Chapter 1

Vorticity and Potential Vorticity

In this chapter we use potential vorticity to unify a number of the concepts we have previously encountered. We derive the general form of potential vorticity conservation, and then appropriate approximate forms of the potential vorticity conservation law, corresponding to the familiar quasi-geostrophic and planetary geostrophic equations. In order to keep the chapter essentially self-contained, there is some repetition of material elsewhere.

1.1 Vorticity

1.1.1 Preliminaries

Vorticity is defined to be the curl of velocity, and normally denoted by the symbol $\boldsymbol{\omega}$. Thus

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} \tag{1.1}$$

Circulation is defined to be the integral of velocity around a closed loop. That is

$$C = \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.2}$$

Using Stokes' theorem circulation is also given by

$$C = \int \boldsymbol{\omega} \cdot d\boldsymbol{S} \tag{1.3}$$

The circulation thus may also be defined as the integral of vorticity over a material surface. The circulation is not a field like vorticity and velocity. Rather, we think of the circulation around a particular material line of finite length, and so its value generally depends on the path chosen. If δS is an infinitesimal surface element whose normal points in the direction of the unit vector $\hat{\boldsymbol{n}}$, then

$$\boldsymbol{n} \cdot \nabla \times \boldsymbol{v} = \frac{1}{\delta S} \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} \tag{1.4}$$

where the line integral is around the infinitesimal area. Thus the vorticity at a point is proportional to the circulation around the surrounding infinitesimal fluid element, divided by the elemental area. (Sometimes the curl is *defined* by an equation similar to (1.4).) A heuristic test for the presence of vorticity is then to imagine a small paddle-wheel in the flow: the paddle wheel acts as a 'circulation-meter' and so rotates if vorticity is non-zero.

1.1.2 Simple axisymmetric examples

Consider axi-symmetric motion, in two dimensions. That is, the flow is confined to a plane. We use cylindrical co-ordinates (r, ϕ, z) where z is the direction perpendicular to the plane. Then

$$u_z = u_r = 0 \tag{1.5}$$

$$u_\phi \neq 0 \tag{1.6}$$

(a) Rigid Body Motion

The velocity distribution is given by

$$u_\phi = \Omega r \tag{1.7}$$

Explicitly,

$$\begin{aligned} \boldsymbol{\omega} &= \nabla \times \boldsymbol{v} = \omega_z \cdot \mathbf{k} \\ \omega_z &= \frac{1}{r}\frac{\partial}{\partial r}(r u_\phi) = \frac{1}{r}\frac{\partial}{\partial r}(r^2\Omega) \\ &= 2\Omega \end{aligned} \tag{1.8}$$

That is, the vorticity of a fluid in solid body rotation is twice the angular velocity of the fluid.

(b) The 'vr' vortex

This vortex is so-called because the tangential velocity distribution is such that the product $v\,r$ is constant. That is:

$$u_\phi = \frac{K}{r} \tag{1.9}$$

where K is a constant determining the vortex strength. Evaluating the z–component of vorticity gives

$$w_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\phi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{K}{r}\right) = 0 \tag{1.10}$$

except when $r = 0$, when the expression is singular and the vorticity is infinite. Obviously the paddle wheel rotates when placed at the vortex center.

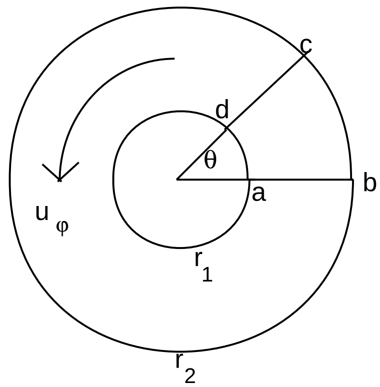

Figure 1.1: Evaluation of circulation in the axi-symmetric vr vortex. The circulation around path $a-b-c-d-a$ is zero. This result does not depend on the size of the path; thus the circulation around any infinitesimal path not enclosing the origin is zero, and thus the vorticity is zero everywhere except at the origin.

We can obtain the result another way by calculating the circulation $\oint \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}$ around an appropriate contour, the contour a-b-c-d-a in fig. 1.1. Over the segments a-b and c-d the velocity is orthogonal to the contour, and so the contribution is zero. Over b-c we have

$$C_{bc} = \frac{K}{r_2}\theta r_2 = K\theta \tag{1.11}$$

and over d-a we have

$$C_{da} = -\frac{K}{r_1}\theta r_1 = -K\theta \tag{1.12}$$

Thus the net circulation $C_{bc} + C_{da}$ is zero. The result is independent of r_1 and r_2 and θ, so we may shrink the contour to an infinitesimal size so that it encloses an infinitesimal surface element. On this, by Stokes' theorem, the vorticity is zero. Thus the vorticity is everywhere zero, except at the origin, and circulation around *any* closed contour not enclosing the origin is zero.

1.2 The Vorticity Equation

Recall that the momentum equation may be written

$$\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{v}\times\boldsymbol{\omega} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{v} \tag{1.13}$$

Taking its curl gives the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla\times(\boldsymbol{v}\times\boldsymbol{\omega}) = \frac{1}{\rho^2}(\nabla\rho\times\nabla p) + \nu\nabla^2\boldsymbol{\omega} \tag{1.14}$$

(For the rest of the chapter we will neglect viscosity.) If the density is constant, or if the density is a function only of pressure (a 'homentropic' fluid) the right-hand-side vanishes, for then

$$\frac{1}{\rho^2}(\nabla\rho \times \nabla p) = \frac{1}{\rho^2}\nabla\rho \times \nabla\rho\,\frac{dp}{d\rho} = 0 \tag{1.15}$$

The second term on the right hand side of (1.13) may be written

$$\begin{aligned} \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) &= (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} \\ &\quad -\boldsymbol{\omega}\nabla\cdot\boldsymbol{v} + \boldsymbol{v}\nabla\cdot\boldsymbol{\omega} \end{aligned} \tag{1.16}$$

As vorticity is the curl of velocity its divergence vanishes, whence we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} - \boldsymbol{\omega}\nabla\cdot\boldsymbol{v} + \frac{1}{\rho^2}(\nabla\rho\times\nabla p) \tag{1.17}$$

The last term on the right-hand-side is sometimes called the *baroclinic* term, but we shall call it the *non-homentropic* term. The divergence term may be eliminated with the aid of the mass-conservation equation

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{v} = 0 \tag{1.18}$$

to give

$$\boxed{\frac{D\tilde{\boldsymbol{\omega}}}{Dt} = (\tilde{\boldsymbol{\omega}}\cdot\nabla)\boldsymbol{v} + \frac{1}{\rho^3}(\nabla\rho\times\nabla p)}. \tag{1.19}$$

where $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}/\rho$. (There is no commonly-used name for this quantity.) For a homentropic fluid $p = p(\rho)$, $\nabla p \times \nabla\rho$ vanishes and we have the simple and elegant form,

$$\frac{D}{Dt}\tilde{\boldsymbol{\omega}} = (\tilde{\boldsymbol{\omega}}\cdot\nabla)\boldsymbol{v}. \tag{1.20}$$

If the fluid is also incompressible, then $\nabla\cdot\boldsymbol{v} = 0$ and, directly from (1.17), the vorticity equation is,

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v}. \tag{1.21}$$

The terms on the right-hand-side of (1.20) or (1.21) are conventionally divided into 'stretching' and 'twisting'. Consider a single Cartesian component of (1.21),

$$\frac{D\omega_x}{Dt} = \omega_x\frac{\partial u}{\partial x} + \omega_y\frac{\partial u}{\partial y} + \omega_z\frac{\partial u}{\partial z}. \tag{1.22}$$

The first term on the right-hand-side acts to intensify the x-component of vorticity if the velocity is increasing in the x-direction — that is, if the fluid is being 'stretched.' In order that mass be conserved, stretching in one direction corresponds to contracting in one or both of the other directions, and, because the conservation of angular momentum demands faster rotation as the moment of inertia falls, the vorticity increases. The second and third terms involve the other components of vorticity and are called twisting terms: vorticity in the x-direction is being generated from vorticity in the y- and z-directions. We return to an informative topological interpretation of vorticity evolution shortly.

The results derived above apply to the absolute vorticity, or the vorticity measured in an inertial reference frame. Since $\boldsymbol{\omega}_a = \boldsymbol{\omega}_r + 2\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}$ is a constant, the vorticity equation in a rotating frame of reference is

$$\frac{D\boldsymbol{\omega}_r}{Dt} = [(\boldsymbol{\omega}_r + 2\boldsymbol{\Omega})\cdot\nabla]\boldsymbol{v} - (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\nabla\cdot\boldsymbol{v} + \frac{1}{\rho^2}(\nabla\rho\times\nabla p) \tag{1.23}$$

or

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}_r}{\rho}\right) = \left(\frac{1}{\rho}(\boldsymbol{\omega}_r + 2\boldsymbol{\Omega})\cdot\nabla\right)\boldsymbol{v} + \frac{1}{\rho^3}(\nabla\rho\times\nabla p) \tag{1.24}$$

1.2.1 Two-dimensional flow

In two-dimensional flow the velocity field is confined to a plane. The velocity normal to the plane, and the rate of change of any quantity normal to that plane, are zero. Let this be the z-direction. Then the velocity, denoted by $\boldsymbol{u}$, is

$$\boldsymbol{v} = \boldsymbol{u} = u\mathbf{i} + v\mathbf{j}, \qquad w = 0. \tag{1.25}$$

Only one component of vorticity non-zero; this is given by

$$\boldsymbol{\omega} = \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{1.26}$$

We let $\eta = \omega_z = \boldsymbol{\omega}\cdot\mathbf{k}$ Both the stretching and twisting terms vanish in two-dimensional turbulence, and the two-dimensional vorticity equation becomes, for incompressible homentropic flow,

$$\frac{D\zeta}{Dt} = 0, \tag{1.27}$$

where $D\zeta/Dt = \partial\zeta/\partial t + \boldsymbol{u}\cdot\nabla\zeta$. Thus, in two-dimensional flow vorticity is conserved following the fluid elements; each material parcel of fluid keeps its value of vorticity even as it is being advected around. Furthermore, specification of the vorticity completely determines the flow field. To see this, we use the incompressibility condition to define a streamfunction ψ such that

$$u = -\frac{\partial\psi}{\partial x}, \qquad v = \frac{\partial\psi}{\partial y}, \tag{1.28}$$

and

$$\zeta = \nabla^2\psi. \tag{1.29}$$

Given the vorticity, the Poisson equation (1.29) is solved for the streamfunction, and the velocity fields obtained through (1.28).

Similarly, numerical integration of (1.26) is a process of time-stepping plus 'inversion.' The vorticity equation may then be written entirely in terms of the streamfunction

$$\frac{\partial \zeta}{\partial t} + J(\psi, \nabla^2 \psi) = 0 \tag{1.30}$$

plus (1.29). The vorticity is stepped forward one time-step using a finite-difference representation of (1.30), and the vorticity 'inverted' using (1.29) and (1.28). The notion that complete or nearly complete information about the flow may be obtained by 'inverting' one field plays a very important role in geophysical fluid dynamics, both in the numerical solution and in diagnostic analysis, as we see in later sections and chapters.

Finally, we note that in the presence of rotation the vorticity equation becomes

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u} \cdot \nabla(\zeta + f) = 0 \tag{1.31}$$

where $f = 2\boldsymbol{\Omega} \cdot \mathbf{k}$. If f is a constant, then (1.31) reduces to (1.30), and background rotation plays no role.

1.3 Kelvin's Circulation Theorem

Kelvin's circulation theorem states that under certain circumstances the circulation around a material fluid parcel is conserved, or the circulation is conserved 'following the flow.' That is

$$\boxed{\frac{D}{Dt} \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} = 0}. \tag{1.32}$$

The primary restrictions are that the flow be inviscid, that body forces be representable as potential forces, and that the flow be homentropic. Of these, the last is the most important for geophysical fluids. The circulation is defined with respect to the inertial frame of reference. That is, the velocity in (1.32) is the velocity relative to the inertial frame; the material derivative D/Dt is reference-frame indifferent. We will give a straightforward proof, beginning with the inviscid momentum equation:

$$\frac{D\boldsymbol{v}}{Dt} = \frac{1}{\rho} \nabla p - \nabla \phi, \tag{1.33}$$

where $\nabla \phi$ represents the body forces on the system. Applying the material derivative in (1.32) through the integral gives

$$\begin{aligned} \frac{D}{Dt} \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} &= \oint \frac{D\boldsymbol{v}}{Dt} \cdot \mathrm{d}\boldsymbol{l} + \boldsymbol{v} \cdot \frac{D\,\mathrm{d}\boldsymbol{l}}{Dt} \\ &= \oint \left[\left(-\frac{1}{\rho} \nabla p + \nabla \phi \right) \cdot \mathrm{d}\boldsymbol{l} + \boldsymbol{v} \cdot d\boldsymbol{v} \right] \end{aligned} \tag{1.34}$$

using the momentum equation and $D(\delta \mathbf{l})/Dt = \delta \boldsymbol{v}$ (see next section). The second and third terms vanish separately, because they are integrals around a closed loop. The first term vanishes for a homentropic fluid; for such a fluid we may define a function G such that $dG/dp \equiv 1/\rho$ and then

$$\begin{aligned} \oint \frac{1}{\rho}\nabla p \cdot \mathrm{d}\boldsymbol{l} &= \oint \frac{dp}{\rho} \\ &= \oint \frac{dG}{dp} dp \\ &= \oint dG = 0. \end{aligned} \tag{1.35}$$

Using Stokes' theorem, the circulation theorem may be written

$$\frac{D}{Dt}\int \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S} = 0 \tag{1.36}$$

That is, the flux of vorticity through a material surface is constant. In some ways this is the more natural form of Kelvin's circulation theorem because it is really a consequence of the topological properties of vorticity.

In a rotating frame of reference, the appropriate forms of the circulation theorem are

$$\frac{D}{Dt}\oint (\boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{l} = 0, \tag{1.37}$$

and

$$\frac{D}{Dt}\int (\boldsymbol{\omega}_r + 2\boldsymbol{\Omega}) \cdot \mathrm{d}\mathbf{S} = 0. \tag{1.38}$$

where v_r and $\boldsymbol{\omega}_r$ are the velocity and relative vorticity as measured in the rotating frame.

In non-homentropic flow, the circulation is not generally conserved. However, it *is* conserved if the material path is in a surface of constant entropy, s, and if $Ds/Dt = 0$. Since the equation of state is of the general form $p = p(\rho, s)$, then if $s =$ constant around the loop the density is a function only of pressure and the integral in (1.35) vanishes. Further, if $Ds/Dt = 0$, entropy remains constant on that same material loop as it evolves and so circulation is preserved.

1.3.1 The 'frozen-in' property of vorticity

At a slightly deeper level Kelvin's circulation can be seen as a consequence of the topological character of the vorticity field, and in particular that vortex lines are tied to material lines. A *vortex-line* is a line drawn through the fluid which is everywhere in the direction of the local vorticity. This definition is analogous to that of a streamline, which is everywhere in the direction of the local velocity. A *material-line* simply connects material fluid elements.

We can draw a vortex line through the fluid. Such a line obviously connect fluid elements and therefore we can define a co-incident material lines. As the fluid moves the material lines deform,

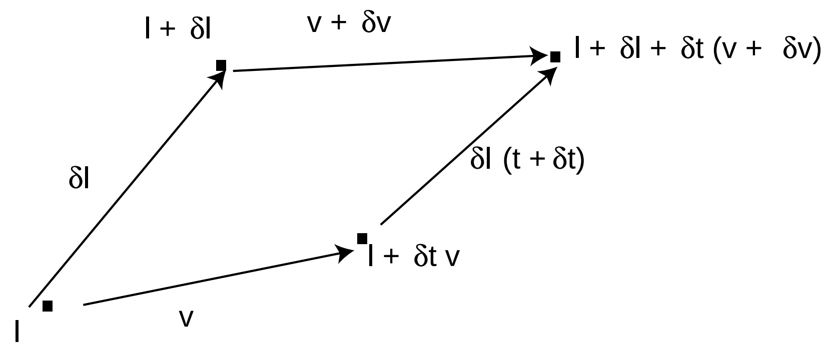

Figure 1.2: Evolution of an infinitesimal material line $\delta\mathbf{l}$ from time t to time $t+\delta t$. It follows from the diagram that $D\,\delta\mathbf{l}\,/Dt = \delta\boldsymbol{v}$. See text for details.

and the vortex lines evolve in a manner determined by the equations of motion. The remarkable property of vorticity is that at later times the vortex lines remain co-incident with the same material lines that they were initially associated. Put another way, a vortex line always contains the same material elements — the vorticity is 'frozen' to the material fluid. Consider how an infinitesimal material line element $\delta\mathbf{l}$ evolves, $\delta\mathbf{l}$ being the infinitesimal material element connecting $\mathbf{l}$ with $\boldsymbol{l}+\delta\mathbf{l}$. The rate of change of $\delta\mathbf{l}$ following the flow is given by

$$\frac{D}{Dt}\delta\mathbf{l} = \frac{1}{\delta t}\big(\delta\mathbf{l}(t+\delta t) - \delta\mathbf{l}(t)\big), \tag{1.39}$$

which follows from the definition of the material derivative in the limit $\delta t \to 0$.

From fig. 1.2 it is apparent that

$$\begin{aligned}\delta\mathbf{l}(t+\delta t) &= \boldsymbol{l} + \delta\mathbf{l} + (\boldsymbol{v}+\delta\boldsymbol{v})\delta t - (\boldsymbol{l}+\boldsymbol{v}\delta t)\\ &= \delta\mathbf{l} + \delta\boldsymbol{v}\delta t\end{aligned} \tag{1.40}$$

Substituting into (1.39) gives

$$\frac{D}{Dt}\delta\mathbf{l} = \delta\boldsymbol{v} \tag{1.41}$$

But since $\delta\boldsymbol{v} = (\delta\mathbf{l}\cdot\nabla)\boldsymbol{v}$ we have that

$$\frac{D}{Dt}\delta\mathbf{l} = (\delta\mathbf{l}\cdot\nabla)\boldsymbol{v} \tag{1.42}$$

Comparing this with (1.19), we see that vorticity evolves in the same way as a line element. To see what this means, at some initial time we can define an infinitesimal material line element parallel to the vorticity at that location, i.e.,

$$\delta\mathbf{l}(\boldsymbol{x},t) = A\boldsymbol{\omega}(\boldsymbol{x},t) \tag{1.43}$$

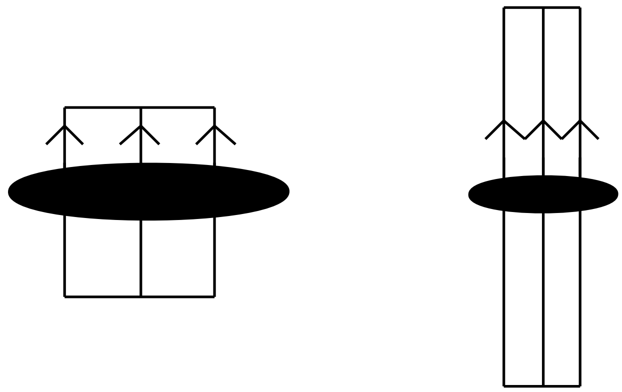

Figure 1.3: Stretching of material lines leads to vorticity amplification. However, because the volume of fluid is conserved, the material lines converge, the surface shrinks, and the integral of vorticity over a material surface (the circulation) remains constant.

where A is a (dimensional) constant. Then, for all subsequent times, the magnitude of the vorticity of that fluid element (wherever that particular element may be in the fluid) remains proportional to its length, and is in that direction, i.e.,

$$\boldsymbol{\omega}(\boldsymbol{x}', t') = A^{-1}\delta\mathbf{l}(\boldsymbol{x}', t'). \tag{1.44}$$

1.3.2 Implications for circulation.

The vorticity equation tells us that vorticity is enhanced if the velocity is in the direction of vorticity is increasing. We can now see why. If, say $\partial u/\partial x > 0$, then material lines in the x-direction are being stretched. Since the vorticity is amplified in proportion to the length of the physical line element, vorticity is amplified. (This mechanism is important in the dynamics of tornadoes.) Now consider a vortex tube (a collection of vortex lines) passing through a surface whose normal vector is parallel to the direction of vorticity (that is, the plane of the surface is orthogonal to the vorticity. See fig. 1.3). Let the volume of a small material box around the surface be δV, the length of the material lines be δl and the surface area be δS. Then

$$\delta V = \delta l \, \delta S \tag{1.45}$$

But the vorticity through the surface is proportional to the length of the material lines. That is $\omega \propto \delta l$, and

$$\delta V \propto \omega \delta S \tag{1.46}$$

The right hand side is just the circulation around the surface. If the corresponding material tube is stretched, then of course vorticity is amplified. But the volume of the material, δV, must remain constant if the fluid is incompressible (recall that the constancy of density implies the constancy of a material volume), and the circulation remains constant. Thus, although vorticity has been amplified by stretching, the vortex lines are closer together and the product $\omega\delta S$ remains constant and circulation is conserved.

We will verify the circulation theorem for incompressible, homentropic flow one other way, directly from the vorticity equation. At some point choose the z-direction to be parallel to the vorticity. Thus,

$$\boldsymbol{\omega} = (0, 0, \zeta) \tag{1.47}$$

Then the vorticity equation is

$$\frac{D\zeta}{Dt} = \zeta\frac{\partial w}{\partial z} \tag{1.48}$$

which, using the mass conservation equation, may be written

$$\frac{D\zeta}{Dt} = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = -\zeta\nabla_2 \cdot \boldsymbol{u} \tag{1.49}$$

where $\nabla_2 \cdot \boldsymbol{u} = \partial u/\partial x + \partial v/\partial y$ is the two-dimensional divergence. Multiply (1.48) by the area of the infinitesimal material surface element δS

$$\delta S\frac{D\zeta}{Dt} = -\delta S\zeta\nabla_2 \cdot \boldsymbol{u} \tag{1.50}$$

Now, the rate of change of the surface area is given by

$$\frac{D}{Dt}\delta S = \delta S\nabla_2 \cdot \boldsymbol{u} \tag{1.51}$$

(This is analogous to the result for rate of change of a volume element, $D\delta V/Dt = \delta V\nabla \cdot \boldsymbol{v}$.) Using (1.51) in (1.50) gives

$$\delta S\frac{D\zeta}{Dt} = -\eta\frac{D\delta S}{Dt} \tag{1.52}$$

or

$$\frac{D}{Dt}(\zeta\delta S) = 0 \tag{1.53}$$

verifying the circulation theorem.

1.4 * Vortex Stretching and Viscosity: a Simple Example

In this section we take a slight detour from our main theme and give a simple example of how vortex stretching and the associated amplification of vorticity can be balanced by dissipation (viscous effects) leading to a steady state. This kind of balance occurs in tornadoes and other similar natural phenomena, including the common-or-garden bath plug vortex.

1.4.1 Posing the Problem

We suppose that there is an axi-symmetric swirling flow, around some center, or core. The flow is converging toward the center, and to satisfy mass continuity it is accelerating upward at a constant rate. The azimuthal velocity is unspecified, except that we shall suppose that it is only a function of the radial co-ordinate. If the given rate of convergence is α, then the velocity field is in cylindrical co-ordinates (r, ϕ, z):

$$\boldsymbol{v} = (-\frac{1}{2}\alpha r, u_\phi, \alpha z) \tag{1.54}$$

The first term represents the radial converge, the second term the 'swirl,' and the third term the upward motion which will produce vortex stretching. The vortex stretching term will produce vortex amplification, and we may expect that only through the introduction of viscosity may be achieve a steady state. Our goal is to find the form of u_ϕ produced by a balance between vortex stretching and dissipation.

Mass Conservation

This velocity field is consistent in that it satisfies the constant density form of the mass conservation equation. Explicitly,

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(u_r r) \\ &= 0 + \alpha - \frac{1}{2r}\frac{\partial}{\partial r}(\alpha r^2) = 0 \end{aligned} \tag{1.55}$$

This holds no matter the form of the azimuthal field, provided it is a function only of r.

Vorticity

The vorticity is only non-zero in the vertical direction:

$$\begin{aligned} \omega_\phi &= \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) = 0 \\ \omega_r &= \left(\frac{1}{r}\frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z}\right) = 0 \\ \omega_z &= \left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\phi) - \frac{\partial u_r}{\partial \phi}\right) = \zeta(r) \end{aligned} \tag{1.56}$$

In words, the radial velocity depends only on the radial co-ordinate and the vertical velocity depends only on the vertical co-ordinate; therefore only the radial dependence of the azimuthal velocity produces a vorticity, this in the vertical direction.

1.4.2 The solution

The vertical component of the vorticity equation contains only the stretching term, and is,

$$\frac{D\zeta}{Dt} = \zeta\frac{\partial u_z}{\partial z} + \nu\nabla^2\zeta \tag{1.57}$$

or

$$\frac{\partial\zeta}{\partial t} + u_r\frac{\partial\zeta}{\partial r} + \frac{u_\phi}{r}\frac{\partial\zeta}{\partial\phi} + u_z\frac{\partial\zeta}{\partial z} = \zeta\frac{\partial u_z}{\partial z} + \nu\nabla^2\zeta \tag{1.58}$$

In the steady state and using (1.54) this simplifies to

$$-\frac{1}{2}\alpha r\frac{\partial\zeta}{\partial r} = \zeta\alpha + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\zeta}{\partial r}\right) \tag{1.59}$$

Combining the vortex stretching and advection terms, we find

$$-\frac{1}{2}\alpha\frac{\partial}{\partial r}(\zeta r^2) = \nu\frac{\partial}{\partial r}\left(r\frac{\partial\zeta}{\partial r}\right) \tag{1.60}$$

which integrates once to

$$-\frac{1}{2}\alpha\zeta r^2 = \nu r\frac{\partial\zeta}{\partial r} + C \tag{1.61}$$

To avoid a singularity at the origin, the constant C must be zero, whence

$$\zeta = \zeta_0\exp\left[-\frac{\alpha r^2}{4\nu}\right] \tag{1.62}$$

This is the steady state, radially symmetric, vorticity distribution in which amplification due to vortex stretching is balanced by viscous dissipation. Vorticity falls very quickly away from a rotational core whose thickness r_o, as determined by an e-folding scale, is given by

$$r_o = 2\left(\frac{\nu}{\alpha}\right)^{1/2} \tag{1.63}$$

As viscosity tends to zero, or as the intensity of the convergence increases, the flow becomes irrotational away from a delta function at the origin. Might this model apply to a tornado? Using a molecular viscosity, $\nu \sim 10^{-5}\,\mathrm{m^2\,s^{-1}}$, and supposing a convergence rate of $\alpha = 1\,\mathrm{cm\,s^{-1}}$ per meter, then we find $r_o \sim 3$ cm, which is rather small. Aside from uncertainties in the convergence rate, the main error here is supposing that vortex stretching is balanced by molecular diffusion of momentum. In fact, there are likely to be small scale turbulent motions which in some ill-defined way greatly enhance the effective viscosity and thicken the rotational core.

Velocity Field

Integrating the expression (1.56) for vorticity, we find the radial velocity field to be

$$u_\phi(r) = -\frac{1}{r}\frac{2\nu}{\alpha}\zeta_0 \exp\left[-\frac{\alpha r^2}{4\nu}\right] + \frac{A}{r} \tag{1.64}$$

where A is a constant of integration. Requiring $u_\phi = 0$ at the origin gives $A = 2\nu\zeta_0/\alpha$. Since the first term decays much faster than the second, for $r >> r_o$ (or for $\zeta_0 = 0$) the swirling velocity field goes as $1/r$, which is indeed irrotational.

Interestingly, the distributions (1.62) and (1.64) turn out to be the final steady solutions of the initial value problem for almost any initial vorticity distributions. The main restrictions are that $\zeta \to 0$ faster than $1/r^2$ as $r \to \infty$, and that the initial circulation is finite and non-zero (its value determining ζ_0). Thus, no matter how diffuse the initial vorticity, the process of convergence and the associated vortex stretching produce a very tightly bound vorticity distribution. The process of vortex stretching is very common. It is important not only in obvious cases of vortex intensification, like tornadoes, but is believed to be responsible for the cascade of energy to small scales in three-dimensional turbulent motion.

1.5 Potential Vorticity Conservation

Although Kelvin's circulation theorem is a general statement about vorticity conservation, it is not always very *useful* statement for two reasons. First, it is a not a statement about a *field*, such as vorticity itself. Second, it is not satisfied for non-homentropic flow, such as is found in the atmosphere and ocean. (Of course non-conservative forces and viscosity also lead to circulation non-conservation, but this applies to virtually all conservation laws and does not diminish them.) It turns out that it is possible to derive a beautiful conservation law that overcomes both of these failings and one, furthermore, that is extraordinarily useful in geophysical fluid dynamics. This is *potential vorticity.* The basic idea is that we can use a scalar field that is being advected by the flow to keep track of, or to take care of the evolution of fluid elements. For non-homentropic flow this scalar field must be chosen in a special way (it must be a function of the entropy alone), but the restriction to homentropic flow can then be avoided. Then using the scaler evolution equation in conjunction with the vorticity equation gives us a scalar conservation equation.

1.5.1 Homentropic flows.

(i) From Kelvin's theorem

for an infinitesimal volume we write Kelvin's theorem as:

$$\frac{D}{Dt}\left[(\boldsymbol{\omega}\cdot\boldsymbol{n})\delta A\right] = 0 \tag{1.65}$$

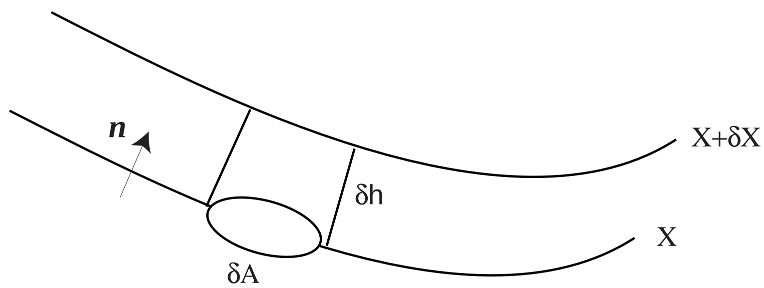

Figure 1.4: An infinitesimal fluid element, bounded by two isosurfaces of the conserved tracer χ.

where $\boldsymbol{n}$ is a unit vector normal to an infinitesimal surface δA (see fig. 1.4). The volume is bounded by two isosurfaces of values χ and $\chi + \delta\chi$, where χ is a conserved tracer satisfying

$$\frac{D\chi}{Dt} = 0. \tag{1.66}$$

Since $\boldsymbol{n} = \nabla\chi/|\nabla\chi|$ and the infinitesimal volume $\delta V = \delta h \delta A$ we have that

$$\boldsymbol{\omega} \cdot \boldsymbol{n}\delta A = \boldsymbol{\omega} \cdot \frac{\nabla\chi}{|\nabla\chi|}\frac{\delta V}{\delta h} \tag{1.67}$$

Since χ is conserved on material elements, and the mass $\rho\delta V$ of the volume element is also conserved, (1.65) becomes

$$\frac{\rho\delta V}{\delta\chi}\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla\chi\right) = 0 \tag{1.68}$$

or

$$\frac{D}{Dt}\left(\tilde{\boldsymbol{\omega}} \cdot \nabla\chi\right) = 0 \tag{1.69}$$

where $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}/\rho$.

(ii) From the frozen-in property

From (1.66), the difference between χ between two infinitesimally close fluid elements is conserved, that is

$$\frac{D}{Dt}(\chi_1 - \chi_2) = \frac{D\delta\chi}{Dt} = 0 \tag{1.70}$$

But $\delta\chi = \nabla\chi \cdot \delta\mathbf{l}$ where $\delta\mathbf{l}$ is the infinitesimal vector connecting the two fluid elements. Thus

$$\frac{D}{Dt}\left(\nabla\chi \cdot \delta\mathbf{l}\right) = 0 \tag{1.71}$$

But since the line element and the vorticity (divided by density) obey the same equation, we can replace the line element by vorticity (divided by density) in (1.71) to obtain again

$$\frac{DQ}{Dt} = 0 \tag{1.72}$$

where $Q = (\tilde{\boldsymbol{\omega}} \cdot \nabla\chi)$ is the *potential vorticity*.

The vorticity in the potential vorticity conservation law is the *absolute* vorticity. In a rotating frame of reference, the conserved potential vorticity is $Q = (\boldsymbol{\omega} + 2\boldsymbol{\Omega})/\rho$ where $\boldsymbol{\omega}$ is the relative vorticity and $\boldsymbol{\Omega}$ the rate of rotation of the coordinate system.

1.5.2 Nonhomentropic flow

For nonhomentropic flow the potential vorticity equation becomes,

$$\frac{DQ}{Dt} = \frac{1}{\rho^3}\nabla\chi \cdot (\nabla\rho \times \nabla p). \tag{1.73}$$

However, if the equation of state can be written in the form $p = p(s, \rho)$ where s is the entropy, and if we demand that the conserved scalar be a function of entropy s alone then we recover

$$\boxed{\frac{D}{Dt}\left(\tilde{\boldsymbol{\omega}} \cdot \nabla\theta\right) = 0}, \tag{1.74}$$

where

$$\frac{D\theta}{Dt} = 0 \tag{1.75}$$

and $\theta = \theta(s)$. For atmospheric and oceanic applications, (1.74) is the most useful and form of potential vorticity conservation, and the equation has profound consequences. It turns out that potential vorticity is a much more useful quantity for *nonhomentropic* flow than for homentropic flow, because the required use of a special conserved scalar imparts more information to the potential vorticity conservation law. In homentropic flow the very generality and ubiquity of the conservation law seems to make it less useful.

1.5.3 Potential vorticity in isentropic coordinates

Following Salmon (1998) suppose there are three independent scalars χ_i, $i = 1, 2, 3$ that each satisfy

$$\frac{D\chi_i}{Dt} = 0. \tag{1.76}$$

Then, for a homentropic fluid, there are three conserved potential vorticities,

$$Q_i = (\tilde{\boldsymbol{\omega}}) \cdot \nabla \chi_i \tag{1.77}$$

.

Now, if we use the $\nabla \chi_i$ to define a coordinate system, and let

$$\boldsymbol{v} = A_1 \nabla \chi_1 + A_2 \nabla \chi_2 + A_3 \nabla \chi_3 \tag{1.78}$$

then (see problem 2) the potential vorticity field (1.77) may be equivalently written

$$\boldsymbol{Q} = (Q_1, Q_2, Q_3) = \nabla_\theta \times \boldsymbol{A} \tag{1.79}$$

where

$$\nabla_\theta = \left(\frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial \theta_3} \right). \tag{1.80}$$

The conservation law may then be written

$$\frac{D}{Dt}(\nabla_\theta \times \boldsymbol{A}) = 0 \tag{1.81}$$

That is, the conserved potential vorticity is the curl of the velocity in θ coordinates.

For a nonhomentropic fluid, the conservation law is lost. However, take the entropy s as one of the coordinates. Then, the potential vorticity in direction of increasing s (i.e., the direction parallel to ∇s) is still conserved, and

$$\frac{D}{Dt}\left(\nabla_\theta \times \boldsymbol{A} \cdot \nabla s\right) = 0. \tag{1.82}$$

Often, atmospheric and oceanic models are cast in a ‘semi-Lagrangian’ form, in which the vertical coordinate is a function of entropy (typically density in the ocean or potential temperature in the atmosphere). In this case the potential vorticity conservation takes a useful and simple form, and we shall discuss this case more later on.

1.6 Potential Vorticity Conservation in Approximate Models

As we saw in previous chapters, the full Navier-Stokes equations are far too complex to be of direct use for ocean-atmosphere modelling. However, appropriate approximate forms of potential vorticity are conserved by the respective approximate models — Boussinesq, hydrostatic and so on. We consider a few such cases here, and leave others as problems for the reader.

1.6.1 The Boussinesq equations

Recall that the Boussinesq equations are an incompressible set (by our definition, that $\nabla \cdot \boldsymbol{v} = 0$.) Thus, density need not play a role in the definition of a conserved potential vorticity, since the equation for vorticity itself is isomorphic to that for a line element if volume is conserved.

However, the Boussinesq equations are not homentropic, for the equation of state is typically of the form

$$\rho = \rho_0(1 - \alpha(T - T_0)). \tag{1.83}$$

The thermodynamic equation is often written as an equation for the buoyancy $b = -g\delta\rho/\rho_0$, i.e.,

$$D\rho/Dt = 0 \tag{1.84}$$

(in the absence of heat sources).

From these two considerations, it is plain that the appropriate form for potential vorticity is

$$Q = \boldsymbol{\omega} \cdot \nabla b, \tag{1.85}$$

or equivalently

$$Q = \boldsymbol{\omega} \cdot \nabla T \tag{1.86}$$

and that this is a Lagrangian conserved quantity for unforced, inviscid, adiabatic flow. Expanding (1.85) in Cartesian coordinates we obtain,

$$Q = (v_x - u_y)b_z + (w_y - v_z)\rho_x + (u_z - w_x)\rho_y \tag{1.87}$$

1.6.2 The hydrostatic Boussinesq equations

In the hydrostatic approximation the vertical momentum equation is

$$\frac{\partial \phi}{\partial z} = b \tag{1.88}$$

where $\phi = p/\rho_0$ and $b = -g\delta\rho/\rho_0$ is the buoyancy.

It can be shown that (problem 3) that the appropriate potential vorticity is, in Cartesian coordinates,

$$Q_h = (v_x - u_y)\rho_z - v_z\rho_x + u_z\rho_y, \tag{1.89}$$

and that this is quasi-conserved (i.e., conserved if the flow is inviscid and adiabatic) when advected by the three-dimensional velocity.

This can be transformed into a more revealing form by writing it as

$$Q_h = \rho_z\left[\left(v_x - v_z\frac{\rho_x}{\rho_z}\right) - \left(u_y - u_z\frac{\rho_x}{\rho_z}\right)\right]. \tag{1.90}$$

But the terms in the inner brackets are just the horizontal velocity derivatives at constant ρ. To see this, note that

$$\begin{aligned} \left.\frac{\partial v}{\partial x}\right|_\rho &= \left.\frac{\partial v}{\partial x}\right|_z + \left.\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right|_\rho \\ &= \left.\frac{\partial v}{\partial x}\right|_z - \left.\frac{\partial v}{\partial z}\frac{\partial \rho}{\partial x}\right|_z \Big/ \frac{\partial \rho}{\partial z}, \end{aligned} \tag{1.91}$$

with a similar expression for $\partial u/\partial y|_\rho$. (These relationships follow from standard rules of partial differentiation. Derivatives with respect to z are implicitly taken at constant x and y.) Thus, we obtain

$$\begin{aligned} Q_h &= \frac{\partial \rho}{\partial z}\left(\left.\frac{\partial v}{\partial x}\right|_\rho - \left.\frac{\partial u}{\partial y}\right|_\rho\right) \\ &= \frac{\partial \rho}{\partial z}\zeta_\rho \end{aligned} \tag{1.92}$$

Thus, potential vorticity is simply the horizontal vorticity evaluated on a surface of constant density, multiplied by the vertical derivative of density. Note that we could replace density by buoyancy b in the above derivation. This is a 'semi-lagrangian' version of potential vorticity, since x and y are Eulerian coordinates whereas the 'vertical' coordinate, ρ or b, is Lagrangian.

1.7 Quasi-geostrophy, revisited

Rather than deriving the quasi-geostrophic approximation from the momentum equations, it is revealing (and easier) to begin with potential vorticity, and make the corresponding approximations.[1] We will do this in the Boussinesq approximation, although the extension to the compressible case is straightforward, and also assume hydrostasy. As before, the three assumptions we will have to make are, written informally,

(i) Small Rossby number.

(ii) Basic state stratification that does not vary in the horizontal.

(iii) Variations in the Coriolis parameter are small.

The second and third assumptions are peculiar to quasi-geostrophy. Other geostrophic models (e.g., planetary geostrophy, frontal geostrophy) may be derived with other assumptions replacing (ii) and (iii).

The basic idea is that we begin with the potential vorticity equation and make approximations both to the potential vorticity and to the advecting velocity. Scaling theory guides the approximations, and these may be formalized by the use of nondimensionalization and asymptotics, although some find such formalities obfuscating. Since in chapter **??** we proceeded asymptotically, here we shall be more informal.

Without approximation, we write the stratification as:

$$b = \hat{b}(z) + b'(x, y, z, t). \tag{1.93}$$

Then, in Cartesian coordinates (purely for simplicity) in a rotating frame of reference the potential vorticity is

$$\begin{aligned} Q &= (f_0 + \beta y + \zeta)(\hat{b}_z + b'_z) - (v_z b'_x - u_z b'_y) \\ &= \left[f_0 \hat{b}_z \right] + \left[(\beta y + \zeta)\hat{b}_z + f_0 b'_z \right] + \left[(\beta y + \zeta) b'_z - (v_z b'_x - u_z b'_y) \right]. \end{aligned} \tag{1.94}$$

We will subsequently drop the primes on the perturbation b, and write $N^2(z) = -\hat{b}_z$. From the assumptions above, we anticipate that the terms on the second line are in decreasing order of size, and that, since the first term is constant in space and time, the dynamically important part of potential vorticity is given by,

$$q = (\beta y + \zeta)N^2 + f_0 b'_z, \tag{1.95}$$

and indeed this is correct. More formally we scale the equations and perform an asymptotic expansion in powers of the Rossby number. At low Rossby number geostrophic balance ($\boldsymbol{f} \times \boldsymbol{u} \approx -\nabla\phi$) gives the for pressure scaling

$$\phi \sim fUL \tag{1.96}$$

Using geostrophic and hydrostatic balance ($\partial\phi/\partial z = b$) gives thermal wind balance

$$\boldsymbol{f_0} \times \frac{\partial \boldsymbol{u}}{\partial z} = -\nabla b \tag{1.97}$$

with associated scaling

$$b \sim \frac{fUL}{H} \tag{1.98}$$

Note that this only applies to the perturbation buoyancy, because the basic state is unvarying in the horizontal. Geostrophic and thermal wind balance imply if we define a streamfunction ψ by

$$\frac{\psi = \phi}{f_0} \tag{1.99}$$

then

$$\boldsymbol{u}_g = \mathbf{k} \times \nabla\psi \tag{1.100}$$

and

$$b = f\frac{\partial\psi}{\partial z}. \tag{1.101}$$

The ratio of the size of the perturbation buoyancy to the basic state buoyancy is then given by

$$\left|\frac{b_z}{\hat{b}_z}\right| = \frac{|b_z|}{N^2} \sim \frac{fUL}{H^2N^2} = \frac{F^2}{R_o} \tag{1.102}$$

where $F = U/NH$ is the *Froude number*. [This expression is sometimes written

$$\frac{|b_z|}{N^2} \sim \frac{1}{R_o R_i} \tag{1.103}$$

where $R_i = N^2/(\partial u/\partial z)^2$ is the *Richardson number*. However, I prefer to think of this expression in terms of the Froude number, because it not the shear of the flow that is central here. Rather, it is the effects of stratification.] To derive the quasi-geostrophic approximation, F^2/R_o is assumed small, as indeed it is for large-scale flows in the ocean and atmosphere, although to demand that it be small may seem a little unmotivated for it has little to do with geostrophic balance; rather, it is a demand that the stratification be strong. We return to this point shortly.

Assuming that $\beta y \sim \zeta/f \sim R_o$ then, in the low Rossby number limit, the potential vorticity is:

$$Q = \left[f_0N^2\right] + \{R_o\}\left[(\beta y + \zeta)N^2 + \left\{\frac{F^2}{R_o^2}\right\} f_0 b'_z\right] + \{R_o\}\left\{\frac{F^2}{R_o^2}\right\}\left[(\beta y + \zeta)b_z - (v_z b'_x - u_z b'_y)\right], \tag{1.104}$$

where the terms in curly brackets indicate the scales of the terms multiplying them relative to the first term. This can be formalized by nondimensionalizing all of the variables using (1.96), (1.98), (1.102). The terms in (1.104) are then all $O(1)$ nondimensional quantities, save for the nondimensional factors in curly brackets.

Since the first term in (1.104) is a constant, the lowest order approximation to Q that will contribute to its dynamical evolution is,

$$q = \left[(\beta y + \zeta)N^2 + \left\{\frac{F^2}{R_o^2}\right\} f_0 b_z\right] \tag{1.105}$$

We see that if F^2/R_o^2 is not $O(1)$, that is if F^2/R_o is not small, then the term $f_0 b_z$ will dominate the evolution of potential vorticity. In this limit (which arises in the planetary geostrophic approximation) all dynamics involving relative vorticity are absent. This approximation relates to the scale of the motion. The condition $F^2/R_o \sim R_o$ is equivalent to

$$\frac{R_o L^2}{\lambda^2 = O(R_o).} \tag{1.106}$$

where $\lambda = NH/f$ is the *radius of deformation*. Thus, condition on stratification is equivalent to demanding that the horizontal scale of the motion must be the deformation scale or smaller; it cannot be larger than the deformation scale by $O(R_o^{-1}$.

Similarly, if $|\beta y| \sim f$, then the dominant dynamically active term is $\beta y N^2$ and the leading order term in the evolution equation would become $\beta v = 0$. Although this graphically illustrates the constraining effect that the β–effect has on meridional motion, it is not a very dynamically rich equation.

Let us now consider how the advecting velocity should be approximated. The potential vorticity equation is thus

$$\frac{Dq}{Dt} + w\frac{\partial \hat{q}}{\partial z} = 0 \tag{1.107}$$

where $\hat{q} = f_0 N^2$ and the advective derivative is (at this stage) fully three-dimensional. Now, since f is nearly constant, the horizontal velocity is nearly non-divergent. That is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \sim R_o\frac{U}{L} \ll \frac{U}{L} \tag{1.108}$$

Thus the proper scaling for the vertical velocity is,

$$w \sim R_o\frac{UH}{L}. \tag{1.109}$$

and the vertical advection in (1.107) is important only in advecting the basic state potential vorticity $\hat{q}$. This equation becomes, after divinding by N^2,

$$\frac{\partial q}{\partial t} + \boldsymbol{u}_g \cdot \nabla q + w\frac{\partial \hat{q}}{\partial z} = 0 \tag{1.110}$$

where

$$q = (\beta y + \zeta) + \frac{f_0}{N^2}b_z. \tag{1.111}$$

is the approximation to (perturbation) potential vorticity in the quasi-geostrophic limit.

We can simplify this further with the help of the thermodynamic equation,

$$\frac{\partial b}{\partial t} + \boldsymbol{u}_g \cdot \nabla b + w\frac{\partial \hat{b}}{\partial z} = 0 \tag{1.112}$$

Eliminating w between (1.110) and (1.112) (left as an exercise for the reader: problem 4) gives the quasi-geostrophic potential vorticity equation,

$$\frac{\partial q_g}{\partial t} + \boldsymbol{u}_g \cdot \nabla q_g = 0 \tag{1.113}$$

where

$$q_g = (\beta y + \zeta) + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}b\right) \tag{1.114}$$

or, in terms of streamfunction,

$$q_g = (\beta y + \nabla^2 \psi) + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) \tag{1.115}$$

There are a number of important points to be made:

1. The so-called quasi-geostrophic potential vorticity given by (1.114) or (1.115) is *not* an approximation to the Ertel potential vorticity. That approximation is given by (1.111).

2. Nevertheless, (1.114) (or (1.115)) is more useful than (1.111) because it is conserved when advected by the horizontal geostrophic velocity and because

3. q_g may be written as the three-dimensional divergence of a vector $\boldsymbol{J}$ where $\boldsymbol{J} = \nabla\psi + \mathbf{k}\,(f_0^2/N^2)\partial\psi/\partial z$.

4. The streamfunction may be diagnosed from the potential vorticity by the solution of the three-dimensional elliptic equation:

$$\left(\nabla^2 + \frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\psi = q_g - \beta y. \tag{1.116}$$

5. From the streamfunction, the velocity and buoyancy are easily diagnosed as elements of the vector:

$$(u, v, b) = (-\psi_y, \psi_x, f\psi_z). \tag{1.117}$$

Notes

1. Charney and Stern (1961) were the first to clarify the relationship between quasi-geostrophic potential vorticity and Ertel potential vorticity, and derive quasi-geostrophic theory in isentropic coordinates. Vallis (1996) gives a derivation similar to, if a little more formal, that here.

Further Reading

Batchelor, G. K. 1967. *An Introduction to Fluid Dynamics.* Cambridge University Press.
This contains an extensive discussion of vorticity and vortices.

Salmon, R. S. 1998. *Geophysical Fluid Dynamics.* Oxford University Press.
Chapter 4 contains a brief discussion of potential vorticity, and chapter 7 a longer discussion of Hamiltonian fluid dynamics, in which the particle relabelling symmetry that gives rise to potential vorticity conservation is discussed.

Problems

1. For the vr vortex, choose a contour of arbitrary shape (e.g. a square) with segments neither parallel nor orthogonal to the radius, and not enclosing the origin. Show explicitly that the circulation around it is zero. (Some may think this problem is perverse.)

2. Show that potential vorticity defined by (1.79) is the same as that defined by (1.77). *Hint:* begin with (1.77).

3. Show, beginning with the momentum equations, that in the hydrostatic Boussinesq approximation the quantity given by (1.89) is quasi-conserved (i.e., conserved if the flow is adiabatic and inviscid) when advected by the three-dimensional flow.

4. Eliminate the vertical velocity between the thermodynamic and potential vorticity equations ((1.110) and (1.112)) to obtain the so-called quasi-geostrophic potential vorticity equation (1.113).

5. Derive the quasi-geostrophic potential vorticity equation appropriate in the atmospheric case — i.e., allow density to vary by $O(1)$ in the vertical. You may do this either using pressure coordinates, or the 'quasi-Boussinesq' approximation in height coordinates, or some other way.

Chapter 2

Basic Theory of Incompressible Turbulence

A turbulent flow is one that is dominated by eddies with a spectrum of sizes between some upper and lower bounds.[1] The individual eddies come and go, and are intrinsically unpredictable. Loosely, turbulence is high Reynolds number fluid flow, dominated by nonlinearity, with both spatial and temporal disorder.

The circulation of the atmosphere and ocean is of course, to a large degree, simply motion of a forced-dissipative fluid subject to various constraints such as rotation and stratification. The scales of much motion of interest are certainly significantly larger than the dissipation scale (the scale at which molecular viscosity becomes important) by several orders of magnitude and, at may if not all scales, the motion is highly nonlinear. This seems to accord with our definition of turbulence. What this means precisely, and how the motion of the atmosphere and ocean connects to the standard and some more modern theories of turbulence is the subject of the next two chapters. First, and before considering turbulence in a geophysical context, in section 2.1, we ask 'what is the problem?' Then we will consider the classical scaling theories of turbulence in both two and three dimensions. These have their origins in classical papers by Kolmogorov and Oboukhov in 1941, and, for the two-dimensional case, Kraichnan in 1967. We won't consider many of the variations and subtleties associated with the classical theory (such as the effects of intermittency) except in so far as they may affect geophysical flows. Following this we will discuss geophysical issues.

2.1 The Closure Problem

Although it is impossible to predict the detailed motion of each eddy, the mean state may not be changing. For example, consider the weather system, in which the storms, anti-cyclones, hurricanes, fronts etc.—constitute the eddies. Although we cannot predict these very well, we certainly have some skill at predicting their mean state—the climate. For example, we know that next summer will be warmer than next winter, and that in California summer will be drier than winter. We know that next year it will be colder in Canada than in the Mexico, although there might be an occasional day when this is not so.

We would obviously like to be able to predict the mean climate without necessarily trying to predict or even simulate all the eddies. We might like to know what the climate will be like one hundred years from now, without trying to know what the weather will be like on February 9 2056, plainly an impossible task. Even though we know what equations determine the system, this task proves to be very difficult because the equations are nonlinear, and we come up against the 'closure' problem.

Let us try to write an equation for the mean flow. The program would be to first decompose the velocity field into mean and fluctuating components,

$$\boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}' \tag{2.1}$$

Here $\overline{\boldsymbol{v}}$ is the mean velocity field, and $\boldsymbol{v}'$ is the deviation from that mean. The mean may be a time average, in which case $\overline{\boldsymbol{v}}$ is a function only of space and not time. It might be a time mean over a finite period (e.g a season if we are dealing with the weather). Most generally it is an ensemble mean. Note that the average of the deviation is, by definition, zero; that is $\overline{\boldsymbol{v}'} = 0$. We then substitute into the momentum equation and try to obtain a closed equation for $\overline{\boldsymbol{v}}$.

To visualize the problem most simply, we carry out this program for a model nonlinear system which obeys

$$\frac{du}{dt} + uu + \nu u = 0 \tag{2.2}$$

The average of this equation is:

$$\frac{d\overline{u}}{dt} + \overline{uu} + \nu\overline{u} = 0 \tag{2.3}$$

The value of the term $\overline{uu}$ is not deducible simply by knowing $\overline{u}$, since it involves correlations between eddy quantities $\overline{u'u'}$. That is, $\overline{uu} = \overline{u}\,\overline{u} + \overline{u'u'} \neq \overline{u}\,\overline{u}$. We can go to next order to try (vainly!) to obtain a value. First multiply (2.2) by u to obtain an equation for u^2, and then average it to yield:

$$\frac{1}{2}\frac{d\overline{u^2}}{dt} + \overline{uuu} + \nu\overline{u^2} = 0 \tag{2.4}$$

This equation contains the undetermined cubic term $\overline{uuu}$. An equation determining this would contain a quartic term, and so on in an unclosed hierarchy.

Most current methods of 'closing the hierarchy' make assumptions about the relationship of (n+1)'th order terms to n'th order terms, for example by supposing that:

$$\overline{uuuu} = \overline{uu}\;\overline{uu} - \alpha\overline{uuu} \tag{2.5}$$

where α is some parameter. Such assumptions require additional, and sometimes dubious, reasoning. Nobody has been able to close the system without introducing physical assumptions not directly deducible from the equations of motion.

In the constant density (say $\rho = 1$) Navier-Stokes equations, the x-momentum equation for an averaged flow is

$$\frac{\partial \overline{u}}{\partial t} + (\overline{\boldsymbol{v}} \cdot \nabla)\overline{u} = -\frac{\partial \overline{p}}{\partial x} - \nabla \cdot \overline{\boldsymbol{v}'u'}. \tag{2.6}$$

Written out in full in Cartesian co-ordinates, the last term is

$$\nabla \cdot \overline{\boldsymbol{v}'u'} = \frac{\partial}{\partial x}\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'} \tag{2.7}$$

These terms, and the similar ones in the y- and z- momentum equations, represent the effects of eddies on the mean flow. They are known as *Reynolds stress* terms. The 'problem' of turbulence might be considered to be to find a representation of such Reynolds stress terms in terms of mean flow quantities. However, it is not at all clear that any reasonable general solution (or parameterization) even exists, short of computing the terms explicitly.

2.2 The Kolmogorov Theory

2.2.1 The physical picture

Consider high Reynolds number incompressible flow which is being maintained by some external force. Then the evolution of the system is governed by

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \boldsymbol{F} + \nu\nabla^2\boldsymbol{v} \tag{2.8}$$

and

$$\nabla \cdot \boldsymbol{v} = 0 \tag{2.9}$$

Here, **F** is some force we apply (i.e., we are stirring the fluid). A naïve scale analysis of these equations indicates that the relative sizes of the inertial terms on the left-hand-side to the viscous term is the Reynolds number UL/ν. To be explicit let us consider the ocean, and take $U = 0.1$ m/s, $L = 1000$ km and $\nu = 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$. Then $R_e \sim 10^{11}$, and apparently we can neglect the viscous term on the right hand side of (2.8). But this can lead to a paradox. The fluid is being forced, and this forcing is likely to put energy into the fluid. We obtain the energy budget for (2.8) by multiplying by $\boldsymbol{v}$ and integrating over a domain. If there is no flow into or out of our domain, the inertial terms in the momentum equation conserve energy and we obtain,

$$\frac{d}{dt}\int \rho\boldsymbol{v}^2\,\mathrm{d}V = \int \left(\boldsymbol{F}\cdot\boldsymbol{v} + \nu\boldsymbol{v}\cdot\nabla^2\boldsymbol{v}\right)dV \tag{2.10}$$

or

$$\frac{d\hat{E}}{dt} = \int \left(\boldsymbol{F}\cdot\boldsymbol{v} - \nu\boldsymbol{\omega}^2\right)dV \tag{2.11}$$

where $\hat{E}$ is the total energy. If we neglect the viscous term we are led to an inconsistency, since the forcing term puts energy in ($\mathbf{F} \cdot \mathbf{v} > \mathbf{0}$), but there is nothing to take it out! Thus, energy keeps on increasing.

What is amiss? It is true that for motion with a 1000 km length scale and a velocity of a few centimetres per second we can neglect viscosity when considering the balance of forces in the momentum equation. But this does not mean that there is no motion at much smaller length scales—indeed we are led to the inescapable conclusion that there *must* be some motion at smaller scales in order to remove energy. Where and how does this motion occur? Boundaries are one important region. If there is high Reynolds number flow above a solid boundary, for example the wind above the ground, then viscosity *must* become important in bringing the velocity to zero in order that it can satisfy the no-slip condition at the surface.

In the 'boundary layer' the viscous terms must be at least of the same order as the inertial terms, that is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \sim \nu\frac{\partial^2 u}{\partial y^2} \tag{2.12}$$

which implies the viscous terms may be important in boundary layer of approximate size

$$L \sim \frac{\nu}{U} \tag{2.13}$$

This is a very small number for geophysical fluids, of order millimetres or less.

The other way that energy may be dissipated is through energy dissipation in the interior of the fluid. In order to do this the fluid must somehow generate motion on very small time and space scales. How might this happen? Suppose the forcing acts only at large scales, and its direct action is to set up some correspondingly large scale flow, composed of eddies and shear flows and such-like. Then typically (a mathematician would like to say 'generically') there will be an instability in the flow, and a smaller eddy will grow. At first it will grow exponentially, because during the period the eddy is small the large scale flow may be treated as an unchanging shear flow, and the disturbance while still of small amplitude will obey linear equations of motion similar to those applicable in idealized Kelvin-Helmholtz instability. The exponential instability clearly must be drawing energy from the large scale quasi-stationary flow, and it will eventually saturate at some *finite* (as opposed to infinitesimal) amplitude. Although it has grown in intensity, it is still typically smaller than the large scale flow which fostered it (remember how the growth rate of the shearing instability got larger as wavelength of the perturbation decreased). As it reaches finite amplitude, the perturbation itself may become unstable, and smaller eddies will feed off its energy and grow, and so on. The process has been encapsulated in the following ditty, attributed to L. F. Richardson,

Greater whorls have lesser whorls,
which feed on their velocity.
And lesser whorls have smaller whorls,
and so on to viscosity.

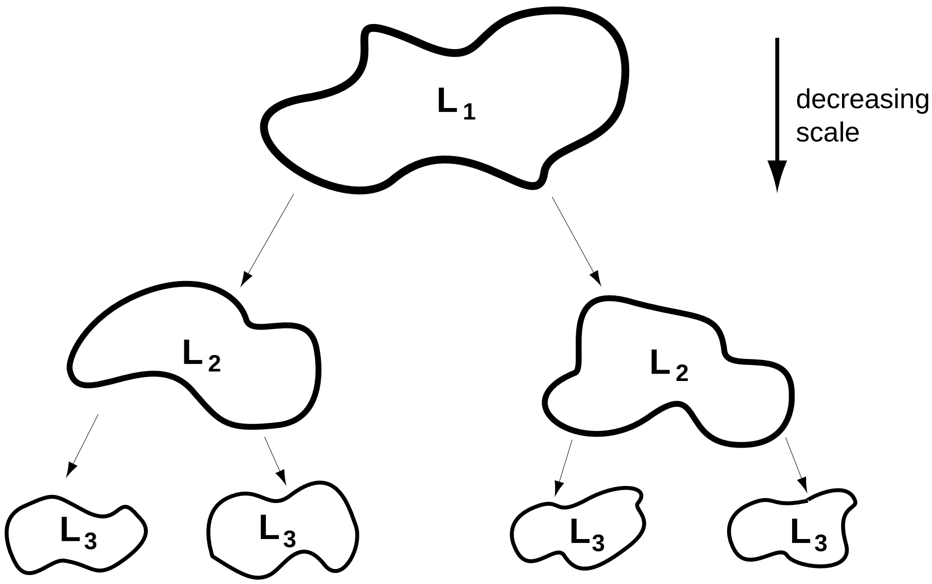

Figure 2.1: Schema of a ‘cascade’ of energy to smaller scales: eddies at a large scale break up into smaller scale eddies, thereby transferring energy to smaller scales. If the transfer occurs between eddies of similar sizes (i.e., it is ‘spectrally local’) the transfer is said to be a cascade. The eddies in reality are embedded in each other.

The picture which emerges if of a large scale flow which is unstable to eddies somewhat smaller in scale. These eddies grow, and develop still smaller eddies. Energy is transferred to smaller and smaller scales in a cascade-like process (fig. 2.1). Finally, eddies are generated which are sufficiently small that they feel the effects of viscosity, and energy is drained away. There is a flux of energy from the large scales to the small scales, where it becomes dissipated.

It seems that this picture of turbulence was envisioned by Richardson in the first part of this century, and was quantified by Kolmogorov and Oboukhov.[2]

2.2.2 Inertial range theory

Given the above picture, it is possible to predict what the energy spectrum is, i.e. the intensity of the motion as a function of wavenumber. Let us suppose that the flow is statistically isotropic and homogeneous; the latter condition precludes the presence of solid boundaries but can be achieved in a periodic domain (see e.g. fig. 2.2). (This naturally puts an upper limit, sometimes called the outer scale, on the size of eddies which can be achieved.)

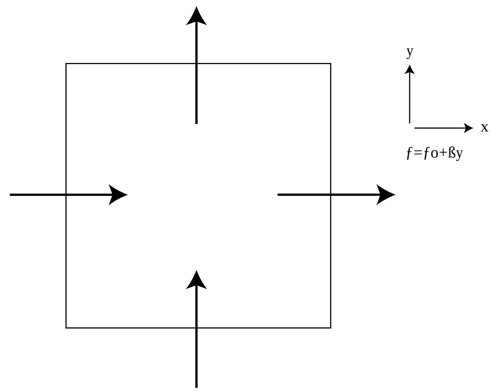

Figure 2.2: A Cartesian periodic domain, in two dimensions. Fluid that leaves on one side enters with the same properties on the other side, allowing a statistically homogeneous flow. If $\beta \neq 0$ the flow will not be isotropic, but may still be homogeneous.

If we decompose the velocity field into Fourier components, then we may write

$$u(\boldsymbol{x}, t) = \int \tilde{u}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\boldsymbol{x}} \, dk \tag{2.14}$$

where $\tilde{u}$ is the coefficient of the k'th wavenumber —i.e. the Fourier transformed field—with similar identities for v and w. The energy in the fluid is given by (assuming density is unity)

$$\begin{aligned} \hat{E} = \int E \, dV &= \frac{1}{2} \int (u^2 + v^2 + w^2) \, dV \\ &= \frac{1}{2} \int (\tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2) \, dk \end{aligned} \tag{2.15}$$

using Parseval's theorem. Here $\hat{E}$ is the total energy, E the energy per unit volume, and $\mathcal{E}$ is the energy spectral density. We suppose that the turbulence is homogeneous and isotropic. We write (2.15) as

$$\hat{E} = \int \mathcal{E}(k) \, dk \tag{2.16}$$

where $\mathcal{E}(k)$ is the modal energy, and, because of the assumed isotropy, k is a scalar wavenumber given by $k^2 = k_x^2 + k_y^2 + k_z^2$.

We calculate the energy spectrum of a turbulent fluid by a simple physical assumption, first articulated by Kolmogorov. The idea is to suppose that the forcing scale is much larger than the dissipation scale. Then we assume that there is a range of scales intermediate between the large scale and the dissipation scale where neither forcing nor dissipation are explicitly important. This assumption, known as the locality hypothesis, depends on the nonlinear transfer of energy being

Table 2.1:

Quantity	**Dimension**
Wavenumber, k	1/L
Energy per unit mass, E	$U^2 \sim L^2/T^2$
Energy spectrum, $\mathcal{E}(k)$	$EL \sim L^3/T^2$
Energy Flux, ε	$E/T \sim L^2/T^3$

sufficiently local (in spectral space). Given this, this intermediate range is known as the inertial range, because the inertial terms and not the forcing of dissipation must dominate in the momentum balance here. If the rate of energy input per unit volume by stirring is equal to ε, then if we are in a steady state there must be a flux of energy from large scales to small also equal to ε, and an energy dissipation rate, also ε.

In the inertial range, by assumption, no physical processes are important except inertial ones. The energy spectrum must, therefore, be a function *only* of the energy flux ε and the wavenumber itself. That is,

$$\mathcal{E}(k) = f(\varepsilon, k). \tag{2.17}$$

Dimensional analysis then gives us the form of this function (see table 2.1) . In (2.17), the left hand side has dimensionality L^3/T^2; the dimension T^{-2} on the left-hand-side can only be balanced by $\varepsilon^{2/3}$ since k has no time dependence, i.e.

$$\begin{aligned} \mathcal{E}(k) &\sim \varepsilon^{2/3} g(k) \\ \frac{L^3}{T^2} &\sim \frac{L^{4/3}}{T^2} g(k). \end{aligned} \tag{2.18}$$

We see that $g(k)$ must have dimensions $L^{5/3}$ and the functional relationship we *must* have, if the assumptions are right, is

$$\boxed{\mathcal{E}(k) = \mathcal{K}\varepsilon^{2/3}k^{-5/3}}. \tag{2.19}$$

This is the famous ‘Kolmogorov -5/3 law’, enshrined as one of the cornerstones of turbulence theory (see fig. 2.3). The parameter $\mathcal{K}$ is a dimensionless constant, undetermined by the theory. It is known as Kolmogorov’s constant and experimentally is found to be approximately 1.5.

Another, perhaps slightly more physical, way to derive this is to note that we may define an eddy turn-over time $\tau(k)$, which is the time taken for a parcel with energy $\mathcal{E}(k)$ to move a distance $1/k$. Thus,

$$\tau(k) = (k^3\mathcal{E}(k))^{-1/2} \tag{2.20}$$

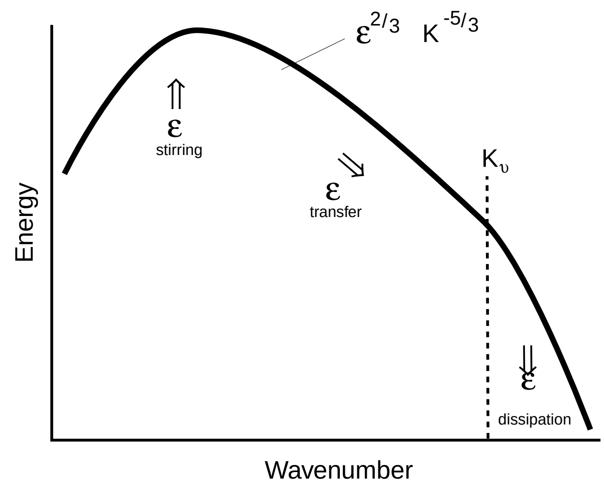

Figure 2.3: The energy spectrum in three-dimensional turbulence, in the theory of Kolmogorov. Energy is supplied at some rate ϵ; it is transferred ('cascaded') to small scales, where it is ultimately dissipated by viscosity.

Kolmogorov's assumptions are then equivalent to setting

$$\epsilon \sim \frac{k\mathcal{E}(k)}{\tau(k)} \tag{2.21}$$

which, since we demand that ϵ be constant, again yields:

$$\mathcal{E}(k) = \mathcal{K}\epsilon^{2/3}k^{-5/3}, \tag{2.22}$$

This spectral form has been verified many times observationally, the first time using some very high Reynolds number oceanographic observations [17].

Now, the assumptions of homogeneity and isotropy are really ansatzes — we make them because we want to have a tractable model of turbulence. Certainly we can conceive of a thought experiment which is homogeneous and isotropic. The essential physical assumption is that there exists an inertial range in which the energy flux is constant. Lacking a more comprehensive or fundamental theory we can test it only through experiment. It requires that the energy be cascaded from large to small scales in a series of steps, for then the energy spectra will be determined by spectrally local quantities. Without this, we could conceivably have

$$\mathcal{E}(k) = C\varepsilon^{2/3}k^{-5/3}h(k/k_o) \tag{2.23}$$

where h is an unknown function and k_o the wavenumber at the forcing scale. This is just as dimensionally consistent as (2.19). We essentially postulate that at some wavenumber much smaller than the forcing scale there is no functional dependence of the energy spectra on the forcing scale, and $h(k/k_o) = 1$. In fact in high Reynolds turbulence, the $-5/3$ spectra is observed to a fairly high degree of accuracy — it is perhaps better observed than one might expect.

The viscous scale

At some small length-scale we should expect viscosity to become important and the scaling theory we have just set up will not work. The Kolmogorov theory allows us to determine this scale.

In the inertial range friction is unimportant because the timescales on which it acts are too long for it be important—dynamical effects dominate instead. However, at smaller timescales the viscous timescale decreases. In the momentum equation we have

$$\frac{\partial u}{\partial t} + \ldots = \nu \nabla^2 u \tag{2.24}$$

and so a viscous or dissipation timescale is

$$\tau_d \sim \frac{1}{k^2 \nu} \tag{2.25}$$

The eddy turnover time in the Kolmogorov spectrum is

$$\tau(k) = \varepsilon^{-1/3} k^{-2/3} \tag{2.26}$$

The wavenumber at which dissipation becomes important is given by equating these, yielding the dissipation wavenumber,

$$k_d \sim \left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \tag{2.27}$$

or the associated length-scale

$$L_d \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \tag{2.28}$$

L_d is sometimes called the Kolmogorov scale. It is the *only* quantity which can be created from the quantities ν and ε with the dimensions of length.

These are very sensible formulae; as the stirring rate (ε) increases or ν decreases the frictional scale decreases. For $L >> L_d$, $\tau_i << \tau_d$ and inertial effects dominate. For $L << L_d$, $\tau_d << \tau_i$ and frictional effects dominate. (In fact for length-scales smaller than the dissipation scale, (2.26) is inaccurate; the energy spectrum falls off more rapidly than $k^{-5/3}$ and the 'inertial' timescale falls off less rapidly than (2.26) implies, and dissipation dominates even more.)

How big is L_d in the atmosphere? It is rather complicated because the largest scales of motion in the atmosphere are two-dimensional, and (2.19) does not apply. Further, we need to estimate the value of ε. In the atmosphere ε is ultimately determined by how much energy is received by the sun; however, not all of this goes to stir the atmosphere—some of it is simply radiated back to space as infra-red radiation. A very crude estimate of ε, likely to be wrong by a factor of ten, comes from noting that ε has units of U^3/L. At length-scale of 100m in the atmospheric boundary layer we estimate velocity fluctuations of order 1 cm/s, giving $\varepsilon \approx 10^{-8}\text{m}^2\text{s}^{-3}$. Using (2.28) we find the dissipation scale to be $L_d \approx 1 - 10$ mm. So dissipation becomes important at millimetre scales in the atmosphere. If we try to simulate the atmosphere on a computer by resolving all scales from the global to the Kolmogorov scale, we would end up with about 10^{27} degrees of freedom—a number greater than Avogadro's number. Thus trying to model turbulence is akin to trying to model an ideal gas by following the motion of each individual molecule. How should we model it? This, in a nutshell, is the (unsolved) problem of turbulence.

2.2.3 Scaling arguments for inertial ranges

Kolmogorov's spectrum, as well as some other useful scaling relationships, can be obtained in a slightly different, but essentially equivalent, way as follows. If we for the moment ignore viscosity, the Euler equations are invariant under the following scaling transformation:

$$x \Longrightarrow x\lambda \quad v \Longrightarrow v\lambda^r \quad t \Longrightarrow t\lambda^{1-r} \tag{2.29}$$

where r is an arbitrary scaling exponent. So far there is no physics. Now make the following physical assumptions: First we make the locality hypothesis, namely that the energy flux through a wavenumber k depends only on local quantities, namely the wavenumber itself and the energy $\mathcal{E}(k)$ or velocity $v(k)$. Second, the flux of energy from large to small scales is assumed finite and constant. Third we assume that the scale invariance (2.29) holds, on a time-average, in the intermediate scales between the forcing scales and dissipation scales. This is likely to be strictly valid only in the limit of infinite Reynolds number, but for finite Reynolds number it is made plausible by the locality hypothesis. (It is important to note that the infinite Reynolds number limit *is* a limit, and is different from simply neglecting the viscous term in (1.1), which gives the so-called Euler equations. This is because, as we shall see, this term contributes even in the zero-viscosity limit.) The time average in practice need be no longer than a few longest eddy turn-over times, and depending on how local the energy transfer actually is we do not need an infinite Reynolds number for the scaling to be valid in the inertial range.

Dimensional analysis then tells us that the energy flux scales as

$$\epsilon \sim \frac{v^3}{l} \sim \lambda^{3r-1} \tag{2.30}$$

from which the assumed constancy of ϵ gives $r = 1/3$. This has a number of interesting consequences. The velocity then scales as

$$v \sim \epsilon^{1/3}k^{-1/3}, \tag{2.31}$$

and he velocity gradient scales as $\nabla v \sim \varepsilon^{1/3}k^{2/3}$, as does the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. These quantities thus blow up (i.e., become infinite) at very small scales, but this is in fact avoided in any physical situation by the presence of viscosity.

We can now recover (2.22) easily since dimensionally

$$\mathcal{E} \sim v^2 k^{-1} \sim \varepsilon^{2/3} k^{-2/3} k^{-1} \sim \varepsilon^{2/3} k^{-5/3}. \tag{2.32}$$

The structure functions S_m of order m, which are the average of the m'th power of the velocity difference over distances $l \sim 1/k$, scale as $(\delta v_l)^m \sim \epsilon^{m/3} l^{m/3} \sim \epsilon^{m/3} k^{-m/3}$. In particular the second-order structure function, which is the Fourier transform of the energy spectra, scales as $S_2 \sim \epsilon^{2/3} k^{-2/3}$.

The viscous effects become important at a range given by equating the viscous and inertial terms in the momentum equation, that is when

$$\nu k^2 v \sim k v^2 \tag{2.33}$$

which yields

$$k_\nu \sim \left(\frac{\epsilon}{\nu^3}\right)^{1/4}. \tag{2.34}$$

The scale $l_\nu \sim k_\nu^{-1}$ is called the *Kolmogorov scale*. In the limit of viscosity tending to zero, l_ν tends to zero, but the energy dissipation does not. The energy dissipation is given by

$$\dot{E} = \int \nu \boldsymbol{v} \cdot \nabla^2 \boldsymbol{v} \, d\boldsymbol{x} \tag{2.35}$$

Since the length at which dissipation acts is the Kolmogorov scale, using (2.31) this expression scales as (for a box of unit volume)

$$\dot{E} \sim \nu k_\nu^2 v^2 \sim \nu \frac{\varepsilon^{2/3}}{k_\nu^{2/3}} k_n u^2 \sim \varepsilon \tag{2.36}$$

with $k = k_\nu$. This result is demanded by the phenomenology. Energy is input at some large scales, and the magnitude of the stirring largely determines the energy input and cascade rate. The scale at which viscous effects become important is determined by the value of the molecular viscosity by (2.34). If viscosity tends to zero, this scale becomes smaller and smaller in such a way as to preserve the constancy of the energy dissipation.

Finally, the time scales as $t \sim l^{2/3}$ implying that for smaller scales the 'eddy-turnover time' on which structures at that scale deform becomes smaller and smaller.

2.3 Two-Dimensional Turbulence

In two dimensions the situation is complicated by another quadratic invariant, the enstrophy. In two dimensions, the vorticity equation is:

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u} \cdot \nabla \zeta = \nu \nabla^2 \zeta \tag{2.37}$$

where $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$ and $\zeta = \mathbf{k} \cdot \nabla \times \boldsymbol{u}$. It is easily verified that when $\nu = 0$ (4.7) conserves not only the energy but also the enstrophy $Z = \int \frac{1}{2}\zeta^2 \, d\boldsymbol{x} = \int k^2 \mathcal{E}(k) dk$.

We now ask, how does the distribution of energy and enstrophy change in a turbulent flow?

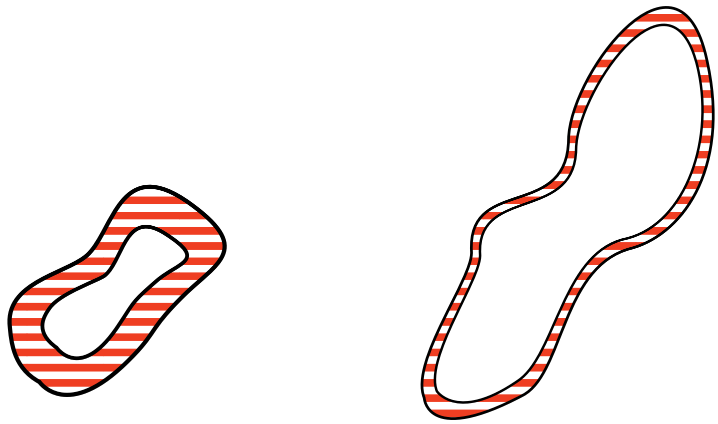

Figure 2.4: In incompressible two-dimensional flow, a band of fluid (hatched region) will generically be elongated, but its area (proportional to the fluid mass) will be preserved. Since vorticity is tied to fluid parcels, the vorticity over the hatched area is also preserved; thus, vorticity gradients must increase and the enstrophy distribution is thereby moved to smaller scales.

2.4 Energy and Enstrophy Transfer in Two-Dimensional Turbulence

Two-dimensional turbulence differs from three-dimensional turbulence in that energy is expected to be transferred to *larger scale*. This behaviour arises from the twin integral constraints of energy and enstrophy. The following three arguments illustrate why this should be so.

I. Vorticity Elongation

Consider a band or a patch of vorticity, as in fig. 2.4, in a nearly inviscid fluid. The vorticity of each element of fluid is conserved as the fluid moves. Now, we should expect that quasi-random motion of the fluid will act to elongate the band but, since its area must be preserved, vorticity gradients will increase. This is equivalent to the enstrophy moving to smaller scales. However, the energy in the fluid is

$$E = -\frac{1}{2}\int \psi\zeta \, d\boldsymbol{x} \tag{2.38}$$

Now the streamfunction is obtained by solving the Poisson equation $\nabla^2\psi = \zeta$, and if the vorticity is elongated primarily only in one direction (as it must be to preserve area) the integration involved in solving the Poisson equation is such that scale of the streamfunction will normally become larger

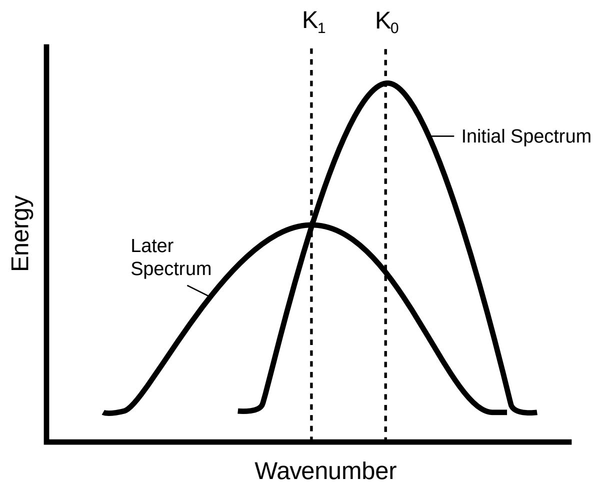

Figure 2.5: In two-dimensional flow, the ‘center of gravity’ of the energy spectrum will move to large scales (smaller wavenumber) provided that the width of the distribution increases, which can be expected in a nonlinear, eddying flow

in all directions. As a consequence, the cascade of enstrophy to small scales is accompanied by a transfer of energy to large scales.

II. An energy-enstrophy conservation argument

A moments thought will reveal that the distribution of energy and enstrophy in wavenumber space are are analogous to the distribution of mass and moment of inertia of a lever respectively, with wavenumber playing the role of distance from the fulcrum. Any rearrangement of mass such that its distribution also becomes wider must be such that the center of mass moves toward the fulcrum. Thus, energy would move to *smaller* wavenumbers and enstrophy to larger. We may formalize this argument, as follows. Suppose we start with some given distribution of energy in wavenumber space and by way of nonlinear interactions we redistribute the energy. The total energy and enstrophy are, respectively

$$E = \int E(k)\,dk \tag{2.39}$$

$$Z = \int k^2 E(k)\,dk \tag{2.40}$$

Any rearrangement which preserves both quantities, and which causes the distribution to spread out in wavenumber space, will tend to move energy to small wavenumbers and enstrophy to large.

Let

$$k_e = \frac{\int k E(k)\, dk}{\int E(k)\, dk} \tag{2.41}$$

and, for simplicity, we normalize units so that the denominator is unity. The spreading out of the energy distribution is formalized by setting

$$\frac{dI}{dt} = \frac{d}{dt} \int (k - k_e)^2 E(k)\, dk > 0. \tag{2.42}$$

Expanding out the integral gives

$$\begin{aligned} I &= \int k^2 E(k)\, dk - 2k_e \int k E(k)\, dk + k_e^2 \int E(k)\, dk \\ &= \int k^2 E(k)\, dk - k_e^2 \int E(k)\, dk, \end{aligned} \tag{2.43}$$

where the last equation follows because $k_e = \int k E(k) dk$ is, from (2.41), the energy-weighted centroid. Because both energy and enstrophy are conserved, (2.43) gives

$$\frac{dk_e^2}{dt} = -\frac{1}{\int E(k)\, dk} \frac{dI}{dt} < 0. \tag{2.44}$$

Thus, the centroid of the distribution moves to smaller wavenumber and to larger scale (see fig. 2.5).

An appropriately defined measure of the center of the enstrophy distribution, on the other hand, moves to higher wavenumber. The proof of this[3] is analogous, except that we work with the inverse wavenumber, which is a direct measure of length. Let $q = 1/k$ and assume that the enstrophy distribution spreads out by nonlinear interactions, so that

$$\frac{dJ}{dt} > 0, \tag{2.45}$$

where

$$J = \int (q - q_e)^2 Z(q)\, dq, \tag{2.46}$$

where $Z(q)$ is the enstrophy distribution and

$$q_e = \frac{\int q Z(q)\, dq}{\int Z(q)\, dq}. \tag{2.47}$$

Expanding the integrand in (2.46) using (2.47) gives

$$J = \int q^2 Z(q)\, dq - q_e^2 \int Z(q)\, dq, \tag{2.48}$$

But $\int q^2 Z(q) dq$ is conserved, because this is the energy. Thus,

$$\frac{dJ}{dt} = -\frac{d}{dt}\int q_e^2 Z(q)\, dq \tag{2.49}$$

whence

$$\frac{dq_e^2}{dt} = -\frac{1}{\int Z(q)\, dq}\frac{dJ}{dt} < 0 \tag{2.50}$$

Thus, the length scale characterizing the enstrophy distribution gets smaller, and the corresponding wavenumber gets larger.

III A similarity argument

This argument is based on two physical requirements:

1. That there is no externally imposed length-scale in the problem;
2. That the energy is conserved.

It is the first of these that suggests a similarity argument be used.

Write the total energy (per unit mass) of a fluid as

$$U^2 = \int \mathcal{E}(k)\, \mathrm{d}k \tag{2.51}$$

where $E(k)$ is the energy spectrum — i.e., $E(k)\delta k$ is the energy in the small wavenumber interval δk. Thus, solely on dimensional considerations, we can write

$$\mathcal{E}(k,t) = U^2 L \hat{\mathcal{E}}(\hat{k}, \hat{t}) \tag{2.52}$$

where $\hat{\mathcal{E}}$, and its arguments, are nondimensional quantities, and L is a length-scale. However, on physical considerations, the only parameters available to determine the energy spectrum are U, the (square root of the) total energy, t, the time, and k, the wavenumber. There is *a priori* no 'external' length-scale, and so E should not depend explicitly on L. To make this so, let

$$\mathcal{E}(k,t) = \hat{t} g(\hat{k}\hat{t}) \tag{2.53}$$

where g is an arbitrary function of its arguments. Then

$$\begin{aligned} \hat{\mathcal{E}} &= \frac{Ut}{L} g(kLt\frac{U}{L}) \\ &= \frac{Ut}{L} g(Ukt). \end{aligned} \tag{2.54}$$

Therefore,

$$\mathcal{E}(k,t) = U^2 L\hat{\mathcal{E}} = U^3 t g(Ukt). \tag{2.55}$$

which is independent of L.

It is the choice $\hat{\mathcal{E}} = \hat{t}g(\hat{k}\hat{t})$ is crucial here. If we were to choose (say) $\hat{\mathcal{E}} = \hat{t}g(\hat{k}/\hat{t})$ then we would not obtain a form for (2.55) that is independent of L xxxx. This choice is a *similarity* solution, and is due to Batchelor (1969). A little thought will show that it is the most general solution for the energy spectrum that does not depend on the parameter L.

Conservation of energy now implies that the integral

$$I = \int t g(Ukt)\,\mathrm{d}k \tag{2.56}$$

not be a function of time. Defining $\vartheta = Ukt$ then this requirement is met if

$$\int_0^\infty g(\vartheta) d\vartheta = \text{Constant.} \tag{2.57}$$

(The constant may be chosen to be one.) The spectrum is a function of k only through the combination $\vartheta = Ukt$. Thus, as time proceeds features in the spectrum moves to smaller k. Suppose that the energy is initially peaked at some wavenumber. If the wavenumber of the energy peak is denoted k_p, then tk_p is preserved, and k_p must diminish with time, and the energy move to larger scales. Similarly, the energy weighted mean wavenumber (say $\overline{k}$) moves to smaller wavenumber, or larger scale. To see this explicitly, we have

$$\begin{aligned} \overline{k} &= \frac{\int k\mathcal{E}\,dk}{\int \mathcal{E}\,dk} \\ &= \frac{\int ktg(Ukt)\,dk}{\int tg(Ukt)\,dk} \\ &= \frac{1}{Ut}\frac{\int \vartheta g(\vartheta)\,d\vartheta}{\int g(\vartheta)\,d\vartheta} \\ &= \frac{1}{Ut}A \end{aligned} \tag{2.58}$$

where all the integrals are over the interval $(0,\infty)$ and $A = \int \vartheta g(\vartheta)\,d\vartheta / \int g(\vartheta)\,d\vartheta$ is a constant. Thus, the mean wavenumber decreases with time. Defining a characteristic scale $\overline{l} = 1/\overline{k}$ then a measure of the rate of increase of scale is given by

$$\frac{1}{\overline{l}}\frac{d\overline{l}}{dt} = Uk(B/A). \tag{2.59}$$

The quantity $1/Uk$ is the 'eddy-turnover time' of an eddy of characteristic size $1/k$, that is the time taken for a parcel of fluid to travel a distance $1/k$. Thus, modulo the constant A, the eddy turnover time is also the typical time it takes for the eddy containing scales to double in size.

The total enstrophy in this similarity argument is *not* conserved.

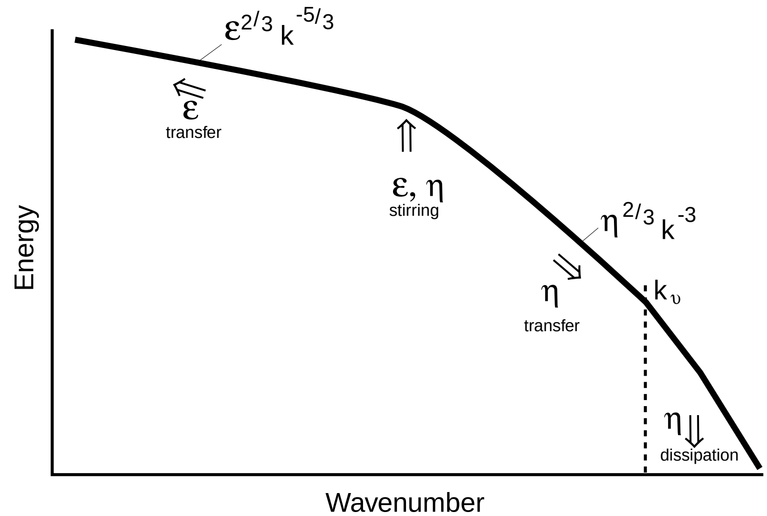

Figure 2.6: The putative energy spectrum in two-dimensional turbulence. Energy supplied at some rate ϵ is transferred to *large* scales, whereas enstrophy is transferred to small scales at a rate η where it may be dissipated by viscosity. If the forcing is localized at a scale $k)f$ the $\eta \approx k_f^2 \eta$.

2.4.1 Inertial ranges in 2D turbulence

We now consider how energy and enstrophy might be distributed in forced-dissipative two-dimensional flow. It is easy to show that energy dissipation goes to zero as Reynolds number rises. The total dissipation of energy is, from (2.37),

$$\frac{dE}{dt} = -\nu \int \zeta^2 \, \mathrm{d}x \tag{2.60}$$

Since vorticity itself is bounded from above we see that energy dissipation goes to zero as viscosity goes to zero, and hence also in the infinite Reynolds number (but finite energy) limit. Thus, unlike the three dimensional case, there is no mechanism for the dissipation of energy at small scales in high Reynolds number two-dimensional turbulence. On the other hand, we do expect enstrophy to be dissipated at large wavenumbers.

These arguments lead one to propose the following scenario in two-dimensional turbulence (fig. 2.6). Energy and enstrophy are input at some scale L_I and energy is transferred to larger scales (toward the fulcrum) and enstrophy is cascaded to small scales where ultimately it is dissipated. In the enstrophy inertial range the enstrophy cascade rate η is assumed constant. Using the dimensionally correct scaling

$$\eta \sim \frac{k^3 \mathcal{E}(k)}{\tau(k)} \tag{2.61}$$

yields the prediction

$$\mathcal{E}(k) = \mathcal{K}'\eta^{2/3}k^{-3}, \tag{2.62}$$

where $\mathcal{K}'$ is also, it is supposed, a universal, order one, constant. It is of course also quite possible to obtain (2.62) from scaling arguments identical to those following equation (2.29). The scaling transformation (2.29) still holds, but now instead of (2.34) we assume that the enstrophy flux is constant with wavenumber. Dimensionally we have

$$\eta \sim \frac{v^3}{l^3} \sim \lambda^{3r-3} \tag{2.63}$$

which gives $\lambda = 1$. The exponent n determining the slope of the inertial range is given, as before, by $n = -(2r+1)$ yielding the -3 spectra of (4.12). The velocity now thus scales as $v \sim \eta^{1/3}k^{-1}$, and the time scales with distance as

$$t \sim l/v \sim \eta^{-1/3}. \tag{2.64}$$

Thus, it is length-scale invariant. The appropriate Kolmogorov scale is given by equating the inertial and viscous term in (1.1) or (2.37), which gives, analogously to (2.34)

$$k_\nu \sim \left(\frac{\eta^{1/3}}{\nu}\right)^{1/2} \tag{2.65}$$

The energy dissipation is easily calculated to got to zero as $\nu \to 0$. The enstrophy dissipation, analogously to (2.36) goes to a finite limit given by

$$\begin{aligned} \dot{Z} &= \frac{d}{dt}\int \frac{1}{2}\zeta^2 \,\mathrm{d}x = \nu \int_0^{k_\nu} \zeta \nabla^2 \zeta \\ &\sim \nu k_\nu^4 v^2 \sim \eta \end{aligned} \tag{2.66}$$

2.4.2 Difficulties with the phenomenology

The phenomenology of two-dimensional turbulence is unfortunately not quite as straightforward as it might seem. We will not delve into these issues, but we will mention only the most egregious. To begin, note that timescale (2.64) is apparently independent of scale. If the spectra were any steeper then turnover times would actually increase with wavenumber, which seems unphysical. A useful refinement of the estimate of the inertial timescale is:

$$\tau = \left\{\int_{k_0}^{k} (p^2\mathcal{E}(p))dp\right\}^{-1/2}, \tag{2.67}$$

where k_0 is a lower wavenumber cut-off. This formula recognizes the straining effects of all velocity scales larger than the scale of interest. Using this in (2.61) yields the log-corrected range

$$\mathcal{E}(k) = \mathcal{K}'\eta^{2/3}(\log(k/k_0))^{-1/3}k^{-3}. \tag{2.68}$$

This spectrum is nevertheless likely to be observationally indistinguishable from the uncorrected range.

However, this has not fixed the underlying problem with the two-dimensional phenomenology, which is as follows. The inertial range predictions are based on the assumption of locality, in spectral space, of energy and enstrophy transfers. Now, a useful measure of this locality is given by the straining at a particular wavenumber, say k, from other wavenumbers. The total strain $T(k)$ at k is given by

$$T(k) = \left\{ \int_0^k \mathcal{E}(p) p^3 d \log p \right\}^{1/2} \tag{2.69}$$

The contributions to the integrand from each octave are given by

$$\mathcal{E}(p) p^3 \Delta \log p \tag{2.70}$$

In three dimensions, use of the $-5/3$ spectra indicates that the contributions from each octave below k increase with wavenumber, being a maximum close to k, implying locality and *a posteriori* being consistent with the locality hypothesis. However, in two-dimensions each octave makes the same contribution. The strain, and possibly the enstrophy transfer, are hardly local after all! This very heuristic result implies that the two-dimensional phenomenology is on the verge of not being self-consistent, and suggests that the -3 spectral slope is the shallowest limit that is likely to be actually achieved in nature or in any particular computer simulation, rather than a very robust result. Why? Well, suppose the detailed dynamics attempt in some way to produce a shallower slope; using (2.70) the strain is then local and the shallow slope is forbidden by the Kolmogorovian scaling results. However, if the dynamics organizes itself into structures with a steeper slope (say k^{-4}) the strain is quite nonlocal. The fundamental assumption of Kolmogorov scaling is not satisfied, and there is no inconsistency. In fact numerical simulations do reveal a slope steeper than k^{-3}, often dominated by isolated vortices. However, the dynamical processes leading to their formation, and their precise relationship with the enstrophy cascade, are not at this time fully understood.

There is one other aspect of the phenomenology which is superficially a problem. In the limit of zero viscosity (2.66) implies that enstrophy dissipation remains constant, and therefore that palinstrophy (mean square curl of the vorticity) is infinite somewhere. However, it has been shown rigorously that the inviscid equations — (2.37) with the right-hand-side set to zero — have no singularities and enstrophy dissipation remains zero. This is not really a contradiction, firstly because we are concerned with the zero viscosity *limit* in (2.66). Even if we were to suddenly ‘turn off’ the viscosity in an infinitely high resolution simulation of (2.37), then the enstrophy inertial range (assuming it exists) would slowly spread to larger and larger wavenumbers; during this period of adjustment the fluid has indeed zero enstrophy dissipation. It takes the fluid an infinite time to come to equilibrium with an infinitely long inertial range. Only then is the enstrophy dissipation non-zero, which is not an inconsistency with the rigorous results which only prohibit singularities forming in finite time.

2.5 Predictability of Turbulence

One of the central properties of turbulence is its unpredictability due to nonlinear interactions. Some authors will draw a distinction between 'sensitive dependence on initial conditions' and 'unpredictability'. The former's meaning is unambiguous, and it is normally applied to deterministic systems. The latter is sometimes applied only to indeterminism arising out of stochasticity, when the equations of motion are not known. However, here we take them to be synonymous, and use the latter (since it is but one word) to mean unpredictability arising from chaos. Actually, the difference between chaos and stochasticity lies not so much in the underlying dynamics, but in our knowledge of them. Whereas chaos is essentially but a word for deterministic unpredictability and 'randomness', stochasticity describes randomness arising from incomplete knowledge of the system, as for example in Brownian motion. In most cases the difference between stochasticity and chaos is merely a difference in our knowledge of the dynamics. For example, most computers have 'random number generators' built in, and these are often used in the simulation of stochastic systems. However, the algorithm producing the random numbers is completely deterministic, and if we regard that algorithm as part of the system, we have chaos, not stochasticity.

The modern ideas of nonlinear dynamics and chaos have not, interestingly enough, had at this time much impact on theories of, or ideas of how to cope with, strong turbulence. Even prior to the classical paper of Lorenz (1963) and Ruelle and Takens(1971) it was believed that turbulence was truly unpredictable (see e.g., Thompson 1959, and Novikov 1961) notwithstanding the picture of Landau of turbulence as a large collection of periodic, and presumably predictable, motions. The unpredictability was thought to arise from the utter complexity of the flow. The reasons for the loss of predictability were probably only properly understood when it was realized that even systems with a small number of degrees of freedom could be unpredictable. Assuming that the dynamical systems arguments applicable to weak turbulence apply to strong turbulence, and hence that a turbulent fluid *is* in fact unpredictable, then just using the scaling laws we can heuristically obtain estimates of the predictability time for a turbulent fluid.

The physical space fields $\zeta(\boldsymbol{x})$ may be expressed as an infinite Fourier sum or integral, for example $\zeta = \sum \hat{\zeta}_{\mathbf{k}} \exp(i\mathbf{x}\cdot\mathbf{k})$ or $\zeta = \int \hat{\zeta}_{\mathbf{k}} \exp(i\mathbf{x}\cdot\mathbf{k})\mathrm{dk}$. The former is appropriate in a bounded domain (where the wavenumbers are quantized), the latter in an infinite domain. We are usually concerned with a finite domain, but will nevertheless often replace sums by integrals where it will simplify things. In two-dimensions (for simplicity) the inviscid vorticity equation may be written, in spectral form

$$\frac{\partial \zeta_k}{\partial t} + \sum a_{kpq}\zeta_p\zeta_q = -\nu k^2 \zeta_k \tag{2.71}$$

where a_{kpq} are geometrical coupling coefficients which arise when (2.37) is Fourier transformed. The hats over transformed quantities have been dropped. At any given instant the equation of motion may be linearized about its current state, and the subsequent motion would then be described by an equation, valid for short times, of the form:

$$\frac{\partial \zeta_k'}{\partial t} + A_{kq}\zeta_q' = 0 \tag{2.72}$$

and the eigenvalues of the matrix A_{kq} (whose explicit form does not concern us here) determine the short term growth of errors in the system. Because the system is chaotic, A_{kq} has positive (growing) eigenvalues. If spectral interaction in the inertial range are sufficiently local, it becomes meaningful to inquire as to the growth of errors at any particular scale k, for then the matrix A_{kq} is dominated by terms close to its diagonal. In particular, the rate of error growth at any particular scale is then given by the size of the appropriate coefficient of A_{kq}, which is ku_k where u_k is just a typical velocity at scale k. This of course is just the inverse of the eddy turnover time (4.2). After a time τ_k, errors will have grown sufficiently that a linear approximation is no longer valid; at that scale errors will saturate but at the same time will begin to contaminate the 'next larger' (in a logarithmic sense) scale, and so on. Thus, errors initially confined to a scale k at $t = 0$ will contaminate the scale $2k$ after a time τ_k. The total time taken for errors to contaminate all scales from k' to the largest scale k_0 is then given by, treating the wavenumber spectrum as continuous,

$$\begin{aligned} T &= \int_{k_0}^{k'} \tau_k \, d(\ln k) \\ &= \int_{k_0}^{k'} \frac{d(\ln k)}{\sqrt{k^3 E(k)}} \end{aligned} \tag{2.73}$$

If the energy spectrum is a power law of the form $E = C'k^{-n}$ this becomes

$$T = [C'k^{(n-3)/2}]_{k_0}^{k'} \frac{2}{(n-3)}. \tag{2.74}$$

As $k' \to \infty$ the estimate diverges for $n > 3$, but converges if $n < 3$.

What does these heuristic results mean? Taken them at face value they imply that two-dimensional turbulence is indefinitely predictable; if we can confine the initial error to smaller and smaller scales of motion, the payoff is that the 'predictability time' (the time taken for errors to propagate to all scales of motion) can be made longer and longer, indeed infinite. This is consistent with what has been rigorously proven about the two-dimensional Navier-Stokes equations, with or without viscosity, namely that they exhibit 'global regularity', meaning they stay analytic for all time provided the initial conditions are sufficiently smooth. This does *not* mean that two-dimensional flow is in practice necessarily predictable. Two-dimensional turbulence is almost certainly chaotic and an arbitrarily small amount of noise will render a flow truly unpredictable sometime in the future. It is just that we can put off that time indefinitely if we know the initial conditions well enough, and can reduce the amount of external noise sufficiently.

In three dimensions, on the other hand, things are more worrisome. The predictability time estimate from (2.74) converges as $k' \to \infty$ So that even if we push our initial error out to smaller and smaller scales, the predictability time does not keep on increasing. The time it takes for errors initially confined to small scales to spread to the largest scales is simply a few *large* eddy turnover times (because the eddy turnover times of the small scales are so small). This is an indicator that something is awry, either with our methodology or with the Euler equations, since because the system is classical we do not expect such finite time catastrophes. If one were able to prove global regularity for the three-dimensional Euler equations then we would know our analysis were wrong, but such a proof is lacking, and may not exist. The phenomenology thus suggests that the three dimensional Euler equations are not well-posed. If the Euler equations were ill-posed, it would mean that they do not correctly describe a physically realizable system. However, no classical flow is inviscid, and the correct equations for a classical fluid are the Navier-Stokes equations, with viscosity. No matter how small viscosity, if not zero, then at some small wavenumber the local Reynolds number will be small and viscous effects will start to dominate over inertial effects. Beyond the viscous wavenumber, the energy spectrum gets steeper, and as soon as the asymptotic spectra is steeper than -3 we are again assured of indefinite predictability. If the Navier-Stokes equations themselves were shown to have finite-time singularities, it would be a more serious matter.

What does this mean about the weather? In the troposphere the large scale flow behaves more like a two-dimensional fluid, or at least a quasi-geostrophic fluid, than a three-dimensional fluid. At scales smaller than about 100km, the atmosphere starts to behave three-dimensionally. Now the current atmospheric observing system is such that over continents the atmosphere is fairly well observed down to scales of a couple of hundred kilometers. If we knew the enstrophy cascade rate through the atmosphere we could evaluate the predictability time using the formulae derived above, but we may do the sum manually, Fourier transforming in our heads, as it were. Suppose then we have no knowledge of the dynamical fields at scales smaller than 200km. Aside from certain rather intense small scale phenomena, the atmosphere is not especially energetic at these scales and we could estimate a typical velocity of about 1 $\mathrm{m\,s^{-1}}$ giving an eddy turnover time of about 2 days. So in 2 days motion at 400 km scales is unpredictable. The dynamics at these scales is a little more intense, say $U \sim 2\,\mathrm{m\,s^{-1}}$. Coincidentally (?), this also gives a 2 day eddy turnover time, so after 4 days motion at 800 km is unpredictable. Continuing the process, after about 12 days motion at 6000 km is completely unpredictable, and our weather forecasts are essentially useless. This is probably a little better than our experience suggests as to how good weather forecasts are in practice, but of course our models of the atmosphere are certainly not perfect. (Actually, I've fudged the numbers so they come out reasonable; more careful calculations, as well as computer simulations, do give similar results though.) In principle, we could make forecasts better if we could observe the atmosphere down to smaller scales of motion. Observing down to 100, 50 and 25 kilometres would (if the atmosphere remained two-dimensional) each add about a couple of days to our forecast times.

However, at small scales of motion the atmosphere starts behaving three-dimensionally. Here, the eddy turnover times decrease rapidly with scale and the predictability time is largely governed by the predictability time of the largest scale of motion. Thus, the *theoretical* limit to predictability is governed by the scale at which the atmosphere turns three-dimensional, probably about 100 km. So we see that we can't increase the length of time we can make good weather forecasts for longer than about two weeks, no matter how good our models and no matter how good our observing system. This is the theoretical predictability limit of the atmosphere. The so-called butterfly effect has its origins in this argument: a butterfly flapping its wings over the Amazon is, so it goes, able to change the course of the weather a week or so later.

One other point may be apposite. The predictability of a system is sometimes characterized by its spectrum of Lyapunov exponents. In a turbulent system the largest Lyapunov exponent is likely be associated with the smallest scales of motion, and the error growth associated with this effectively saturates at small scales. The timescales of error growth affecting the larger scales, which are the timescales of most interest, are determined by slower, larger scale processes whether or not the cascade-like growth of error described above is correct. This means that the largest Lyapunov exponents probably have nothing whatever to do with the growth of error at the larger scales in a turbulent fluid.

2.6 Spectrum of a Passive Tracer

Let us now consider, phenomenologically, the spectrum of a passive tracer, such as a dye, that obeys the equation

$$\frac{D\phi}{Dt} = F + \kappa\nabla^2\phi, \tag{2.75}$$

where F is the 'forcing' or injection of the dye, and κ is its diffusivity. In general κ differs from the kinematic molecular viscosity ν; their ratio is the *Prandtl number* $\sigma = \nu/\kappa$. We assume that the tracer is injected as some well-defined scale k_0, and that κ is sufficiently small that dissipation only occurs at very small scales. (Note that dissipation only reduces the tracer *variance*, not the amount of tracer itself.) The turbulent flow will generically tend to stretch patches of dye into elongated filaments, in much the same way as vorticity in two-dimensional turbulence is filamented — note that fig. 2.4 applies just as well to a passive tracer in either two or three dimensions as it does to vorticity in two dimensions. Thus we expect a transfer of tracer variance from large-scales to small. If the dye is injected at a rate χ then, by analogy with our treatment of the cascade of energy, we have

$$\chi \propto \frac{P(k)k}{\tau(k)} \tag{2.76}$$

where $P(k)$ is the spectrum of the tracer, k is the wavenumber and $\tau(k)$ is the timescale of the turbulent flow. If the spectral slope of the turbulence is -3 or shallower, then

$$\tau(k) = [k^3 E(k)]^{-1/2} \tag{2.77}$$

Suppose that the turbulent spectrum is given by

$$E(k) = Ak^{-n} \tag{2.78}$$

then

$$\chi \propto \frac{P(k)k}{[Ak^{3-n}]^{-1/2}} \tag{2.79}$$

and

$$\boxed{P(k) = BA^{-1/2}\chi k^{(n-5)/2}} \tag{2.80}$$

where B is a constant. Note that the steeper the energy spectrum the shallower the tracer spectrum.

If the energy spectrum is steeper than -3 then the estimate (2.77) should be replaced by

$$\tau(k) = \left[\int_{k_0}^{k} p^2 E(p)dp\right]^{-1/2} \tag{2.81}$$

where k_0 is the low-wavenumber limit of the spectrum. If the spectrum is shallower than -3 then the integrand is dominated by the contributions from high wavenumbers and (2.81) effectively reduces to (2.77). If the spectrum is steeper than -3 then the integrand is dominated by contributions from low wavenumbers. Indeed for $k \gg k_0$ then we can approximate the integral by $[k_0^3 E(k_0)]^{-1/2}$, or the eddy-turnover time at large scales, $\tau(k_0)$. The energy spectrum then becomes

$$\boxed{P(k) = C\chi\tau(k_0)^{-1}k^{-1}} \tag{2.82}$$

where C is a constant.

In all these cases the tracer cascade is to smaller scales even if, as may happen in two-dimensional turbulence, energy is cascading to larger scales.

The scale at which diffusion becomes important (the diffusive microscale) is given by equating the turbulent time-scale $\tau(k)$ to the diffusive time-scale $(\kappa k^2)^{-1}$. This is independent of the flux of tracer, χ, essentially because the equation for the tracer is linear, Determination of expressions for these in two and three dimensions are left as problems for the reader.

2.6.1 Examples

1. *Inertial range flow in three dimensions*

   Consider a range of wavenumbers over which neither viscosity nor diffusivity directly influence the turbulent motion and the tracer. Then in (2.80) $A = C\varepsilon^{2/3}$ where ε is the rate of energy transfer to small scales and C the Kolmogorov constant, and $n = 5/3$. The tracer spectrum becomes

   $$P(k) = D\varepsilon^{-1/3}\chi k^{-5/3}. \tag{2.83}$$

   (This result dates to Oboukhov (1949) and, independently, Corrsin (1951).) It is interesting that the $-5/3$ exponent appears in both the energy spectrum and the passive tracer spectrum. Using (2.77), this is the only spectral slope for which this occurs. Experiments show that this range does, at least approximately, exist with a value of D of about 0.5.

2. *Enstrophy range in two-dimensional turbulence* In the forward (enstrophy) inertial range the timescale is just

$$\tau(k_\nu) = \eta^{-1/3} \tag{2.84}$$

(assuming of course that classical phenomenology holds). Directly from (2.76) the corresponding tracer spectrum is then

$$P(k) = B\eta^{-1/3}\chi k^{-1}. \tag{2.85}$$

The passive tracer spectrum now has the same slope as the spectrum of vorticity variance (i.e., the enstrophy spectrum), which is perhaps comforting since the tracer and vorticity obey the same equation in two dimensions.

3. *Inverse energy-cascade range in two-dimensional turbulence* Now we are supposing that the energy injection occurs at a smaller scale than the tracer injection, so that there exists a range of wavenumbers over which energy is cascading to larger scales while tracer variance is simultaneously cascading to smaller scales. The tracer spectrum is then

$$P(k) = D'\varepsilon^{-1/3}\chi k^{-5/3}, \tag{2.86}$$

the same as (2.83), although ε is now a cascade to larger scales and D' does not necessarily equal D.

4. *The viscous range of large Prandtl number flow*

If $\sigma = \nu/\kappa \gg 1$ then there may exist a range of wavenumbers in which viscosity is important but not tracer diffusion. The energy spectrum is then very steep, and (2.82) will apply. The appropriate low wavenumber is the dissipation wavenumber, so that in three dimensions

$$k_0 = k_\nu = \left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \tag{2.87}$$

and

$$\tau(k_\nu) = \left(\frac{\nu}{\varepsilon}\right)^{1/2} \tag{2.88}$$

The tracer spectrum is then

$$P(k) = D\left(\frac{\nu}{\varepsilon}\right)^{1/2}\chi k^{-1}. \tag{2.89}$$

This is sometimes called the *Batchelor spectrum*.

In two dimensions the spectral slope in the corresponding (high wavenumber) viscous but non-dissipative range is the same. However, the appropriate timescale at the dissipation scale is given not by (2.88) but by (2.84). Even in the viscous range then, the tracer spectrum is given by (2.85) and the phenomenology predicts that there is no break in the spectral slope of a passive tracer in two dimensions when the viscous range is reached at the dissipation wavenumber,

$$k_\nu = \left(\frac{\eta}{\nu^3}\right)^{1/6} \tag{2.90}$$

5. *Small Prandtl number flow*

   For small Prandtl number the energy inertial range may co-exist with a range over which tracer variance is being dissipated. The flux of the tracer is now no longer constant; rather it diminishes according to

$$\frac{d\chi}{dk} = -2\kappa k^2 P(k). \tag{2.91}$$

   However, we may still assume that χ and $P(k)$ are related by

$$\chi = \frac{P(k)k}{\tau(k)}. \tag{2.92}$$

   In the energy inertial range of three dimensional flow we have

$$\tau(k) = (k^2\varepsilon)^{-1/3} \tag{2.93}$$

   so that

$$P(k) = \chi(k)\varepsilon^{-1/3}k^{-5/3} \tag{2.94}$$

   and (2.91) becomes

$$\frac{d\chi}{dk} = -2\kappa\chi\varepsilon^{-1/3}k^{1/3}. \tag{2.95}$$

   Solving this gives

$$\chi = \chi_0 \exp(-\frac{3}{2}\kappa\varepsilon^{-1/3}k^{2/3}), \tag{2.96}$$

   where χ_0 is the tracer flux at the beginning (low wavenumber end) of the tracer dissipation range. The tracer spectrum is then

$$P(k) = \chi_0\varepsilon^{-1/3}k^{-5/3} \exp(-\frac{3}{2}\kappa\varepsilon^{-1/3}k^{2/3}). \tag{2.97}$$

   The tracer spectrum thus falls exponentially in this range.

The spectra in a number of these cases are illustrated in figure (xxx) (not yet available).

2.7 Spectra in the Time Domain

Let $\tilde{\mathcal{E}}(\omega)$ be the frequency spectrum in the time-domain, such that the total energy (per unit mass) is given by

$$E = \frac{1}{2}\boldsymbol{u}^2 = \int \mathcal{E}(\omega)\, d\omega. \tag{2.98}$$

We can proceed analogously to the case in wavenumber space, supposing that the energy spectrum is a function of ε, the energy transfer rate. (Note that the dimensions of ε, L^2/T^3, makes no specific reference to it being an energy transfer through *wavenumber* space.) For the energy inertial range, in either two or three dimensions, dimensional analysis then yields

$$\mathcal{E}(\omega) \sim \frac{L^2}{T} \sim \varepsilon\omega^{-2} \tag{2.99}$$

So the spectrum is

$$\tilde{\mathcal{E}}(\omega) = C\varepsilon\omega^{-2} \tag{2.100}$$

where C is a non-dimensional constant. We can also obtain this result, along with an estimate for C in terms of the Kolmogorov constant, if we suppose that the angular frequency corresponding to k is given by

$$\omega = \left[\mathcal{E}(k)k^3\right]^{1/2} = \mathcal{K}^{1/2}\varepsilon^{1/3}k^{2/3} \tag{2.101}$$

where $\mathcal{K}$ is Kolmogorov's constant. Then $\tilde{\mathcal{E}}(\omega) = \mathcal{E}(k)\partial k/\partial\omega$ and we find that

$$\tilde{\mathcal{E}}(\omega) = \frac{3}{2}\mathcal{K}^{3/2}\varepsilon\omega^{-2} \tag{2.102}$$

However, the equality in (2.101) cannot really be justified.

In the enstrophy inertial range of two dimensional turbulence such dimensional analysis is of little use. For if we suppose that the energy spectrum is a function of the enstrophy cascade rate η and the frequency ω we have that

$$\tilde{\mathcal{E}}(\omega) \sim \frac{L^2}{T} \sim \eta^a\omega^b \sim T^{-3a}T^{-b} \tag{2.103}$$

where a and b are putative powers to make the equation dimensionally consistent. Clearly, this is not possible. Physically, the problem arises because each wavenumber in the enstrophy inertial ranges has the same time-scale.

2.7.1 The space-time spectrum

We might suppose that there exists the general space-time spectrum $\hat{\mathcal{E}}$ such that

$$E = \hat{\mathcal{E}}(k,\omega)\, d\omega\, \mathrm{d}k \tag{2.104}$$

If we write

$$\hat{\mathcal{E}}(\omega, k) = \varepsilon^{2/3} k^{-5/3} f(\omega, k) \tag{2.105}$$

where $f(\omega, k)$ is some function of its arguments, then by simple similarity (i.e., dimensional) arguments we must have

$$f(\omega, k) = g(\omega t_k) t_k \tag{2.106}$$

where t_k is the time-scale at wavenumebr k. We further require that

$$\int_0^\infty g(\omega t_t) t_k \, d\omega = 1 \tag{2.107}$$

which is satisfied if

$$\int_0^\infty g(\alpha)\, d\alpha = 1, \tag{2.108}$$

$\alpha (= \omega t_k)$ is just the argument of g.

The time-spectrum $\tilde{\mathcal{E}}$ is given by

$$\tilde{\mathcal{E}}(\omega) = \int \varepsilon^{2/3} k^{-5/3} g(\alpha) t_k \, \mathrm{d}k. \tag{2.109}$$

Using turbulence phenomenology, for the energy inertial range $t_k = (k^2 \varepsilon)^{-1/3}$, so that $dt_k/dk = -2/3 k^{-5/3} \varepsilon^{-1/3}$ and

$$\frac{1}{\omega}\frac{d\alpha}{dk} = -2/3 k^{-5/3} \varepsilon {-} 1/3 \tag{2.110}$$

Thus,

$$\begin{aligned} \tilde{\mathcal{E}}(\omega) &= \int \varepsilon^{2/3} k^{-5/3} g(\alpha) t_k \, \mathrm{d}k \\ &= \int -\frac{3}{2}\frac{1}{\omega} \varepsilon g(\alpha) t_k \, d\alpha \\ &= \int -\frac{3}{2}\frac{1}{\omega} \varepsilon g(\alpha) \frac{\alpha}{\omega} \, d\alpha \\ &= \ = \int \varepsilon \omega^{-2} \int h(\alpha) d\alpha \end{aligned} \tag{2.111}$$

where $h(\alpha)$ is an undetermined function.

2.7.2 Eulerian Spectra

Suppose that a probe that measures velocity fluctuations is put in a turbulent fluid. What energy spectrum would result from its measurements? It would not be the ω^{-2} derived above, because the velocity variations at a fixed point are due to the sweeping past the probe of small eddies by large eddies. That is, the probe serves to map the spatial frequency of the turbulence into the time-frequency by the relationship

$$\omega = Uk \tag{2.112}$$

where U is the rms velocity of the energy containing scales. Note that this does not violate the locality hypothesis, which says that small eddies are torn apart by eddies of comparable scale. Here we are saying that small eddies are swept by eddies at the energy containing scales. The difference is between the contribution to the integral of the shear spectrum

$$\text{Shear}^2 \propto \int_{k_0}^{l} \varepsilon^{2/3} p^{1/3}\, dp \tag{2.113}$$

which is dominated by contributions to the integrand from near k itself, and the velocity spectrum

$$\text{Velocity}^2 \propto \int_{k_0}^{l} \varepsilon^{2/3} p^{-5/3}\, dp \tag{2.114}$$

which is dominated by contributions from near k_0. Using (2.112) is equivalent to Taylor's *frozen field* hypothesis, since one is supposing that the small-scale turbulent structure is frozen as it is swept past the probe. Using it in the Kolmogorov $-5/3$ spectrum (2.19) gives

$$\tilde{\mathcal{E}}(\omega) = \mathcal{K}\varepsilon^{2/3} U^{2/3} \omega{-}5/3 \tag{2.115}$$

In the enstrophy inertial range of two-dimensional turbulence a similar argument gives

$$\tilde{\mathcal{E}}(\omega) = \mathcal{K}\eta^{2/3} U^{2/3} \omega{-}3 \tag{2.116}$$

Using these transformations is often the simplest way to measure the energy spectrum in a fluid. To directly measure the spectrum in wavenumber space requires measurements of the velocity correlation between two points, a more difficult measurement.

If (2.115) is the spectrum measured at a point, what does the ω^{-2} spectrum physically represent? Tennekes and Lumley argue that this is a 'lagrangian' time spectrum, related to the temporal evolution 'seen by an observer moving with the turbulent velocity fluctuations.' This is a little unsatisfactory, and a reader who gives a better explanation will win a small prize.

Notes

1. A precise definition of turbulence is hard to come by, and in that sense turbulence is like pornography: hard to define, but easy to recognize when we see it!

2. Lewis Fry Richardson (1881–1953) was a British scientist best known as the person who envisioned weather forecasting in its current form — that is, numerical weather prediction. However, instead of an electronic computer performing the calculations, he envisioned a hall full of people, performing calculations in unison all directed by a conductor at the front. He is also well-known for the 'Richardson number.' Remarkably, he also made contributions to the theory of war. He was a known as a pacifist.

   A. N. Kolmogorov was a Russian theoretical physicist, who made seminal contributions to turbulence (in three papers in 1941 and another in 1962), statistics, and classical mechanics through the well-known but less-understood Kolmogorov-Arnol'd-Moser theorem.

3. This form of the proof arose in discussion with Isaac Held.

Further Reading

Tennekes and Lumley 1972. *A First Course in Turbulence.*
This book remains the classical introduction to the subject.

Monin and Yaglom 1966. *Statistical Fluid mechanics.*
This two volume book is encyclopaedic in content and contains a wealth of information. It is the ultimate reference on the subject.

Frisch 1995. *Turbulence: the legacy of A. N. Kolmogorov.*
This is modern account of turbulence, written in a slightly personal but readable style.

Doering and Gibbon.
This is perhaps as readable account as one can get on the mathematics of regularity and well-posedness of the Navier-Stokes and Euler equations.

Chapter 3

Geostrophic Turbulence

Geostrophic turbulence is, loosely speaking, turbulent flow that exists in flows that are in near-geostrophic balance. Typically, such motions are described by the quasi-geostrophic equations but this is not strictly necessary; we can envision highly nonlinear flow in the frontal geostrophic equations and perhaps even in the planetary geostrophic equations. Nevertheless, the quasi-geostrophic equations have been the home for most theoretical and numerical studies, and we focus on them. The two physical effects that pervade these notes, namely rotation and stratification, continue to provide basic constraints on the flow; indeed their effects are so pervasive that, perhaps ironically, it becomes easier to say something interesting about geostrophic turbulence than about incompressible two- or three-dimensional turbulence. In those problems, there is nothing else to understand other than the problem of turbulence itself; the basic problem rears its head immediately, and unless one can 'solve' that problem there is little else to say. On the other hand, rotation and stratification give one something else to hang to, and it becomes possible to address geophysically interesting phenomena without having to solve the whole problem. Our plan is consider the effects of rotation first, then stratification.

3.1 Rotational Effects in Two-Dimensional Turbulence

We have seen that one of the effects of rapid rotation on a fluid is its 'two dimensionalisation,' captured by the Taylor-Proudman theorem. In the limit of motion of a scale much shorter than the deformation radius, the quasi-geostrophic potential vorticity equation reduces to the two-dimensional equation,

$$\frac{Dq}{Dt} = 0 \tag{3.1}$$

where $q = \zeta + f$. This is the simplest equation with which to study the effects of rotation. The effects of rotation are of course already playing a role in enabling us to reduce a complex three-dimensional flow to two-dimensional flow. Further, suppose that the Coriolis parameter is constant. Then (3.1) becomes simply the two-dimensional vorticity equation

$$\frac{D\zeta}{Dt} = 0. \tag{3.2}$$

Thus, ironically, constant rotation has *no effect* on purely two-dimensional motion. Flow which is already two-dimensional—flow on a soap film, for example—is unaffected by rotation.

Suppose, though, that the Coriolis parameter is variable, as in $f = f_0 + \beta y$. Then we have

$$\frac{D}{Dt}(\zeta + \beta y) = 0 \tag{3.3}$$

or

$$\frac{D\zeta}{Dt} + \beta v = 0 \tag{3.4}$$

If the dominant term in this equation is the one involving β, then we obtain $\beta v = 0$. That is, there is no flow in the meridional direction. The flow, presuming it exists, will be organized into flow in the *zonal* direction. [*An aside:* From the point of view of the quasi-geostrophic asymptotics used in deriving (3.3), one assumes that variations in Coriolis parameter are small, i.e., that $\beta y = O(R_o) f$. However, this does *not* preclude the β term being the dominant one in any subsequent equation, so long as it is not supposed to be $1/O(R_o)$ bigger than the other terms. Alternatively, one might have posited two-dimensionality, and (3.1), *ab initio* in which there is no asymptotic restriction on the size of f.] This constraint may be interpreted as a consequence of angular momentum and energy conservation, as discussed further in chapter (xx). A ring of fluid encircling the earth at a velocity u has an angular momentum per unit mass $a \cos\theta(u + \Omega \cos\theta)$ where θ is the latitude. Moving this ring of air polewards (i.e., giving it a meridional velocity) while conserving its angular momentum requires its velocity and hence energy to increase. Unless there is a source for that energy the flow is constrained to remain zonal.

3.1.1 Organization of turbulence into zonal flow

Heuristics

Let us now consider how flow can become organized into zonal bands, from the perspective of two-dimensional turbulence. Re-write (3.1) in full as

$$\frac{\partial\zeta}{\partial t} + \boldsymbol{u} \cdot \nabla\zeta + \beta v = 0. \tag{3.5}$$

If $\zeta \sim U/L$ and if $t \sim T$ then the terms in this equation scale as

$$\frac{U}{LT} \quad : \quad \frac{U^2}{L^2} \quad : \quad \beta U \tag{3.6}$$

How time scales (i.e., advectively or with a Rossby wave frequency scaling) is determined by which of the other two terms dominates, and this in turn is scale dependent. For large scales the β-term will be dominant, and at smaller scales the advective term is dominant. The cross-over scale, or the 'β-scale' L_β, is given by

$$L_\beta = \sqrt{\frac{U}{\beta}}. \tag{3.7}$$

This is not a unique definition of the cross-over scale, since we have chosen the same length scale to connect vorticity to velocity and to be the β-scale, and it is by no means *a priori* clear that this should be so. If the scale is different, the various terms scale as

$$\frac{U}{LT} \quad : \quad \frac{U\mathcal{Z}}{\mathcal{L}} \quad : \quad \beta U \tag{3.8}$$

where $\mathcal{Z}$ is the scaling for vorticity (i.e., $\zeta \sim \mathcal{Z}$). Then

$$L_{\beta_Z} = \frac{Z}{\beta}. \tag{3.9}$$

In any case, (3.7) and (3.9) both indicate that at some *large* scale the vorticity equation Rossby waves are likely to dominate whereas at small scales advection, and turbulence, dominates.[1] The cross-over is reasonably sharp, as indicated in fig. 3.1.

Another heuristic way to derive (3.7) is by a direct consideration of timescales. The Rossby wave frequency is β/k and an inverse advective timescale is Uk, where k is the wavenumber. Equating these two gives the well-known equation for the β-wavenumber

$$\boxed{k_\beta = \sqrt{\frac{\beta}{U}}} \tag{3.10}$$

This equation is the inverse of (3.7), but note that factors of order unity, and even π, cannot be revealed by simple scaling arguments such as these.

Turbulence Heuristics

Can we be more precise about the scaling using the phenomenology of turbulence? Let us suppose that the fluid is stirred at some well-defined scale k_f, producing an energy input ε. Then, energy cascades to large scales at that same rate. At some scale, the β term in the vorticity equation will start to make its presence felt. By analogy with the procedure for finding the dissipation scale in turbulence, we can find the scale at which linear Rossby waves dominate by equating the inverse of the turbulent eddy turnover time to the Rossby wave frequency. The eddy-turnover time is

$$\tau(k) = \varepsilon^{-1/3} k^{-2/3} \tag{3.11}$$

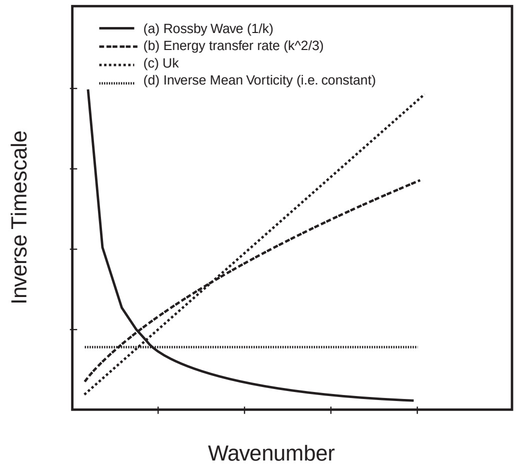

Figure 3.1: The β–turbulence cross-over. The thick solid curve is the frequency of Rossby waves, proportional to $1/k$. The other curves are various estimates of the inverse turbulence timescale, or 'turbulence frequency.' These are the turbulent eddy transfer rate, proportional to $k^{2/3}$ in a $k^{-5/3}$ spectrum; the simple estimate Uk where U is an rms velocity; and the mean vorticity, which is constant. Where the Rossby wave frequency is larger (smaller) than the turbulent frequency, i.e., at large (small) scales, then Rossby waves (turbulence) dominates the dynamics.

Equating this to the inverse Rossby wave frequency k/β gives the β-scale

$$\boxed{k_\beta = \left(\frac{\beta^3}{\varepsilon}\right)^{1/5}}. \tag{3.12}$$

From a practical perspective this is less useful than (3.10), since it is generally much easier to measure velocities than energy transfer rates, or even vorticity. Nonetheless, it is a little more fundamental from the point of view of turbulence since one can often imagine that ε is determined by processes largely independent of the β, whereas the magnitude of the eddies (i.e. U) at the energy containing scales is likely to be a function of β.

Generation of anisotropy

None of the measures discussed so far take into account the anisotropy inherent in Rossby waves, nor do they suggest how the flow might organize itself into zonal structures. Now, energy transfer will be relatively inefficient at those scales where linear Rossby waves dominate. But the wave-turbulence boundary is not isotropic; the Rossby wave frequency is quite anisotropic, being given by

$$\omega_\beta = -\frac{\beta k_x}{k_x^2 + k_y^2}. \tag{3.13}$$

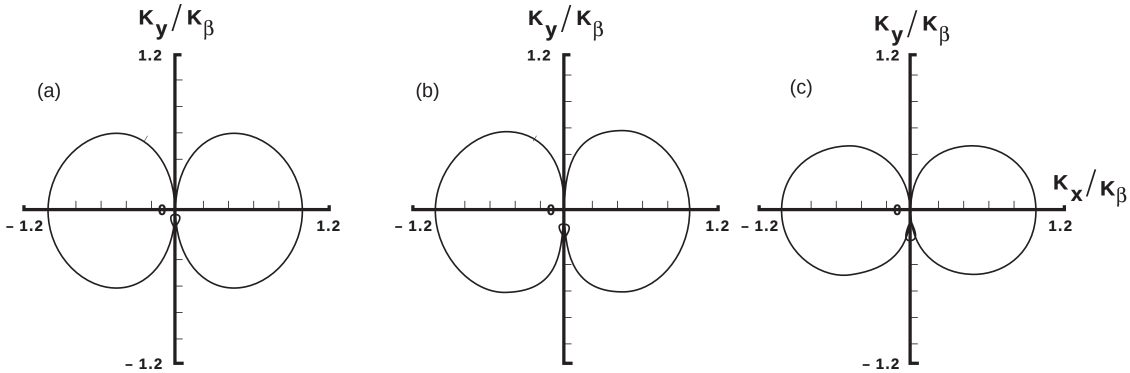

Figure 3.2: The anisotropic wave-turbulence boundary k_β, in wave-vector space calculated three different estimates of the turbulent frequency: (a) The turbulent eddy transfer rate, proportional to $k^{2/3}$ in a $k^{-5/3}$ spectrum; (b) the simple estimate Uk where U is an rms velocity; (c) the mean vorticity, which is constant. All produce qualitatively similar shapes. Within the 'dumb-bells' Rossby waves dominate and enstrophy transfer is inhibited. The inverse cascade thus leads to a predominance of zonal flow.

If, as a first approximation, we suppose that the turbulent part of the flow remains isotropic, the wave turbulence boundary is given from the solution of

$$\varepsilon^{1/3}k^{2/3} = -\frac{\beta k_x}{k^2} \tag{3.14}$$

where k is the isotropic wavenumber. Solving this gives differing expressions for the x- and y-wavenumber components of the wave-turbulence boundary, namely

$$\begin{aligned} k_{x_\beta} &= \left(\frac{\beta^3}{\epsilon}\right)^{1/5} \cos^{8/5}\theta \\ k_{y_\beta} &= \left(\frac{\beta^3}{\epsilon}\right)^{1/5} \sin\theta \cos^{3/5}\theta \end{aligned} \tag{3.15}$$

where the polar co-ordinate is parameterized by the angle $\theta = \tan^{-1}(k_y/k_x)$.

This rather uninformative-looking formula is illustrated in fig. 3.2. If the 'turbulence frequency' is parameterized by the simple expression Uk then the wave turbulence boundary is given from

$$Uk = -\frac{\beta k_x}{k^2}, \tag{3.16}$$

which has solutions

$$\begin{aligned} k_{x_\beta} &= \left(\frac{\beta}{U}\right)^{1/2} \cos^{3/2}\theta \\ k_{y_\beta} &= \left(\frac{\beta}{U}\right)^{1/2} \sin\theta \cos^{1/2}\theta \end{aligned} \tag{3.17}$$

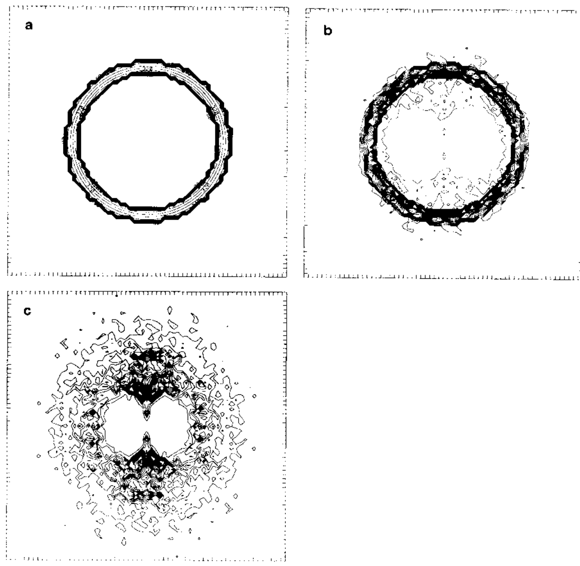

Figure 3.3: Evolution of the energy spectrum in an unforced, inviscid two-dimensional simulation on the β-plane. The panels show contours of energy in wavenumber (k_x, k_y) space. The initial spectrum (a) is isotropic. Panels (b) and (c) show the spectrum at later times.

The region inside the dumb-bell shapes in fig. 3.2 is dominated by Rossby waves, where the natural frequency of the oscillation is *higher* than the turbulent frequency. If the flow is stirred at a wavenumber higher than this, energy cascades to larger scales, but it will be unable to efficiently excite modes in the dumb-bell. Nevertheless, there is still a natural tendency of the energy to seek the gravest mode, and it may do this by cascading toward the $k_x = 0$ axis, i.e., toward zonal flow. In this way a turbulent flow can produce zonally elongated structures.

Does this putative mechanism actually work? Fig. 3.3 shows the freely evolving (unforced, inviscid) energy spectrum in a simulation on a β–plane, with an initially isotropic spectrum. The energy cascades to larger scales, 'avoiding' the region inside the dumb-bell and piling up at $k_x = 0$.

Consistently, forced-dissipative simulations show a robust tendency to produce zonally-elongated structures and zonal jets (fig. 3.4).

3.2 Stratified Geostrophic Turbulence

3.2.1 Quasi-geostrophic flow as an analogue to two-dimensional flow

Now let us consider stratified effects in a simple setting, namely the quasi-geostrophic equations with constant Coriolis parameter and constant stratification. The (dimensional) unforced and inviscid

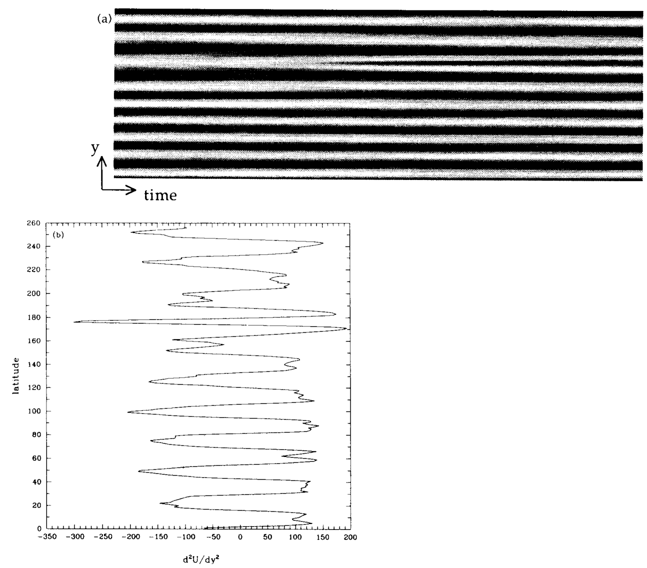

Figure 3.4: (a) Gray-scale image of zonally average zonal velocity as a function of time and latitude, produced in a simulation forced around wavenumber 80 and with $k_\beta = \sqrt{\beta/U} \approx 10$. (b) Values of $\partial^2 U/\partial y^2$ as a function of latitude, where U is the zonally averaged zonal velocity.

governing equation is then

$$\begin{aligned} \frac{Dq}{Dt} &= 0 \\ q &= \nabla^2\psi + \lambda^2 \frac{\partial^2\psi}{\partial z^2} \end{aligned} \tag{3.18}$$

where $\lambda = f_0/N$ and $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\nabla$ is the two-dimensional material derivative. If we rescale the vertical co-ordinate by H, the height of the domain, then $\lambda = (f_0/NH)$ is the inverse first deformation radius.

These equations are strongly analogous to the equations of motion for purely two-dimensional flow. In particular, with appropriate boundary conditions there are two quadratic invariants of the motion, the energy and the enstrophy, which are obtained by multiplying (3.18) by ψ and q and integrating over the domain. The conserved quantities are

$$E = \int \left\{ (\nabla\psi)^2 + \lambda^2 \left(\frac{\partial\psi}{\partial z} \right) \right\}^2 dV \tag{3.19}$$

and

$$Z = \int q^2 dV = \int \left\{ \nabla^2\psi + \lambda^2 \left(\frac{\partial^2\psi}{\partial z^2} \right) \right\}^2 dV \tag{3.20}$$

where the integral is over a *three-dimensional* domain. (However, we continue to use the convention that $\nabla\psi = \mathbf{i}\partial\psi/\partial x + \mathbf{j}\partial\psi/\partial y$ and $\nabla^2\psi = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2$). Horizontal boundary conditions of either no-normal (ψ = constant) or periodic flow will suffice for this. The energy integral in addition requires a horizontal boundary condition at the top and bottom of the domain, and setting $\partial\psi/\partial z = 0$ will suffice. The enstrophy integral in fact holds for each horizontal layer, because potential vorticity in the quasi-geostrophic equations is conserved when advected by the horizontal flow.

The analogy with two-dimensional flow is even more transparent if we further rescale the vertical co-ordinate by $1/\lambda$; i.e., let $z \to z/\lambda$. Then the energy and enstrophy invariants are:

$$E = \int (\nabla_3\psi)^2 dV \tag{3.21}$$

and

$$Z = \int q^2 dV = \int \nabla_3^2\psi \, dV \tag{3.22}$$

where the subscript '3' makes explicit the three-dimensional nature of the derivative. The invariants have exactly the same form as the two-dimensional invariants.

Given that, we should expect that any dynamical behaviour that occurs in the two-dimensional equations, *and that depends solely on the energy/enstrophy constraints*, should have an analogy in quasi-geostrophic flow. The arguments surrounding the transfer of energy to large-scale, and enstrophy to small scale, are based on the existence of such constraints. Thus, classical quasi-geostrophic turbulence will be characterized by a cascade of energy to large-scale with a $k^{-5/3}$ spectrum and a cascade of enstrophy to small-scales with a k^{-3} spectrum. However, the wavenumber is the now *three-dimensional* wavenumber, appropriately scaled by the deformation radius in the vertical. Interestingly, the energy cascade to larger horizontal scales is accompanied by a cascade to larger vertical scales—a *barotropization* of the flow. We will come across this again, for it is an important and robust process in geostrophic turbulence.

In two-dimensional turbulence the equation of motion is isotropic in those two-dimensions. In quasi-geostrophic turbulence, the governing equation (3.18) is decidedly non-isotropic even though some of invariants—the energy and enstrophy—are. Thus, the dynamics of quasi-geostrophic turbulence cannot in general be expected to be isotropic in three-dimensional wavenumber.

Very small horizontal scales

Consider very small horizontal scales, such that

$$L_H^2 \ll \frac{L_Z^2 N^2}{f_0^2} \tag{3.23}$$

where L_H and L_Z are the horizontal and vertical scales of motion, respectively. The governing equation now reduces to the two-dimensional vorticity equation $D\zeta/Dt = 0$ where $\zeta = \nabla^2\psi$. If the vertical scale of motion remains that of the domain size, then this criterion is satisfied for scales much smaller than the deformation radius, but this cannot be guaranteed, especially given the notion of a forward cascade of enstrophy to smaller horizontal and vertical scales.

Very large horizontal scales

Now consider scales for which

$$L_H^2 \ll \frac{L_Z^2 N^2}{f_0^2}. \tag{3.24}$$

The evolution equation is

$$\frac{D}{Dt}\left\{\lambda^2 \frac{\partial^2 \psi}{\partial z^2}\right\} = 0 \tag{3.25}$$

Temperature advection at the boundary. More here....

To examine the detailed dynamical behaviour of quasi-geostrophic turbulence, we turn to a simpler model, that of two-layer flow.

3.2.2 Two-layer quasi-geostrophic flow

We will consider flow in two layers, governed by the quasi-geostrophic equations:

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0 \qquad (1 = 1, 2) \tag{3.26}$$

where

$$\begin{aligned} J(a,b) &= \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial y}\frac{\partial a}{\partial x} \\ q_1 &= \nabla^2\psi_1 + \lambda_1^2(\psi_2 - \psi_1) \\ q_2 &= \nabla^2\psi_2 + \lambda_2^2(\psi_1 - \psi_2) \end{aligned} \tag{3.27}$$

These equations may be considered as either a crude finite difference approximation to the continuous equations (3.18), with a rigid-lid boundary condition ($w = 0$) at the top and bottom. This is probably the most appropriate interpretation for the atmosphere, where the sharp increase in stratification at the tropopause inhibits vertical motion.

Alternately, they may be considered a physical model of two immiscible fluid layers of differing density, which is perhaps a better interpretation for the ocean where the thermocline separates the two fluids. In this case the two layers should be on unequal thickness, since the depth of the thermocline is rarely greater than 1 km. In either case the parameters λ_i are given by the expressions

$$\lambda_i^2 = \frac{f_0^2}{g' D_i} \qquad (i = 1, 2) \tag{3.28}$$

where D_i is the layer depth and g' is the reduced gravity.

Let us further simplify to the case of two equal layer depths, D, possibly thereby foregoing for the moment an easy oceanic relevance. Then $\lambda_1 = \lambda_2 = \lambda = f_0/\sqrt{g'D} = (f_0/ND)$. Thus, λ^{-1} is simply related to the first baroclinic or internal radius of deformation.

The equations then conserve the total energy,

$$E = \int (\nabla\psi_1)^2 + (\nabla\psi_2)^2 + \lambda^2(\psi_1 - \psi_2)^2 \, dA \tag{3.29}$$

and the enstrophy

$$Z = \int \left(\nabla^2\psi_1 + \lambda(\psi_1 - \psi_2)\right)^2 dA, \tag{3.30}$$

both of which may be thought of as finite difference analogs of the continuous invariants. The first two terms in the energy expression represent the kinetic energy, and the last term is the potential energy, proportional to the variance of temperature. The enstrophy is conserved layer-wise, as well as volume integrated.

Baroclinic and barotropic decomposition

Define the barotropic and barotropic streamfunctions by

$$\begin{aligned} \psi &:= \frac{1}{2}(\psi_1 + \psi_2) \\ \tau &:= \frac{1}{2}(\psi_1 - \psi_2) \end{aligned} \tag{3.31}$$

Then the potential vorticities for each layer may be written:

$$q_1 = \nabla^2\psi + (\nabla^2 - \lambda^2)\tau \tag{3.32}$$

$$q_2 = \nabla^2\psi - (\nabla^2 - \lambda^2)\tau \tag{3.33}$$

and the equations of motion may be rewritten as evolution equations for ψ and τ as follows:

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) + J(\tau, (\nabla^2 - \lambda^2)\tau) = 0 \tag{3.34}$$

$$\frac{\partial}{\partial t}(\nabla^2 - \lambda^2)\tau + J(\tau, \nabla^2\psi) + J(\psi, (\nabla^2 - \lambda^2)\tau) = 0 \tag{3.35}$$

We immediately note the following:

1. ψ and τ are vertical modes. ψ is the barotropic mode with a ‘vertical wavenumber,’ k_z, of zero, and τ a baroclinic mode with a ‘vertical wavenumber,’ k_z, of one.

2. Just as purely two dimensional turbulence can be considered to be a plethora of interacting triads, whose two-dimensional vector wavenumbers sum to zero, geostrophic turbulence may be considered to be similarly comprised of a sum of interacting triads, whose vertical wavenumbers must also sum to zero. The types of triad interaction are:

$$(\psi, \psi) \to \psi \tag{3.36}$$

$$(\tau, \tau) \to \psi \tag{3.37}$$

$$(\psi, \tau) \to \tau \tag{3.38}$$

3. Wherever the Laplacian operator acts on τ it is accompanied by $-\lambda^2$. That is, it is *as if* the effective horizontal wavenumber (squared) of τ is shifted, so that $k^2 \to k^2 + \lambda^2$.

Conservation properties

Multiply (3.35) by ψ and (3.35) by τ and horizontally integrating over the domain, assuming once again that the domain is either periodic or has solid walls, gives

$$\frac{dT}{dt} = \int \psi J(\tau, (\nabla^2 - \lambda^2)\tau) dA \tag{3.39}$$

$$\frac{dC}{dt} = \int \tau J(\psi, (\nabla^2 - \lambda^2)\tau) dA \tag{3.40}$$

where

$$T = \int (\nabla\psi)^2 \, dA \tag{3.41}$$

is the energy associated with the barotropic flow (the 'barotropic energy')

$$C = \int (\nabla\tau)^2 \, dA \tag{3.42}$$

is the baroclinic energy.

An easy integration by parts shows that

$$\int \psi J(\tau, (\nabla^2 - \lambda^2)\tau) dA = -\int \tau J(\psi, (\nabla^2 - \lambda^2)\tau) dA \tag{3.43}$$

and therefore

$$\frac{d}{dt} E = \frac{d}{dt}(T + C) = 0 \tag{3.44}$$

or total energy is conserved.

The enstrophy invariant may be expressed in terms of ψ and τ as:

$$Z = \int (\nabla^2\psi)^2 + \left((\nabla^2 - \lambda^2)\tau\right)^2 dA \tag{3.45}$$

$$\frac{dZ}{dt} = 0. \tag{3.46}$$

Just as for two-dimensional turbulence, we may define the spectra of the energy and enstrophy. Thus spectrally as:

$$T = \int T(k)dk \tag{3.47}$$

and similarly for $C(k)$. The enstrophy spectrum $Z(k)$ is then related to the energy spectra by

$$Z = \int Z(k)dk = k^2T(k) + (k^2 + \lambda^2)C(k)dk. \tag{3.48}$$

which is analogous to, but because of the presence of λ^2 not exactly the same as, the relationship between energy and enstrophy in two-dimensional flow. Thus, we begin to suspect that the phenomenology to two-layer turbulence is somehow related to, but perhaps richer than, that of two-dimensional turbulence.

Phenomenological analysis

Two types of triad interactions are possible:

I. Barotropic triads:

An interaction that is purely barotropic (i.e., as if $\tau = 0$) conserves T, the barotropic energy, and the associated enstrophy $\int k^2T(k)dk$. Thus, purely barotropic flow is exactly the same as purely two-dimensional flow. Explicitly, the conserved quantities are

$$\text{Energy:} \qquad \frac{d}{dt}\left(T(k) + T(p) + T(q)\right) = 0 \tag{3.49}$$

$$\text{Enstrophy:} \qquad \frac{d}{dt}\left(k^2T(k) + p^2T(p) + q^2T(q)\right) = 0 \tag{3.50}$$

II. Baroclinic triads:

Baroclinic triads involve two baroclinic wavenumbers (say p, q) interacting with a barotropic wavenumber (say k). Their vector sum is zero. The energy and enstrophy conservation laws for this triad are

$$\frac{d}{dt}\left(T(k) + C(p) + C(q)\right) = 0 \tag{3.51}$$

$$\frac{d}{dt}\left(k^2T(k) + (p^2 + \lambda^2)C(p) + (q^2 + \lambda^2)C(q)\right) = 0 \tag{3.52}$$

Consider the following cases of baroclinic triad:

1. $(p.q, k) \gg \lambda$. Then neglect λ^2 in (3.51) and (3.52). A baroclinic triad then behaves as if it were a barotropic triad. Alternatively, reconsider the layer form of the equations,

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0 \tag{3.53}$$

where

$$\begin{aligned} q_i &= \nabla^2\psi_1 + \lambda^2(\psi_j - \psi_i) \qquad [i = (1, 2),\ j = 3 - 1] \\ &\approx \nabla^2\psi_i \end{aligned} \tag{3.54}$$

In this case, each layer is decoupled from the other. Enstrophy is cascaded to small scales and energy is transferred to larger scales, until such a time as the scale becomes comparable with the deformation scale at which time the dynamics are no longer quasi-barotropic. Note that the transfer of enstrophy to small scales in a purely two-dimensional fashion depends on the two-layer nature of the flow. In reality, the small scales of a continuously stratified flow may not be representable by a two-layer model: remember that in a continuously stratified quasi-geostrophic model the enstrophy cascade occurs in *three-dimensional* wavenumber. Thus, as the horizontal scales become smaller, so does the vertical scale and the first internal deformation radius is not the relevant parameter.

2. $(p.q, k) \ll \lambda$. Then the energy and enstrophy conservation laws collapse to:

$$\frac{d}{dt}\left(C(p) + C(q)\right) = 0 \tag{3.55}$$

That is to say, energy is conserved in the baroclinic field, with the barotropic mode k acting the mediate the interaction. There is no constraint on the transfer of baroclinic energy to smaller scales, and no production of barotropic energy at $k \ll \lambda$.

3. $(p.q, k) \sim \lambda$. This is the most general case, and baroclinic and barotropic modes are both important. Suppose that we define the quasi-wavenumber k' by $k'^2 := k^2 + \lambda^2$ for a baroclinic mode and $k'2 = k^2$ for a barotropic mode, and similarly for p' and q'. Then energy and enstrophy conservation can be written

$$\begin{aligned} \frac{d}{dt}\left(E(k) + E(p) + E(q)\right) &= 0 \\ \frac{d}{dt}\left(k'^2E(k) + p'^2E(p) + q'^2E(q)\right) &= 0 \end{aligned} \tag{3.56}$$

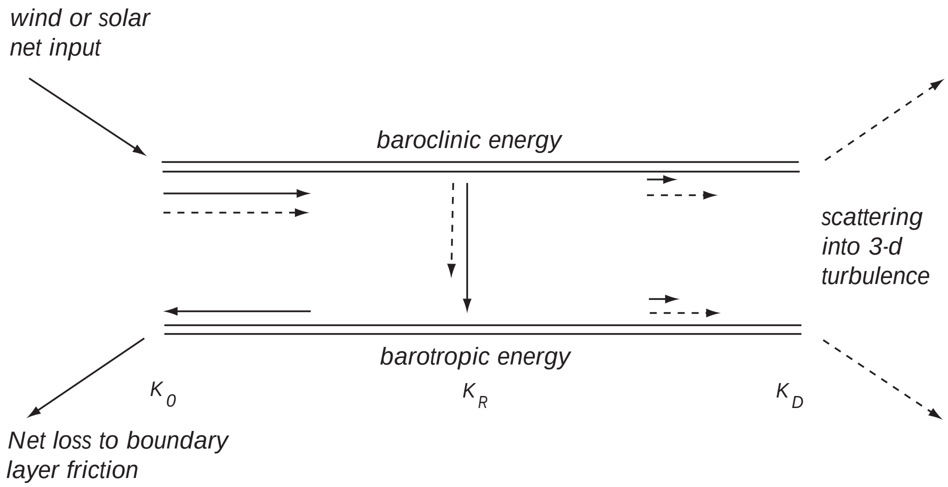

Figure 3.5: Schema of two-layer baroclinic turbulence (after Salmon 1980)

where $E(k)$ is the energy (barotropic or baroclinic) of the particular mode. This are formally identical with the conservation laws for purely two-dimensional flow. By analogy with two-dimensional flow, we expect energy to seek the gravest (smallest quasi-wavenumber) mode. Since the gravest mode has $\lambda = 0$ this implies a *barotropization* of the flow.

4. *Baroclinic Instability* Baroclinic instability in the classic Phillips problem concerns the instability of a flow with vertical but no horizontal shear. We can approximate this in a triad interaction for which $p \ll (k, q, \lambda)$. Then $k^2 \approx q^2$ and the conservation laws are:

$$\begin{aligned} \frac{d}{dt}\left(T(k) + C(p) + C(q)\right) &= 0 \\ \frac{d}{dt}\left(k^2 T(k) + \lambda^2 C(p) + (k^2+\lambda^2)E(q)\right) &= 0 \end{aligned} \tag{3.57}$$

Then

$$\begin{aligned} \dot{C}(p) &= -\left[\dot{C}(q) + \dot{T}(k)\right] \\ \dot{\lambda}^2 C(p) &= -\left[(k^2+\lambda^2)\dot{C}(q) + k^2\dot{T}(k)\right] \end{aligned} \tag{3.58}$$

whence

$$\dot{C}(q) = (\lambda^2 - k^2)\dot{T}(k) \tag{3.59}$$

Baroclinic instability requires that both $\dot{C}(q)$ and $\dot{T}(k)$ be positive. This can only occur if

$$k^2 < \lambda^2. \tag{3.60}$$

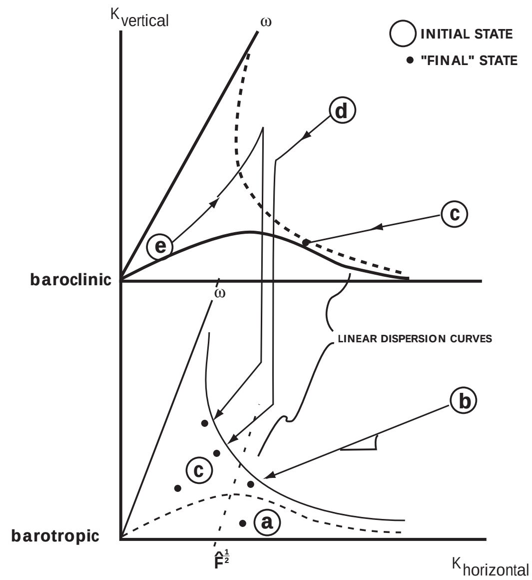

Figure 3.6: Energy flow in two-layer baroclinic turbulence (after Rhines 1977)

Thus, there is a *high wave-number cut-off* for baroclinic instability. This cut-off arises solely from considerations of energy and enstrophy conservation, and is not dependent on linearizing the equations and looking for normal mode instabilities (although of course it is consistent with such a calculation (section **??**).

Phenomenology of Baroclinic Turbulence

Putting together considerations above leads to the following picture of baroclinic turbulence in a two-layer system (see fig. 3.5 and fig. 3.6). At large horizontal scale we imagine some source of baroclinic energy, which in the atmosphere might be the differential heating between pole and equator, or in the ocean might be the wind. Baroclinic instability effects a nonlocal transfer of energy to the deformation scale, where both baroclinic and barotropic modes are excited. From here there is an enstrophy cascade in each layer to smaller and smaller scales, until eventually the scale is small enough so that non-geostrophic effects become important and enstrophy is scattered by three-dimensional effects. At scales larger than the deformation radius, there is an inverse barotropic cascade of energy to larger scales, which causes the excitation of large-scale barotropic modes. the energy at large scales is dissipated by boundary layer effects: Ekman drag, for example, is a scale independent dissipation mechanism.

3.3 †A Scaling Theory for Geostrophic Turbulence

We now construct a phenomenological, but quantitative, theory of two-layer geostrophic turbulence.[2]

Small Scales

For small scales, i.e., $k^2 \gg \lambda^2$, the potential vorticity in each layer is, with $\beta = 0$,

$$\begin{aligned} q_1 &= \nabla^2\psi_1 + \lambda^2(\psi_2 - \psi_1) \approx \nabla^2\psi_1, \\ q_2 &= \nabla^2\psi_2 + \lambda^2(\psi_1 - \psi_2) \approx \nabla^2\psi_2. \end{aligned} \tag{3.61}$$

Thus, each layer is decoupled from the other. Thus, enstrophy will cascade to smaller scales and, should there be an energy source at scales smaller than the deformation scale it will cascade to larger scales. However, baroclinic instability (of the mean flow) occurs at scales *larger* than the deformation radius. Thus, energy extracted from the mean flow is essentially trapped at scales larger than the deformation scale.

Large Scales

For large scales, $k^2 \ll \lambda^2$ we eliminate terms involving k^2 if they appear along with terms involving λ^2. The baroclinic and barotropic equations respectively become

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) = -J(\tau, \nabla^2\tau) - U\frac{\partial}{\partial x}\nabla^2\tau + D_\psi \tag{3.62}$$

and

$$\frac{\partial \tau}{\partial t} + J(\psi, \tau) = -U\frac{\partial}{\partial x}\psi + D_\tau. \tag{3.63}$$

In the barotropic equation, we shall further argue that, in the energy containing scales, $|\psi| \gg |\tau|$ and that we can then ignore the nonlinear term on the right-hand-side of (3.62). This takes the form of a self-consistency argument. Suppose that the main effect of the baroclinic terms, including the forcing terms involving the mean shear, act to supply energy to the barotropic mode, and that the terms on the left-hand-sideof (3.62) indeed dominate at large scales. Then at these large scales the barotropic streamfunction obeys the two-dimensional vorticity equation, and we may expect an energy cascade to large scales with energy spectrum:

$$\mathcal{E}_\psi = C_1\varepsilon^{2/3}k^{-5/3}, \tag{3.64}$$

where C_1 is the Kolmogorov-Kraichnan constant appropriate for the inverse cascade and ε is the as yet undetermined energy flux through the system. We may suppose that this cascade holds for wavenumbers $k_0 < k \ll \lambda$. The wavenumber k_0 is the halting scale of the inverse cascade, determined by one or more of frictional effects, the β–effect, or the domain size.

Now, from the baroclinic streamfunction is being advected as a passive tracer — it is being stirred by ψ. Thus, any energy that is put in at large scales by the interaction with the mean flow (via the term proportional to $U\psi_x$) will be cascaded to smaller scales. Thus, we expect the baroclinic energy spectrum to be that of a forward cascade a passive tracer in a $-5/3$ spectrum, given by (c.f., (2.86))

$$\mathcal{E}_\tau = C_2 \varepsilon_\tau \varepsilon^{-1/3} k^{-5/3} \tag{3.65}$$

where C_2 is the Kolmogorov constant appropriate for a forward passive tracer cascade, $\varepsilon_t au$ is the transfer rate of baroclinic energy and ε is the same quantity appearing in (3.64). Now, since energy is not lost to small scales, we have that $\varepsilon_\tau = \varepsilon$. Thus, the energy in the barotropic and baroclinic modes are comparable at sufficiently large scales. Since the energy density in the former is $(\nabla\psi)^2$ and in the latter $(\nabla\tau)^2 + \lambda^2\tau^2 \sim \lambda^2\tau^2$ the magnitude of ψ must then be much larger than that of τ. Specifically, at the energy containing wavenumber $k_0 \ll \lambda$ we expect

$$|\psi| \sim \frac{\lambda|\tau|}{k_0}. \tag{3.66}$$

The barotropic equation (3.62) then becomes

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) = -U\frac{\partial}{\partial x}\nabla^2\tau + D_\psi. \tag{3.67}$$

3.3.1 Scaling properties

The baroclinic equation may be written as

$$\frac{\partial \tau}{\partial t} + J(\psi, \tau - Uy) = D_\tau. \tag{3.68}$$

That is, τ is stirred by ψ in a mean gradient provided by the shear U. Scaling arguments would suggest that at the scale k_0^{-1} the magnitude of τ is given by

$$\tau \sim \frac{U}{k_0} \tag{3.69}$$

with associated velocity (proportional to the vertical shear of the eddies) at this scale being

$$v_\tau \sim U. \tag{3.70}$$

Using (3.66) the magnitude of the barotropic streamfunction at this scale is given by

$$\psi \sim \frac{\lambda U}{k_0^2}, \tag{3.71}$$

with asscoiated barotropic velocity at the energy containing scale being given by

$$v_\psi \sim \frac{\lambda U}{k_0}. \tag{3.72}$$

Multiplying (3.63) by τ and integrating over the domain, the energy input to the system is given by the polewards heat flux. That is,

$$\varepsilon = U\lambda^2 \overline{\psi_x \tau} \sim \frac{U^3\lambda^3}{k_0^2}. \tag{3.73}$$

The correlation between ψ_x and τ cannot be determined by this argument. Nevertheless, we have produced a physically based 'closure' for the flux of energy through the system in terms only of the mean shear and other external (at least to quasi-geostrophic theory) parameters such as the deformation scale and the halting scale k_0.

Finally, we calculate the 'eddy diffusivity' defined by

$$\kappa \equiv \frac{\overline{v'b'}}{\frac{\partial \bar{b}}{\partial y}} = \frac{\overline{\psi_x \tau}}{\frac{\partial \bar{\tau}}{\partial y}} \tag{3.74}$$

Using (3.69) and (3.71) gives

$$\kappa \sim \frac{\lambda U}{k_0^2} \tag{3.75}$$

which, if the mixing velocity is the barotropic stirring velocity, implies a mixing length of k_0^{-1}. (Note also that the eddy diffusivity is just the magnitude of the barotropic streamfunction at the energy containing scales.)

3.3.2 The β–effect

As discussed in section 3.1 the β–effect provides a soft barrier for the inverse cascade, at the scale

$$k_\beta \sim \left(\frac{\beta^3}{\varepsilon}\right)^{1/5} \tag{3.76}$$

The energy containing scale is not *a priori* the same as k_β, because in the absence of frictional processes energy will still seek to cascade to larger scales, and in doing so become anisotropic. However, the cascade becomes much less efficient and friction does have more time to act to halt the cascade. Using (3.73) and (3.76) we obtain

$$k_\beta = \frac{\beta}{U\lambda}. \tag{3.77}$$

Now using (3.75) and (3.77) we obtain for the eddy diffusivity,

$$\kappa \sim \frac{\lambda^3 U^3}{k_\beta^2} \tag{3.78}$$

The magnitudes of the eddies themselves are easily given using (3.71) and (3.69), namely

$$\tau \sim \frac{U^2 \lambda}{\beta}, \tag{3.79}$$

$$v_\tau \sim U, \tag{3.80}$$

and

$$\psi \sim \frac{U^3 \lambda^3}{\beta^2}, \tag{3.81}$$

$$v_\psi \sim \frac{U^2 \lambda^2}{\beta}. \tag{3.82}$$

Again, the magnitude of the barotropic streamfunction is equal to the eddy diffusivity.

Clearly, in this model, the eddies become *less* energetic with increasing β, although the cause is slightly different from the linear case in which the presence of a mean gradient of planetary vorticity usually acts to stabilize a flow, and reduce the growth rate of baroclinic instability. Furthermore, the eddy amplitudes increase more rapidly with the mean shear than previously. The reason for these is that as βdecreases, the inverse cascade can extend to larger scales, thereby increasing the overall energy of the flow. Similarly, as U increases not only does the eddy amplitude increase as a direct consequence (as is (3.69) and (3.71)) but ssalso $k_\beta$falls (see (3.77)), giving rise to a superlinear increase of the eddy magnitudes with U.

3.3.3 Discussion

Notes

1. The ‘wave-turbulence’ boundary harks back to Rhines (1975) in GFD.
2. The theory and related numerical simulations was expounded in a pair of papers by V. Larichev and I. Held (see bibliography).