Reflections on a Christmas-tree bauble

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1 Introduction

Totally reflecting spheres are most commonly encountered as decorations hanging from Christmas trees, and as steel ball-bearings. In earlier times they were hung in the windows of country cottages to ward off evil spirits, and known as witch balls; these are now highly prized as antiques. On the Continent large silvered glass globes are sometimes employed to decorate gardens, because of the attractive images of cloudscapes that they reflect. The imaging properties of these spheres must surely have been analysed many times during the past few hundred years (if not before), but a fairly extensive search through the literature has failed to uncover any published work on the subject, except for the book by Minnaert (1940) which contains a brief description of the properties of garden globes, together with a number of applications to the study of atmospheric optical phenomena. Every book on optics deals with spherical mirrors, of course, but the treatments seem universally restricted to the paraxial rays from mirrors of small aperture; even in these works, it is often the properties of *concave* reflectors that are emphasized (because of their usefulness as magnifiers), convex mirrors being relegated to brief mention as a special case to which the formulae remain applicable if the sign of the radius of curvature is reversed.

As far as the teaching of optics at an elementary level is concerned, this neglect of the properties of complete spheres is a pity. Certainly the restriction to paraxial rays is necessary in simple optical instruments, but we shall show in §3 that it is unnecessary when dealing with observations with the naked eye alone. The main purpose of this article is to draw the attention of science teachers to a number of visually appealing phenomena, which are analysed in the first part of §2 in a way that should be understandable by most pupils at O level or even earlier; there is no need to introduce mathematical approximations, such as 'sin θ can be replaced by θ', which may confuse the application of the law of specular reflection—perhaps the most intuitively immediate of all the topics in elementary physics. The deeper analysis of the structure of the images, in §3, might help A level pupils to understand more precisely why the paraxial approximation is necessary for telescopes, etc. (the contents of that section could perhaps form the basis of a 'research project').

2 The appearance of the images

Consider a sphere of radius r (figure 1) whose centre C lies at a distance d from the eye E of an observer. We shall ignore diffraction and stereoscopic effects throughout, and defer until §3 the implications of the finite size of the pupil of the eye. The rays reaching E from the surface must lie within a cone of semiangle $\phi_{\max}$, given by

$$\sin \phi_{\max} = r/d. \qquad (1)$$

Rays making an angle ϕ at the eye, measured from the 'forward direction' EC, have been reflected at a point P after coming from a direction making an angle θ with EC. As ϕ *decreases* from its maximum value $\phi_{\max}$, θ *increases* from a minimum, equal to $\phi_{\max}$ (the case of grazing incidence), to a maximum value of 180° (when ϕ is zero). This means that rays from *all incident directions* θ are imaged in the sphere, except those lying in the cone surrounding the forward direction which is intercepted by the sphere itself.

It is this *wide-angle property* of the reflection from a complete sphere which is responsible for the fascination of the images that may be seen in it (see figure 2, and two paintings by Escher (1967)). By holding the sphere against the window of a room, for instance, an observer looking outwards can see the interior of the room (which is behind him) reflected in the central regions, surrounded by the (weirdly curved) image of the window frame, which in turn is surrounded by most of the scene outside the room, reflected at near-grazing incidence around the periphery of the sphere (which may therefore be thought of as a sort of 'topological synthesiser', enabling the inside and outside of a room to be seen together). Out-of-doors, the whole sky can be seen reflected in a portion of the globe, and the resulting increase of contrast gradients often enables cloud detail to be discerned which is not immediately apparent to the unaided eye (this is an interesting example, inasmuch as it shows the occasional advantage of a minifier over a magnifier—the detail concerned would not be visible in a telescope). The spheres can also be used to take wide-angle photographs over a range of nearly 360° (actually $360°-2\phi_{max}$), whereas conventional wide-angle lenses (also used in spyholes in front doors to inspect potentially unwelcome callers) have a range that rarely exceeds 180°; the only disadvantage of the reflectors is the unavoidable appearance of the camera lens in the centre of the picture. The motion of objects is dramatically distorted by reflection in the spheres—a car approaching from the front appears on the edge, moves inwards, turns around, and disappears by receding into the centre of the sphere, while overhanging foliage seems to 'pour' into the centre as the observer passes underneath.

Of course an incomplete sphere will still show the wide-angle property to a certain degree, which explains the common use of convex mirrors for rear viewing on cars and bicycles, as an anti-theft precaution in supermarkets, and simply for room decoration (for an early example, see the painting 'The Arnolfini Wedding', by Jan van Eyck 1434, in the National Gallery, London).

To proceed further, it is necessary to know the precise relation between the angle of incidence θ and the angle ϕ at the eye (figure 1). An elementary trigonometrical calculation, based on the law of specular reflection at P, leads to the result

$$d \sin \phi = r \cos\left(\frac{\theta-\phi}{2}\right). \tag{2}$$

As d increases, and the eye recedes from the sphere, the images (which are localized within the sphere, see §3) appear to be distributed across a disc—the projection of the sphere on a plane perpendicular to EC. In this limit, when ϕ_{max} is small, virtually the whole sphere of incident directions θ is imaged onto this disc. We specify positions on the disc by their distance a from the centre; from figure 1, we have

$$a = EP \times \sin\phi \simeq d \sin \phi \quad (d \gg r)$$

and substitution into (2), together with neglect of ϕ compared with θ on the right-hand side of (2) leads to

$$a = r \cos\frac{\theta}{2}. \tag{3}$$

This relation, which may also be derived directly, shows how the image moves from the edge of the disc to the centre as the object moves from the forward to the backward directions. The limit we are discussing, which is valid in the commonly-occuring situation where the spheres subtends a small angle at the eye, is not the paraxial case, because although the rays

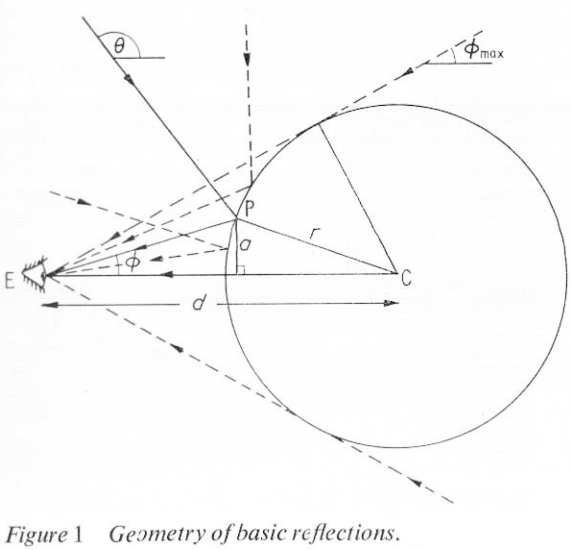

Figure 1 *Geometry of basic reflections.*

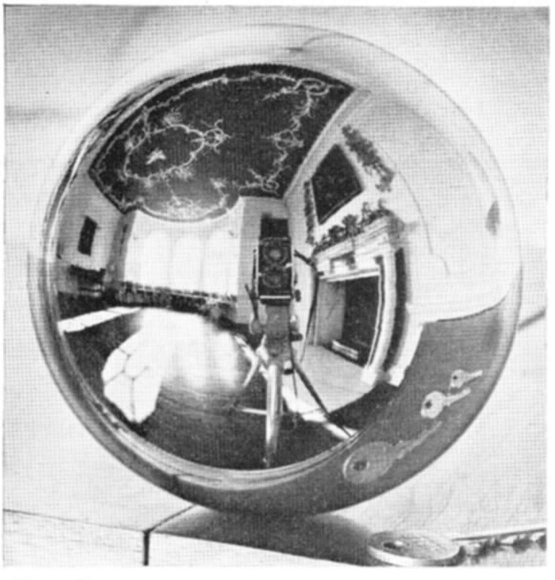

Figure 2

entering the eye are nearly parallel, the rays incident on the sphere come from all directions. An interesting special case of (3) occurs when looking down on a globe out-of-doors (figure 3). The horizon, which is the locus of points making an angle θ equal to 90° with the vertical EC, is imaged as a circle whose radius is

$$a = r\cos 45° = \frac{r}{\sqrt{2}}$$

within this circle is the reflection of the sky, while outside it can be seen the ground.

To include the three-dimensional aspects of the problem, we rotate figure 1 about the axis EC, specifying the rotation by an azimuth angle ψ which runs from 0° to 360°. We can then regard the reflection as *mapping* a *solid angle* $\mathrm{d}\Omega$ (figure 4) of incident rays, given by

$$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\psi$$

onto an *area* $\mathrm{d}A$ of the image disc, given by

$$\mathrm{d}A = a\,\mathrm{d}a\,\mathrm{d}\psi.$$

The *mapping ratio* from solid angle to area is found, by making use of (3), to be

$$\left|\frac{\mathrm{d}A}{\mathrm{d}\Omega}\right| = \frac{a}{\sin\theta}\left|\frac{\mathrm{d}a}{\mathrm{d}\theta}\right| = \frac{r\cos(\theta/2)\times r\sin(\theta/2)}{2\sin\theta} = \frac{r^2}{4} \quad (4)$$

a relation which can be checked by noting that the total area of the disc is πr^2, while the total solid angle of the sphere of directions is 4π. The important point is that the mapping is *uniform*—equation (4) is independent of θ and ψ, so that a given $\mathrm{d}\Omega$ is mapped onto the same $\mathrm{d}A$, no matter where the incident ray bundle is situated in relation to the forward direction.

This uniformity of mapping has interesting implications: for instance, we know that any variations of sky brightness that are observed in a globe are real, and are not artefacts of the reflection process. Further, the strikingly distorted images seen in the globe nevertheless subtend the same relative areas as do the original objects, only angles and the straightness of lines being changed. This suggests that an *area-preserving* map of the whole world would be seen on the image disc if a small silvered globe were situated at the centre of the earth (assumed transparent), those countries being mapped with least distortion which lie on OE produced (ie behind the observer). Figure 5 shows the parallels and meridians for polar and equatorial viewing positions. This mapping is known to cartographers as 'Lambert's azimuthal equal-area projection', introduced in 1772 (see Raisz 1948); it is widely employed to show world air routes.

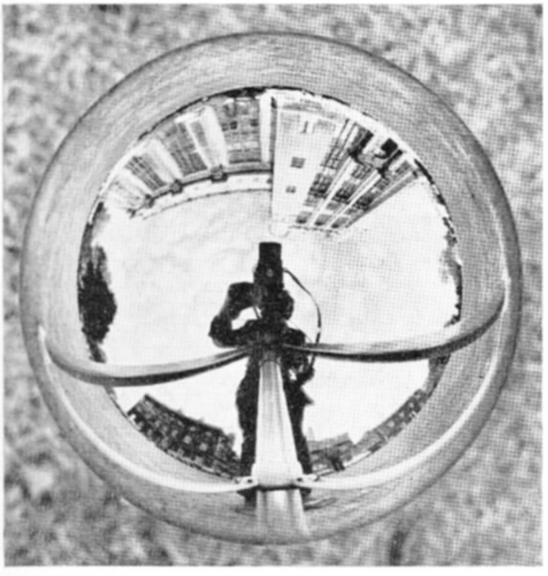

Figure 3

3 The structure of the images

We now ask: where are the images localized, that are formed by reflection in a silvered globe? This question played no part in our treatment of §2, because there we considered the eye pupil as a point, which in general admits only a single ray from a point object, whose image is therefore not localized at all (this is why a hole, pierced with a fine needle in a piece of metal foil held close to the eye, enables the long-sighted to read in comfort at close range, and the short-sighted to see distant scenes clearly). Consider all those rays hitting the sphere which have emerged from a point object O (see figure 6, which is the same as figure 1 with the arrows on the rays reversed). They will be reflected into all directions, and if the emergent rays are produced backwards into the sphere, they will intersect, not at a single image point, but on a *caustic surface*, so named because on the corresponding surface for a concave reflector or convex lens the sun's heat may be concentrated.

The eye will see the 'image' of O at different points

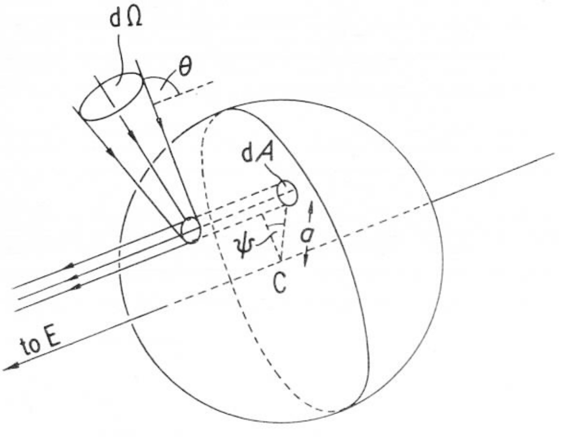

Figure 4 *Mapping of incident rays onto image disc.*

on this surface (or curve, if we restrict ourselves to a plane), depending (figure 6) on the angle of observation θ, which has the same meaning as in §2. It is easily seen that the caustic always lies within the sphere (touching it only for the grazing rays), so that the images are all virtual. The point nearest the centre is the cusp K on the line CO corresponding to the directly-reflected ray ($\theta = 180°$). The closest possible approach of this cusp to the centre occurs when the object lies at infinity, in which case KC is $r/2$—the well-known value for the focal length of a convex mirror for paraxial rays. Thus, whatever the positions of object and eye, we know that the image will always lie between two spheres, centred on C, whose radii are r and $r/2$.

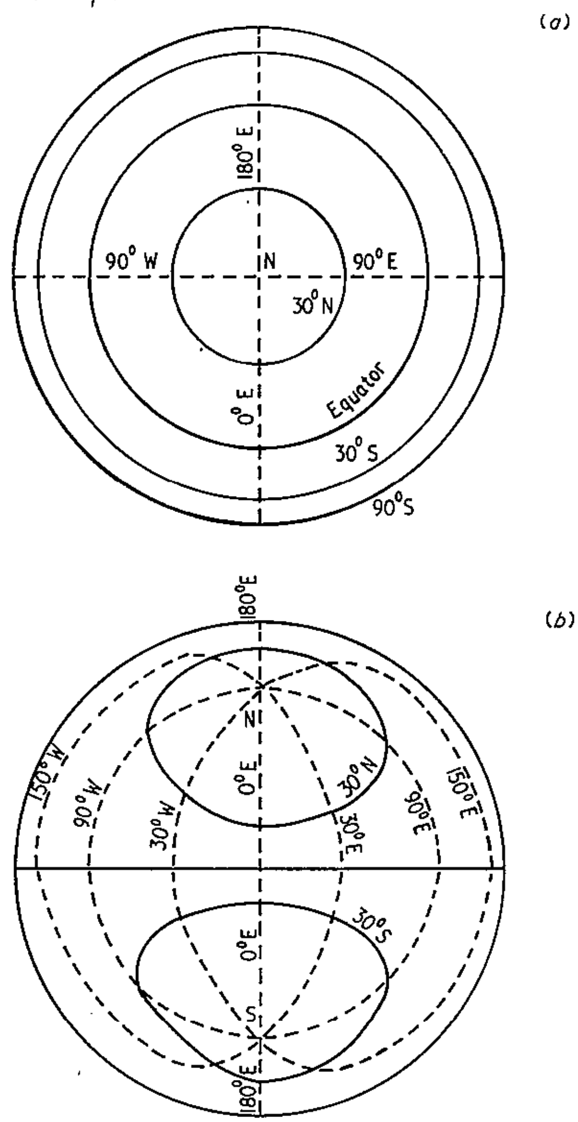

Figure 5 *Sketch of parallels (———) and meridians (— — —) on an equal-area map of the world for (a) polar projection (b) equatorial projection.*

This blurring-out into caustics of the images of point objects in wide-aperture mirrors is the reason why most treatments are restricted to the paraxial case. Yet the images seen in spheres—and indeed in the 'crazy mirrors' of varying curvature found at fairgrounds—are frequently crystal clear, so that we must investigate why it is that caustics are not often observed directly with the unaided eye. (The caustics seen in teacups are no exception to this rule, because they are observed not by looking along the rays whose envelope they are, but by scattered light.) The basic reason is that the small aperture e of the eye pupil admits rays from only a short arc-length dS of the caustic, while the least distance of distinct vision g sets a limit to the closeness with which this length can be examined.

To estimate the angle α subtended at the eye by the 'image' of height h which this portion of the caustic essentially constitutes, we consider figure 7, which shows (greatly exaggerated) those rays with angles of emergence ranging from θ to $\theta + d\theta$ which enter the eye. The size h is clearly given by

$$h = \frac{dS d\theta}{2} = \frac{dS}{d\theta}\frac{d\theta^2}{2} = R\frac{d\theta^2}{2}$$

where R is the radius of curvature of the caustic at the point considered. The angle α subtended at the eye is

$$\alpha = \frac{h}{g} = \frac{Rd\theta^2}{2a} \qquad (5)$$

while $d\theta$ is given by (figure 7)

$$e = h + gd\theta \approx gd\theta \qquad (6)$$

(the term h has been neglected because it only becomes

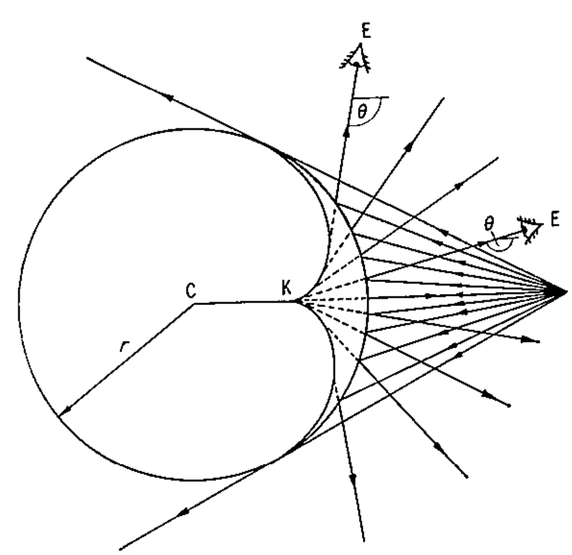

Figure 6 *Formation of virtual caustic of reflection.*

comparable with e for huge spheres exceeding several metres in diameter—in other words, the 'image' is generally much smaller than the eye pupil, whose diameter it can never exceed under any circumstances). Substituting for $d\theta$ in (5) gives

$$\alpha = \frac{Re^2}{2g^3}.$$

Caustics are observable if α exceeds the smallest angle β that the eye can resolve, which is given by the Rayleigh criterion as

$$\beta = 1{\cdot}22\lambda/e$$

λ being the wavelength of the light used. The critical ratio α/β is therefore

$$\frac{\alpha}{\beta} = \frac{R}{2{\cdot}44\lambda}\left(\frac{e}{g}\right)^3. \tag{7}$$

This formula applies to observation in any curved surface. We can estimate R for a sphere by inspecting figure 6, from which it is clear that R is of order r; an exact calculation gives

$$R = \frac{3r}{4}\cos\frac{\theta}{2}$$

for the case where the object point is very far away. The ratio (7) now becomes

$$\frac{\alpha}{\beta} = \frac{0{\cdot}31r}{\lambda}\left(\frac{e}{g}\right)^3\cos\frac{\theta}{2} \tag{8}$$

from which we can immediately see that the cusp K is unobservable (this corresponds to back reflection of the paraxial rays from the front of the sphere, with $\theta = 180°$).

Under normal daytime conditions the aperture e of the pupil is about 3×10^{-3}m, while the least distance g of comfortably distinct vision is about 0·2m; taking the wavelength of yellow light as 5×10^{-7}m, we get

$$\frac{\alpha}{\beta} \simeq 2r\cos\frac{\theta}{2}$$

where r, the sphere radius, is measured in metres. This ratio only exceeds unity for spheres more than about a metre across, when the additional complication arises that the caustic is buried deep inside the sphere (except at near-grazing incidence), and cannot be approached to within the distance g. Thus we do not expect to see caustics easily in daylight.

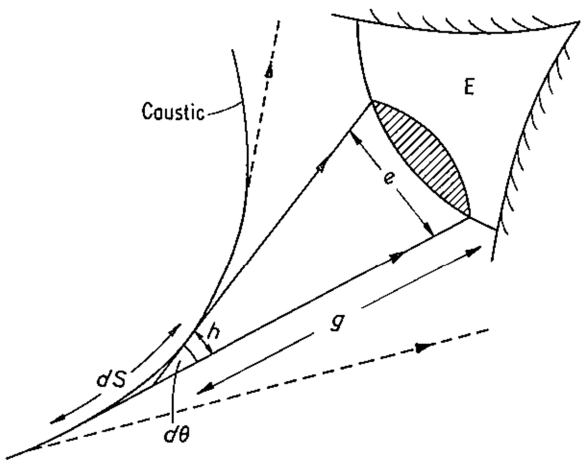

Figure 7 Part of the caustic contributing to rays admitted by the pupil.

A dramatic increase in the magnitude of α/β can be obtained by using the facts that a dark-adapted pupil can expand to at least double its normal size, while most people can with a certain discomfort focus on objects as close as 0·12m. The ratio α/β now becomes

$$\frac{\alpha}{\beta} \simeq 75\, r\cos\frac{\theta}{2}$$

and we predict that in these new circumstances the distortion of the images of point objects should be observable in spheres down to a few centimetres across.

These conclusions are borne out by experiment. In a darkened room the caustic is clearly seen in a sphere of 0·1 m radius, but it vanishes rapidly—ie the image contracts almost to a point—when a light is switched on, causing the pupil to contact. Observation is best carried out at about $\theta = 90°$, when $\cos\theta/2 \simeq 0{\cdot}7$, the slight improvement in resolution at more glancing angles (smaller θ) being outweighed by the greater effects of surface imperfections, etc. An adequate point source is a pinhole in a piece of kitchen foil, wrapped over the shade of a desk lamp.

In optical instruments, such as telescopes, where the image is greatly magnified by an eyepiece, the caustic would be clearly visible, and would constitute a serious aberration of the system, if observation were not restricted to the narrow bundle of paraxial rays for which $\cos\theta/2$ is negligible (of equation (8)).

4 Conclusions

It seems that silvered globes may be a useful aid in the teaching at various levels of a wide range of concepts, including the laws of reflection, plane trigonometry, solid angle, mapping of a sphere, image formation, and caustics. An easy way of producing the globes is to silver the inside surface of a round-bottomed flask, whose neck provides a useful handle (care should be taken to select a flask whose glass is relatively free from blemishes, and it is advisable to stop up the end of the neck with a rubber bung to exclude dust.)

Acknowledgments

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References

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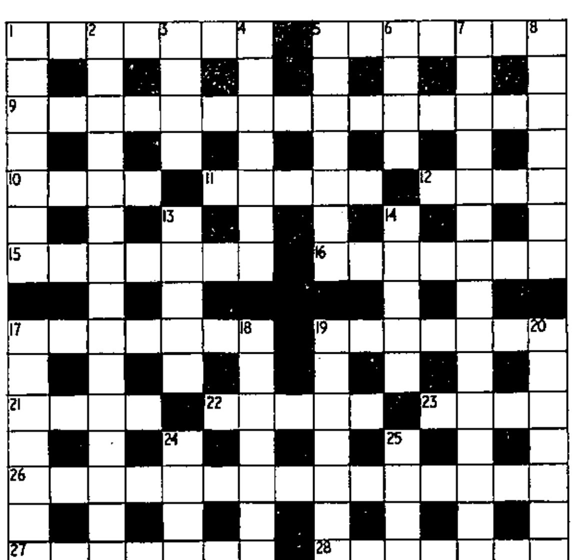
Project Technology 1970–72

The Schools Council has produced a booklet *Schools Council Project Technology: The Next Two Years* which is a progress report of the Schools Council's Project Technology as well as an outline of the Project's plans for 1970–2. It explains the present position of the project in relation to grants from the Council and the program of work over the two-year extension period, which began in 1970, for the Project's regional groups and associated teachers.

Copies of *Schools Council Project Technology: The Next Two Years* are available (25p for a single copy, £3 for 20 copies, £6·05 for 50 copies—all inclusive of postage) from the Despatch Section, Schools Council, 160 Great Portland Street, London W1N 6LL. The correct money must be sent with the order, and cheques made payable to Schools Council Publications.

Physics crossword



Solution on p52

Clues across

1 This kind of physics gets us right into the middle of things (7)
5 Three-legged laboratory animals? (7)
9 Extravagant notions that can rarely be tolerated nowadays . . . (10, 5)
10 . . . for if I go in a backward rail union they may lead to this (4)
11 Twisted ropes can cause a problem (5)
12 Energy after an old mass gives a single sound (4)
15 French degrees—of meaning only (7)
16 Bodies at the tops of universities (7)
17 Does this if disturbed from unstable equilibrium (7)
19 Cattle men with senior scouts after optical measure (7)
21 A sweet potato back for the old American (4)
22 Aspect of a hill that doesn't mean what it says . . . (5)
23 . . . though a gash may reveal more in the escarpment (4)
26 Watching television may produce a dark current (7, 8)
27 Statutes of the highest state (3, 4)
28 Brunel's work did this to the gorge (7)

Clues down

1 Weighty but ineffective lump (7)
2 Aspect of our subject that leans into a neighbouring lab. (8, 7)
3 Loving little planet (4)
4 Unruly appearance of South American city to US (7)
5 Adjustor capacitors without a million make mini-plastics (7)
6 Horticultural centre of all eyes (4)
7 The last drop in the test-tube of an adventurous chemist (3, 3, 3, 6)
8 All may be go, but their momentum is still conserved (7)
13 Essence of music from populous region round French water-spout (5)
14 The positive one in the cell (5)
17 Boredom experienced by one of those heavenly twins? (4–3)
18 Ag makes the mirror a mirror (7)
19 Treacherous faults which can make a lattice useful (7)
20 Little bit of light in past fast motion went all over the place (7)
24 Mountains commemorated in Mesozoic rocks (4)
25 What counts, we hear—10 to 9 times (4)

Eric Deeson