Chapter 19 - Gunter’s Quadrant

Introduction

In 1623, Edmond Gunter (1581-1626) published a description of a quadrant in *The Description and Use of the Sector, Cross-Staff and other Instruments* that is much easier to use than the *quadrans novus* for finding the time. Gunter’s quadrant is not a folded astrolabe, although it is partly based on the stereographic projection, and many of its functions are familiar to astrolabe users, particularly those familiar with the *quadrans novus*.

It is interesting to speculate on whether Gunter’s quadrant would have been developed if the Prophatius quadrant had been used in Europe, since it is no easier to use and does not provide any additional information.

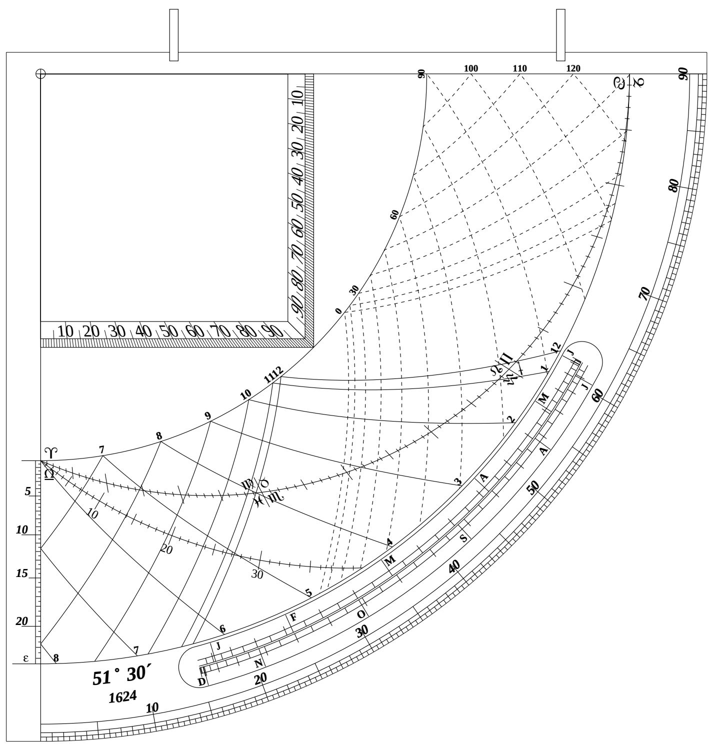


Figure 19-1. Gunter’s Quadrant for London, 1624

Gunter was a mathematics teacher who, in 1619, was named the third Professor of Astronomy at Gresham College, London, a position he held for the rest of his life. Gresham College was founded in 1597 by Sir Thomas Gresham, founder of the London stock exchange, and whose father had been Lord Mayor, to provide free public lectures to the citizens of London. The lecturers were paid (100/ pa) from rents from shops around the Royal Exchange bequeathed for this purpose. The Royal Society was founded at Gresham College in 1660. Gresham was unique in that lectures were given in both English and Latin, and were intended for practical men. Other Gresham lecturers include Christopher Wren and Robert Hooke. It is still in operation.

Gunter's treatise on the sector was originally written in Latin and circulated among his acquaintances privately. He is also remembered for "Gunter's Line", a logarithmic rule that anticipated the slide rule, contributions to surveying, and he was the first to use the terms cosine, cotangent, etc. to describe trigonometric operations on complements of angles (i.e., 90 degrees - angle).

There are few sources for information about Gunter's quadrant. Two are Gunter's own description of the instrument and the English translation of Nicolas Bion's, *The Construction and Principal Uses of Mathematical Instruments* published by Edmund Stone in 1758. Both sources are a bit obscure for the modern reader.

Following is a paragraph from Gunter:

> *"Thus in our latitude of 51 gr. 30.Northward, the Sunne hauing 23 gr.30m.of North declination, if it ſhall be required to finde the altitude of the Sunne for ſeuen in the morning,; here becauſe the latitude and declination are both alike to the Northward, and the houre propoſed falleth betweene noone and ſix, you may take 23 gr.30m.the arke of the declination out of 78 gr.22m.the fourth arke belonging to the fift houre from the meridian, ſo there will remaine 54 gr.52m.for your fift arke. Then working according to the Canon, you ſhall find, ..."*

The same subject from Stone is:

> *"But if the Sun be not in the Equator, you muſt say, As the Co-ſine of the Hour from the Meridian is to the Radius, So is the Tangent of the Latitude to the Tangent of a 4th Arc. Then conſider the Sun's Declination, and the Hour propoſed; if the Latitude and Declination be both alike, and the Hour fall between Noon and Six, ſubtract the Declination from the aforeſaid 4th Arc, and the Remainder will be a 5th arc."*

Suffice it to say, it takes a measure of dedication to extract the essence of the instrument from these sources.

Description

Gunter's quadrant contains the essential scales for finding the time with a simple Sun sight, and can be used to solve a number of problems related to time and the position of the Sun. A few stars may be included for use at night. Some of the arcs and scales are based on a stereographic projection onto the equator. These arcs are drawn with the projection from both the South Celestial Pole and North Celestial Pole. They serve two functions depending on the projection origin. Like an astrolabe plate, the Gunter quadrant must be made for a specific latitude. As with other quadrants, Gunter's quadrant is equipped with sights and a weighted thread with a sliding bead.

An example of a representative Gunter's quadrant is shown in Figure 19-1, which implements the quadrant in Gunter's treatise for the latitude of London in 1624. This figure provides a reference if it is desired to reproduce the instrument described by Gunter. The quadrant scales are shown in Figure 19-2. The quadrant in Figure 19-3 is for the latitude of Washington, DC and modern time. Gunter's example included five stars that are not shown on this sample.

The degree scale on the limb is the normal quadrant scale of altitudes. This scale is also used to show right ascension. It is divided by half degrees.

The arc inside the degree scale is the stereographic projection of the Tropic of Cancer or the Tropic of Capricorn. The interpretation depends on the half of the year under consideration. This arc represents the Tropic of Capricorn when the Sun's declination is negative in the autumn and winter, and the Tropic of Cancer when the Sun's declination is positive in the spring and summer. This arc represents a quadrant of the Tropic of Capricorn when the projection origin is at the South Celestial Pole and the Tropic of Cancer when the origin is at the North Celestial Pole.

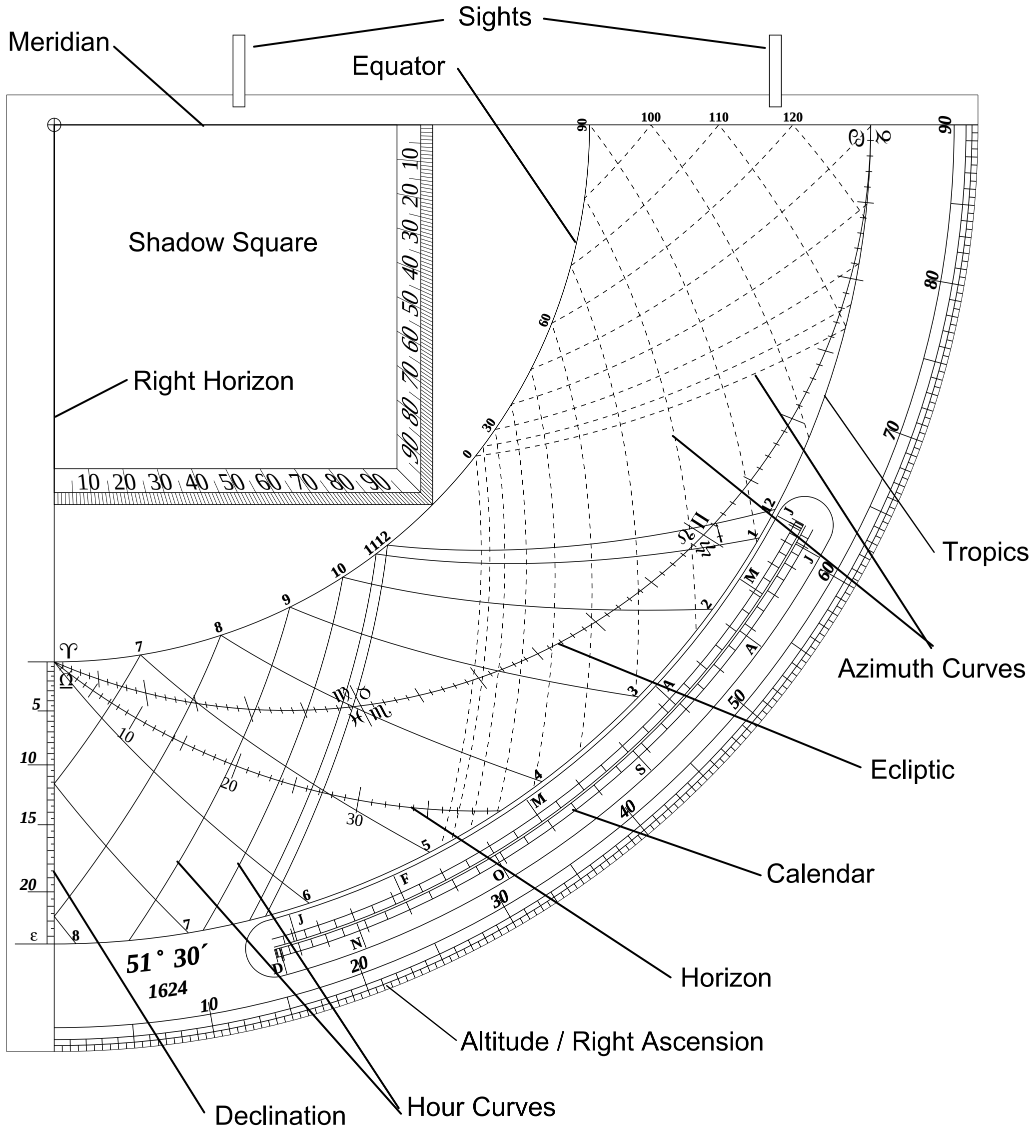


Figure 19-2. Gunter Quadrant Scales

The inner arc is the stereographic projection of the equator. The equator projection is the same regardless of the projection origin.

A scale of declination is in the left margin. The sign of the declination must be inferred from the time of year with positive declination in the spring and summer and negative declination in the autumn and winter.

The calendar scale just inside the degree scale on the limb associates the Sun's maximum altitude with each day of the year. When the thread is stretched over a date it shows the Sun's meridian altitude for the date on the limb scale and vice versa. The tic marks on the calendar scale are for noon of the date to simplify reading the Sun's altitude.

The portion of the horizon lying between the tropics is shown. This is the only part of the horizon needed to solve problems related to sunrise/sunset. The horizon scale is divided in degrees of azimuth from east or west.

The ecliptic is drawn between the tropics and is divided by the zodiac. The Sun moves over the scale twice in a year, once from the equator to the Tropic of Cancer, then back to the equator, then from the equator to the Tropic of Capricorn and back to the equator. The zodiac signs shown indicate which direction the Sun is moving by the position of the sign. The degrees of longitude are generally not labeled since each tic represents four longitudes. The value assigned to a particular tic mark depends on the Sun's current sign and the direction the Sun is moving along the arc.

The hour curves represent equal hours. Each hour curve has two sections originating at the equator. The section used depends on the time of year. The section that arcs to the left is used in the autumn and winter, and the section to the right is used in the spring and summer. Each curve represents two times depending on whether it is morning or afternoon. The hour labels along the equator are used in the morning, and the hour labels along the tropic arc are used in the afternoon. The hour curves are not stereographically projected and are not arcs of circles, but are calculated from the latitude, declination and Sun's hour angle.

Notice a line drawn from the apex through the point where the noon curve intersects the equator shows the co-latitude of the instrument on the limb degree scale.

The azimuth curves show the Sun's azimuth from south. Each azimuth angle is represented by two curves. Azimuth curves that angle to the left from the equator are used in the spring and summer and the curves angling to the right are used in the autumn and winter. The azimuth arcs are not stereographically projected and are not arcs of circles, but are calculated from latitude and declination. The Sun's altitude corresponding to a given azimuth is represented as co-altitude (90° - altitude) on the limb degree scale to reduce the congestion on the left side of the instrument. Note the quadrant does not measure azimuth using the modern astronomical convention of the angle from north, increasing to the east. East and West on the quadrant are shown as 90°. East is 90° azimuth and west is 270° azimuth using the astronomical convention. An obsolete convention measured azimuth from south, increasing to the west.

A shadow scale of 100 divisions is normally included. This scale provides the tangent or cotangent of an angle. Tangents and cotangents greater than one must be inferred from the angle being measured (i.e., tangents of angles greater than 45° are greater than one).

Gunter included five stars on his example, apparently in an attempt to make the quadrant more useful at night. It is highly questionable whether any use was made of the stars since they are rather difficult to use, and there are so few of them that it is problematical any of them would be in view when needed. Stars are not included in the examples shown. Gunter also suggested including a nocturnal on the back to make the total instrument usable at night, so he apparently did not have much confidence in using the stars to find the time.

A Gunter's quadrant for Washington, DC for 2003 is shown in Figure 19-3. Close comparison of this figure with Figure 19-1 will illustrate the effect of latitude on the quadrant. Washington is quite a bit south of London, so the Sun's altitude over the course of the year will be much higher in the sky and the horizon is much flatter. The azimuth curves are not included in Figure 19-3, because they would overlap the hour curves making the quadrant too congested. The calculations for this figure use modern values. Most significantly, the obliquity of the ecliptic is about 1/15 of a degree less than it was in 1624. This may appear to be a small value, but it affects the entire instrument.

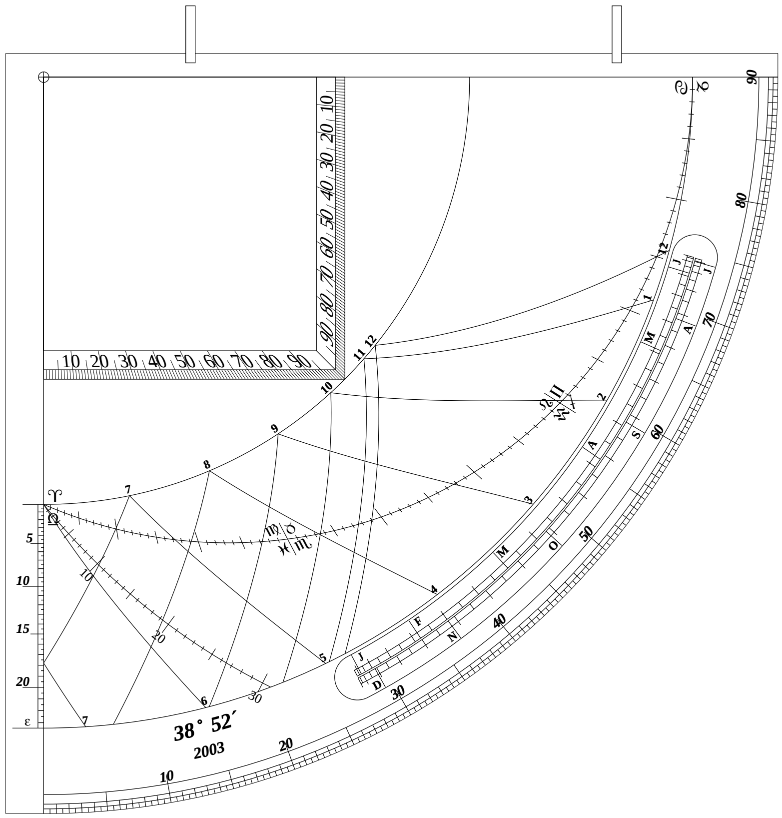


Figure 19-3. Gunter Quadrant for Washington, DC

The functions of the ordinary astrolabe and *quadrans novus* derive directly from the stereographic projection of the celestial sphere and timekeeping definitions. In a sense, the Gunter quadrant is not as pure an instrument as the astrolabe because the scales are not geometric results of the projection but are artificially calculated to provide the desired function. In one sense, it is a rather clever instrument combining the classic projection with calculations quite advanced for the time. On the other hand, it does not have the geometric appeal of the older instruments. It is, however, much easier to use and requires almost no astronomical knowledge to operate.

Using Gunter's Quadrant

The following examples use the example quadrant for 51° 30′, 1624 (Figure 19-4). The numbers in parentheses after an example are the true values. In general, it is not possible to read the quadrant to this level of precision.

Note, the Julian calendar was still in use in England in 1624, even though most of continental Europe switched to the Gregorian calendar in 1582. The date of the vernal equinox in England in 1624 was March 10, and it was March 20 elsewhere in Europe. The use of the Julian calendar will make some of the results seem at odds with experience. This type of consideration is not unusual when working with old instruments and treatises. The quadrant showing the thread and bead for each sample problem is shown in Figure 19-4. The number next to each line is the number of the problem.

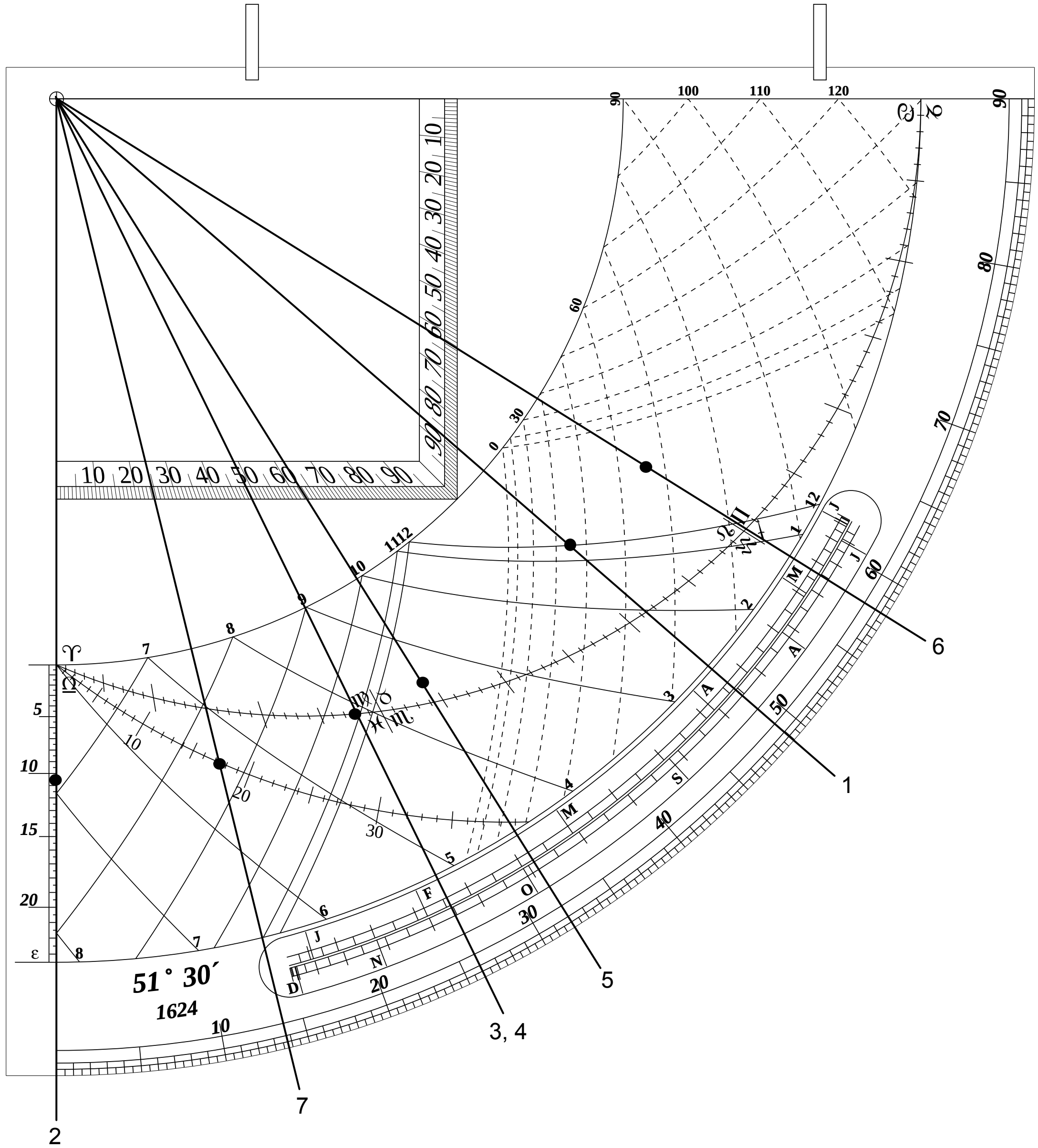


Figure 19-4. Sample Problems

1. Find the Sun’s maximum altitude for a date.

Lay the thread over the date on the calendar scale and read the Sun’s noon altitude from the degree scale on the limb.

If the date is August 15, lay the thread over this point on the calendar scale and read 49° as the Sun’s noon altitude on the limb degree scale (49.096°).

2. Find the Sun’s declination for a date.

Lay the thread over the date on the calendar scale. Move the bead to the 12-hour curve. Rotate the thread to the left margin and read the Sun’s declination from the declination scale.

Continuing with August 15, the Sun's declination is found to be about 10.5° (10° 36′).

3. Find the Sun's position in the zodiac for a date.

Lay the thread over the date on the calendar scale. Move the bead to the 12-hour curve. Rotate the thread until the bead is on the zodiac scale. Read the Sun's longitude from the zodiac scale considering the season.

For example, for August 15, the Sun's longitude is Virgo 2° (152.5°). The section of the zodiac arc to use is inferred from the date.

This problem can be worked in reverse to find the date corresponding to a given solar longitude.

4. Find the Sun's right ascension for a date.

Position the thread at the Sun's longitude in the zodiac and read the right ascension from the degree scale on the limb.

Continuing the example for August 15, set the thread on Virgo 2° and read 26° on the limb. Some interpretation of this result is needed. The Sun has moved from the vernal equinox, where the right ascension is zero, through the summer solstice (90°) and is moving from right to left on the ecliptic. So, the Sun has moved through 90° + (90° - 26°) = 154°. Convert to time by dividing by 15 to get 10.27 hrs or 10 h. 16 m. (10h 18m).

5. Find the time of day.

Set the bead on the thread to the day's declination as in Problem 2. Measure the Sun's current altitude by letting the Sun shine through the holes in the sights and orienting the quadrant directly at the Sun. Read the Sun's altitude from the degree scale on the limb. Read the time from the hour curves being careful to use the correct section of the hour curves.

For August 15, the bead is set for a declination of 10.5°. If the Sun's altitude in the afternoon of August 15 is measured as 32°, the bead falls not quite halfway between the 3 and 4 hour curves using the afternoon labels at the bottom of the curves (3:24PM).

6. Find the Sun's azimuth from the meridian.

This problem is very similar to the previous problem except the azimuth curves are used instead of the hour curves. The difference is that the position on the azimuth is found using the co-altitude (90° - altitude) on the degree scale.

For example, in problem 5, the Sun's altitude was measured as 32°. We lay the thread on the limb angle of 90° - 32° = 58°. The bead will fall part way between the 60° and 70° curves at about 64° (63.8°). The azimuth is to the west since it is afternoon.

7. Find the time of sunrise (sunset).

The horizon arc is used to find the time of sunrise or sunset. There are two steps. First, the hour angle from east or west of sunrise/sunset is found and converted to time. Consider sunrise. Set the bead to the declination for the day and move the thread to the horizon arc.

Read the hour angle from east from the degree scale on the limb (Gunter calls this the *ascensional distance*). Convert the hour angle to time by dividing by 15.

For example, on August 15, set the bead to the declination of 10.5° and move the thread so the bead is on the horizon. The thread falls on about 13.75° on the limb. Multiply 13.75° by 4 min./deg. to get 55 minutes. It is summer, so sunrise is before 6AM. Therefore, sunrise is 55 min. before 6AM or 5:05AM (5:05AM). Gunter acknowledges this method is only approximate since it is difficult to read the hour angle with great precision.

The azimuth of sunrise from east is read from the tic marks on the horizon arc (17.3°).

Making Gunter's Quadrant

A different instrument is required for every latitude. It is prudent to calculate the astronomical constants such as the obliquity of the ecliptic for the year of the instrument before beginning the layout.

- **Basic design parameters.** Determine the size of the quadrant. The side dimension of most old Gunter quadrants was about six inches (15 cm), which gives a very useable instrument. A slightly larger instrument of say, eight inches (20 cm) would be a bit more accurate. The example quadrant has an overall size of 7.75 inches (19.7 cm).
- **Margins.** Margins outside the quadrant proper are needed for the declination scale and azimuth labels. About 1/2 inch (13 mm) is sufficient. The two margins need not be equal and it may be necessary for the margin on the side with the sights to be a bit larger to keep the sights from intruding on the scales.

  The intersection of the margins defines the center of the limb and tropic arcs and the hole for the thread. The limb, equator and tropics are drawn from this center. The radius of the limb is less than the length of the sides. Therefore, there must be a straight section the width of the margin from the tangent to the limb at the margin to the edge of the quadrant.

- **Limb division.** Divide the limb by degrees. It is possible to show half degrees on most quadrants. It is customary to box the degree scale for this type of instrument. Almost all 17th and 18th century instruments used boxed scales.
- **Tropics.** The radius of the tropics (R_{cap}) will be defined next, since this dimension determines all of the other scales. Room must be left for the calendar scale between the tropic and the limb. The example quadrant leaves 0.9 inch (2.3 cm) for the limb and calendar scales.
- **Equator.** The equations for drawing the stereographically projected elements are the same as for the astrolabe. Draw the equator arc: $R_{eq} = R_{cap} \tan [(90 - \varepsilon)/2]$. Set the compass to this radius and draw the arc from the right horizon to the meridian.
- **Declination scale.** The declination scale is needed for the rest of the construction, so it needs to be drawn as accurately as possible. It should be divided into half-degree segments. The divisions need to be calculated for negative declinations from 0 to -ε. Your instrument will not be accurate if you do not use the exact obliquity for the year of the quadrant. The position of each declination point is:

  $$R_\delta = R_{eq} \tan [(90 - \delta)/2]$$

- **Ecliptic.** The radius of the ecliptic arc can be calculated from:

  $$R_{ec} = (R_{cap} + R_{can}) / 2 = R_{cap} / 2 \cos^2[(90 - \varepsilon)/2] = R_{cap} / (1 + \sin \varepsilon)$$

The position of the ecliptic center is:

$$\mathbf{y_{ec} = R_{eq} \tan \varepsilon}$$

Measure down the meridian y_{ec} units and draw an arc of a circle from the intersection of the equator with the right horizon, to the intersection of the tropic with the meridian.

The ecliptic arc is divided in the same way as the ecliptic on an astrolabe and the same graphical method can be used. It is, however, simpler to calculate the Sun's declination for each longitude and mark the longitude using an arc from the declination scale. The Sun's longitude, λ, is found from spherical astronomy as, $\mathbf{\sin \lambda = \sin \delta / \sin \varepsilon}$. Solve this equation for δ for each required longitude. Note that "only" 88 longitudes need to be calculated. Each calculated point is marked at the point where δ intersects the ecliptic.

- **Horizon**. The horizon radius is calculated from:

  $$\mathbf{R_h = R_{eq} / \sin \varphi}$$

  The horizon center is:

  $$\mathbf{y_h = R_{eq} / \tan \varphi}$$

  Draw the horizon circle from the intersection of the equator and the right horizon until it intersects the tropic arc.

  Divide the horizon by degrees of azimuth from east/west. We know how to calculate the azimuth of sunrise/sunset, A, from spherical astronomy: $\cos A = \sin \delta / \cos \varphi$. Here, the azimuth angle is measured from east/west so we need $\cos(90 - A) = \sin A$. We need to mark specific azimuths so, to draw the tic marks, solve for the declination from $\mathbf{\sin \delta = \sin A \cos \varphi}$. Calculate the declination for each tic, measure down the declination scale to the value and mark where the arc crosses the horizon. It is customary to mark each degree. The maximum azimuth shown on the horizon is the azimuth at the solstices and is calculated from $\mathbf{\sin A_{max} = \sin \varepsilon / \cos \varphi}$.

- **Calendar scale**. This scale ties the Sun's meridian altitude to the date. It should be divided as finely as possible, but five days is sufficient for most purposes. Gunter suggests finding the Sun's declination for each day of the year from an almanac and calculating the Sun's meridian altitude from: $\mathbf{h = 90 - \varphi + \delta}$. This is fine if you have an almanac for the year of the instrument, which is not always possible.

  The Sun's declination for each day of the year can be calculated fairly easily on a computer. The Sun's declination for a date is $\mathbf{\sin \delta = \sin \varepsilon \sin \lambda}$, where λ is the Sun's true geocentric longitude. The Sun's true longitude is calculated from:

  True Longitude = True Anomaly + Mean Longitude - Mean Anomaly

  See Chapter 25, Astronomical Calculations, for details on calculating these parameters. The meridian altitude is calculated using the equation above. Make sure the longitude is calculated at local noon for each day. The Sun's declination changes fairly rapidly during parts of the year and there is enough difference in half a day to affect the accuracy of the scale.

  If possible, you might consider drawing the calendar, hour curves and azimuth curves in color with one color representing spring/summer and afternoon (red) and a contrasting color (blue) for autumn/winter and morning.

  The calendar scale in the examples is calculated for noon, UT. The calculations would be somewhat more precise if corrected for the time difference from your location to UT to account for the change in solar longitude due to the time difference. For example, the west coast of North America is a third of a day from the Greenwich meridian and there is a sensible change in longitude in that time for parts of the year. This correction may not seem to be required given the inherent lack of precision in the instrument, but there is some personal gratification in making it as accurate as possible.

- **Hour curves**. The hour curves are calculated from the standard equation from spherical astronomy that relates altitude to latitude, declination and hour angle. These curves are not arcs of circles, so it is necessary to calculate a number of points and connect the points with a smooth curve. It is adequate to calculate the Sun's altitude for each degree of declination and connect the points with straight lines.

  The equation for calculating the Sun's altitude is:

  $$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos H$$

  where H is the hour angle of the time = hours from noon × 15°, e.g., the hour angle of 10AM/2PM = 30°. The calculations must be done for $-\varepsilon \le \delta \le \varepsilon$.

  Each point calculated is plotted on the quadrant at the polar coordinates (δ, h) if plotting manually; set the bead to δ and rotate the thread to h and locate the point.

  The point's rectangular coordinates can also be calculated from the declination arc for δ and the altitude, h. Let d be the distance inside the tropic arc for the day's declination: $\mathbf{c = R \tan[(90-\delta)/2]}$. The x, y coordinates of the point to plot is then:

  $$x = c \cos (90-h) = c \sin h \text{ and } y = c \sin(90-h) = c \cos h$$

  Plot using (90-h) because we are drawing the quadrant in QIV.

  A considerable number of calculations are required to define the hour curves. The calculations can easily be done with a simple computer program, a programmable calculator or with a spreadsheet program.

- **Azimuth curves**. The azimuth curves show the Sun's azimuth for a given declination and altitude. The azimuth curves will overlap the hour curves for latitudes less than 45°, so you may not want to include them for an instrument made for most of the US. The loss is not great as the azimuth curves are not used as often. They are, however, useful for some applications and may be desired. You may need to include only those curves that do not overlap the hour curves or you may be able to find a way to make the instrument useable with the overlapping curves.

  The Sun's altitude given the azimuth, declination and latitude are needed to locate the points on the curves. It is not possible to calculate the Sun's altitude directly, but Gunter solved the problem in a clever way (see below for an outline of the proof).

  For a given azimuth, calculate the Sun's altitude when the declination is zero; i.e., the Sun is on the equator, from:

  $$\tan h_0 = \cos A / \tan \varphi$$

  Then calculate an auxiliary angle, $\mathbf{x}$ from:

  $$\sin x = \cos h_0 \sin \delta / \sin \varphi$$

  Calculate the Sun's altitude, h, for the azimuth and declination:

  If A < 90°, $\mathbf{h = x + h_0}$

  If A > 90°, $\mathbf{h = x - h_0}$

  Once again, a significant number of calculations are required, but this is not difficult with a spreadsheet program or a simple computer program. The curves are drawn using the same technique as the hour curves, remembering to plot h on the (90-h) division on the altitude scale.

Each point calculated is plotted on the quadrant at the polar coordinates (90-h, δ) if plotting manually; i.e., set the bead to δ and rotate the thread to 90-h and locate the point.

The point's rectangular coordinates can also be calculated from the declination arc for δ and the altitude, h. Let c be the distance inside the tropic arc for the day's declination: $c = R \tan[(90-\delta)/2]$. The x, y coordinates of the point to plot are then:

$$x = c \cos h,\ y = c \sin h$$

- **Shadow Square.** Draw the shadow square if desired. The square can be positioned according to taste. Each side of the square is divided into 100 equal divisions with each 10 divisions labeled. The labels should be 'sheared' so the slope of the characters matches the slope of the division.
- **Stars.** You may or may not want to include stars on your quadrant. If so, the star's position is defined by its declination and right ascension, considering the appropriate quadrant. Be sure to precess the stars to the date of the quadrant. You will need to include a list of the stars and their coordinates, perhaps on the back or inside the shadow square, in order for them to be useful.

Notes:

1. Gunter (Bion) outlines a method for the hour curves similar to the method shown above for the azimuth curves. They apparently did not realize the curves could be calculated directly without resorting to a trigonometric trick.
2. It would certainly be possible to make a Gunter's quadrant with hour curves reflecting modern time zones. It would only be necessary to adjust the hour angle of the hour curves to include the longitude correction of your location from the center of the time zone. Note, however, it would also be necessary to include a time zone adjustment when calculating the calendar scale to account for the Sun's declination change from UT to the local time zone. It would be very difficult to include the equation of time in the hour curves since each curve section is used for both positive and negative declination.
3. If you refer to Gunter's or Bion's treatise, you need to work each example carefully. There are many numerical errors. It is not clear whether the errors are computational or typesetting. One is, however, left with a sense of admiration for the huge amount of hand calculation required in the 17th century to make an instrument of this type, particularly since the available trigonometry tables also had many errors. It is possible Gunter's quadrants were made with circular hour and azimuth arcs located by three points. This method would certainly reduce the effort required to make the instrument, but the result would be incorrect.
4. A proof of Gunther's method for finding the Sun's altitude for a given azimuth and declination is outlined below.

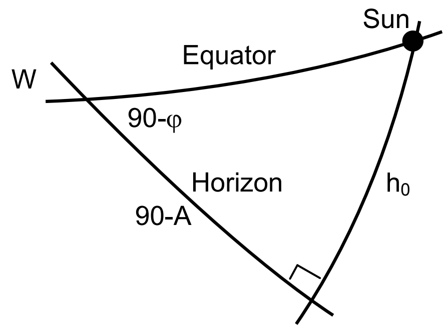
   The problem to be solved is to calculate h given δ, φ and A.

   There is a standard equation from spherical trigonometry that at first glance appears to solve the problem:

   $$\sin \delta = \sin \varphi \sin h - \cos \varphi \cos h \cos A$$

   However, this equation cannot be solved directly for h, although it can be solved by successive approximations.

Gunter needed a way to solve directly for h, and he solved the problem by breaking it into two parts that can be solved and the results combined. First, the Sun's altitude when its declination is zero is solved. Then the additional altitude of the Sun when the declination is not zero is found, and the two components are added. The Sun's altitude when it is on the equator can be found using the construction below:



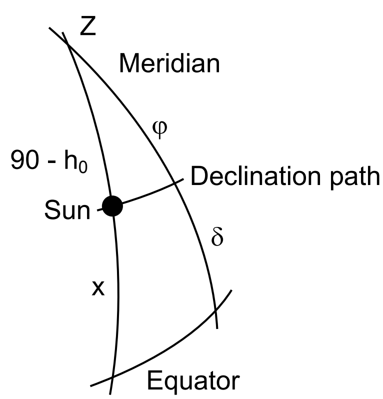
The construction is for the Sun when it is above the horizon, on the equator, and has an azimuth less than 90°.

This is a right spherical triangle that can be solved using Napier's rules as:

$$\sin(90 - A) = \tan h_0 \tan \varphi \quad \text{or} \quad \cos A = \tan h_0 \tan \varphi$$

The second part finds the additional altitude of the Sun when the declination is non-zero. The construction below is for the case of a positive declination. The case for negative declination is similar.

Call the Sun's altitude above its altitude when the declination was zero. The construction uses two spherical triangles, both of which have the zenith as the apex, one side on the meridian and the other side the Sun's hour circle. The interior angle at the apex is the Sun's azimuth. The base of the larger triangle is the equator, and the sides are φ on the meridian and ($90-h_0$) on the Sun's hour circle. The base of the smaller triangle is the declination path of the Sun. The angular distance from the equator to the declination path on the meridian is the Sun's declination. The distance from the equator to the Sun on the Sun's hour circle is the unknown, x.



The two triangles are similar since the declination path and the equator are parallel and the triangles share a common apex. Therefore, x : ($90-h_0$) :: δ : φ. And,

$$\sin x / \sin(90 - h_0) = \sin x / \cos h_0 = \sin \delta / \sin \varphi \quad \text{or} \quad \sin x = \cos h_0 \sin \delta / \sin \varphi$$