the claim. That is, learning an explicit integer list representation is not merely learning words for symbols

already represented. To see this, let us consider Gallistel and Gelman's (1992) arguments for the strong proposal and the problems that arise.

There are many different ways analog magnitude representations of number might be constructed. The earliest proposal was Meck and Church's (1983) accumulator model. The idea is simple—suppose the nervous system has the equivalent of a pulse generator that generates activity at a constant rate, and a gate that can open to allow energy through to an accumulator that registers how much as been let through. When the animal is in a counting mode, the gate is opened for a fixed amount of time (say 200 msec.) for each individual to be counted. The total energy accumulated then serves as an analog representation of number. Meck and Church's model seems best suited for sequentially presented individuals, such as bar presses, tones, light flashes, or jumps of a puppet. Gallistel (1990) proposed, however, that this mechanism functions as well in the sequential enumeration of simultaneously present individuals.

Gallistel and Gelman (1992) argue that the accumulator model is formally identical to the integer list representational system of positive integers, with the successive states of the accumulator serving as the successive integer values, the mental symbols that represent numerosity. They point out that the accumulator model satisfies all the principles that support verbal counting: States of the accumulator are stably ordered, gate opening is in 1–1 correspondence with individuals in the set, the final state of the accumulator represents the number of items in the set, there are no constraints on individuals that can be enumerated, and individuals can be enumerated in any order. Thus, Gelman and Gallistel (1992) argue that the Meck and Church (1983) analog magnitude system is continuous with and is likely to be the ontogenetic underpinnings of an explicit integer list representational system and counting. This is the strong position Dehaene (*TNS*, précis) seems to endorse when he says that the verbal system provides a list of words to express the numerical meanings captured by states of the analog magnitude representations.

Unfortunately for this proposal, there is considerable evidence that suggests that the Church and Meck model is false, and that analog magnitude representations of number are not constructed by any iterative process. In particular, the time that subjects require to discriminate two numerosities depends on the ratio difference between the numerosities but not on their absolute value (Barth, Kanwisher and Spelke, under review). In contrast, time should increase monotonically with N for any iterative, counting process. Moreover, subjects are able to discriminate visually presented numerosities under conditions of stimulus size and eccentricity in which they are not able to attend to individual elements in sequence (Intrilligator, 1997). Their numerosity discrimination therefore could not depend on a process of counting each entity in turn, even very rapidly.

Problems such as these led Church and Broadbent (1990) to propose that analog magnitude representations of number are constructed quite differently, through no iterative process. Focusing on the problem of representing the

numerosity of a set of sequential events (e.g., the number of tones in a sequence), they proposed that animals perform a computation that depends on two timing mechanisms. First, animals time the temporal interval between the onsets of successive tones, maintaining in memory a single value that approximates a running average of these intervals. Second, animals time the overall duration of the tone sequence. The number of tones is then estimated by dividing the sequence duration by the average intertone interval. Although Church and Broadbent did not consider the case of simultaneously visible individuals, a similar non-iterative mechanism could serve to compute numerosity in that case as well, by measuring the average density of neighboring individuals, measuring the total spatial extent occupied by the set of individuals, and dividing the latter by the former. Dehaene and Changeux (1989) described an analog magnitude model that could enumerate simultaneously presented visual individuals in a different manner, also through no iterative process.

The analog magnitude representational system of Church and Broadbent (as well as that of Dehaene and Changeux) differs from the original Meck and Church accumulator model in a number of important ways. Because the processes that construct these representations are not iterative, the analog magnitudes are not formed in sequence and therefore are less likely to be experienced as a list. Moreover, the process that establishes the analog magnitude representations does not require that each individual in the set to be enumerated be attended to in sequence, counted, and then ticked off (so that each individual is counted only once). These mechanisms do not implement any counting procedure.

Furthermore, none of the analog magnitude representational systems, even Church and Meck's accumulator system, has the power to represent natural number in the way an integer list representational system does. For one thing, analog magnitude systems have an upper limit, due to the capacity of the accumulator and/or the discriminability of the individuals in a set, whereas base system integer list systems do not (subject to the coining of new words for new powers of the base). But the problem is much worse than that. Consider a finite integer list, like the body counting systems. Because it is finite, this system is also not a representation of the natural numbers, but it is still more powerful than analog magnitude representations, for it provides an *exact representation* of the integers in its domain.

Thus, all analog magnitude representations differ from any representation of the natural numbers, including integer list representations, in two crucial respects. Because analog magnitude representations are inexact and subject to Weber fraction considerations, they fail to capture small numerical differences between large sets of objects. The distinction between 7 and 8, for example, cannot be captured reliably by the analog magnitude representations found in adults. Also, non-iterative processes for constructing analog magnitude representations, such as those proposed by Dehaene and Changeux (1989) and by Church and Broadbent (1990), include nothing that corresponds to the

successor function, the operation of 'adding one.' Rather, all analog magnitude systems positively obscure the successor function. Since numerical values are compared by computing a ratio, the difference between 1 and 2 is experienced as different from that between 2 and 3, which is again experienced as different from that between 3 and 4. And of course, the difference between 7 and 8 is not experienced at all, since 7 and 8, nor any higher successive numerical values, cannot be discriminated.

In sum, analog magnitude representations are not powerful enough to represent the natural numbers and their key property of discrete infinity, do not provide exact representations of numbers larger than 4 or 5, and they do not support any computations of addition or multiplication that build on the successor function.

4. A Second Core System of Number Representation: Parallel Individuation of Small Sets

In Section 3, I argued that analog magnitude representations are not powerful enough to represent the natural numbers, even the finite subset of natural numbers within the range of numbers these systems handle. A second reason to doubt that analog magnitude representations are the cognitive foundation of integer list representation is that they are unlikely to underlie most of the spontaneous representations of number that have been found in infancy. Rather, a distinct system of core knowledge is likely to do so, and this system is a better candidate to be the number-relevant cognitive foundation of the explicit integer list representational system.

In *TNS* and the précis, Dehaene reviews data from habituation and violation of expectancy looking time paradigms that demonstrate that infants distinguish small sets on the basis of number of individuals in them. He writes as if these data provide evidence for number line (analog magnitude) representations in preverbal infants. However, before we draw that conclusion, we need evidence that analog magnitude representations underlie the infant's performance in these number discrimination tasks. Many researchers (Scholl and Leslie, 1999; Simon, 1997; Uller, Carey, Huntley-Fenner and Klatt, 1999) have suggested that a very different representational system might support infants' number sensitivity in these experiments. In the alternative representational system, number is only implicitly encoded; there are no symbols for number at all, not even analog magnitude ones. Instead, the representations include a symbol for each individual in an attended set. Thus, a set containing one apple might be represented: '0' or 'apple,' and a set containing two apples might be represented '0 0' or 'apple apple,' and so forth. Because these representations consist of one symbol (file) for each individual (usually object) represented, they are called 'object-file' representations. Furthermore, several lines of evidence identify these symbols with the object–file representations

studied in the literature on object-based attention (see Carey and Xu, in press, and Scholl and Leslie, 1999).

For reasons of space limitations, here I present just one knock-down argument in favor of object-file representations over analog magnitude representations as underlying performance in most of the infant number studies (see Uller et al., 1999, for a review of several other lines of evidence). Success on many spontaneous number representation tasks do not show the Weber–fraction signature of analog magnitude representations; rather they show the set-size signature of object file representations. That is, the number of individuals in small sets (1 to 3 or 4) can be represented, and numbers outside of that limit cannot, even when the sets to be contrasted have the same Weber–fraction as those small sets where the infant succeeds.

The set-size signature of object-file representations is motivated by evidence that even for adults there are sharp limits on the number of object-files open at any given moment, that is, the number of objects simultaneously attended to and tracked. The limit is around 4 in human adults. The simplest demonstration of this limit comes from Pylyshyn and Storm's (1988) multiple object tracking studies. Subjects see a large set of objects on a computer monitor (say 15 red circles). A subset is highlighted (e.g., 3 are turned green) and then become identical again with the rest. The whole lot is then put into motion and the observer's task is to track the set that has been highlighted. This task is easy if there are 1, 2 or 3 objects, and performance falls apart beyond four. Trick and Pylyshyn (1994) demonstrate the relations between the limit on parallel tracking and the limit on subitizing—the capacity to directly enumerate small sets without explicit internal counting.

If object-file representations underlie infants' performance in number tasks, then infants should succeed only when the sets being encoded consist of small numbers of objects. Success at discriminating 1 vs. 2, and 2 vs. 3, in the face of failure with 3 vs. 4 or 4 vs. 5 is not enough, for Weber–fraction differences could equally well explain such a pattern of performance. Rather, what is needed is success at 1 vs. 2 and perhaps 2 vs. 3 in the face of failure at 3 vs. 6—failure at the higher numbers when the Weber fraction is the same or even more favorable than that within the range of small numbers at which success has been obtained.

This set–size signature of object–file representations is precisely what is found in some infant habituation studies—success at discriminating 2 vs. 3 in the face of failure at discriminating 4 vs. 6 (Starkey and Cooper, 1980). Although set-size limits in the infant addition/subtraction studies have not been systematically studied, there is indirect evidence that these too show the set-size signature of object file representations. Robust success is found on 1 + 1 = 2 or 1 and 2 - 1 = 2 or 1 paradigms (Koechlin, Dehaene and Mehler, 1998; Simon, Hespos and Rochat, 1995; Uller et al., 1999; Wynn, 1992a). In the face of success in these studies with Weber fraction of 1:2, Chiang and

Wynn (2000) showed repeated failure in a 5 + 5 = 10 or 5 task, also a Weber fraction of 1:2.

Two parallel studies (one with rhesus macaques; Hauser, Carey and Hauser, 2000; one with 10- to 12-month-old infants; Feigenson and Carey, under review) provide a vivid illustration of the set-size signature of object–file representations. Both studies also address a question left open by the infant addition/subtraction studies and by the infant habituation studies, and both studies address an important question about object–file models themselves. The question about infant number representation: is it the case that nonverbal creatures merely discriminate between small sets on the basis of number, or do they also compare sets with respect to which one has more? The question about object–file models themselves: Is the limit on set sizes a limit on *each* set represented, or a limit on the total number of objects that can be indexed in a single task?

In these studies, a monkey or an infant watches as each of two opaque containers, previously shown to be empty, is baited with a different number of apple slices (monkeys) or graham crackers (babies). For example, the experimenter might put two apple slices (graham crackers) in one container and three in the other. The pieces of food are placed one at a time, in this example: 1 + 1 in one container and then 1 + 1 + 1 in the other. Of course, whether the greater or less number is put in first, as well as whether the greater number is in the leftmost or rightmost container, is counterbalanced across babies/monkeys. Each participant gets only one trial. Thus, these studies tap spontaneous representations of number, for the monkey/baby does not know in advance that different numbers of pieces of food will be placed into each container, or even that they will be allowed to choose. After placement, the experimenter walks away (monkey) or the parent allows the infant to crawl toward the containers (infant). The dependent measure is which container the monkey/baby chooses.

Figures 1 and 2 show the results from adult free-ranging rhesus macaques and 10- to 12-month-old human infants, respectively. What one sees is the set-size signature of object–file representations. Monkeys succeed when the comparisons are 1 vs. 2; 2 vs. 3, and 3 vs. 4, but they fail at 4 vs. 5, 4 vs. 8, and even 3 vs. 8. A variety of controls ensured that monkeys were responding to the number of apple slices placed in the containers, rather than the total amount of time apple was being placed into each container, the differential attention being drawn to each container, or even the total volume of apple placed into each container (even though that surely is what monkeys are attempting to maximize). For instance, performance is no different if the comparison is 2 apple slices and a rock into one container vs. 3 apple slices, even though now the total time placing entities into each container and the total amount of attention drawn to each container is equal. Also, monkeys go to the container with 3 when the choice is one large piece ($\frac{1}{2}$ apple) vs. three small pieces (which sum to $\frac{1}{2}$ apple). We assume that although the monkeys

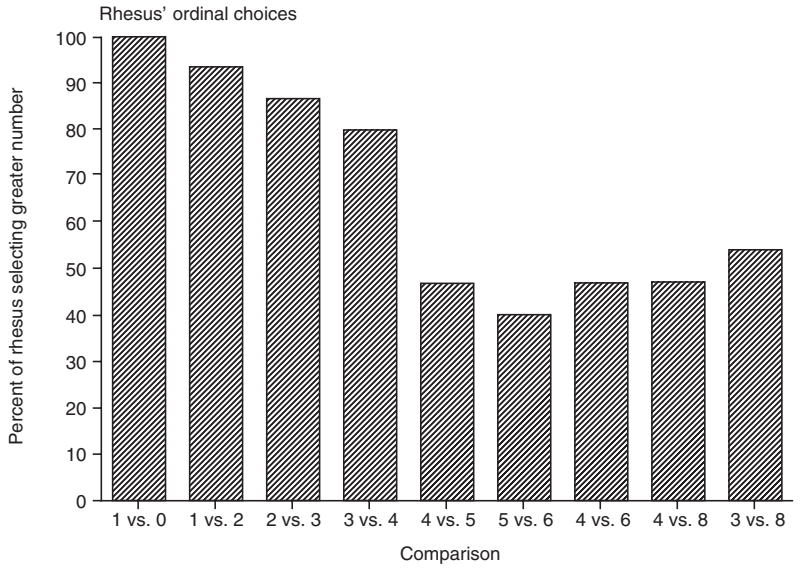

Figure 1 *Adult Rhesus Macaques. Percent choice of the box with more apple slices.*

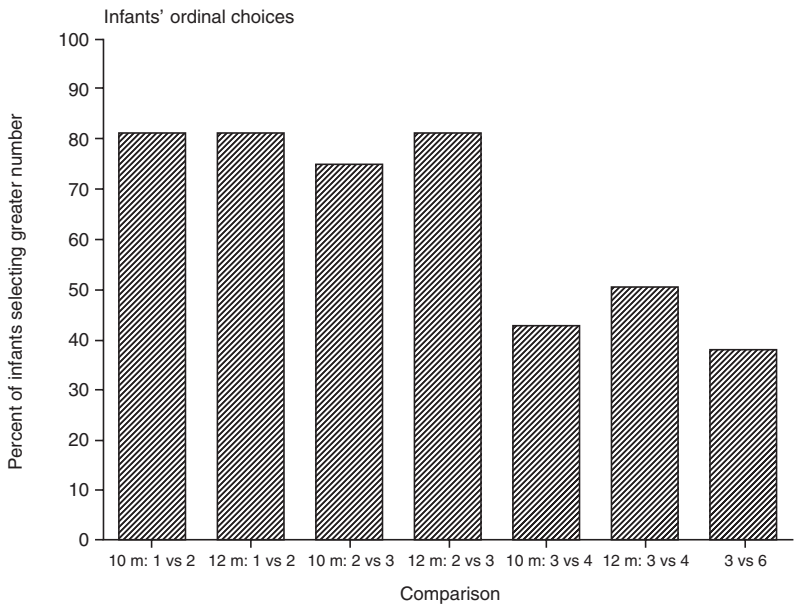

Figure 2 *10- and 12- month-old infants. Percent choice of the box with more graham crackers.*

are trying to maximize the total amount of apple stuff, they are making an equal volume assumption and using number to estimate amount of stuff. (From 10 feet away and with the slices shown briefly as they are placed into the container, apparently monkeys cannot encode the volume of each piece).

These data show that rhesus macaques spontaneously represent number in small sets of objects, and can compare them with respect to which one has more. More importantly to us here, they show the set-size signature of object-file representations; monkeys succeed if both sets are within the set-size limits on parallel individuation (up to 3 vs. 4), and fall apart if one or both of the sets exceeds this limit. Also, it is of theoretical significance to the object-file literature that monkeys succeed in cases where the total number represented (7, in 3 vs. 4) exceeds the limit on parallel individuation. Apparently, monkeys can create two models, each subject to the limit, and then compare them in memory.

As can been seen from Figure 2, the infant data tell the same story exactly, except that the upper limit is 3 instead of 4. The lower limit in human babies than in adult rhesus macaques is not surprising, given maturational considerations. The set-size signature of object-file representations rules out the possibility that analog magnitude representations of number underlie the baby's choices. It is not ratio differences between the sets that is determining success (success at 1 vs. 2, 1:2; 2 vs. 3; 2:3; in the face of failure at 3 vs. 6, 1:2), but rather the absolute size of the largest set (performance falls apart when one of the sets exceeds the limits on object-file representations).

Object-file representations are numerical in five senses. First, the opening of new object files requires principles of individuation and numerical identity; models must keep track of whether this object, seen now, is the same *one* as that object seen before. Spatio-temporal information must be recruited for this purpose, because the objects in many experiments are physically indistinguishable from each other, and because, in many cases, property/kind changes within an object are not sufficient to cause the opening of a new object file (Kahneman, Triesman and Gibbs, 1992; Pylyshyn, in press; Xu and Carey, 1996). Second, the opening of a new object file in the presence of other active files provides a natural representation for the process of adding one to an array of objects. Third, object-file representations provide implicit representations of sets of objects; the object-files that are active at any given time as a perceiver explores an array determine a set of attended objects. Fourth, if object-file models are compared on the basis of 1–1 correspondence, the computations over object file representations provide a process for establishing numerical equivalence and more/less. Fifth, object files represent numerosity exactly for set sizes up to about 4 and are not subject to Weber's Law.

Notice also that object-file representations are a system of core knowledge in all the senses analog magnitude number representations are. They are evolutionarily ancient, available to preverbal infants, have a dedicated neural substrate (involving, interestingly, the inferior parietal cortex, just as analog magni-

tude representations do; e.g., Culham et al., 1998), and continue to underlie object-based attention throughout the life span.

Unlike the analog magnitude system of number representations, the object-file system is not dedicated to number representations. Number is only implicitly represented in it, as it contains no symbols for numbers. It does not have the power to represent natural numbers, for two reasons. Most importantly, object-file models contain no symbols for cardinal values. The only symbols in such models represent the individual objects themselves. Second, object file models have an upper bound at very low set sizes indeed.

Object-file representations cannot account for all the evidence from studies of number representations in infants. In particular, such representations cannot account for infants' successful discrimination of 8 from 16 dots (Xu and Spelke, 1999). These numbers are out of the range of object-file representations, and as mentioned above, infant discrimination of large numbers is subject to Weber-fraction constraints; the infants in Xu and Spelke's studies failed to distinguish 8 from 12 dots. Also, object-file representations cannot account for infants' success at discriminating sets of events (e.g., jumps of a puppet; Wynn, 1996) or sounds (e.g., syllables in words; Bijeljac–Babic, et al., 1991) on the basis of number, although it is not yet known whether such stimuli also are subject to the set-size limitations of parallel individuation. Like Xu and Spelke, I conclude that infants have two systems of number-relevant representations; the object-file representations that are deployed with small sets of individual objects and analog magnitude representations that are deployed with large sets of objects, and perhaps with sequences of events or sounds.

5. On the Use of the Term *number* in *TNS*

To summarize the argument so far: The analog magnitude representational system is unlikely to be *the* core knowledge system that underlies the ontogenesis of arithmetical concepts for two distinct reasons. First, representations of natural numbers are the most likely candidate for this role, and analog magnitude systems do not have the power to represent natural numbers. Second, analog magnitude representational systems are unlikely to underlie infants' behaviors in most of the studies taken to show that infants represent number. Thus, there are other candidates, namely object-file representations, for the ontogenetic underpinning of the explicit integer list representations, the developmentally earliest system for representing natural numbers.

When we say that infants or non-verbal animals represent *number*, it is very important to be clear on what we are claiming. It is necessary to specify the precise nature of the symbol systems that underlie the number sensitive behavior, and ask in what senses they are representations of number—what numbers do they have the capacity to represent and what number-relevant computations do they support? I have argued above that neither of the candidate representational systems that underlie the behavior on nonlinguistic num-

ber tasks represent *number* in the sense of *natural number* or *positive integer*. Nonetheless, both systems support number-relevant computations, and the analog magnitude system contains symbols for approximate number, so they deserve to be called representations of number, as long as one does read too much into the term 'representation of number.'

In sum, even if the infant is endowed with both analog magnitude and object file systems of representation, the infant's capacity to represent number will be markedly weaker than that of the child who commands the integer-list representation. Evolution did not make the positive integers. Neither object files nor analog magnitudes can serve to represent large exact numerosities: object files fail to capture number concepts such as 'seven' because they exceed the capacity limit of four, and analog magnitude representations fail to capture such concepts because they exceed the limits on their precision (for infants, a 1:2 ratio; Xu and Spelke, 1999).

6. The Child Constructs an Integer List Representation of the Natural Numbers

Because analog magnitude representations cannot represent the natural numbers, they are not the cognitive foundation of arithmetic knowledge in the strong sense. But what of the weaker claim? When the child constructs the integer list representation of number, does he/she build it out of analog magnitude representations? This would leave the weaker version of claim that analog magnitude representations are the cognitive foundation of arithmetical knowledge intact. Or does the child construct the integer list representation from object-file representations, or from both systems, or neither? In the précis, Dehaene says that core knowledge is enriched and combined to create new mathematical knowledge. What exactly are the processes through which such enrichment and combination takes place?

Answering these questions is far out of the scope of the present commentary (see Carey and Spelke, in press, for an attempt to do so). I assume that bootstrapping processes, in the sense of Quine (1960) are required. Quine's bootstrapping metaphors (e.g., scrambling up a chimney supported by the sides one is creating through noting the interrelations among terms of the language; Quine, 1960) have the essential property that structures are initially built, at least partially, without being interpreted in terms of antecedently available concepts. Although Quine does not emphasize this, analogical mapping is often one part of the process through which this happens. Carey and Spelke, in press, outline four different bootstrapping processes that could, in principle, accomplish the construction of the integer list system of number representation. They differ in the primitives, the systems of core knowledge, that they draw on. In the rest of my comments, I sketch some data that suggest that two systems of core knowledge not yet considered are part of the bootstrapping process, and then end with some reasons to expect that object-file represen-

tations, rather than analog magnitude representations, are the number-relevant system drawn upon.

7. Linear Order

The problem of how the child builds an integer list representation decomposes into three related sub-problems. The first: how does the child learn the ordered list itself, 'one, two, three, four, five, six…?' The second: how does the child learn what each symbol on the list means? The third: how does the child learn how the list itself represents number, such that the child can infer the meaning of a newly mastered integer symbol (e.g., 'eleven')?

Children have the capacity to learn meaningless ordered lists: 'eeny, meeny, miny, mo,' 'a, b, c, d, e…' 'Sunday, Monday, Tuesday, Wednesday, Thursday…' This capacity is part of core knowledge. It is available early in childhood, and other primates have it as well. For example, Schwartz, Chen and Terrace (1991) showed that rhesus macaques can be taught to press four arbitrary symbols, appearing on a touch-screen in random positions on each trial, in a given order. The capacity to learn meaningless ordered lists obviously remains available throughout the life cycle. As of now, there is no known specialized neural substrate for this ability.

There is extensive evidence that initially the list of number words ('one, two, three, four, five…') is learned as a meaningless ordered list (Fuson, 1988). Children know the list, and can even engage in the counting routine, for over a year before they learn what the word 'four' means, or how the integer list works (Wynn, 1990, 1992b).

That the initial meaning of the list of number words is exhausted by their serial order illustrates one feature of Quinian bootstrapping. In learning a new linguistic structure, initially the terms are sometimes defined only in terms of each other, and not yet interpreted in terms of any antecedently available concepts.

8. The Semantics of Natural Language Quantification

Natural language syntax/semantics is a system of core knowledge, and natural language syntax/semantics contains representations of number. That is, all languages structure sentences, in part, in terms of number. All languages have syntactic/morphological devices for quantification, although languages differ in which ones they express. These devices include quantifiers and determiners, singular/plural distinctions, count/mass subcategorization on nouns, the *is* of numerical identity, and so forth. Of course, these numerical representations do not express the natural numbers. Nonetheless, they may provide some of the relevant conceptual apparatus from which a representation of natural numbers is bootstrapped. The semantics of quantifiers, for example, involves con-

cepts of *set* and *individual*, as does the semantics of the singular/plural distinction.

One reason to believe that core semantic knowledge plays such a role is that in the earliest stages of the learning process, the quantificational distinction between singular/plural organizes the meanings children assign to number words. Given a pile of toys (say pigs) and asked to give the experimenter, 'one pig,' young two year olds succeed. But when asked to give 'two pigs,' or 'four pigs' or 'six pigs,' two-year-olds typically grab a random number (more than one) and hand them to the experimenter. The number grabbed is not related to the number asked for, except that it contrasts with one. The number word is doing some of the work here; the same results obtain if it is 'two fish' or 'four fish' or 'six fish' (Wynn, 1990, 1992b). Conversely, in production, young two-year-olds use 'two' as a generalized plural marker. Shown an array of consisting of 1 bee, they call it 'one bee,' and shown arrays of 2, 3, 4, 5, 6, 7 or 8 bees, they call it 'two bees.' Importantly arrays of 8 bees are called 'two' just as frequently as are arrays of 2 or 3 (LeCorre, in preparation; Van de Walle, Le Corre and Brannon, in preparation.) In this earliest stage of the process then, number words other than one refer to pluralities, and 'two' is produced as if it were a synonym of the plural marker '-s' or the quantifier 'some.' Notice, at this point in learning, analog magnitude representations clearly are playing no role whatsoever. Analog magnitude representations provide no principled distinction between 1, (—), on the one hand, and 2, 3, 4, 5 …8, etc. (——, ———, ————, ————, … ————————, etc.), on the other. If lexical items had been mapped onto rough regions of analog magnitude representations, then words for larger numbers should pick out bigger sets, even if the mapping were wildly approximate. There is no hint of such a pattern in the currently available data.

9. Object-file Representations, Analog Magnitude Representations, Both?

Wynn (1990, 1992b) detailed several steps between this initial stage and the full mastery of the integer list representation of number. Children next learn the precise meaning of 'two,' as a dual marker, and take all higher number words to contrast with both 1 and 2, that is, to refer to sets of 3 or greater. Longitudinal studies found children in that state of knowledge often for several months. Then children work out 'three' as referring to sets with precisely three individuals, and some take higher number words as synonyms that contrast with three. At approximately 3 years of age, children learn what 'four' means. Wynn found no children who knew what 'four' meant who had not worked out the meaning of all the numbers in their count list. That is, at 3 years of age, children made the induction, for any word in the count list that refers to sets with cardinality n, the next word in the list refers to sets with cardinality n + 1.

As of now the data are not available to address whether analog magnitude representations play a role in the later stages of this process, as the weak form of the hypothesis that they are the ontogenetic foundation of number representations requires. There are several reasons to doubt that they do. First, as stressed earlier, all analog magnitude representations, but especially those created by non-iterative processes, obscure the successor (+1) relation between adjacent numbers. However, the induction the child makes about how counting works instantiates this relation. Object-file representations do better in this regard, for the operation of opening another object-file is a natural analog of adding one, and infant addition/subtraction studies (Koechlin, et al., 1998; Simon, et al., 1995; Uller et al., 1999; Wynn, 1992a) and the choice studies (1 + 1 vs. 1 + 1 + 1; Figure 2) show that infants represent the results of adding or subtracting an object from a set encoded in memory. Second, in the learning of number words, representations of small sets (1, 2 and 3) play a privileged role. As we saw in Section 3, infants spontaneously represent small sets with object-file representations; analog magnitude representations do not appear to be drawn upon until the numbers in the set get relatively large. For these (and other) reasons, I believe that object-file representations are a better candidate than analog magnitude representations for the system of core knowledge that underlies the learning of the explicit integer list representation of number (see Carey and Spelke, in press, for an extended treatment of these issues).

We do not know, as of now, how the child constructs the integer list representation of number out of the building blocks provided by core knowledge. It may still turn out that analog magnitude representations play a role in this process. An important empirical issue is whether any evidence can be found that toddlers who have not yet worked out how '1, 2, 3, 4, 5…' represents number have mapped later items on the list onto higher regions of number line representations. The data available to date, from Wynn (1990, 1992b), suggest that the answer is no, but more sensitive probes can be devised. If this pattern of data holds up, then we can safely conclude that mappings between regions of the analog magnitude representations and words for positive integers are constructed only *after* the child has constructed the latter, and cannot be part of the cognitive foundation of that construction.

10. Conclusions

Specialized input analyzers ensure that the symbols that articulate core knowledge pick out the relevant entities in the world. Further, innately specified computations are defined over these symbols. For core knowledge, therefore, the two components of what specifies the meaning of any given symbol (its extension and its conceptual role) are at least partially innately given. As Dehaene convincingly argues, analog magnitude representations of number are part of core knowledge. In adulthood, these representations are automatically activated in numerical reasoning, even in tasks where they are logically

unnecessary (or even counterproductive). Adults have mapped integer words onto analog magnitude representations, and since analog magnitude representations are innately interpreted, this mapping provides some of the meaning of integer words. In a very real sense, then, analog magnitude representations are part of the evolutionary foundation of numerical concepts.

The argument I have developed in my commentary on Dehaene's précis is paradoxical. A system of numerical core knowledge, analog magnitude representations of number, may not be part of the ontogenetic foundation of the first representational system with the power to express natural number. *Evolutionary* cognitive foundations and *ontogenetic* cognitive foundations are conceptually distinguishable, and as a matter of empirical fact, appear to be distinct. Some system of knowledge may be innate, and eventually integrated with other relevant systems in the representation of a given domain of knowledge, but still play no role in the learning of the systems of representation with which it is ultimately integrated.

At the very least, the process of constructing a mapping between number word representations and analog magnitude representations is not simply one of coining lexical items for prelinguistic symbols. Rather, the integer list system of number representation itself transcends any known system of number representations in core knowledge. As Dehaene remarks, in the course of the development of mathematics, core knowledge is extended and different domains of core knowledge are related to each other. I argued here that constructing the integer list representation of number (even a small finite part of it) requires extension and interrelation of different systems of core knowledge. A major challenge to the naturalistic project Dehaene has called for is to specify the bootstrapping mechanisms that extend and relate systems of core knowledge, resulting in representational systems with more expressive power than any that were antecedently available.

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