QUESTIONS IN PHYSICAL INTERDETERMINATION

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The point of view has become prominent in recent generalisations that space and time are not conditions prior to the formulation of physical change, but that rather each group of located physical events constructs an appropriate space-time for itself as part of its own evolution. If that proves not to be consistent or tenable ground, we must fall back on space and time being already essentially there, as a foundation prior to thought; though the evolution of events may happen to be best described as one involving change in the mode adopted for the expression of the continuum space-time, as well as change in phenomena located therein. The unresolvable essence of space-time would be a fourfold ordered array of points : body could be given to it by constructing various kinds of space, and various networks of reference in each kind. It has to be considered how far such a compound specification is pleonastic. The conditions requisite for physical determinacy in general, and for unambiguous continuation of physical fields into hitherto unexplored regions, demand further attention in this and other regards.

The purpose of the considerations which follow is mainly critical interpretation and simplification of ideas in relation to existing difficult analytical formulations, which have established an essential authority by agreement with crucial facts of physical nature. The degree of confidence in recent theories resting on perhaps ambiguous or only partial determination in space-time, or sometimes overdetermination, which prompted two difficult eclipse expeditions and is strengthened by their remarkable results, invites consideration of such theories from all sides. They have even diverted attention from the steady output of fundamental atomic laws by radioactive experiment. The doctrine that strict adherence to algebraic analysis avoids the pit-falls of intuition is not always a sound one; it may unduly restrict the horizon, however hazy, of science.

As the aim of this study is to bring the underlying principles into closer touch with the familiar historical development of the physics of an extended medium, so far as its fourfold analytical symmetry will permit, the distraction of the very complex analysis which seems to be inseparable from the analysis is avoided; that is indeed fully accessible in the literature, where it is expounded from many sides. It is sufficient for critical purposes to consider the ideas of the ordinary aether-theory placed in relation to the new, and illustrate by its simpler argument. The exposition becomes mainly a study in the implications of the dynamical Principle of Action, and the possibilities of an extended scope for it which shall be more than an analytical formulation : the results that emerge perhaps suggest that the new chains of analytical ideas in physical science should be capable of further clarification.

We shall be led to consider the possibilities as regards a universe of permanent atoms, gravitational and electric, interacting entirely in modes expressed by extremal types of modification of intervening generalized space, taking the place of modes of transmission across an aethereal medium, of the perhaps more crude ordinary elastic type, in the space. This view, which is a modified aether-theory, would not admit displacement of solar spectral lines.

The general trend of the gravitational theory has been assumed to be known : readers unfamiliar with it will find a very complete informing discussion from the relativist point of view in A. S. Eddington's recent brilliant and intensive book *Time Space and Gravitation.*

Contrast of the algebraic and geometrical methods.

In the formulation of general dynamical science in the hands of Lagrange the specifications had become purely analytical. In the words of his famous pronouncement in the preface to the *Mécanique Analytique,* « On ne trouvera point de figures dans cet ouvrage. » This has proved to be a feature of strength for the evolution of dynamics, by allowing and even suggesting the utmost formal generality of the frame of the science, transcending all limitations imposed by intuitive prepossessions regarding force and movement. Thus the beginning even of the train of ideas leading to the Riemann differential geometry have been traced back to this work. On the other hand this detached analytical abstraction of the argument naturally introduced, at first, a cleavage betwen its analysis and the concrete thought of the physical investigators, who were probing the order of nature by cultivation of the simpler dynamical ideas and their mutual experimental relations. Accordingly the evolution of appropriate wider conceptions in the science, largely under the hands of Kelvin and Maxwell, preluded by the introduction of such explicit concepts as degrees of freedom with their generalised co-ordinates and component momenta in Thomson and Tait's *Natural Philosophy* in 1867, followed by the dynamical imagery inherent everywhere in Maxwell's *Electricity and Magnetism* a few years later and partially adopted later by Helmholtz, heralded a new era as regards modes of general physical thought. Perhaps at that time the further advantage on the side of generalisation, inherent in the wholly unrestricted outlook of the unvisualized Lagrangian analysis, was not always adequately appreciated.

Lagrange was the most resourceful of analysts, powerful because concise. For instance, his discrimination between the differential of a function and its displacement or variation, indicated by the contrast of symbols d and δ, has in many respects doubled the power and range of infinitesimal analysis. The function of the pure analyst is to go ahead in his own abstract world of related symbols, with no diagrams and no acquired intuitions to chain him to earth. *Point de figures.* His instinct is to generalise continually. But that he may not wander into an artificial world out of range of ordinary human mentality, the intervention of the geometer is necessary from time to time, to connect his formal constructions with the mechanism of actual thought. Riemann was essentially a geometer : so was Hamilton. Since their time a new doctrine of invariance has arisen, from special beginnings of linear transformation in pure algebra, but developing into wider ranges as an affair of permanence of relations between the members of a pair of groups when change is made from one member in either group to another: this also is an integral or geometric relation, and has even provided a criterion for degrees of permanence and substantiality. The most interesting problem presented by the recent analyses of degrees of relativity is how to elucidate their algebraic procedure and results to the ordinary geometric apprehension.

Lagrange seems to have delighted to dwell on the enlargement which accrued to the powers of analysis from the simple device of extension of its symbolism. In the lectures in the *Calcul des Fonctions*, which contained much of his final thought in simplication of the process of the calculus, he has devoted a chapter (*Leçon 21*) to an exposition of the previous efforts of the Bernoullis, of Brook Taylor, and of Euler towards a general method of variations. « Mais dans l'état actuel de l'analyse, on peut regarder ces discussions comme inutiles, parce qu'elles regardent des méthodes oubliées, comme ayant fait place à d'autres plus simples et plus générales. Cependant elles peuvent avoir encore quelque intérêt pour ceux qui aiment à suivre pas à pas les progrès de l'analyse, et à voir comment les méthodes simples et générales naissent des questions particulières et des procédés indirects et compliqués. » Then in a final lecture he translates definitively the theory which he had evolved in his youth in the *Turin Memoirs* of 1758, in terms of differentials, into the modern form of gradients of the functions and of their variations.

The final remark at the end of the lectures, referring to the « Equations aux limites qui n'étaient pas connues avant le calcul des variations, et sans lesquelles on n'aurait que des solutions incomplètes » may be regarded as an indication towards the future, when the extremal formula of varying Action, in the hands of its discoverer Hamilton, was to open up yet another world of analysis, of crucial importance for the full development of general principles in theoretical physical science. Considering the innate simplicity of the source of this great chain of ideas, merely the introduction of variations alongside differentials, one is ready to pardon the ingenuous remark of Thomas Young, the vast range of whose own attainments generated an impatience of any but the most direct and time-saving expositions, in his sketch of the life of Lagrange, that « the merit of the invention is none the less... because it might have occurred to a less distinguished mathematician ».

The dynamical equations of Lagrange, in general coordinates, certainly had their source in the Principle of Action as treated by the new calculus of variations as early as 1758. It may now seem strange that in the *Mécanique Analytique* twenty years later the method of Action was put aside in favour of a formulation by Virtual Work. Its restriction to what have more recently been named holonomous coordinates was one cause. The method returned with vastly increased power in Hamilton's two dynamical memoirs (*Phil. Trans.* 1834-5), introducing the entirely new and most fruitful aspect of the Action as an extremal function connecting directly the initial and final stages of the course of change in the material system.

The idea of invariance, which in connexion with the Minkowski fourfold analysis acquired, on a narrow basis, the designation of relativity, had already pervaded, of course, the ordinary formal theory of isotropic elasticity as initiated long ago by Navier and Cauchy. When Green transformed it in 1837 into a theory of energy by introduction of Lagrangian variations, the expression for the energy-function was limited and determined by the condition of permanence of form when the co-ordinates were changed in various ways, though it was perhaps implied rather than exhibited that this invariance was complete. When Mac Cullagh soon after, tacitly adopting this method from Green in his brilliant and final formulation of optical dynamics, was guided to the square of the curl (or rotation) of the displacement as the proper expression for the energy leading to the desired optical results, he recognised fully the difficulty which his critics over-pressed, the absence of any overt mechanical explanation of his abstract formulae, of the nature of elastic strain. He fell back upon the recognition — for the first time in mathematical science — that the curl function of the displacement is thoroughly appropriate, as being in fact a differential invariant vector as regards the isotropic group of changes of frame of reference : his energy function thus satisfied the condition for relativity in its most modern form, and was abstractly permissible though independent — even possibly because independent — of any quality such as material elastic strain. Finally, Stokes in his memoir of 1849 on the Dynamical Theory of Diffraction of rays, in which he evolved completely the general laws of spreading of propagated effects, incidentally placed isotropic elastic theory in three dimensions on its final basis, and even prepared the vectorial frame for one of the most suggestive advances of the age, the Helmholtz theory of permanent vorticity in frictionless fluid media.

The inverse method of potential functions (1) had been fully developed long before this time, by Green in 1828 in his *Essay* on the Mathematical Theory of Electricity and Magnetism. Here, for the first time, the idea of a general spatial function as determined solely by its poles arose, including complete discussion of the different distributions of poles outside a region that would give the same potential function within the region. His ideas and methods, and the reaction which they promoted in abstract mathematics by leading up to the theory of functions, have remained fundamental ever since. Their author was an isolated genius, cast up sporadically outside the range of schools and universities, at the village of Sneinton near Nottingham. How did he get inoculated with mathematical research? The terms of an early manuscript prospectus, drawn up by Babbage, of the famous Analytical Society founded at Cambridge in 1813 mainly by John Herschel, Peacock and himself, as a missionary body to introduce the cultivation of the inverse Continental analysis by study of the French models, alongside (if not instead of) the fluctional and vectorial methods of Newton, have recently thrown an interesting light on this question. For reference is made to Mr Bromhead of Caius College, then a Bachelor of Arts, as the Secretary of the Analytical Society, from whom information regarding its operations could be obtained : and the same Sir Edward Ffrench Bromhead turns up in the list of the original subscribers to Green's *Essay* in 1828, and as the communicator of Green's subsequent memoirs to the Cambridge Philosophical Society, and also as the patron of Green who introduced him to membership of Gonville and Caius College in 1834. We are entitled to infer that it was the stimulus of access in the country to his friend's collection — apparently very complete and kept up to date, under the influence of the Cambridge movement — of the classical French mathematics, that gave his opportunity to one of the greatest analysts of modern times.

Interdetermination.

The principle named after Dirichlet asserts that the value of a potential V given over a boundary determines its value outside in free space devoid of sources :

(1) The introduction of the name *potential* is usually ascribed to Green, but it was employed by Gauss independently. It would seem that it is to be traced back to the idea of *vis potentialis*, in the sense of potential energy, in the hands of D. Bernoulli and Euler.

thus it determines the gradient $\partial V/\partial n$ over the boundary, and arbitrary values of V and $\partial V/\partial n$ would be inconsistent with empty space outside. But they are quite consistent with a wider view which includes such sources. According to a proposition first asserted by Gauss, who rediscovered many of Green's special results in potential theory ten years after him, if the value of a Newtonian potential is given in any region of free space, the potential is thereby determined in all places beyond that region which can be reached from it without passing across attracting matter. Its value is thus uniquely determined by the process of gradual continuation, after the manner of Taylor's expansion theorem, in the space surrounding the unknown sources or masses of which it is the potential : and the process can be prolonged towards them, either until it encounters actual regional sources or until their simplest possible and most condensed form is reached, as limiting singular surfaces, curves, or points at which the potential becomes infinite or has other type of singularity. But the problem of tracing a limited field of potential to its singularities seems, though definite, to be practically intractable so far as regards those outside the region (1).

As regards the very remarkable property thus asserted by Gauss, two related questions arise. How small may the region be, for a complete knowledge of it thus to determine the whole universe of Newtonian attraction? For a given such region, how minute must the knowledge of the potential be, in order to justify a practical extension to distant places? For example, we are accustomed to infer from constancy of potential in a small region inside a hollow charged conductor, that it must be constant everywhere inside until the charged surface is reached : but we are now asked to recognise that a more minute knowledge for that small region would involve complete knowledge of the details of the distribution of the electrons (if they were at rest) which make up the charge in the conductor. In the criticism and amendment to which principles of mathematical determination, including the cognate one named after Dirichlet, have been subjected in more recent times, this proposition of unlimited continuation seems to have escaped scrutiny. We propose to adopt it provisionally as a basis for *physical* discussion.

If we accept this result without limitation, it carries with it much more. For an isotropic uniform elastic medium, whether a material solid or an electric aether, or a medium partaking of both their kinds of elasticity, a cognate assertion can be made. Here the sources are regions within which extraneous force straining the medium is applied, or else are centres or poles of innate intrinsic strain such as electrons can be considered to be in an aether. If we know the state of strain in any small region free from extraneous force, it is held that the state of strain

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(1) This problem formed the subject, long ago, of an epistolary correspondence between Kelvin and Stokes, which still exists.

everywhere adjacent to it is thereby rendered definite : we can thus extend the known region so as to surround all the centres of extraneous force or intrinsic strain, and so by closing in upon them determine their nature, so far as it regards the medium, as before. This statement holds, because the same Laplacian differential operator that determines the distribution of a Newtonian potential — the only possible isotropic form — is involved, as *infra*, in the equations of this problem as well.

A more intuitive physical system, which throws light on the nature of the result, is that of a freely jointed frame of rigid bars, such as a girder or bridgework. If an increase of tension is put into one of its members, the tension or thrust thereby produced in every other member, however distant, can be determined uniquely by a process of statical continuation applied at the joints. This, however, implies that the frame is just stiff, and no more. If there is any slackness the result, of course, could not hold : if the frame is over-stiff, owing to presence of redundant members, the result becomes indeterminate until the elastic qualities and yielding of its members are brought into consideration. Comparing with the result for a continuous medium, we can recognise that an elastic solid, even when in internal intrinsic strain, has complete elastic inter-connections, but not redundant ones, so long as the relation of stress to strain remains of linear type.

Such complete interdetermination is a very remarkable mode of relation. It involves that knowledge of change in any small part of a system determines the change throughout : that each part is in a sense the cause of the whole. The familiar relation of cause and effect has thus vanished, along with all its metaphysical perplexities : every part of the set of concurrent events is now the determining cause of all the remainder. It is a necessary condition for this result that the circumstances should be strictly statical; elastic or other transmission in time must not enter into the scheme.

Interdetermination for statics of an isotropic medium.

The proof of the extension above stated is readily indicated. The equations of static frictionless strain in an isotropic solid medium, are, in Lamé's notation, of type, as determined from the isotropy alone, in terms of displacement $\xi\eta\zeta$,

$$(\lambda + \mu)\frac{\partial\theta}{\partial x} + \mu\nabla^2\xi = X,$$

where

$$\theta = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z},$$

leading to

$$(\lambda + 2\mu)\nabla^2\theta = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}.$$

Thus the expansion of volume represented by θ is determined by the last equation, of Laplacian type, in terms of places of concentration of applied force X, Y, Z : if θ is known over any small region, it is thus determinate by continuation beyond it, and all around these sources which are thus themselves determined. And θ being thus ascertained throughout the field, the same argument applies to ξ as given by the first equation. Complete interdetermination is thus established : knowledge of a small portion determines the whole, including even the extraneous forces that are disturbing the medium beyond.

There is no formal restriction here to circumstances of stability, such as will enter later in more general questions. Similar analysis applies in all cognate problems of static isotropic extension.

Interdetermination fails in a pseudo-space.

If, however, we deal with an Euclidean threefold pseudo-space in which one dimension is imaginary, the spatial invariant is

$$\delta x^2 + \delta y^2 - \delta z^2;$$

and the differential form contravariant to it now gives an invariant equation correlative to that of Laplace,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) V = 0.$$

But if z represents time, this equation connotes propagation of effect by waves in the two dimensional space x, y : thus obviously the state of affairs in a limited region for a limited time cannot here determine the state everywhere for all time. In fact, the quasi-radial form for V, here $(x^2 + y^2 - z^2)^{-\frac{1}{2}}$, does not now belong to a point source in the threefold extension, but to a singular conical surface $x^2 + y^2 = z^2$, of unlimited extent, on which it is everywhere infinite; which marks another fundamental difference showing the hasard of physical extension by analogy from a spatial continuum to a pseudo-spatial one. It may be of interest to note conversely in passing, that the equation of propagation in three dimensions of space is satisfied formally by

$$V = A(x^2 + y^2 + z^2 - t^2)^{-1},$$

which belongs to an expanding spherical surface-source expressed by the equation $r = t$, the unit of time being chosen so that the velocity of transmission is unity. Similar considerations apply to the fourfold world-history of space-time of the modern relativity theories. It is only to real spaces that complete static interdetermination can belong. Freedom demands that at least one dimension shall be imaginary : otherwise the world would be a hard rigid construction with no initiative anywhere. But even with this freedom, data over a complete boundary in the fourfold ought, along with the sources inside, to determine the activity throughout the interior, and in exterior space as well, provided there are no sources there.

In particular, this absence of complete interdetermination in the fourfold pseudo-space saves the Einstein Principle of Equivalence, which asserts that *locally* a field of gravitation and a field of acceleration relative to the frame of reference are not distinguishable. For if each could be continued without ambiguity, the poles of the gravific field would at length emerge and reveal the distinction. A gravific field given in a limited region can be continued into the region beyond, only when there are no poles or centres of attraction beyond or when such poles are all given in advance.

Minimum property related to continuation.

What prescribes these limitations of determinacy, as regards continuation in Cauchy's manner, of a scheme of functions given by differential equations (for example, the Einstein space potentials) from a region in which they are given to the unexplored extension beyond? In such a case if the value of the functions and their successive gradients(1) are given along the boundary, up to the order just below the highest order occurring in the characteristic differential equations, then these latter gradients are determined at that surface, and uniquely if they occur in the equations only in the first power. And by successive differentiations all the still higher gradients at the surface may be determinable. Then by the expansion of Taylor all the functions concerned can be continued from the boundary, along any linear paths we may choose, into the region outside that surface. Let us regard them as so continued, along direct paths, to an adjacent outside surface enclosing the original boundary, over which a distribution of values of the functions is thus obtained. This implies that the two surfaces are not too far apart, and that no poles or other

(1) The first gradient along the normal to the boundary necessarily suffices as the other first gradients are already involved. [The argument in this section applies *primarily* to the case in which a number of functions are controlled by an equal number of differential equations.]

singularities are encountered between them in the process. It is just an analogous ranging of the eye, along various interlacing paths in the space around us, that provides our conception of it and its contents.

But there is insufficient assurance, as yet, that, for the functions as determined by this process, there will be proper continuity in directions transverse to these linear paths of continuation. If, however, the differential equations on which this continuation is based are the conditions of extremal(1) variation of a continuous integral of a function which is essential positive within the limits of the region, the position seems to be much more simple. For the boundaries being near together, and the gradients between them being regular, there must be some distribution of the functions with the assured continuity which makes the integral minimum for the given values over the two boundaries. In this case, at any rate, we may presume that the functions, thus conditioned by a set of extremal differential equations, can be continued uniquely beyond the original boundary by successive short stages, and so continually onwards, so long as singular regions can be avoided in the process. The minimal property affords assurance, such as would not otherwise exist(2), that the continuations, necessarily made at first along isolated outward linear paths, blend into a continuous result in all paths over the region.

In general statical theory, stability with its minimum energy would thus be an essential condition for complete interdetermination.

The essentials of representation in space and time.

The mode of arriving at a conception of Euclidean space, when the Cartesian systematic analysis has replaced the Euclidian synthetic way, is to find all the frames of reference with respect to which the essential internal properties of a permanent or rigid system of bodies retain the same expression. To assert that these relations are invariant with respect to this group of similar frames is to say that mobility of a frame, changing it into adjacent ones in the group, makes no difference; or, owing to the mutual relationship, it is to assert that the rigid system

(1) A positive connotation, on Hamiltonian lines, is here proposed for the term *extremal*, already in use in a different and less defined sense. An integral is extremal when its variation is expressible in terms of changes over the *boundary alone* of the region of integration. On Hamilton's principle of Variation of Action the extremal form of the integral becomes expressed as a function of initial and final configurations or other *data*, alone. This fertile remark involves the whole group of reciprocal relations expressing the mutual influences of changes at pairs of elements of the boundary, however far apart, when under extremal conditions. We may also speak of a variation as being extremal in this sense.

(2) But if there are no sources, or other singular points of the functions, in the region outside, the continuation is definite.

is itself freely mobile with regard to any one of the continuous group of similar frames of reference.

We may have two solid systems thus freely mobile, one of them for example being a network of polar or other curvilinear coordinates : it follows that a mutual relation of free mobility is established between the two systems, and we can thus refer the rigid system to a curvilinear frame instead of a Cartesian one. We may even generalise by transforming algebraically to a frame which mixes time with space, as has often been done in physical problems. It is only one step further to refer the system to an extended collocation of point-relations which cannot be interpreted as a flat space at all, except possibly as regards its infinitesimal parts : it must, however, be expressible in a differentially continuous extension, unless we are prepared to atomise extension itself. These ideas are suitably illustrated by the statical variational theory for a stressed framework of bars, as readily developed in terms of the relations between the invariant lengths of the bars alone, without any reference to space. The unresolvable essence of space is mere extension in threefold continuity : the frame which particularises it may be Euclidean or elliptic or heterogeneous at choice : the difference will be that the relations of nature will be expressed in different forms, and it is the search for simple and direct laws of physical connection that prescribes the most natural frame, which is thus adjoined, so to say, to these physical laws or relations. The final stage of such advance would be to the gravific fields where, after Einstein, the space-time frame is no longer uniform.

Scientific geometry led early to the recognition of rigid bodies freely mobile in flat threefold space, though the implications involved in their existence were left for Helmholtz and Lie to unravel. The same tendency of thought leads naturally to enquiry as to the continuous group of frames, for all of which the vast scheme of travelling radiations which establish interconnections in the universe presents the same aspect : this group has been identified by Minkowski with the frames of a fourfold flat space-time. Complication thus arises from the emergence of two groups of equivalent frames of reference, constructed on similar ideas, one for rigid bodies, another for radiation : one of the two must absorb the other and be universal. It is a surprise that it is the more recondite group that seems to prevail : and another surprise that gravitation remains refractory to it until the flatness, which originally recommended it as an invariant scheme of nature, is broken down, though not in the direction of a Newtonian attraction suitable to rigid bodies. Thus are theories which have served their turn transcended by new views founded upon themselves, in the continued adaptation of the scheme into which we are compelled to adjust our survey of physical nature.

Non-uniform invariance founded on local characteristics.

All characteristic equations in continuous physics, such as, for example, the differential equations of the gravific potentials in the Einstein formulation, are essentially relations prescribing local constitution; they are available, as above, only for gradual continuations from elements of extension to adjacent elements in succession, provided such continuation in any direction fits in without inconsistency for the directions transverse. The essential quality of all such equations, — the main example is that of Laplace, — is that they retain their forms, are invariant, when the co-ordinates of reference are changed *locally* to any other scheme of coordinates, algebraically related of the appropriate group. This secures that if the continuation of the functions is formally possible, the result is a finite extended scheme with *intrinsic* internal relations of its own ramifying all over it, independent of the particular frame of the relevant group, in which it happens to be set. For example, the symmetric fourfold radial field of a gravific point-centre proves to be thus uniquely determined, virtually by a process of continuation from the core, provided we postulate that the coordinate functions are to be isotropic as regards each differential region of it in the three elements of extension. At some stage in the process of continuation the field may, however, begin to be disturbed sensibly by the presence of another gravific centre : that disturbance cannot be predicted, but by exploring it around that new centre we can determine its source, in the only respect that is relevant, which is just so far as regards its influence on the adjacent space : and so on by including other centres, until all space and its contents have been explored. To clear the ground for such an exploration of a less restricted spatial structure, we proceed, following Riemann, by use of the simplest definite apparatus of continuation that will allow greater freedom than a Euclidean frame, merely a differential measuring rod imagined as retaining unchanging length in all the isotropic elements of extension into which it may be successively transferred. Does not the process, then, imply that the space is already there, that it is a matter of exploring or organising a pre-existing space, with its poles, if there are such, by this new extraneous apparatus, rather than of creating afresh a space for our use as we proceed? If the space is postulated to be there already, the question as to whether the process of continuation is self-consistent, and therefore possible, need hardly arise, if only the machinery of measurement is self-consistent. Thus space and time have not been transcended in the new developments; we have only agreed to measure (that is, describe) the same essential continuum according to a different, but still intrinsic, plan, such as may permit more coherent expression of some parts of the scheme of physical nature, at the expense possibly of other parts. It is a question of pure analytical mathematics, important for our schemes of physical representation, but without ultimate philosophical implication. The Newtonian frame of space and separate time is absolute, or pre-eminent, only by its simplicity and the extreme closeness of its adaptation to wide ranges of physical investigation : there can be no meaning in the term absolute except universality of convenience.

The control which makes a radial field of fourfold space-time extension definite for gravific purposes as above, is not merely that the elements of it shall be isotropic or flat, and that the scale of length shall be imagined as an invariable transferable differential rule materialising in fact a locally invariant quadratic differential expression : it must also be possible to construct a function of Hamiltonian extremal variation from it. An alternative procedure would be to retain the same ordered distribution of the points as corresponds to a flat fourfold, and attempt to strain the length of the differential measuring rule in passing from place to place. But such variation of the standard of length in an extremal extension, without loss of directional isotropy in its application, appears to have been shown by H. Weyl to amount precisely to the introduction of an electrodynamic field operating in and slightly modifying the extremal spatial (or gravific) field. In the case of radial symmetry the results seem to be still definite, if there is an electric charge of unrestricted amount at the origin superposed on the mass there. The gravific pole involves no electric field and so no charge at the origin; and an electric charge involves no gravific field, in this radial case, though it alters the space around it according to a different law of its own. Thus in the more general case when the matter is regarded as a continuous distribution, an electric field seems to require an accompanying residual field of the gravific type (1). The atomic view brings greater precision. A gravific pole deforms the space, without any electric accompaniment. But an electric pole in addition to its electrodynamic field involves also deformation of the space, residual and very slight, yet according to a different law from the gravific pole; in fact it involves a slight deformation such as could only be produced by a volume distribution of gravific poles over the region and of infinitely increasing density towards the centre. But the definite value of the atomic charge remains to be accounted for, as also the definite masses of the atoms with which it is associated.

Can an aether be eliminated?

Another point cognate to determination also deserves further consideration. For reasons which are readily understood, of the ten gravific potentials introduced by Einstein four can be quite arbitrary; so that of their ten differential equations

(1) De Donder, *Teyler archives* (April 1916).

determining their mode of distribution, any four are consequences of the other six. This in fact connotes the arbitrary choice *in the space* of the four space-time parameters or co-ordinates in terms of which the equations of variation of the space are formed. If there is an electric field also present, specified by the four components of its vector potential, there are four more differential equations : and any four of the fourteen are now determined by the ten others. The significance of the resulting four identities, which must be necessarily involved in the extremal character of the Action principle, as expressing essential convergence of stress-tensors, is elucidated in terms of direct physical variation *infra*. But the present remark is that it has been emphasised (compare Hilbert) that if an electric field is given, its presence removes the indeterminateness or relativity of the space, for it determines all ten gravific potentials in addition to its own poles. This looks very like a reversion to the position that an electric field requires a fixed aether for its specification — so fugitive are the ideas belonging to a region of imperfect determination. Or, conversely, if the ten potentials are given arbitrarily and so cannot belong to a pure gravific space, they require and determine not only a definite fourfold extension but also an electric field in that continuum with its own sources or poles, which will usually be a continuous distribution. And, as above noted, a set of centres or poles, electric or gravific, can hardly be conceived without their fields of influence being given with them, which requires an *à priori* space in which to exhibit them. The presence of a space manifold, imagined as a definite unique scheme of marked extension, seems to be prior to all systems, however general, for its representation whether by flat or elliptic or other modes of laying it out.

Radiation as the fundamental means of physical connection : necessity for three dimensions of space.

In the world of physical nature changes are produced and controlled mainly, perhaps entirely, by free radiation across space. We have learned from Maxwell that when an electric current in one coil induces a current in a neighbouring coil, it does so by means of aethereal pulses or undulations, sent out from it swiftly across the intervening space, which are absorbed by the atomic structure of this second coil where it meets them and produce their effect so smoothly as to escape direct detection. In telegraphy the message is really carried from source to receiver by electric waves in the surrounding space, which are merely guided and prevented from spreading away sideways by the deficiency of electric elasticity due to atomic freedom in the conductor. The waves of unguided telegraphy in free space, whether of small or great length, are the normal mode of intercommunication in nature :

guidance by a telegraph wire is as artificial as concentration of sound by a tube, as Lord Kelvin used to emphasise. Even in chemistry, when one molecule stimulates another the effect is probably to be regarded as transmitted across by intimate electric fields, and part of the energy must go off in radiation.

With what degree of latitude is radiation to be specified, if it is thus to be recognised as the basic means of physical connection between systems out of direct contact? We recognise in it the idea of clean, sharp propagation of effect by waves and pulses, travelling unchanged even across celestial spaces, leaving no trail of disturbance along their paths such as would drain away part of the energy during its transmission. Such pulses, issuing from a small molecular source, must in an isotropic medium spread out at sufficient distance as spherical shells. They must thus be expressible by functions such as φ, and the gradients of such functions, restricted to the form

$$\varphi = F(r)f(ct - r),$$

where c is the velocity of their propagation. In n isotropic dimensions of space and one of time, the differential equation determining the nature of φ in every locality must, irrespective of dynamical considerations, if it does not contain gradients above the second, be of form

$$\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \dots + \frac{\partial^2\varphi}{\partial x_n^2} = c^{-2}\frac{\partial^2\varphi}{\partial t^2};$$

for it must be linear, and no other form seems to have the necessary isotropic invariance in the flat continuum. To enable this equation to have a solution of the radial type expressed above, it is not difficult (1) to prove that we must have

$$n = 1 \text{ and } F(r) = A, \text{ or else } n = 3 \text{ and } F(r) = Ar^{-1}.$$

If time has one dimension, space must thus be restricted to three dimensions, if atomic vibrations are to be able to send out pure radiation travelling across celestial spaces absolutely without degradation. For example, this would not be possible in two dimensions : a line source in three dimensions corresponds to a point source in two, but the elements of the line are at different distance from an outside point, so that, as Lord Rayleigh has explained, the vibrations emerging from a simultaneous impulse all along the line take different times to travel to that point and

(1) [This result has been given by P. Duhem in *Hydrodynamique*, *Élasticité*, *Acoustique*, Cours professé en 1890-91, tome II; as appears from an important memoir by V. Volterra, *Acta Mathematica* 18, 1894, pp. 161-232 (cf. p. 221) in which the characteristics of the various types of differential equations in the text are discussed from the point of view of the principles of Huygens and Green by strict mathematical analysis. Cf. also Volterra, Clark University Lectures, 1909].

could not arrive together : instead of a clear cut cylindrical shell of radiation expanding outward, all the space inside the outward boundary would be filled with a trail which the radiation had left behind in its progress.

The simplicity and disentanglement of physical phenomena, as dominated by radiation between the portions of permanent matter, seems thus to be intimately bound up with the three dimensions of our space. This characteristic equation of local determination of all functions such as potentials that are connected with radiation, thus necessarily restricted to three dimensions, namely

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}-c^{-2}\frac{\partial^2\varphi}{\partial t^2}=0,$$

will remain invariant when the variables x, y, z, t are changed by the appropriate continuous group of linear transformations. As the sum of differential squares

$$\delta x^2+\delta y^2+\delta z^2-c^2\delta t^2$$

is contravariant to the cognate differential operator acting on φ, that operator will remain invariant for all transformations within the group which maintains the latter differential expression invariant. That is, it will be invariant for any transference of the frame of reference in a flat fourfold or pseudo-space with three real dimensions x, y, z and one imaginary one t. Such displacements form the fundamental fourfold Minkowski group. Radiations by electrodynamic agency are the main case in point : and the (Lorentz) correspondence is well-known and simple, which connects their vectors and their poles when expressed in terms of one such frame and when the corresponding (though different) vectors but the *same poles* are expressed in another(1).

Close up to a radiant source, for example an electric bipolar (Hertzian) vibrator, the radiation is of course not pure in this sense of sharp transmission. In addition to amplitudes of vibration varying as r^{-1} which carry away radiant energy, there are terms varying as r^{-2} which represent alternating readjustment of the local field of the electric poles and effect merely local fluctuations of energy in that field. In dealing with energy purely radiant this latter part must be assigned to the molecular local fields : it does not come practically into our field specifications at all, except among the molecules of dense matter.

(1) Cf. *Aether and Matter*, Cambridge, 1900, chapter XI.

Conditions for physical continuation illustrated by modes of formulation for radiation.

A previous chapter of physical determinacy will now be reviewed, as its method may be of wide application in newer fields. A train of elastic disturbance, due to sources in any region, is transmitted across an arbitrary boundary surrounding them : its value outside ought to be expressible in terms of the succession of circumstances at that boundary for the duration of the disturbance. In the case of light this is the problem of diffraction, as formulated by Fresnel. In an isotropic uniform medium of unlimited extent it is resolvable in various ways. The effect of the changing traction exerted on the medium beyond, at each element of the surface, can be computed : or the effect of the changing displacement thus imposed (with no displacement at other elements) can be determined : and the result in either case derived by integration. But, as Lord Rayleigh noted long ago, the expressions so obtained for the effect emitted from an element do not agree. Thus the question arises whether either of them can be regarded as the actual effect for which that element is responsible; though the total effect integrated over the whole surface for all the elements must be the same in both cases.

The most graphic and suggestive mode of treatment of such a question is by the inverse general method of Green, applied by him in his *Essay* of 1828 to electrostatic fields, either steady or in which the velocity of adjustment is taken to be infinite. This order of ideas can be extended to the most general types of disturbance (1). If there is a system of poles A inside and another B outside, we can determine the surface distributions of poles, in simple and double sheets spread over it, which with A will produce the actual field inside and no field outside, or with B will produce the actual field outside and no field inside. The former is the case now in question : we can choose B arbitrarily, and for each choice obtain a different distribution of surface poles, and all of these systems produce outside the actual effect propagated from A. They will also produce an effect inside, namely that due to the system B with changed signs, which can be made what we please by choice of B. The distribution of surface poles that is equivalent *in the aggregate* to given inside sources is thus, in the case of disturbances of general type, widely indeterminate, in the absence of more definite knowledge than the state of affairs at the surface alone.

But now we restrict the disturbance to pure radiation, advancing in the outward direction only, and so at a distance from its molecular sources : it is also to be cohe-

(1) Cf. a paper *On the Mathematical Expression of the Principle of Huygens* (Proc. London Math. Soc. Ser. 2, Vol. I, p. 1, 1903).

rent radiation, otherwise its energy would be an accumulation of irregular elements, and we could not treat of its alternating stresses at all except as a residual average, for instance in considering mechanical effects or the radiation pressure which is everywhere equal to density of energy.

For such pure outward radiation the disturbance propagated up to the boundary surface or interface is wholly transmitted beyond it by actions occurring at that surface : no part of it is sent backwards by these actions. It is this restriction that renders the problem of the mode of transmission from the surface definite : the secondary sources on that interface must be such as to give the actual disturbance in the medium just outside it and no disturbance at all just inside it. There must be no sources B. The function of each element of surface being thus now precise, the rate of transmission of energy across each element of the surface, and its ray paths, are also definite and are readily estimated : the result should of course agree with the Poynting flux, if the radiation is electric. The solution, so restricted, is in accord with the well-known formula hit upon indirectly by Kirchhoff in 1883, apparently without recognition of the indetermination of the problem; which was, however, also restricted, though in exhaustive analytical detail, to the case of pressural waves such as sound (¹) depending on one scalar potential φ. That formula is in fact frequently claimed, but as we now note with only partial justification, to be a final definitive formulation of the Principle of Huygens.

But it seems to have never yet been recognised that this problem of continuation of coherent radiation was adequately and properly treated without any ambiguity long previously, in the classical memoir by Stokes (²) which in the historical order formed a chief foundation of the modern general theory of wave-transmission, thus illustrating from yet another side its comprehensive range. He there introduces for this purpose the refined principle that the influence propagated up to the surface in a very small interval of time τ is represented beyond it immediately after by a shell of the radiation just outside, of thickness $c\tau$, and that therefore the subsequent effect at greater distance is represented as that sent on from this thin shell of volume distribution of disturbance in the medium (³). His previous general expressions for the disturbance transmitted from local volume distributions of displacement and velocity, imposed on an unlimited uniform medium, then enabled him to reach an unequivocal result for the disturbance emerging from the surface in terms of its local data. It should be verifiable from these formulae that no disturbance is transmitted backward. But the immediate aim of Stokes was

(¹) G. Kirchhoff, *Zur Theorie der Lichtstrahlen* (Wied. Ann. 18, 1883).

(²) G. G. Stokes, *On the Dynamical Theory of Diffraction* (Cambridge Transactions, Vol. IX, 1849; reprinted in Math. and Phys. Papers, Vol. II, pp. 243-328).

(³) *Loc. cit.*, § 33.

to obtain data for the change of plane of polarisation to be anticipated in diffracted light, such as could be put to experimental test as a criterion between the types of theories of elastic media that were competing at that stage of optical science ; accordingly it was not within his purpose to sum up the effects transmitted from all the surface elements and verify that the result vanishes in the backward region (1).

Recently without knowledge of Stokes's discussion, the investigation (2) by aid of secondary sources after the manner of the electrostatic theory of Green has been carried through for general electric radiation, arriving at remarkably simple results for the amplitudes actually contributed by the elements of surface and for their polarisations. On comparison with Stokes's conclusions of 1849, discrepancy seems, however, to be revealed. This is probably to be traced to the cause that, though the equations of propagation must be the same for all types of optical theory, yet the stress-tensors of the medium are different for the electric (rotational) and the elastic solid (strain) specifications, this being in fact the source of the differing laws of reflection for the waves of light. This involves also that radiation is propagated differently from an electric bipolar (Hertzian) vibrator and from a source involving alternating strain, which would account for the discrepancy. But the procedure of Stokes for propagation by strain could, of course, be carried through on the same principles for the different problem of electric propagation : if this were done the aggregate secondary sources in his layer of disturbance of breadth $c\tau$ ought to agree with the superficial distribution derived from Green's method, which identifies the sources with the analytic singularities of the functions concerned.

Extremal variation of Action as the foundation of all relations in general physics.

The variation of the total intrinsic Action of a physical system is postulated to be of extremal character. As it is essentially an integral taken over the spatial extension alone, the widest possible variation, in the Einstein type of theory, arises from changes in the potentials, ten or less, of the type of space belonging to the problem and changes in the components of the electric vector potential; for these are all the independent variables. A particular type of variation is that due to the changes in these potentials, taken at the same points, but arising from displacement of some frame to which the poles of the material system can be regarded as attached. This

(1) This statement must be corrected : the verification appears explicitly near the end of § 34.

(2) *On the Mathematical Representation of the Principle of Huygens Pt* II (Proc. London Math. Soc., november 1919).

type of variation must be extremal because it is included in the general extremal type : and the analytic expression of this fact leads immediately to a stress-tensor, which has no convergence, provided the Action does not involve the coordinates or potentials absolutely, but only their local gradients. This seems to be the source of all the identities that turn up in the analytical developments. The conclusion holds good, however the co-ordinates be transformed; but if they are such that the fourfold element of extension is not equal to the product of differentials of the coordinates, the necessary factor, a function of the coordinates expressed by the familiar symbol $\sqrt{-g}$ of Einstein, will enter into the form of the tensor and destroy its descriptive simplicity. There are as many forms of tensor as there are systems of coordinates, all of them equally effective, and indeed fundamentally identical. It has long been recognised that the Maxwell stress was not shown to be unique : but any addition to its tensor must be locally invariant and must introduce no new poles, and would apparently have to come from the addition of another scalar invariant, of higher complexity, to the Action-density. The system of coordinates might express any network of curves, however complex, or even belong to any frame of bars, to which the moving sources or poles can be regarded as attached : but to avoid complication, we refer this imaginary material framework to the same coordinates to which the potentials of the space are referred. The influence exerted between a source or congeries of poles and the spatial medium around may then be identified with the resultant of the tensor of this material framework integrated over any barrier surface which cuts out the source from the region. As regards a gravific source the procedure of Einstein shows, and it is easy (*cf.* p. 24) to verify directly, that the result is equivalent to an added tensor which is a mass-constant multiplied by the quadratic tensor of its component velocities as in gas-theory. This appears to be the origin of the claim that the laws of conservation of mass, momentum, and energy of particles are implicit in the Einstein theory of gravity (1). From our point of view we should rather adhere to the historical claim that they are implicit in the all-embracing Hamiltonian Extremal Variation of Action, to which all physical theory dealing with relations across a distance, including gravitation, has to conform. Recognition of a space or an aether whose character is subject to variation, and the artifice of an imagined framework of coordinates in that space to which the material bodies remain attached in their motion, and the postulate of extremal variation of an Action expressed as the integral of a local Action-density which must be invariant with regard to local differential change of the frame, make up the outfit for the whole theory. The circumstance that the spatial part of the Action involves second

(1) The fact that when the Action is made extremal, its density vanishes everywhere except at the poles, has even been advanced as a difficulty in the application of the method of Action, whereas it is the source of all the simplification.

gradients does not destroy the scheme, though it renders it somewhat abnormal in comparison with the quadratic functions of first gradients that belong to ordinary dynamics. The variety of possible tensor forms that has been evolved by different writers has here its natural origin.

It has been a favourite position, in the enforcement of special relativist ideas, to assert that physical reality can only consist of the meetings of particles tracing their various paths in space-time, and that all that is essential is the order of these points of encounter along these various curves, the intervening void being filled in conveniently by any form of space whatever that we may choose. Though graphic as an illustration, this is perhaps hardly intended as a serious expression of the scope of physical reality. Let us observe what takes its place in the formulation here sketched. A space is to be chosen by means of its potentials, such as may prove most efficient as regards the range of phenomena to be formulated in it. The moving sources or masses are attached to an imagined mobile framework in space as thus specified, representable by the same coordinates, which may be chosen as we please subject only to those masses being attached in its virtual displacements : and it is this unlimited generality of the coordinate framework holding the masses and also measuring the space that now replaces the filling in of an intervening void as above. For displacement, conveniently represented by variation of the coordinates employed (1), of this network in the space, the Lagrangian process must evolve an extremal variation of the distribution of Action, thus introducing a tensor, which integrated over a barrier surface expresses the mechanical effects transmitted to each source or mass. The notion of colliding material points is no longer relevant : the intimate nature, perhaps infinitely complex, of each source remains unexplored : it is certainly far more than a point or particle, but its relations to the spatial extension, in which the activities reside so as to pass on by its mediation to other sources, are adequately expressed by the stress-tensor appropriate to the particular set of coordinates in the selected form of space, to which the material framework has also been regarded as attached. In ordinary electrodynamic theory this can be held to be the suitable dynamical variational setting of the Maxwell stress, originally introduced sporadically, whose place in the scheme has been an unresolved puzzle since its initiation. When the uniform mixed space-time frame of Minkowski is introduced,

(1) That is, the set of coordinates that is arbitrarily chosen to represent the space (or medium), but yet being in the space must vary along with it, is also conveniently chosen to describe the imaginary material framework that is supposed to connect the poles. If a stress tensor for this framework approximates to Maxwell's form, the coordinates must approximate to Cartesian. In other terms, the variation by displacement of xyz to $x+p$, $y+q$, $z+r$ implies that there is something at the point to displace : but the displacement of the medium of transmission is fully represented by variation of its potentials : thus the other must belong to an imagined material frame attached to the arbitrary set of coordinates and also to the poles.

that stress-form should expand in the well-known striking way into a symmetric fourfold stress-energy-momentum tensor which embraces all the principles of conservation. When with Einstein the fourfold uniformity is only in the differential element of extension, while its scale varies from element to element, the Action with its extremal variation enables the tensor still to connect by mutual relations the parts at a distance. Thus are evolved physical principles suitable to our perceptions, through the reciprocal relations that arise from the proved possibility of expression of the Action in terms of the extremal data alone. These Hamiltonian principles of extremal variation, first detected by him on optical lines for ray-systems, afterwards extended to the general Newtonian dynamical world, would thus be the ultimate universal formulation, putting in evidence the selection that extracts or renders possible an ordered physical science, so far as regards reversible phenomena, including relations of entropy, in an unresolvable atomic cosmos.

On this view it would not be correct procedure to adjoin to the Action special terms for the matter and for the electricity in the region, as has been done. The seat of the activity is the region, and the Action is solely a volume integral throughout it. But in approaching a material pole or atom, or rather the singular curve which is its path in the fourfold, the integrand tends to infinity : so it is cut out by a barrier surface, and its effect in the variation is replaced by that of a surface integral over that tubular barrier, which, reducing to a line-integral in the Action, represents the atom in so far as it interacts with the medium (1). Similarly for an electric atom or electron. In each case the atom is a mere nucleus keying up activity in the surrounding spatial threefold extension, not an active source of activity itself (2).

Source of the linear integrals along orbits of poles that are made extremal in the Action.

The derivation of the linear integral in the reduced extremal Action, which in its own extremal form determines the orbital curve of a particle, from the formula expressing the volume-distribution of the Action throughout the region, seems to be strikingly simple and direct as regards the gravific terms. For it appears that even when an influencing electric field is present, and therefore the extension does not belong to the type of an Einstein purely gravific space, the gravific part G of the

(1) This result seems to follow, as will appear immediately, from the remarkable fact that the distribution of extremal gravific Action in the medium is all concentrated at the poles.

(2) This is the point of view of *Aether and Matter* (1900) Chapter vi, variation of strain in aether being now replaced by variation of structure of space.

Action-density, when made extremal, reduces to zero everywhere throughout free extension. Therefore this Action is made up only of discrete parts, becoming concentrated, when the extremal condition is satisfied, at each pole. On account of the determinate distribution around each pole, regarded as at rest, owing to the radial and differential isotropic symmetry of coordinates, the integration over the three dimensions of space, in coordinates relative to the pole as origin, leads to a value constant and characteristic of the pole which we may call its mass : and the element of the Action attached to the pole is therefore of the form $mc\delta t$, involving the remaining differential, that of time. When transformation is made to a coordinate frame with respect to which the pole is moving with velocity u, v, w this element $mc\delta t$ transforms by the Lorentz correspondence to $m\delta t(c^2 - u^2 - v^2 - w^2)^{\frac{1}{2}}$, which is $mc\delta s$, where $\delta s^2 = c^2\delta t^2 - \delta x^2 - \delta y^2 - \delta z^2$, being the square of the fourfold interval of linear extension, regarded as a time-interval. The orbital part of the Action, with distribution rendered extremal throughout the extension, is therefore $c\int mds$: but this is not a new term appended to the Action, and the orbit derived from it is really controlled, as it ought to be, by the activity in the extension or medium around the particle. A cognate instantaneous deduction of the orbital terms for an electric pole $\int(mds + e\mathrm{F}dx + e\mathrm{G}dy + e\mathrm{H}dz - e\mathrm{V}cdt)$ may possibly be feasible on Weyl's formulation(1). Incidentally the reason for the preference for a change of sign as above in δs^2, making it positive for a time-interval, becomes manifest : for otherwise the lapse of time in an orbit would be an imaginary quantity. The interval from one point-time to another is to be measured by $c\delta t$ (or by δt in altered units) diminished by correction for the change of position. The actual orbit from one point in time-space to another is that for which the sum of the elementary intervals along it is thus diminished as little as possible. The principle that a free line (geodesic) is the path of *maximum* extension (not spatial length) for constant time between every two adjacent points of it, derived by Robb in his synthetic construction of space in terms of the relation of time, is thus recognised as an essential statement involved in the reality of time and not a mere paradox.

The variational ideas of Hilbert.

The first to attempt systematic consolidation, into a deductive system, of the tentative constructions of tensors and vectors, which seem to have been originally

(1) But a ray of light is not a singular curve in the fourfold, thus the possession of the inertia of energy is not sufficient to constitute mass. It is not even a free or extremal path : but when the analysis is carried over into a correlative Newtonian space and time, it does become a hyperbolic orbit.

inspired by generalisation of the Maxwellian stress-formulation, appears to have been Hilbert. In an exposition with the comprehensive title "Die Grundlagen der Physik", *Göttinger Nachrichten*, November 20, 1915, he has identified, following on a brief indication by Einstein himself, a function whose integral extended over the fourfold domain will by variation evolve the Einstein scheme of differential equations. The integrand ('Welt Function') must be invariant as regards change of system of coordinates ('weltparameter') : thus if its form is not to be unreasonably complex it must be restricted to the single Riemann invariant of local quasi-curvature appropriate to the fourfold extension : and it is verified that this in fact gives the required form.

But if an electrodynamic scheme is to be included as well as gravitation, an electric potential which is a fourfold vector must appear as contributing a part to the integrand, in addition to the part involving the ten gravitational potentials that define the linear interval. This will lead by variation to a scheme, involving interdependence of electric and gravitational fields. But Hilbert stresses the property that in such a problem in a fourfold four of the differential equations, in this case fourteen in number, that arise from the extremal variational property of the integral, are not independent but are involved in the others. He points out that these four may be chosen to be the electrodynamic relations : and he concludes with emphasis that in this sense electrodynamic phenomena are manifestations ('Wirkungen') of gravitation, and that Riemann's aspiration to establish connection between light and gravitation has thus been at length realised.

Though intuition rebels against such sweeping specialisation, this analytical relationship ought to be capable of direct physical interpretation. If we fall back on our guiding principle that the gravitational field is determined by its singular points, poles of various kinds which stand for the Newtonian attracting matter, it might be held to mean that where that field is free of singularities the electrodynamic field will be free of singularities also, for the four electric field-equations are involved in the ten gravitational ones. In that case the Hilbert principle would require that the electrical and gravitational poles coincide, that the ultimate Newtonian attracting element is also an electric element, and conversely. Is then every electron, and every other type of ion, also in another but correlative aspect a particle of matter of definite gravitating mass? What is the relation between these two properties of the pole? For there should be one or more types of such relation corresponding to electron, ion. Are we here in sight of the strict relation of electricity to matter? The answer, if any, would have to come by minute constructive analysis of the part of the fields that closely surrounds the structure of the pole, possibly in ways yet unimagined. If the pole is of infinitesimal extent that local field will be of intense curvature, and abstruse relations must be involved. The exact solution of Schwarzschild carries this gravitational field close up to the pole, and that of

Nordström (cf p. 28) the electric field : but the nature of the pole itself, recognised always as really a structure essentially unknown, may still escape us even in these respects of merely physical configuration.

Here we come up against fundamentals. The ultimate poles are known to be either all alike as regards both mass and charge, namely electrons, or they are of a limited number of definite types, as regards mass and spectrum associated with negative or positive multiples of the fundamental charge, namely atoms. What contribution will this type of theory, suggesting relation of charge to gravific mass, manage to make to the expression of this ultimate fact?

These questions will be approached again when the radial field is examined in detail. It appears that an electric source involves a space around it of definite constitution different from the Einstein species of distance-spaces which represent fields purely gravific. It is thus hardly appropriate or exact to describe an electric pole as involving a gravific as well as electric field, though its spatial field could be imitated by a continuous regional distribution of mass or gravific sources except that the density would be concentrated in a higher degree at the centre. On transition to the Maxwell relations, it must be this distribution that represents the intrinsic inertia of the electron.

It may be noted, as a feature characteristic of the wayward evolution and consolidation of theories, that the gravific part of the Action has been determined as a scalar distribution, invariant for a general Riemann space, and then an electric part separately invariant as regards its own fundamental vector has been added, which destroys the Einstein extremal type of that space in which pure gravitation was found to subsist : an electric field interacts with and disturbs a gravific field, because they both involve changes in space.

In a continuation of date December 1916 Hilbert seems to recognise that the principle of physical determinacy could no longer be relied upon in general in a fourfold whose elements are, as he terms it, pseudo-Euclidean, that is flat but with some of the four dimensions imaginary. He thus replaces the imaginary variables by real space-time coordinates, and expresses a set of conditions betwen the space potentials that are necessary if the scheme is to correspond with real space and time, in the respect that when the coordinates are changed effects are still to succeed their antecedents without ambiguity, that, as he phrases it, to this degree the principle of causality is to obtain. The change to a system of coordinates in which this property is preserved he characterises as a proper space-time transformation : the condition involved (p. 67) seems to be that there shall everywhere be only *one* imaginary dimension in the fourfold (1). When this condition is attended to it is

(1) The conditions ensure (cf. p. 10) that the spatial part of δs^2 shall be essentially positive.

made out, by means of approximate Laplacian analysis, that knowledge of the distribution of the fourteen potentials, ten spatial and four electric, at the 'present' makes their course of change in the 'future' determinate and unique, so far as these conditions hold and therefore *this course of change has physical meaning.* The terms present and future (or the reversible past) are here defined with reference to that coordinate of the pseudo-Euclidean fourfold which is the expression of time. If more (or less) than one imaginary dimension arises anywhere in an analytical transformation of the fourfold, that transformation has to be rejected as being beyond the determinate world of physics, or perhaps as unsuitable to express it. The scheme then could *not* claim to evolve reality, with its determination of the future from the whole of the present, which appears as something absolute in the background to which the analysis has to be careful to conform (1).

Thus one can concur with Hilbert and with Klein that the essential feature of the new analysis, which has merged gravitation in the scheme of space and time, is that the extremal relations that determine extension after Riemann on the basis of distance are now involved in altered form along with the extremal relations of Action that determine dynamical sequence. They could not be extricated except in the limiting case corresponding to an infinite velocity of radiant propagation, when space becomes determined separately and the sequence of schemes of events is then determined in that space. The new feature is that the formulation of space-time, as well as the dynamics, is modified by, and so itself now helps to determine, the distribution of material and electric poles that constitutes the permanent essence of matter, and without which it has even been argued that space itself must vanish.

But Hilbert seems to prefer, as usual in algebraic analysis, to ignore the atomic clue, by constructing (p. 70) an exposition in which there shall be no singular points in the fourfold, but in place of them a continuous distribution of electric fourfold potential which he aims at identifying with material manifestations. The idea of a duplex medium, aether *plus* material atoms (cf. p. 36) is thus lost.

The radial type of combined gravific and electric field.

The simplest case of concomitant gravitational and electric fields is that of a single point-source of mass m and charge e. This case has been worked out by Nordström (2) with a result which, like the Schwarzschild field confined to a mass-centre, is exact; namely that

$$\delta s^2 = F^{-1}\delta r^2 + (r\delta\theta)^2 + (r\sin\theta\delta\varphi)^2 - c^2F\delta t^2$$

(1) The intermixture of real and imaginary coordinates has been criticized on lines essentially similar by R. A. Sampson : cf. the *Halley Lecture*, Oxford, 1920.

(2) *Amsterdam Proceedings* 1918, p. 1241.

where with his units

$$F = 1 - \frac{km}{4\pi r} + \frac{ke^2}{8\pi r^2}.$$

This in ordinary c. g. s. units with e in electrostatic measure seems to correspond to

$$F = 1 - \frac{2\gamma m}{c^2 r} + \frac{\gamma e^2}{2c^4 r^2},$$

in which γ is the constant of gravitation $6 \cdot 10^{-8}$ c. g. s. The second term is now that obtained in the solutions by Schwarzschild and by Dröste for a central gravitation alone; and the dimensions of the third term have been made to correspond with it by introducing the appropriate power of c which had been chosen to be unity.

In this form of δs^2, r is merely a radial parameter which can be replaced by any function of r. Thus in this case of conical symmetry, the presence of an electric field still leaves the spatial potentials with an arbitrary feature; but it can be readily shown that this indetermination is removed if the coordinate system as expressed by algebraic functions is postulated to be everywhere locally isotropic.

The equation of a free orbit of an uncharged body in this field is determined (cf. p. 24) by $\delta \int ds =$ extremal. Now approximately

$$-ic\frac{\delta s}{\delta t} = 1 - \frac{\gamma m}{c^2 r} + \frac{\gamma e^2}{4c^4 r^2} - \frac{1}{2c^2}v^2 + \frac{1}{2c^2}(1 - F^{-1})\left(\frac{\partial r}{\partial t}\right)^2 + \dots$$

where v is the velocity of the body with reference to the frame, the relatively very small last term being a main cause of the Einstein progression of the apse of a planetary orbit. But the ordinary variational equation of a free orbit in the Newtonian frame is

$$\delta \int \left(\frac{1}{2}v^2 + V\right) dt = \text{extremal},$$

in which V is the potential of the field of force per unit mass, thus for example the radial component of attraction being $-m\partial V/\partial r$. On comparison it appears that the free orbit of the previous specification, by $\delta \int ds =$ extremal, is described as if under a central Newtonian attraction

$$-\frac{\partial}{\partial r}\left(-\frac{\gamma m}{r} + \frac{\gamma e^2}{4c^2 r^2}\right), \text{ which is } \frac{\gamma m}{r^2} - \frac{\gamma e^2}{2c^2 r^3},$$

but is, as before, subject to the disturbance arising from the last term in $\delta s/\delta t$.

The first term of this central force represents Newtonian gravitation and gives an elliptic form to the orbit. The second or disturbing term, inversely as the cube of the distance, was shown by Newton for the purposes of his lunar theory to produce precession of the apse of the ellipse : in fact the usual equation of the central orbit, in terms of u equal to r^{-1}, is

$$\frac{\partial^2 u}{\partial \theta^2} + u = \frac{P}{h^2 u^2} = \frac{\gamma m}{h^2} - \frac{\gamma e^2}{2c^2 h^2} u,$$

or

$$\frac{\partial^2 u}{\partial \theta'^2} + u = \text{constant}, \qquad \text{where } \theta'^2 = \theta^2 \left(1 + \frac{\gamma e^2}{2c^2 h^2}\right),$$

representing an ellipse rotating through an angle $2\pi \frac{\gamma e^2}{4c^2 h^2}$ in each revolution of the planet.

This electric progression of the apse of the orbit, being the whole effect of the electric field, might conceivably compete in magnitude with the gravitational progression which is only a secondary effect, a point which we shall now examine. The latter is by the Einstein formula $2\pi \frac{3}{h^2} \left(\frac{\gamma m}{c}\right)^2$ for an orbit nearly circular, the expression being as usual here adapted to practical units by altering the power of c so as to make it of zero dimensions.

The ratio of the electric to the gravitational precession is $\frac{1}{12\gamma} \left(\frac{e}{m}\right)^2$. If the central source of activity is an electron e/m is of order $\frac{7}{4} 10^7 c$: if an atom of hydrogen it is of order $10^4 c$: and γ is $6 . 10^{-8}$. Thus if the source is an atom of matter with ionic charge the electric progression overwhelms the other part. The intra-atomic theories of rings of electronic satellites will here possibly be very different from the customary calculations.

It is not unlikely that the Sun has a small positive charge due to the expulsion of electrons by molecular disintegration, reaching a limit determined by its resulting field [1]. If it were charged so that the average field at its surface is as much as one volt per cm., and if a is its radius, its charge E would be given by

$$\frac{E}{a^2} = \frac{10^8}{c}; \quad \text{and} \quad M = \frac{4}{3} \pi a^3 \rho,$$

so that for the sun E/M would then be of order $4 . 10^{-4} c^{-1}$.

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(1) Thus electrons could be discharged with radio-active speeds from the Sun, and its positive charge thus continually increased, until that charge would be sufficient to drive out positive ions beyond recall by gravitational attraction : this would put a limit of the solar field at 900 volts per cm., and very much less if the ions have the initial velocity of α rays.

As the planet that is under the influence of the Sun would be a conductor, there is the effect of direct attraction due to the displacement of its own electricity to be added. For a planet of radius b this is the attraction between E on the Sun and an image $-\mathrm{E}b/r$ in the planet, which is a force E^2b/r^3. The total effect of this disturbing force is again a progression of the apse, by Newton's theorem, amounting for a planet of mass m to a fraction $-2c^2b/m\gamma$ of the above indirect electric effect due to supposed electric warping of the space.

The Einstein effect, the direct electric effect, and the indirect electric effect on the planetary apse must thus be represented by

$$2\pi\frac{3}{h^2}\left(\frac{\gamma \mathrm{M}}{c}\right)^2, \qquad -\pi\frac{b\mathrm{E}^2}{m\gamma h^2}, \qquad 2\pi\frac{\gamma \mathrm{E}}{4c^2h^2},$$

which are as

$$\frac{3\gamma^2}{c^2}\left(\frac{\mathrm{M}}{\mathrm{E}}\right)^2, \qquad \frac{b}{2m}, \qquad \frac{\gamma}{4c^2}.$$

For a solar field of n volts per cm. these ratios are for the planet Mercury of order

$$1, \quad 2 \cdot 10^{-30}n^2, \quad 2 \cdot 10^{-22}n^2,$$

showing how absolutely negligible the electric apsidal effects are for matter in bulk in comparison with the direct gravitational, and also how vastly the spatial effect of an electrified Sun on an electrically neutral planet exceeds that of its direct electric attraction.

License in choice of physical units.

This occasion is not inopportune to consider how the present irritating uncertainty of the units adopted in abstract mathematical writings relating to physical science might be mitigated. For purposes of practical physics and engineering there is only one recognised set of units, the original electromagnetic c. g. s. system with its practical multiples, volts, ohms, coulombs, amperes and so on. In abstract algebraic analysis it is now usual to gain symmetry as regards the velocity factor c by a mixed electrostatic magnetostatic system, instead of that of Maxwell's *Treatise*. It would be less confusing to choose a provisional unit of time so that c becomes unity and so disappears altogether : the final results could then be translated into the practical unitary system by inserting the power of c in each term that is required to give it the proper dimensions. Again, some writers are accustomed, following Heaviside, to absorb the frequent factor 4π by also altering the units of electric and magnetic quantity; so that there is added uncertainty in the final result as to whether this has been done or not. We are now liable in writings on astronomical relativity to be abruptly confronted with the expression of the mass of the sun in kilometres, involving yet another new orientation. The result of such unitary license is to make it impossible, without an exploration which consumes time, for a person who has ceased to be a specialist to pass with security from one writer's work to that of another. This promotes impatience and discouragement, and intensifies that narrow insulated specialism which it is in the interest of all to circumvent as far as possible. It is a slight matter, but one in which international settlement such as was so important for technical electrical science seems almost called for.

Nature of radial and other spaces.

If the function $F(r)$ in the expression above for a radial space is essentially positive, that is if e/m merely exceeds $(2\gamma)^{\frac{1}{2}}$ or $12^{\frac{1}{2}} . 10^{-1}$, or a rather different value for the isotropic scheme whereas it is about $10^4 c$ for an α particle, the singular region near the origin can shrink to a point. Otherwise there would be also two spherical surface sheets of singularity, of extremely minute radius : and in the space inside the force may be repulsion from the centre, while in the different universe outside the force is attraction.

It is recalled above that in the differential theory for radial space the function of r that represents the distance is still arbitrary, even when the centre is an electric pole in addition to being a gravific one. But all essential indetermination disappears if the space is to be specified by integrated coordinates which are everywhere differentially isotropic(1). In general may we assert that a space-time is determined throughout if we know its poles, including their strengths as regards charge and essential mass, and assuming radial symmetry, by the condition that the coordinates shall be everywhere locally isotropic? To the free mobility of Euclidean space this latter condition is of course essential.

As already noted, it is hardly suitable to assign a gravific field as a concomitant to electric manifestations. An electric source modifies radially the free space around it, as does a gravific source : but the two changes follow different specific laws, the electric being much more local and usually much smaller.

In one sense this radial space is not self-determined by its central pole alone :

(1) When r is thus transformed to R, chosen as that function of r which renders δs^2 isotropic, the nature of the singular surfaces near the centre will be changed.

the invariant properties which restricted it belonged to space in general. Somewhat analogously an abstract doctrine is often maintained that an isolated particle existing alone could have no field and no mass.

In these formulae the pole has been reduced to rest. But the dynamical tensor belongs to an imaginary material framework in which all the moving poles are attached : this frame, being referred to moving coordinates, therefore involves products of types $\delta x\, \delta t$ in δs^2, and the field near a moving pole is obtained by referring the formula above to coordinates having relative motion. The synthesis of Einstein makes it clear that the local tensor is interpretable in terms of mass and its momentum and energy : but a direct verification from the extremal variation for the radial field (1) would be concise and suitable.

The essential quality of the fourfold pseudo-space has been exhibited without use of imaginaries by Lorentz, by a beautiful variation from the method of Dupin, who invented his indicatrix as descriptive of the neighbourhood of a point of a surface. An ordinary continuous surface is made up of flat elements of area; but this description is not adequate, for it does not bring out the coherence. The feature of curvature must be introduced, and that was exhibited graphically in its local relations by Dupin by utilising a section of the surface parallel and adjacent to the tangent plane at the point : it is in the limit an ellipse or hyperbola whose radii determine the curvatures of the surface in the normal planes. If indicatrices are drawn all over the surface their relations determine the intrinsic character of the curvature. If it could be arranged so that some relations between indicatrices were not altered, by bending of the sheet without changes of length such as stretching would involve, they would express the intrinsic nature of the surface in that respect. It is necessary for this purpose to replace the plane indicatrix of Dupin by the indicatrix of Lorentz, which is a geodesic circle around the point as centre, so infinitesimal that it is adequately expressed by the equation $\int \delta s =$ constant, pushed to the second order. Then the texture of their intersections defines everywhere the intrinsic local continuity of a Gaussian inextensible sheet. In higher dimensions these indicatrices become intersecting hypersurfaces of the second order: and from the permanent geometry of their configuration Lorentz has evolved in more direct intuitive manner the equations of the Einstein theory.

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(1) This seems however to be provided generally by the idea of p. 24.

Special varieties of space-time extension.

The space-time that represents a field of free gravitation, without radiation and resulting electric field, is that of Riemann as controlled by invariance of the quadratic expression

$$\delta s^2 = \Sigma g_{rs} \delta x_r \delta x_s$$

in which the coefficients g_{rs} are implicit functions of some scheme of coordinates or spatial parameters x_r *in that space*, but now restricted in a sixfold manner by the requirements of extremal variation, p. 12. Various cases arise for special investigation, but all apparently very exceptional.

(i) As regards pure gravific space there may be radial symmetry around a centre: the resulting limitations on the potentials have been determined by Schwarzschild and by Dröste : if the integrated spatial coordinates are restricted to be everywhere locally isotropic the expression for δs^2 determining the radial space is unique. A space-time isotropically specified would, if it can exist, thus be determined by its poles, or if limited in extent, by its poles and data over the boundary. Its existence is implied in the approximate form of the gravitation theory.

(ii) The pure gravific space may be isotropic in its coordinates but not also radially symmetrical : then

$$\delta s^2 = P^2(\delta x^2 + \delta y^2 + \delta z^2) - Q^2 \delta t^2,$$

and the differential equations limiting the forms of P and Q may be determined by a similar analysis. It seems that it is only in exceptional cases that this form is possible, the radial field being one of them.

(iii) Or the space may be merely referred to a rectangular network expressible algebraically so that

$$\delta s^2 = g_1 \delta x_1^2 + g_2 \delta x_2^2 + g_3 \delta x_3^2 - g_4 \delta x_4^2,$$

and the inherent limitation similarly obtained.

The extreme limitation of all these types must have been recognized early by Einstein.

But when a free electric field is present the extension is no longer subject to sixfold restriction; though it appears that its single Riemann curvature invariant is restricted to be everywhere zero. This is illustrated (*supra*) by the form for δs^2 around an electric pole as found by Nordström, with its different type of singularity at the pole. Inside a distribution of matter or electricity or both, taken as of uniform *continuous density*, the *fourfold extension is of unrestricted Riemann type* : even its single curvature invariant does not vanish.

Whatever be the type to which a space and its coordinates are restricted, the method of variation remains effective to determine immediately the extremal relation and its form of stress-energy-momentum tensor, which the set of coordinates as constrained to this type requires : so much so that with the method at hand it is almost otiose to select and discriminate among possible forms of tensors. But when the space is limited in advance as above by restricting its potentials, the variation can only arrive as the conditions of invariance for those that remain, that is, for a restricted variation of the nature of the space. Though the equations it leads to will be correct, as in Hilbert's deduction for radial space, other relations may have to be verified before we can know that the hypothesis of a space thus limited is legitimate : for instance, the process of variation now provides a tensor, invariant only for the special group of frames that can be expressed in terms of the restricted number of variables. The determination of the expression for the invariant gravific part of the Action is the crucial operation.

An electric field also must be spatially interpreted if a gravific field is spatial.

We have concluded that the imposition of an electric field breaks up and dissolves the extremal space into which the gravitational effects have been absorbed. The problem thus seems imperative, to specify directly what wider type of spatial continuity then takes its place. In one sense it is defined by extremal variation of the Action as now altered by introduction of an electric vector : but if that vector cannot be associated with space it must be connected with an aether. A most interesting account of it seems to be provided on the lines of the theory developed by H. Weyl, which apparently replaces δs^2 by a wider differential invariant

$$\delta s^2\left(1 + 2\frac{\delta l}{l}\right) \quad \text{where} \quad \frac{\delta l}{l} = F_1\delta x_1 + F_2\delta x_2 + F_3\delta x_3 + F_4\delta x_4,$$

when it turns out that the vector F thus introduced, and partaking in the extremal variation, plays the part of an electric vector potential. The mode of expression of the invariant here adopted implies that the mobile differential isotropic Riemann measuring rule utilized by Einstein must alter its length as it is transferred to new positions, in the differential ratio $\delta l/l$. When it is transferred along a closed circuit back to its starting point its length will thus have returned to a different value unless the expression for $\delta l/l$ is an exact differential. But that is the condition that the electric field shall vanish : the vector potential F could then only be described as (like a residual constant latent momentum in dynamics) possibly a permanent residue left by an electric field after the manner of the 'electrotonic state' of Faraday. This failure of determination as regards the measuring rod has naturally been regarded as a serious obstacle to the reception of such a theory. But on the other hand why may not the local fundamental invariant δs^2 be replaced by a modified one, worked out on parallel lines on its own foundation? The real inconsistency would seem rather to be the insistence on retaining the idea of a *permanent transferable* material scale of intervals of extension in length and time, which is something imposed from outside and out of touch with a theory that develops differentially and so internally(¹). It may be held to be a surviving fragment of the mobile rigid body which is the means of finite measurement in uniform space, but which cannot remain when the space is not uniform, except approximately in a differential domain throughout which uniformity can be assumed.

The controlling local invariant, if thus compounded of the quadratic form δs^2 and the vector F, would be of a special cubic form. From it a spatial synthesis of gravific and electric field relations ought perhaps to be developable by extremal analysis, in which the vector F would now of necessity take part. Here again the permanent intrinsic things would be the gravific and electric poles. The great desideratum, if it be feasible, is a theory of the relation of the field of modified space-time to its sources, following out the methods of Green for the simple case of static Euclidean fields of force, which have already admitted of extension to propagated Euclidean fields of radiation as above indicated, p. 19.

Theory of a binary medium.

As regards the twofold aspect of the subject, propagation in the aether or in quivering space, and mechanical forces sustained by the material poles interspersed in and determining this field of activity, we appear to have gained in comprehension by an enlarged representation. We can adopt any frame of reference at our choice, to which the matter is attached, expressed by coordinates in the space by whose instantaneous values (²) the space also is itself expressed. The property of extremal

(¹) If it be said that the atoms say of hydrogen supply just such mobile scales of length and time throughout the universe, the answer is that they are only approximate in a varying fourfold. See also p. 39.

(²) Compare the Eulerian analytical specification of the motion of a solid body, by reference to coordinates of a frame which is itself in motion along with the body : this likewise is a specification differential as regards time, from which the integrated change of position is to be determined by continuation.

variation of a suitable invariant integral of Action, with regard to the local potentials which specify the gravific and electric fields, then leads to the equations governing and conditioning those interacting fields, as referred to that frame. One special type of variation of the potentials is that arising locally from a Lagrangian virtual displacement of the frame of reference, that is, due to variation of its coordinates, but so as to preserve the region of integration unvaried : for this type the result must be extremal, because it is so for the general variation which includes it. The exhibition of an extremal result for this variation by displacement of coordinates involves immediately the stress-tensor belonging to the coordinate network that had been chosen, whether it be rectangular or curvilinear. It is a self-equilibrated tensor and suitable for transmission of influence, in the sense that its divergence everywhere vanishes, because it is only the gradients of the potentials and not their absolute values that occur in the Action : but this quality breaks down, requires to be supplemented for a complete account, at the singular points or mass-centres at the region. These poles are to be cut out of the region in which the activities are propagated, by any kind of barrier surface closely surrounding them, in Green's manner; and their influence on the region, and its reaction on the poles, are replaced by that of the tensors acting over the barrier surfaces. When the barrier surfaces are taken to be of infinitesimal extent, the integrated effect of the tensors over one of them should in the limit show (as on p. 24) a result interpretable in terms of mass and charge of the included pole, and its momentum and the forces acting on it, in the manner of the tensor adapted by Einstein from gas-theory.

On the elastic-aether theory space is occupied by a medium, and it is the disturbance of the medium, expressed locally by strain involving displacement gradients, which is the variable in the function representing the distribution of Action : in the new mode of representation it is deformation of the space itself, by change of type, that is the foundation of the Action. The isotropically invariant local Lagrangian function of strain, possibly including differential rotation, which expresses the former Action per unit volume, is replaced in the latter specification of an Action by a function that is locally invariant for any assigned type of space, as determined in the Riemann manner by conservation of its group of differential distance relations; and variation of this type of spatial scheme now takes the place of variation of Action in terms of strain. In the former order of ideas, variation of the electric potentials would be taken to mean variation of internal locally balanced latent structure, of which they somehow represent in perhaps unknown way the reduced compendious influence in the local part of the Action, exerted in modes of which hydrodynamic and other illustrations are available. In the latter order of thought there is a heterogeneous ordering of the array of points which impose marked locality on mere spatial extension, and a coordinate network locating a geography in that arrangement; this double ordering is redundant, more than is necessary. The local density of Action in this specification must be some function of the potentials specifying this enlarged type of space, and their gradients with regard to the coordinates, which has to be locally invariant for change of coordinates : naturally it will not involve the absolute local values of those coordinates explicitly, or the absolute local values of the potentials, but only relative gradients of the latter. The redundancy of the data implies in advance identities in the development, which are the feature af the analytical tensor theory.

The principle of an extensional distribution of an Action function, whose integral is postulated to be of extremal variation for every part of the region, poles and other centres of activity being excepted and cut out as above, is the only known type of concatenation assigned by purely local or differential specification, which permits of the deduction of general relations between parts of the medium separated by finite distance(1). A scheme of permanent *relations at a distance* of this kind can alone constitute the direct systematic knowledge, such as it is open to us to acquire, of the course of nature. The abstract Action formulation of Lagrange-Hamilton runs parallel perhaps to the way in which such large-scale knowledge of group-relations becomes constituted for us, in a complex world which we cannot comprehend or even recognise in its atomic detail. The Principle of Extremal Action would thus be the fundamental formula of reversible physical science.

The problem brought into the foreground by Einstein, with such brilliant results in anticipating gravitation, is whether a formulation depending directly on changing concatenations of isotropic quadratic distances in an ultimate array of points is more fundamental, — not more correct but closer to nature, — than one depending on local change of form in a Euclidean scheme of space. It may be said that it is the circumstance that the Lagrangian function T — W, in this case the form of δs^2, is not essentially positive that renders such a question possible *:* for it perhaps saves the physical world from the stark rigidity of complete interdetermination in space and time.

There is the other prior question perhaps more profound, why the fourfold space-time continuum of Minkowski is closer to nature, at any rate as regards each differential element of extension, than a Euclidean space combined with the Newtonian time, although it is far more remote from our means of mental comprehension. Some attempt at an answer has here been offered. It appears that there can hardly be definite radiation, such as disentangles itself radially from the local dynamical disturbance around atomic centres, and is propagated ever after to celestial distances, however great, compactly without change of form or dissipation, unless the universe is organised into space of three dimensions, leaving out the special circumstances

(1) The construction of an invariant tensor implies transmission of stress to a distance, but not also the reciprocal relations that are involved in an extremal distribution of Action.

of only one. It also appears immediately that the vast complex of radiant interchanges, which largely, perhaps completely, determine interrelations in the universe, maintains invariant relations to all frames of reference for space and time that are included in the fourfold group for which $\delta x^2 + \delta y^2 + \delta z^2 - \delta t^2$ is invariant, provided the sources and reflectors of the radiation are compelled to come into invariant relation to this same group of frames instead of the rigid-body group of separated space and time. It is further held that if the energies of dynamic transmission are *regarded as* all resident in the radiation and the spatial or aethereal continuum which carries it, so that the mass-centres and electric centres can be regarded as only passive local intrinsic constraints keying up and thereby determining permanent fields around them, these centres will be responsive freely to the surrounding radiation and no compulsion would be required to adjust their movements into relation with it(1). We are asked then to recognise that in the rare cases in which we can practically distinguish, this formulation in terms of invariant fields of radiant transmission is more flexible and more closely adapted to a view of nature as an interaction of discrete atoms than the familiar formulation in terms of solid bodies maintaining absolutely invariant spatial relations of form. There should be little difficulty about admitting this alternative, if necessary as it appears to be, seeing that in either case the question is as to the construct of a scheme adapted to exhibit the larger relations of an atomic constitution of our universe which is in its particulars beyond detailed investigation.

A criterion for an active æther.

This mode of exposition of the world-history can thus be held to imply a medium of communication, but one too different from our ideas of matter even to be readily expressible in the usual forms with which we clothe space and time. It is at variance with one of the three classical Einstein predictions, the definite displacement of the lines in a solar or stellar spectrum on account of gravitation. According to the strict relativity, an *extramundane* atom, *imagined* to be introduced extraneously into this self-adjusted world-history, would have to carry about with it absolute periods of time of vibration, for it is a self-contained system and the modified space and time around it is a mere local form. Such absolute time has to be identified with the $c'\delta t$ in this invariant expression δs^2, for there is no other way available to introduce it into the theory. But as thus defined, it is *not even integrable*

(1) The argument is set out in *Aether and Matter*, 1900, p. 176. Cf. also International Mathematical Congress, Cambridge, 1912, introduction to lecture dealing with radiant processes as the universal mode of electrical and general physical adjustments.

as regards a moving atom; so it could not be the basis of an independent dynamics of the atomic structure, at any rate unless x, y, z remain constant which seems to locate absolute position. Even then, an extraneous atom moving across from the Earth to the Sun would have no time of its own in which to subsist. Moreover there could be no assurance that the value of the gravitation potential at a distance, thus given directly by spectral observations, would be concordant with the value determined by astronomical methods of continuation. It thus seems difficult to justify the imposition, from without, of an absolute integral scale of time on a world-history that is differentially constituted : its transfer from place to place by an atom may be described as an extramundane fiction. If observations ultimately lead to its acceptance, it will still have to be proved that it is consistent with a self-determination of the universe. Cf. also *Proc. Roy. Soc.*, nov. 1919.

By taking full advantage of the importation of the extremal variational dynamics of Hamilton, practical relativity has here been defined as the establishment of relations between bodies at a distance by continuous interconnexion across intervening space. This scheme works out, after Einstein, into a complete self-consistent theory, before any assertion is made of the identity of all atoms of the same substance. If that identity is to be interpreted absolutely, so that surrounding modified space-time is merely formal and does not count, it gives another universal scale of space and time, in fact, relativity mitigated by absolute knowledge. If these two scales agree, the fact should be capable of the necessary proof : if they are found by spectral observations not to agree, it shows that the modified spatial surroundings of the atom are not a mere formality, that deformation of an active æther is involved, that it must not be transformed *locally* into normal space-coordinates unless the transformation extends outward without limit. The existence of identical atoms, making themselves known by radiation to a distance, is thus held to be something extraneous and absolute, and possibly inconsistent with the extreme relativity that denies an æther in any form : if experiment were to decide otherwise, in accordance with the asserted prediction, it would remain for theory to establish the necessary reconciliation in order to complete itself.