# Toward a Theory of Instruction

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Chapter 3

## Notes on a Theory of Instruction

IN THIS ESSAY I shall attempt to develop a few simple theorems about the nature of instruction. I shall try to illustrate them by reference to the teaching and learning of mathematics. The choice of mathematics as a mode of illustration is not premised on the typicality of mathematics, for mathematics is restricted to well-formed problems and does not concern itself with empirical proof by either experiment or observation. Nor is this an attempt to elucidate mathematical teaching as such, for that would be beyond my competence. Rather, mathematics offers an accessible and simple example for what, perforce, will be a simplified set of propositions about teaching and learning. And there are data available from mathematics learning that have some bearing on our problem.

The plan is as follows. First some characteristics of a theory of instruction will be set forth, followed by a statement of some highly general theorems about the instructional process. I shall then attempt, in the light of specific observations of mathematics learning, to convert these general propositions into workable hypotheses. In conclusion, some remarks will be made on the nature of research in support of curriculum making.

## THE NATURE OF A THEORY OF INSTRUCTION

A theory of instruction is *prescriptive* in the sense that it sets forth rules concerning the most effective way of achieving knowledge or skill. By the same token, it provides a yardstick for criticizing or evaluating any particular way of teaching or learning.

A theory of instruction is a *normative* theory. It sets up criteria and states the conditions for meeting them. The criteria must have a high degree of generality: for example, a theory of instruction should not specify in *ad hoc* fashion the conditions for efficient learning of third-grade arithmetic; such conditions should be derivable from a more general view of mathematics learning.

One might ask why a theory of instruction is needed, since psychology already contains theories of learning and of development. But theories of learning and of development are descriptive rather than prescriptive. They tell us what happened after the fact: for example, that most children of six do not yet possess the notion of reversibility. A theory of instruction, on the other hand, might attempt to set forth the best means of leading the child toward the notion of reversibility. A theory of instruction, in short, is concerned with how what one wishes to teach can best be learned, with improving rather than describing learning.

This is not to say that learning and developmental theories are irrelevant to a theory of instruction. In fact, a theory of instruction must be concerned with both learning and development and must be congruent with those theories of learning and development to which it subscribes.

A theory of instruction has four major features.

First, a theory of instruction should specify the experiences which most effectively implant in the individual a predisposition toward learning—learning in general or a particular type of learning. For example, what sorts of relationships with people and things in the preschool environment will tend to make the child willing and able to learn when he enters school?



Second, a theory of instruction must specify the ways in which a body of knowledge should be structured so that it can be most readily grasped by the learner. "Optimal structure" refers to a set of propositions from which a larger body of knowledge can be generated, and it is characteristic that the formulation of such structure depends upon the state of advance of a particular field of knowledge. The nature of different optimal structures will be considered in more detail shortly. Here it suffices to say that since the merit of a structure depends upon its power for *simplifying information*, for *generating new propositions*, and for *increasing the manipulability of a body of knowledge*, structure must always be related to the status and gifts of the learner. Viewed in this way, the optimal structure of a body of knowledge is not absolute but relative.

Third, a theory of instruction should specify the most effective sequences in which to present the materials to be learned. Given, for example, that one wishes to teach the structure of modern physical theory, how does one proceed? Does one present concrete materials first in such a way as to elicit questions about recurrent regularities? Or does one begin with a formalized mathematical notation that makes it simpler to represent regularities later encountered? What results are in fact produced by each method? And how describe the ideal mix? The question of sequence will be treated in more detail later.

Finally, a theory of instruction should specify the nature and pacing of rewards and punishments in the process of learning and teaching. Intuitively it seems quite clear that as learning progresses there is a point at which it is better to shift away from extrinsic rewards, such as a teacher's praise, toward the intrinsic rewards inherent in solving a complex problem for oneself. So, too, there is a point at which immediate reward for performance should be replaced by deferred reward. The timing of the shift from extrinsic to intrinsic and from immediate to deferred reward is poorly understood and obviously important. Is it the case, for example, that wherever learning involves the integration of a long sequence of acts, the shift should be made as early as possible from immediate to deferred reward and from extrinsic to intrinsic reward?

It would be beyond the scope of a single essay to pursue in any detail all the four aspects of a theory of instruction set forth above. What I shall attempt to do here is to explore a major theorem concerning each of the four. The object is not comprehensiveness but illustration.

### PREDISPOSITIONS

It has been customary, in discussing predispositions to learn, to focus upon cultural, motivational, and personal factors affecting the desire to learn and to undertake problem solving. For such factors are of deep importance. There is, for example, the relation of instructor to student—whatever the formal status of the instructor may be, whether teacher or parent. Since this is a relation between one who possesses something and one who does not, there is always a special problem of authority involved in the instructional situation. The regulation of this authority relationship affects the nature of the learning that occurs, the degree to which a learner develops an independent skill, the degree to which he is confident of his ability to perform on his own, and so on. The relations between one who instructs and one who is instructed is never indifferent in its effect upon learning. And since the instructional process is essentially social—particularly in its early stages when it involves at least a teacher and a pupil—it is clear that the child, especially if he is to cope with school, must have minimal



mastery of the social skills necessary for engaging in the instructional process.

There are differing attitudes toward intellectual activity in different social classes, the two sexes, different age groups, and different ethnic groupings. These culturally transmitted attitudes also pattern the use of mind. Some cultural traditions are, by count, more successful than others in the production of scientists, scholars, and artists. Anthropology and psychology investigate the ways a "tradition" or "role" affects attitudes toward the use of mind. A theory of instruction concerns itself, rather, with the issue of how best to utilize a given cultural pattern in achieving particular instructional ends.

Indeed, such factors are of enormous importance. But we shall concentrate here on a more cognitive illustration: upon the predisposition to explore alternatives.

Since learning and problem solving depend upon the exploration of alternatives, instruction must facilitate and regulate the exploration of alternatives on the part of the learner.

There are three aspects to the exploration of alternatives, each of them related to the regulation of search behavior. They can be described in shorthand terms as *activation, maintenance,* and *direction.* To put it another way, exploration of alternatives requires something to get it started, something to keep it going, and something to keep it from being random.

The major condition for activating exploration of alternatives in a task is the presence of some optimal level of uncertainty. Curiosity, it has been persuasively argued,[1] is a response to uncertainty and ambiguity. A cut-and-dried routine task provokes little exploration; one that is too uncertain may arouse confusion and anxiety, with the effect of reducing exploration.

The maintenance of exploration, once it has been activated,

[1] D. E. Berlyne, *Conflict, Arousal, and Curiosity* (New York: McGraw-Hill, 1960).

requires that the benefits from exploring alternatives exceed the risks incurred. Learning something with the aid of an instructor should, if instruction is effective, be less dangerous or risky or painful than learning on one's own. That is to say, the consequences of error, of exploring wrong alternatives, should be rendered less grave under a regimen of instruction, and the yield from the exploration of correct alternatives should be correspondingly greater.

The appropriate direction of exploration depends upon two interacting considerations: a sense of the goal of a task and a knowledge of the relevance of tested alternatives to the achievement of that goal. For exploration to have direction, in short, the goal of the task must be known in some approximate fashion, and the testing of alternatives must yield information as to where one stands with respect to it. Put in briefest form, direction depends upon knowledge of the results of one's tests, and instruction should have an edge over "spontaneous" learning in providing more of such knowledge.

### STRUCTURE AND THE FORM OF KNOWLEDGE

Any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in a recognizable form.

The structure of any domain of knowledge may be characterized in three ways, each affecting the ability of any learner to master it: the *mode of representation* in which it is put, its *economy*, and its effective *power*. Mode, economy, and power vary in relation to different ages, to different "styles" among learners, and to different subject matters.

Any domain of knowledge (or any problem within that domain of knowledge) can be represented in three ways: by a set of actions appropriate for achieving a certain result (enactive representation); by a set of summary images or graphics that stand for a concept without defining it fully (iconic repre-



sentation); and by a set of symbolic or logical propositions drawn from a symbolic system that is governed by rules or laws for forming and transforming propositions (symbolic representation). The distinction can most conveniently be made concretely in terms of a balance beam, for we shall have occasion later to consider the use of such an implement in teaching children quadratic functions. A quite young child can plainly act on the basis of the "principles" of a balance beam, and indicates that he can do so by being able to handle himself on a see-saw. He knows that to get his side to go down farther he has to move out farther from the center. A somewhat older child can represent the balance beam to himself either by a model on which rings can be hung and balanced or by a drawing. The "image" of the balance beam can be varyingly refined, with fewer and fewer irrelevant details present, as in the typical diagrams in an introductory textbook in physics. Finally, a balance beam can be described in ordinary English, without diagrammatic aids, or it can be even better described mathematically by reference to Newton's Law of Moments in inertial physics. Needless to say, actions, pictures, and symbols vary in difficulty and utility for people of different ages, different backgrounds, different styles. Moreover, a problem in the law would be hard to diagram; one in geography lends itself to imagery. Many subjects, such as mathematics, have alternative modes of representation.

Economy in representing a domain of knowledge relates to the amount of information that must be held in mind and processed to achieve comprehension. The more items of information one must carry to understand something or deal with a problem, the more successive steps one must take in processing that information to achieve a conclusion, and the less the economy. For any domain of knowledge, one can rank summaries of it in terms of their economy. It is more economical (though less powerful) to summarize the American Civil War as a "battle over slavery" than as "a struggle between an expanding industrial region and one built upon a class society for control of federal economic policy." It is more economical to summarize the characteristics of free-falling bodies by the formula $S = \frac{1}{2}gt^2$ than to put a series of numbers into tabular form summarizing a vast set of observations made on different bodies dropped different distances in different gravitational fields. The matter is perhaps best epitomized by two ways of imparting information, one requiring carriage of much information, the other more a pay-as-you-go type of information processing. A highly imbedded sentence is an example of the former (This is the squirrel that the dog that the girl that the man loved fed chased); the contrast case is more economical (This is the man that loved the girl that fed the dog that chased the squirrel).

Economy, as we shall see, varies with mode of representation. But economy is also a function of the sequence in which material is presented or the manner in which it is learned. The case can be exemplified as follows (I am indebted to Dr. J. Richard Hayes for this example). Suppose the domain of knowledge consists of available plane service within a twelve-hour period between five cities in the Northeast—Concord, New Hampshire, Albany, New York, Danbury, Connecticut, Elmira, New York, and Boston, Massachusetts. One of the ways in which the knowledge can be imparted is by asking the student to memorize the following list of connections:

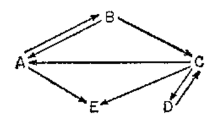
Boston to Concord
Danbury to Concord
Albany to Boston
Concord to Elmira
Albany to Elmira
Concord to Danbury
Boston to Albany
Concord to Albany

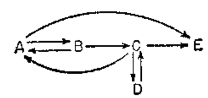


Now we ask, "What is the shortest way to make a round trip from Albany to Danbury?" The amount of information processing required to answer this question under such conditions is considerable. We increase economy by "simplifying terms" in certain characteristic ways. One is to introduce an arbitrary but learned order—in this case, an alphabetical one. We rewrite the list:

Albany to Boston
Albany to Elmira
Boston to Albany
Boston to Concord
Concord to Albany
Concord to Danbury
Concord to Elmira
Danbury to Concord

Search then becomes easier, but there is still a somewhat trying sequential property to the task. Economy is further increased by using a diagrammatic notation, and again there are varying degrees of economy in such recourse to the iconic mode. Compare the diagram on the left and the one on the right.




The latter contains at a glance the information that there is only one way from Albany to Danbury and return, that Elmira is a "trap," and so on. What a difference between this diagram and the first list!

The effective power of any particular way of structuring a domain of knowledge for a particular learner refers to the generative value of *his* set of learned propositions. In the last paragraph, rote learning of a set of connections between cities resulted in a rather inert structure from which it was difficult to generate pathways through the set of cities. Or, to take an example from a recent work,[2] children who are told that "Mary is taller than Jane, and Betty is shorter than Jane" are often unable to say whether Mary is taller than Betty. One can perfectly well remark that the answer is "there" in the logic of transitivity. But to say this is to miss the psychological point. Effective power will, to be sure, never exceed the inherent logical generativeness of a subject—although this is an admittedly difficult statement from the point of view of epistemology. In commonsense terms, it amounts to the banality that grasp of a field of knowledge will never be better than the best that can be done with that field of knowledge. The effective power within a particular learner's grasp is what one seeks to discover by close analysis of how in fact he is going about his task of learning. Much of Piaget's research[3] seeks to discover just this property about children's learning and thinking. There is an interesting relation between economy and power. Theoretically, the two are independent: indeed, it is clear that a structure may be economical but powerless. But it is rare for a powerful structuring technique in any field to be uneconomical. This is what leads to the canon of parsimony and the faith shared by many scientists that nature is simple: perhaps it is only when nature can be made reasonably simple that it can be understood. The power of a representation can also be described as its capacity, in the hands of a learner, to connect matters that, on the surface, seem quite separate. This is especially crucial in mathematics, and we shall return to the matter later.

[2] Margaret Donaldson, *A Study of Children's Thinking* (London: Tavistock Publications, 1963).
[3] Jean Piaget, *The Child's Conception of Number* (New York: Humanities Press, 1952).



SEQUENCE AND ITS USES

Instruction consists of leading the learner through a sequence of statements and restatements of a problem or body of knowledge that increase the learner's ability to grasp, transform, and transfer what he is learning. In short, the sequence in which a learner encounters materials within a domain of knowledge affects the difficulty he will have in achieving mastery.

There are usually various sequences that are equivalent in their ease and difficulty for learners. There is no unique sequence for all learners, and the optimum in any particular case will depend upon a variety of factors, including past learning, stage of development, nature of the material, and individual differences.

If it is true that the usual course of intellectual development moves from enactive through iconic to symbolic representation of the world,[4] it is likely that an optimum sequence will progress in the same direction. Obviously, this is a conservative doctrine. For when the learner has a well-developed symbolic system, it may be possible to by-pass the first two stages. But one does so with the risk that the learner may not possess the imagery to fall back on when his symbolic transformations fail to achieve a goal in problem solving.

Exploration of alternatives will necessarily be affected by the sequence in which material to be learned becomes available to the learner. When the learner should be encouraged to explore alternatives widely and when he should be encouraged to concentrate on the implications of a single alternative hypothesis is an empirical question, to which we shall return.

Reverting to the earlier discussion of activation and the

[4] Jerome S. Bruner, "The Course of Cognitive Growth," *American Psychologist*, 19:1–15 (January 1964).

maintenance of interest, it is necessary to specify in any sequences the level of uncertainty and tension that must be present to initiate problem-solving behavior, and what conditions are required to keep active problem solving going. This again is an empirical question.

Optimal sequences, as already stated, cannot be specified independently of the criterion in terms of which final learning is to be judged. A classification of such criteria will include at least the following: speed of learning; resistance to forgetting; transferability of what has been learned to new instances; form of representation in terms of which what has been learned is to be expressed; economy of what has been learned in terms of cognitive strain imposed; effective power of what has been learned in terms of its generativeness of new hypotheses and combinations. Achieving one of these goals does not necessarily bring one closer to others; speed of learning, for example, is sometimes antithetical to transfer or to economy.

### THE FORM AND PACING OF REINFORCEMENT

Learning depends upon knowledge of results at a time when and at a place where the knowledge can be used for correction. Instruction increases the appropriate timing and placing of corrective knowledge.

"Knowledge of results" is useful or not depending upon when and where the learner receives the corrective information, under what conditions such corrective information can be used, even assuming appropriateness of time and place of receipt, and the form in which the corrective information is received.

Learning and problem solving are divisible into phases. These have been described in various ways by different writers. But all the descriptions agree on one essential feature: that there is a cycle involving the formulation of a testing procedure or trial, the operation of this testing procedure, and the com-



parison of the results of the test with some criterion. It has variously been called trial-and-error, means-end testing, trial-and-check, discrepancy reduction, test-operate-test-exit (TOTE), hypothesis testing, and so on. These "units," moreover, can readily be characterized as hierarchically organized: we seek to cancel the unknowns in an equation in order to simplify the expression in order to solve the equation in order to get through the course in order to get our degree in order to get a decent job in order to lead the good life. Knowledge of results should come at that point in a problem-solving episode when the person is comparing the results of his try-out with some criterion of what he seeks to achieve. Knowledge of results given before this point either cannot be understood or must be carried as extra freight in immediate memory. Knowledge given after this point may be too late to guide the choice of a next hypothesis or trial. But knowledge of results must, to be useful, provide information not only on whether or not one's particular act produced success but also on whether the act is in fact leading one through the hierarchy of goals one is seeking to achieve. This is not to say that when we cancel the term in that equation we need to know whether it will all lead eventually to the good life. Yet there should at least be some "lead notice" available as to whether or not cancelation is on the right general track. It is here that the tutor has a special role. For most learning starts off rather piecemeal without the integration of component acts or elements. Usually the learner can tell whether a particular cycle of activity has worked—feedback from specific events is fairly simple—but often he cannot tell whether this completed cycle is leading to the eventual goal. It is interesting that one of the nonrigorous short cuts to problem solution, basic rules of "heuristic," stated in Polya's noted book[5] has to do with defining the overall problem. To sum up, then, instruction uniquely provides information to the

[5] Gyorgy Polya, *How To Solve It*, 2nd ed. (New York: Doubleday, 1957).

learner about the higher-order relevance of his efforts. In time, to be sure, the learner must develop techniques for obtaining such higher-order corrective information on his own, for instruction and its aids must eventually come to an end. And, finally, if the problem solver is to take over this function, it is necessary for him to learn to recognize when he does not comprehend and, as Roger Brown[6] has suggested, to signal incomprehension to the tutor so that he can be helped. In time, the signaling of incomprehension becomes a self-signaling and equivalent to a temporary stop order.

The ability of problem solvers to use information correctively is known to vary as a function of their internal state. One state in which information is least useful is that of strong drive and anxiety. There is a sufficient body of research to establish this point beyond reasonable doubt.[7] Another such state has been referred to as "functional fixedness"—a problem solver is, in effect, using corrective information exclusively for the evaluation of one single hypothesis that happens to be wrong. The usual example is treating an object in terms of its conventional significance when it must be treated in a new context—we fail to use a hammer as a bob for a pendulum because it is "fixed" in our thinking as a hammer. Numerous studies point to the fact that during such a period there is a remarkable intractability or even incorrigibility to problem solving. There is some evidence to indicate that high drive and anxiety lead one to be more prone to functional fixedness. It is obvious that corrective information of the usual type, straight feedback, is least useful during such states, and that an adequate instructional strategy aims at terminating the interfering state by special means before continuing with the usual pro-

[6] Roger Brown, *Social Psychology* (New York: Free Press of Glencoe, 1965), chapter 7, "From Codability to Coding Ability."

[7] For full documentation, see Jerome S. Bruner, "Some Theorems on Instruction Illustrated with Reference to Mathematics," *Sixty-third Yearbook of the National Society for the Study of Education*, Part I (Chicago: University of Chicago Press, 1964), pp. 306–335.



vision of correction. In such cases, instruction verges on a kind of therapy, and it is perhaps because of this therapeutic need that one often finds therapylike advice in lists of aids for problem solvers, like the suggestion of George Humphrey[8] that one turn away from the problem when it is proving too difficult.

If information is to be used effectively, it must be translated into the learner's way of attempting to solve a problem. If such translatability is not present, then the information is simply useless. Telling a neophyte skier to "shift to his uphill edges" when he cannot distinguish which edges he is traveling on provides no help, whereas simply telling him to lean into the hill may succeed. Or, in the cognitive sphere, there is by now an impressive body of evidence that indicates that "negative information"—information about what something is *not*—is peculiarly unhelpful to a person seeking to master a concept. Though it is logically usable, it is psychologically useless. Translatability of corrective information can in principle also be applied to the form of representation and its economy. If learning or problem solving is proceeding in one mode—enactive, iconic or symbolic—corrective information must be provided either in the same mode or in one that translates into it. Corrective information that exceeds the information-processing capacities of a learner is obviously wasteful.

Finally, it is necessary to reiterate one general point already made in passing. Instruction is a provisional state that has as its object to make the learner or problem solver self-sufficient. Any regimen of correction carries the danger that the learner may become permanently dependent upon the tutor's correction. The tutor must correct the learner in a fashion that eventually makes it possible for the learner to take over the corrective function himself. Otherwise the result of instruction is to create a form of mastery that is contingent upon the perpetual presence of a teacher.

[8] George Humphrey, *Directed Thinking* (New York: Dodd, Mead, 1948).

## SELECTED ILLUSTRATIONS FROM MATHEMATICS

Before turning to the task of illustrating some of the points raised, a word is in order about what is intended by such illustration. During the last decade much work has gone into the mathematics curriculum. One need only mention the curriculum projects that are better known to appreciate the magnitude of the effort—the School Mathematics Study Group, the University of Illinois Committee on School Mathematics, the several projects of Educational Services Incorporated, the Madison Project, the African Mathematics Project, the University of Maryland Mathematics Project, the University of Illinois Arithmetic Project, and the Stanford Project. From this activity, it would be possible to choose illustrations for many purposes. Illustration in such a context in no sense constitutes evidence.

For the fact of the matter is that the evidence available on factors affecting the learning of mathematics is still very sparse. Research on the instructional process—in mathematics as in all disciplines—has not been carried out in connection with the building of curricula. As noted, psychologists have come upon the scene, armed with evaluative devices, only after a curriculum has already been put into operation. Surely it would be more efficient and more useful if embryonic instructional materials could be tried out under experimental conditions so that revision and correction could be based upon immediate knowledge of results.

By means of systematic observational studies—work close in spirit to that of Piaget and of ethologists like Tinbergen[9]—investigators could obtain information sufficiently detailed to allow them to discern how the student grasps what has been presented, what his systematic errors are, and how these are overcome. Insofar as one is able to formalize, in terms of a

[9] Nikolaas Tinbergen, *Social Behavior in Animals* (New York: John Wiley & Sons, 1953).



theory of learning or concept attainment, the nature of the systematic errors and the strategies of correction employed, one is thereby enabled to vary systematically the conditions that may be affecting learning and to build these factors directly into one's curriculum practice. Nor need such studies remain purely observational. Often it is possible to build one's mathematics materials into a programmed form and obtain a detailed behavioral record for analysis.

To make clear what is intended by a detailed analysis of the process of learning, an example from the work of Patrick Suppes[10] will be helpful. He has observed, for example, that the form $3 + x = 8$ is easier for children to deal with than the form $x + 3 = 8$, and while the finding may on the surface seem trivial, closer inspection shows that it is not. Does the difficulty come in dealing with an unknown at the beginning of an expression or from the transfer of linguistic habits from ordinary English, where sentences are easier to complete when a term is deleted from the middle than from beginning of the sentence? The issue of where uncertainty can best be tolerated and the issue of the possible interference between linguistic habits and mathematical habits are certainly worthy of careful and detailed study.

Let me turn now to some illustrations from mathematics that have the effect of pointing up problems raised in the theorems and hypotheses earlier presented. They are not evidence of anything, only ways of locating what might be worth closer study.[11]

[10] Patrick Suppes, "Towards a Behavioral Psychology of Mathematics Thinking," in J. Bruner, ed., *Learning about Learning,* U.S. Office of Education monograph, in press.

[11] For a closer discussion of some of the observations mentioned in what follows, the reader is referred to Bruner, "The Course of Cognitive Growth," and to Jerome S. Bruner and Helen Kenney, "Representation and Mathematics Learning," in L. Morrisett and J. Vinsonhaler, eds., *Mathematical Learning,* Monographs of the Society for Research in Child Development, 30 (University of Chicago Press, 1965), pp. 50–59. The general "bias" on which

Rather than presenting observations drawn from different contexts, I shall confine the discussion to one particular study carried out on a small group of children.[12] The observations to be reported were made on four eight-year-old children, two boys and two girls, who were given an hour of daily instruction in mathematics four times a week for six weeks. The children were in the IQ range of 120–130, and they were all enrolled in the third grade of a private school that emphasized instruction designed to foster independent problem solving. They were all from middle-class professional homes. The "teacher" of the class was a well-known research mathematician (Z. P. Dienes), his assistant a professor of psychology at Harvard who has worked long and hard on human thought processes.

Each child worked at a corner table in a generous-sized room. Next to each child sat a tutor-observer, trained in psychology and with sufficient background in college mathematics to understand the underlying mathematics being taught. In the middle of the room was a large table with a supply of blocks and balance beams and cups and beans and chalk that served as instructional aids. In the course of the six weeks, the children were given instruction in factoring, in the distributive and commutative properties of addition and multiplication, and finally in quadratic functions.

Each child had available a series of graded problem cards which he could go through at his own pace. The cards gave directions for different kinds of exercises, using the materials mentioned above. The instructor and his assistant circulated

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these observations are based is contained in Jerome S. Bruner, *The Process of Education* (Cambridge: Harvard University Press, 1960), and in J. S. Bruner, J. J. Goodnow, and G. A. Austin, *A Study of Thinking* (New York: John Wiley & Sons, 1956).

[12] I am grateful to Z. P. Dienes, Samuel Anderson, Eleanor Duckworth, and Joan Rigney Hornsby for their help in designing and carrying out this study. Dr. Dienes particularly formed our thinking about the mode of presenting the mathematical materials.



from table to table helping as needed, and each tutor-observer similarly assisted as needed. The problem sequences were designed to provide, first, an appreciation of mathematical ideas through concrete constructions involving materials of various kinds. From such constructions, the child was encouraged to form perceptual images of the mathematical idea in terms of the forms that had been constructed. The child was then further encouraged to develop or adopt a notation for describing his construction. After such a cycle, a child moved on to the construction of a further embodiment of the idea on which he was working, one that was mathematically isomorphic with what he had learned, though expressed in different materials and with altered appearance. When such a new topic was introduced, the children were given a chance to discover its connection with what had gone before and shown how to extend the notational system used before. Careful minute-by-minute records were kept of the proceedings, along with photographs of the children's constructions.

In no sense can the children, the teachers, the classroom, or the mathematics be said to be typical of what occurs in third grade. Four children rarely have six teachers, nor do eight-year-olds ordinarily get into quadratic functions. But our concern is with the processes involved in mathematical learning, and not with typicality. It seems quite reasonable to suppose that the thought processes that were going on in the children are quite ordinary among eight-year-old human beings.

### ACTIVATING PROBLEM SOLVING

One of the first tasks faced in this study was to gain and hold the child's interest and to lead him to problem-solving activity. At the same time, there was a specific objective to be achieved—to teach the children factoring in such a way that they would have this component skill in an accessible form to use in the solution of problems. It is impossible to say on the basis of our experience whether the method we employed was the best one, but in any case it appeared to work.

A considerable part of the job of activation had already been done before ever we saw the children. They had working models of exploratory adults in their teachers and their parents. They had no particular resistance to trying out and rejecting hypotheses. The principal problem we faced as teachers who outnumbered the students was to keep the children from converting the task into one where they would become dependent upon us. All of us had had the experience of working with children from less intellectually stimulating backgrounds where there had been less emphasis upon intellectual autonomy, and the contrast was appreciable. Indeed, I can only repeat that where predisposition to learning was concerned, the children in the study were almost specifically trained for the kind of approach we were about to use—an approach with strong emphasis on independence, on self-pacing, on reflectiveness. Had we used a more authoritarian, more mnemonic approach with our group, we would have had to prepare the ground. As it was, the task had already been well begun.

The first learning task introduced was one having to do with the different ways in which a set of cubic blocks could be arranged as "flats" (laid out in rectangular forms on the table, not more than one cube high) and in "walls" and "buildings." The problem has an interesting uncertainty to it, and the children were challenged to determine whether they had exhausted all the possible ways of laying things out. Unquestionably they picked up some zest from the evident curiosity of their teachers as well. After a certain amount of time, the children were encouraged to start keeping a written record of the different shapes they could make, and what their dimensions were. Certain numbers of cubes proved intractable to re-forming (the primes, of course), and others proved com-



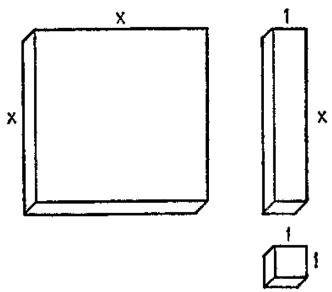
binable in interesting ways—three rows of three cubes made nine, three layers of these nine "flats" had the dimensions of $3 \times 3 \times 3$, and so on. The idea of factoring was soon grasped, and with very little guidance the children went on to interesting conjectures about distributiveness. The task had its own direction built into it in the sense that it had a clear terminus: how arrange a set of cubes in regular two- or three-dimensional forms? It also had the added feature that the idea of alternatives was built in: what are the different ways of achieving such regularity? As the children gained in skill, they shifted to other ways of laying out cubes—in pyramids, in triangles where the cubes were treated as "diamonds," and so on. At this stage of the game, it was necessary to judge in each case whether the child should be let alone to discover on his own.

We shall see, when we come to discuss the balance beam, that the idea of factoring was further deepened by being applied to a "new" problem. I mention the point here because it relates to the importance of *maintaining* a problem-solving set that runs in a continuous direction. It is often the case that novelty must be introduced in order that the enterprise be continued. In the case of the balance beam, the task was to discover the different combinations of rings that could be put on one side of the balance beam to balance a single ring placed on hook 9. In effect, this is the same problem as asking the different ways in which nine blocks can be arranged. But it is in a different guise, and the new embodiment seems capable of stimulating interest even though it is isomorphic with something else that has been explored to the border of satiety.

### STRUCTURE AND SEQUENCE

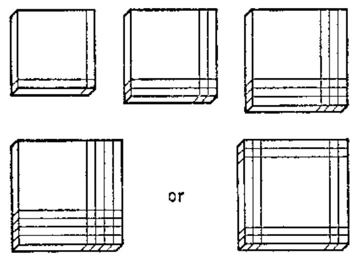
We can best illustrate the points made at the outset by reference to our teaching of quadratic equations to the four children we studied. Each child was provided with building materials. These were large flat squares made of wood whose dimensions were unspecified and described simply as "unknown, or $x$ long and $x$ wide." There were also a large number of strips of wood that were as long as the sides of the square and were described arbitrarily as having a width of "1" or simply as "1 by $x$." And there was a supply of little squares with sides equal to the width "1" of the strips, thus "1 by 1." The reader should be warned that the presentation of these materials is not as simple as all that. To begin with, it is necessary to convince the children that we really do not know and do not *care* what the metric size of the big square is, that rulers are of no interest. A certain humor helps establish in the pupils a proper contempt for measuring in this context, and the snob appeal of simply calling an unknown by the name "$x$" is very great. From there on, the children readily discover for themselves that the long strips are $x$ long—by correspondence. They take on faith (as they should) that the narrow dimension is "1," but that they grasp its arbitrariness is clear from one child's declaration of the number of such "1" widths that made an $x$. As for "1 by 1" little squares, that too is established by simple correspondence with the narrow dimension of the "1 by $x$" strips. It is horseback method, but quite good mathematics.

The child is asked whether he can make a square bigger than






the $x$ by $x$ square, using the materials at hand. He very quickly builds squares with designs like those illustrated below. We ask him to record how much wood is needed for each larger square and how long and wide each square is.




He describes one of his constructed squares: very concretely the pieces are counted out: "an $x$-square, two $x$-strips, and a one square," or "an $x$-square, four $x$-strips, and four ones," or "an $x$-square, six $x$-strips and nine ones," and so forth. We help him with language and show him a way to write it down. The big square is an "$x\Box$," the long strips are "$1x$" or simply "$x$," and the little squares are "one squares" or "one by one" or, better still, simply "1." And the expression "and" can be shortened to $+$. And so he can write out the recipe for a constructed square as $x\Box + 4x + 4$. At this stage, these are merely names put together in little sentences. How wide and long is the square in question? This the child can readily measure off —an $x$ and 2, or $x + 2$, and so the whole thing is $(x + 2)\Box$. Brackets are not so easily grasped. But soon the child is able to put down his first equality: $(x + 2)\Box = x\Box + 4x + 4$. Virtually everything has a referent that can be pointed to with a finger. He has a notational system into which he can translate the image he has constructed.

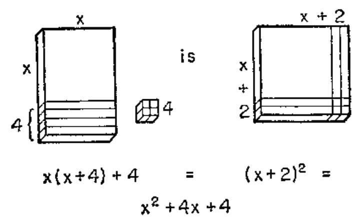
Now we go on to making bigger squares, and each square the child makes he is to describe in terms of what wood went into it and how wide and how long it is. It takes some ruled sheets to get the child to keep his record so that he can go back and inspect it for what it may reveal, and he is encouraged to go back and look at the record and at the constructions they stand for.

Imagine now a list such as the following, again a product of the child's own constructing:

$x^{\square} + 2x + 1$ is $x + 1$ by $x + 1$
$x^{\square} + 4x + 4$ is $x + 2$ by $x + 2$
$x^{\square} + 6x + 9$ is $x + 3$ by $x + 3$
$x^{\square} + 8x + 16$ is $x + 4$ by $x + 4$

It is almost impossible for him not to make some discoveries about the numbers: that the $x$ values go up 2, 4, 6, 8, and the units values go up 1, 4, 9, 16, and the dimensions increase by additions to $x$ of 1, 2, 3, 4. The syntactical insights about regularity in notation are matched by perceptual-manipulative insights about the material referents.

After a while, some new manipulations occur that provide the child with a further basis for notational progress. He takes the square $(x + 2)^2$ and reconstructs it in a new way. One may ask whether this is constructive manipulation, and whether it is proper factoring. But the child is learning that the same amount of wood can build quite strikingly different patterns and remain the same amount of wood—even though it also has a different notational expression. Where does the language begin and the manipulation of materials stop? The interplay is continuous. We shall return to this same example later.

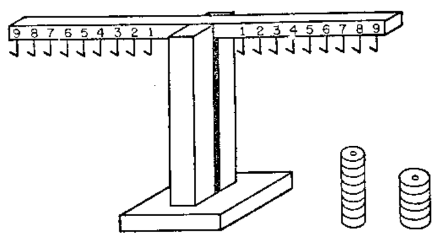





What is now a problem is how to "detach" the notation that the child has learned from the concrete, visible, manipulable embodiment to which it refers—the wood. For if the child is to deal with mathematical properties he will have to deal with symbols per se, else he will be limited to the narrow and rather trivial range of symbolism that can be given direct (and only partial) visual embodiment. Concepts such as $x^2$ and $x^3$ may be given a visualizable referent, but what of $x^n$ ?

How do children wean themselves from the perceptual embodiment to the symbolic notation? Perhaps it is partly explained in the nature of variation and contrast.

The child is shown the balance beam again and told: "Choose any hook on one side and put the same number of rings on it as the number the hook is away from the middle. Now balance it with rings placed on the other side. Keep a record." Recall that the balance beam is familiar from work on factoring and that the child knows that 2 rings on 9 balances 9 on 2 or $m$ rings on $n$ balances $n$ on $m$. He is back to construction. Can anything be constructed on the balance beam that is like the squares? With little effort, the following translation is made. Suppose $x$ is 5. Then 5 rings on hook 5 is $x^2$, 5 rings on hook 4 is $4x$, and 4 rings on hook 1 is 4: $x^2 + 4x + 4$. How can we find whether this is like a square that is $x + 2$ wide by $x + 2$ long, as before? Well, if $x$ is 5, then $x + 2$ is 7, and so 7 rings on hook 7. And nature obliges —the beam balances. One notation works for two strikingly



different constructions and perceptual events. Notation, with its broader equivalency, is clearly more economical than reference to embodiments. There is little resistance to using this more convenient language. And now construction can begin—commutative and distributive properties of equations can be explored: $x(x + 4) + 4 = x^2 + 4x + 4$, so that $x + 4$ rings on hook $x$ plus 4 rings on hook 1 will also balance. The child if he wishes can also go back to the wood and find that the same materials can make the designs illustrated earlier.

Contrast is the vehicle by which the obvious that is too obvious to be appreciated can be made noticeable again. A discovery by an eight-year-old girl illustrates the matter. "Yes, $4 \times 6$ equals $6 \times 4$ in numbers, like in one way six eskimos in each of four igloos is the same as four in each of six igloos. But a venetian blind *isn't* the same as a blind Venetian." By recognizing the noncommutative property of much of our ordinary language, the commutative property of a mathematical language can be partly grasped. But it is still only a partial insight into commutativity and noncommutativity. Had we wished to develop the distinction more deeply we might have proceeded concretely to a contrast between sets of operations that can be carried out in any sequence—like the order in which letters are put in a post box or in which we see different movies—and operations that have a noncommutative order—like putting on shoes and socks—where one must precede the other. The child could be taken from there to a more general idea of commutative and noncommutative cases and to ways of dealing with a notation for them, perhaps by identical sets and ordered identical sets.

We need not reiterate what must be obvious from this sequence. The object was to begin with an enactive representation of quadratics—something that could literally be "done" or built—and to move from there to an iconic representation, however restricted. Along the way, notation was developed



and, by the use of variation and contrast, converted into a properly symbolic system. Again, the object was to start with as economical a representation as possible and to increase complexity only when there was some way for the child to relate the complex instance to something simpler that had gone before.

What was so striking in the performance of the children was their initial inability to represent things to themselves in a way that transcended immediate perceptual grasp. The achievement of more comprehensive insight requires, we think, the building of a mediating representational structure that transcends such immediate imagery, that renders a sequence of acts and images unitary and simultaneous. The children always began by constructing an embodiment of some concept, building a concrete model for purposes of operational definition. The fruit of the construction was an image and some operations that "stood for" the concept. From there on, the task was to provide means of representation that were free of particular manipulations and specific images. Only symbolic operations provide the means of representing an idea in this way. But consider this matter for a moment.

We have already remarked that by giving the child multiple embodiments of the same general idea expressed in a common notation we lead him to "empty" the concept of specific sensory properties until he is able to grasp its abstract properties. But surely this is not the best way of describing the child's increasing development of insight. The growth of such abstractions is important. But what struck us about the children as we observed them is that they not only understood the abstractions they had learned but also had a store of concrete images that served to exemplify the abstractions. When they searched for a way to deal with new problems, the task was usually carried out not simply by abstract means but also by "matching up" images. An example will help here. In going from the wood-blocks embodiment of the quadratic to the balance-beam embodiment, it was interesting that the children would "equate" concrete features of one with concrete features of another. One side of the balance beam "stood for" the amount of wood, the other side for the sides of the square. These were important concrete props on which they leaned. We have been told by research mathematicians that the same use of props—heuristics—holds for them, that they have preferred ways of imaging certain problems while other problems are handled silently or in terms of an imagery of the symbolism on a page.

We reached the tentative conclusion that it was probably necessary for a child, learning mathematics, to have not only a firm sense of the abstraction underlying what he was working on, but also a good stock of visual images for embodying them. For without the latter it is difficult to track correspondences and to check what one is doing symbolically. We had occasion, again with the help of Dr. Dienes, to teach a group of ten nine-year-olds the elements of group theory. To embody the idea of a mathematical group initially, we gave them the example of a four-group made up of the following four maneuvers. A book was the vehicle, a book with an arrow up the middle of its front cover. The four maneuvers were rotating the book a quarter turn to the left, rotating it a quarter turn to the right, rotating it a half turn (without regard to direction of rotation), and letting it stay in the position it was in. They were quick to grasp the important property of such a mathematical group: that any sequence of maneuvers made could be reproduced from the starting position by a single move. This is not the usual way in which this property is described mathematically, but it served well for the children. We contrasted this elegant property with a series of our moves that did *not* constitute a mathematical group—indeed, they provided the counter example themselves by proposing a one-third turn left, one-third turn right, half turn either way, and



stay. It was soon apparent that it did not work. We set the children the task of making games of four maneuvers, six maneuvers, and so on, that had the property of a "closed" game, as we call it—one in which the result of any combination of moves can be achieved by a single move. They were, of course, highly ingenious. But what soon became apparent was that they needed some aid in imagery—in this case an imagery notation—that would allow them to keep track and then to discover whether some new game was an isomorph of one they had already developed. The prop in this case was, of course, the matrix, listing the moves possible across the top and then listing them down the side, thus making it easily possible to check whether each combination of pairs of moves could be reproduced by a single move. The matrix in this case is a crutch or heuristic and as such has nothing to do with the abstraction of the mathematical group, yet it was enormously useful to them not only for keeping track but also for comparing one group with another for correspondence. The matrix with which they started looked like this:

| | s | a | b | c |
|---|---|---|---|---|
| s | s | a | b | c |
| a | a | c | s | b |
| b | b | s | c | a |
| c | c | b | a | s |

s = stay
a = quarter-turn left
b = quarter-turn right
c = half-turn

Are there any four-groups with a different structure? It is extremely difficult to deal with such a question without the aid of this housekeeping matrix as a vehicle for spotting correspondence. What about a game in which a cube can be left where it is, rotated 180° on its vertical axis, rotated 180° on its horizontal axis, and rotated 180° on each of its four cubic diagonals? Is it a group? Can it be simplified to a smaller number of maneuvers? Does it contain the group described above?

In sum, then, while the development of insight into mathematics in our group of children depended upon their development of "example-free" abstractions, this did not lead them to give up their imagery. Quite to the contrary, we had the impression that their enriched imagery was very useful to them in dealing with new problems.

We would suggest that learning mathematics reflects a good deal about intellectual development. It begins with instrumental activity, a kind of definition of things by doing them. Such operations become represented and summarized in the form of particular images. Finally, and with the help of a symbolic notation that remains invariant across transformations in imagery, the learner comes to grasp the formal or abstract properties of the things he is dealing with. But while, once abstraction is achieved, the learner becomes free in a certain measure of the surface appearance of things, he nonetheless continues to rely upon the stock of imagery he has built en route to abstract mastery. It is this stock of imagery that permits him to work at the level of heuristic, through convenient and nonrigorous means of exploring problems and relating them to problems already mastered.

### REINFORCEMENT AND FEEDBACK

With respect to corrective information, there is something particularly happy about the exercises we chose to use. In learning quadratics by the use of our blocks and then by the aid of the balance beam, children were enabled by immediate test to determine whether they had "got there." A collection of square pieces of wood is aggregated in a form that either makes a square or doesn't, and the child can see it immediately. So too with a balance beam: it either balances or it does not. There is no instructor intervening between the learner and the materials.

But note well that the instructor had to enter in several ways. In the first place, he determined within quite constrained



limits the nature of the sequences, so that the children would have the greatest chance of seeing the relation of what went before to what was up now. Whether we succeeded well in these sequences we do not know—save that the children learned some elegant mathematics in a fairly short time. What guided us was some sort of psychological-mathematical intuition, and while that may be satisfactory for such engineering as we did, it is certainly not satisfactory from the point of view of understanding how to do it better.

We failed on several occasions, as judged by the lagging interest of a particular child, when we wanted to be sure that the child had really understood something. Our most glaring failure was in trying to get across in symbolic form (probably too early) the idea of distributiveness—that $a \times (b + c)$ and $(a \times b) + (a \times c)$ could be treated as equal. One of our cleverest young pupils commented at the beginning of an hour, with a groan, "Oh, they're distributing the distributive law again." In fact, our difficulty came from a misjudgment of the importance of giving them a symbolic mode for correcting iconic constructions. We were too eager to be sure that they sensed the notational analogue of the factoring constructions they had been making and which they understood at the iconic level so well that further construction was proving a bore.

We have few fresh observations to report on the matter of overdrive and anxiety. One of our pupils had a rather strong push about mathematics from his father at home. He was the child who, on the first day, had to demonstrate his prowess by multiplying two large and ugly numbers on the board, announcing the while, "I know a lot of math." He was probably our best student, but he made no progress until he got over the idea that what was needed was hard computation. It was he, too, who complained that the blocks used for quadratics *had* to have *some* size. But once he was willing to play with unknowns as "$x$" he showed considerable power. His father was our unwitting ally at this point, for he told him that "$x$'s" were from algebra, which was a subject most children took in high school.

Perhaps the greatest problem one has in an experiment of this sort is to keep out of the way, to prevent oneself from becoming a perennial source of information, interfering with the child's ability to take over the role of being his own corrector. But each classroom situation is unique in this way, and each dyad of teacher and pupil. Some of the teacher-pupil pairs became quite charged with dependency; in others the child or the teacher resisted. But that is another story.

### SOME CONCLUSIONS

A first and obvious conclusion is that one must take into account the issues of predisposition, structure, sequence, and reinforcement in preparing curriculum materials—whether one is concerned with writing a textbook, a lesson plan, a unit of instruction, a program, or, indeed, a conversation with didactic ends in view. But this obvious conclusion suggests some rather nonobvious implications.

The type of supporting research that permits one to assess how well one is succeeding in the management of relevant instructional variables requires a constant and close collaboration of teacher, subject-matter specialist, and psychologist. As intimated earlier, a curriculum should be prepared jointly by the subject-matter expert, the teacher, and the psychologist, with due regard for the inherent structure of the material, its sequencing, the psychological pacing of reinforcement, and the building and maintaining of predispositions to problem solving. As the curriculum is being built, it must be tested in detail by close observational and experimental methods to assess not simply whether children are "achieving" but rather what they are making of the material and how they are organizing it. It is on the basis of "testing as you go" that revision is made. It is this procedure that puts the evaluation process at



a time when and place where its results can be used for correction while the curriculum is being constructed.

Only passing reference has been made to the issue of individual differences. Quite plainly, they exist in massive degree—in the extent to which children have problem-solving predispositions, in the degree of their interest, in the skills that they bring to any concrete task, in their preferred mode of representing things, in their ability to move easily through any particular sequence, and in the degree to which they are initially dependent upon extrinsic reinforcement from the teacher. The fact of individual differences argues for pluralism and for an enlightened opportunism in the materials and methods of instruction. Earlier we asserted, rather off-handedly, that no single ideal sequence exists for any group of children. The conclusion to be drawn from that assertion is not that it is impossible to put together a curriculum that would satisfy a group of children or a cross-section of children. Rather, it is that if a curriculum is to be effective in the classroom it must contain different ways of activating children, different ways of presenting sequences, different opportunities for some children to "skip" parts while others work their way through, different ways of putting things. A curriculum, in short, must contain many tracks leading to the same general goal.

Our illustrations have been taken from mathematics, but there are some generalizations that go beyond to other fields. The first is that it took the efforts of many highly talented mathematicians to discern the underlying structure of the mathematics that was to be taught. That is to say, the simplicity of a mathematics curriculum rests upon the history and development of mathematics itself. But even so glorious an intellectual tradition as that of mathematics was not enough. For while many virtues have been discovered for numbers to the base 10, students cannot appreciate such virtues until they recognize that the base 10 was not handed down from the mountain by some mathematical God. It is when the student learns to work in different number bases that the base 10 is recognized for the achievement that it is.

Finally, a theory of instruction seeks to take account of the fact that a curriculum reflects not only the nature of knowledge itself but also the nature of the knower and of the knowledge-getting process. It is the enterprise par excellence where the line between subject matter and method grows necessarily indistinct. A body of knowledge, enshrined in a university faculty and embodied in a series of authoritative volumes, is the result of much prior intellectual activity. To instruct someone in these disciplines is not a matter of getting him to commit results to mind. Rather, it is to teach him to participate in the process that makes possible the establishment of knowledge. We teach a subject not to produce little living libraries on that subject, but rather to get a student to think mathematically for himself, to consider matters as an historian does, to take part in the process of knowledge-getting. Knowing is a process, not a product.

*Chapter* 4

# Man: A Course of Study

THERE IS a dilemma in describing a course of study. One must begin by setting forth the intellectual substance of what is to be taught, else there can be no sense of what challenges and shapes the curiosity of the student. Yet the moment one succumbs to the temptation to "get across" the subject, at that moment the ingredient of pedagogy is in jeopardy. For it is only in a trivial sense that one gives a course to "get something across," merely to impart information. There are better means to that end than teaching. Unless the learner also masters himself, disciplines his taste, deepens his view of the world, the "something" that is got across is hardly worth the effort of transmission.

The more elementary a course and the younger its students, the more serious must be its pedagogical aim of forming the intellectual powers of those whom it serves. It is as important that a good mathematics course be justified by the intellectual discipline it provides or the honesty it promotes as by the mathematics it transmits. Indeed, neither can be accomplished without the other.

With these things in mind, let me describe the substance or structure of a course in social studies now in the process of construction, parts of which have been taught to children in grade five. What is presented here is a blueprint. It may turn out to be the case, as modifications are made during tryout