The Root-2 Proof as an Example of Non-constructivity

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September 2014

§1. The following neat proof is well known to mathematicians and is often cited by logicians as an example of a non-constructive proof of existence.

Proposition 1 *There exist real numbers a and b, both irrational, such that a^b is rational.*

Proof via $\sqrt{2}$. Consider $\sqrt{2}^{\sqrt{2}}$. If this is rational, choose $a = b = \sqrt{2}$. If not, choose

$$a \;=\; \sqrt{2}^{\sqrt{2}}, \qquad b \;=\; \sqrt{2},$$

and note that in this case

$$a^b \;=\; (\sqrt{2}\,)^{(\sqrt{2}\times\sqrt{2})} \;=\; (\sqrt{2}\,)^2 \;=\; 2.$$

§2. History. As far as I know, the above proof first appeared in 1953 in a short note by Dov Jarden [Jarden 1953]. In 1966 a proof via $\sqrt{3}^{\sqrt{2}}$ was published, [Zeigenfus 1966].[1] Neither of these sources mentioned non-constructivity; indeed the proof's neatness gives it an intrinsic interest of its own.

For the use of this proof to explicitly illustrate non-constructivity, I believe the responsibility dates from 1970 in Swansea University. Peter Rogosinski, a fellow mathematician there, told me the proof, although he did not know a source for it, and I used it as an example of non-constructivity in undergraduate logic courses from 1970 onwards.

In August 1971 Dirk van Dalen gave a course on Intuitionism at a summer school on logic at Cambridge University. I attended and mentioned this 'non-constructive' example to him. He included it, with an acknowledgment, in the type-written course-notes he distributed there [Dalen 1971, p.1]. The summer school was attended by several prominent logicians, and from van Dalen's notes and their (shorter) published version [Dalen 1973], the example seems to have caught the imagination of various writers on constructivity.

Michael Dummett, who was at the summer school, included it in his 1971–72 Oxford University lecture-course on intuitionism[2] and in his book [Dummett 1977, §1.1][3].

It also appeared in [Jon & Top 1973] (which pointed out that $\sqrt{2}^{\sqrt{2}}$ is actually transcendental), in [Lam & Sco 1986, Part II §20 pp.226–227] (which also described

[1] I thank Jeffrey Shallit of the University of Waterloo, Canada, for these references, and for mentioning [Jon & Top 1973]. I am also very grateful to Mark Biggar, Jon Borwein, Dirk van Dalen, Michael Dummett, James P. Jones, Prakash Panangaden and Tim Smith for helpful information.

[2] Attended by Jon Borwein, whom I thank for this information

[3] It was there attributed as "due to Benenson", but Dummett, in correspondence 1991, stated that this was an error. In the 2000 edition of [Dummett 1977], the attribution is "due to Peter Rogosinski and Roger Hindley".

an alternative constructive proof, see below), in [Tro & Dal 1988, Chap.1 §2.3 pp.7–8], and in many later writings and internet items.

§3. Comment. The $\sqrt{2}$-proof is non-constructive because it gives no way of deciding the value of a. Gelfond proved in the 1930s that if x is any algebraic number distinct from 0 and 1, and y is algebraic and irrational, then x^y is irrational, indeed transcendental.[4] So the number a to be chosen in the $\sqrt{2}$-proof must in fact be $\sqrt{2}^{\sqrt{2}}$.

But, without a proof of Gelfond's theorem being added (moreover a constructive proof, if there is an accepted one), the $\sqrt{2}$-proof *as it stands above* is not constructive.

§4. A more constructive proof of Prop. 1.[5] Choose $a = \sqrt{2}$ and $b = 2\log_2 3$. Then

$$a^b \quad = \quad \left(\sqrt{2}\right)^{2\log_2 3} \quad = \quad 2^{\log_2 3} \quad = \quad 3.$$

To see that b is irrational, suppose $b = p/q$ where p and q are integers with no common factor; then

$$p \quad = \quad bq \quad = \quad 2q\log_2 3 \quad = \quad \log_2(3^{2q}),$$

so $2^p = 3^{2q}$, which would contradict the unique factorisation theorem for integers.

§5. Warning. Actually, the $\sqrt{2}$-proof does not illuminate the constructivity concept quite as neatly as it seems to at first, because the concept of irrationality is not completely simple from a constructive viewpoint. A claim that $\sqrt{2}$ is irrational might be interpreted in several different ways, for example:

- ($\forall$ integers p, q) ($\sqrt{2} = p/q$ implies absurdity),
- ($\forall$ integers p, q)($\exists n$) (n-th term in the decimal expansion of $\sqrt{2}$ differs from n-th term in that of p/q).

References

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[4] See [Gelfond 1960, p.106 Thm.2] or [Jon & Top 1973].

[5] This proof is from [Pol & Sze 1976, p.362, answers to Problems 260.1 and 260.2 on p.255]; incidentally, the $\sqrt{2}$-proof is not mentioned there, and these two problems do not occur in earlier editions of [Pol & Sze 1976]. Essentially the same proof also occurs in [Lam & Sco 1986, Part II §20, p.227].

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[Zeigenfus 1966] **Zeigenfus, C.**, *Quickie* Q380, ***Mathematics Magazine*** 39 (1966) p.134. (See also answer A380 on p.111.)