TEACHING NOTE 03-01:
OPTION PRICES AND EXPECTED RETURNS

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In the study of finance, we devote considerable time to deriving and using pricing models. Probably the two best known pricing models are the *Capital Asset Pricing Model* and the *Black-Scholes* (and *Merton*) *Option Pricing Model*.[1] The former, commonly referred to as the CAPM, tells us the required rate of return on the asset. If the asset is correctly priced (which the model assumes must happen in equilibrium), the return expected by investors (the expected rate of return) equals the required rate of return. The model typically appears as follows:

$$E(R_s) = r + [E(R_m) - r]\beta_s,$$

where

$E(R_s)$ = expected and required return on stock s

r = risk-free rate

$E(R_m)$ = expected and required return on the market portfolio consisting of all risky assets

β_s = beta of stock s, obtained as

$$\beta_s = \frac{\operatorname{cov}(R_s, R_m)}{\sigma_m^2}$$

where

$\operatorname{cov}(R_s,R_m)$ = covariance between the return on stock s and market portfolio m

σ_m^2 = variance of the return on market portfolio m

The CAPM is not typically expressed in the form of the asset's price, though that can be done using a model specifying how the asset's price is obtained from its future cash flow. For example, define the one-period return on a stock as

$$R_s = \frac{S'}{S} - 1,$$

[1]Other well-known models are the *Arbitrage Pricing Model* and the *Cost of Carry Forward/Futures Pricing Model* (known as *Interest Rate Parity* in the foreign currency literature).

where S' is the stock price one period later.[2] The expected return would be

$$E(R_s) = \frac{E(S')}{S} - 1.$$

But from the CAPM, we also know that $E(R_s) = r + [E(R_m) - r]\beta_s$. Using the definition of beta stated above and equating these two specifications for the expected return, we have

$$\frac{E(S')}{S} - 1 = r + [E(R_m) - r]\frac{\text{cov}(R_s, R_m)}{\sigma_m^2}.$$

Noting that the covariance of R_s and R_m can be expressed as $(1/S)\text{cov}(S', R_m)$, we can substitute this result and solve for S to obtain:

$$S = \frac{E(S') - \lambda \text{cov}(S', R_m)}{1 + r},$$

where

$$\lambda = \frac{[E(R_m) - r]}{\sigma_m^2}.$$

This equation is the CAPM written in the form of a price. In the numerator is the expected value of the asset at the next date, E(R') minus the risk premium, $\lambda\text{cov}(S', R_m)$. The value λ is considered the market risk premium. It reflects the average level of risk in the market. This risk-adjusted future value is then discounted by the risk-free rate to obtain the price. Financial economists typically refer to this form of the model as a *certainty equivalent*.[3]

The option pricing model is nearly always written in the form of a price, in the following manner

$$c = SN(d_1) - Xe^{-rT}N(d_2),$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and

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[2]We are assuming no dividends. These would not cause any problems, but our approach would vary depending on whether the dividends are known or random.

[3]The notion of a certainty equivalent is that of a value that one would accept for certain in lieu of facing a risky situation. The expected value minus the risk premium that appears in the numerator is a risk-adjusted future value, which can then be discounted at the risk-free rate.

S = current price of the asset

X = exercise price of the option

r = risk-free rate

T = time to expiration

σ = standard deviation of the stock return

The option pricing model is derived in continuous time by forming a risk-free hedge consisting of the stock and the option. The option price formula is obtained as the solution given the constraint that the riskless portfolio must return the risk-free rate to prevent arbitrage.

Seldom is the option pricing model expressed in the form of the option's expected return. In this note we provide this linkage between the notion of an equilibrium expected return and an arbitrage-free price of a call option. The same result can be obtained if the option is a put.

Expected Returns on Options

Consider a hedge portfolio consisting of h shares of the stock and one short option. The portfolio value is hS – c. One period later the portfolio will be worth

$$h(S + \Delta S) - (c + \Delta c)$$

where the symbol Δ means the change in S or c. If this portfolio is hedged, its value should grow at the risk-free rate. Hence, the following condition must hold:

$$(hS - c)(1 + r) = h(S + \Delta S) - (c + \Delta c).$$

Gathering option terms on the left-hand side and stock terms on the right-hand side and dividing by c, we obtain the following useful result:

$$\frac{\Delta c}{c} = r + \left[\frac{\Delta S}{S} - r\right]h\frac{S}{c}.$$

This expression says that the option return is the sum of the risk-free rate and an ex post risk premium, ΔS/S – r, times a risk factor h(S/c). Recall that the Black-Scholes model tells us that

$$h = \frac{\partial c}{\partial S},$$

which should be recognized as the option's delta. Then our risk factor is

$$\frac{\partial c}{\partial S}\frac{S}{c} = \frac{\partial c / c}{\partial S / S}.$$

Economists, of course, recognize any term reflecting the percentage movement in one variable divided by the percentage movement in another as the concept of *elasticity*. Thus, we see that the option's return is related to the stock return by the risk-free rate, a risk premium, and term reflecting the option's elasticity.

Elasticity in this context measures the sensitivity of the option to the stock and, hence, is a reflection of the option's leverage. Elasticity is closely related to the option's delta, $\partial c/\partial S$, but delta is an absolute measure, capturing the movement of the option price relative to the movement in the stock price. Elasticity is a relative measure and, as such, is more appropriate when dealing with rates of return.

The elasticity of a standard European option is at least equal to 1.[4] This means that the absolute value of the option return will exceed the absolute value of the stock return.

The elasticity of an option is usually denoted with the Greek letter omega Ω. Thus, our equation for the return on the option is

$$\frac{\Delta c}{c} = r + \left[\frac{\Delta S}{S} - r\right]\Omega.$$

Armed with this result, we can now examine the expected return on the option. Taking the expectation of this equation, we obtain

$$E\left(\frac{\Delta c}{c}\right) = r + \left[E\left(\frac{\Delta S}{S}\right) - r\right]\Omega.$$

The left-hand side, $E(\Delta c/c)$, is the expected return on the call, which we denote as $E(R_c)$. Within the right-hand side, the term $E(\Delta S/S)$ is the expected return on the stock, which we denote as $E(R_s)$. Now we have a simple equation for the expected return on the call:

$$E(R_c) = r + [E(R_s) - r]\Omega.$$

We see that the expected return on the call equals the risk-free rate plus the risk premium on the stock times the option's elasticity. This functional form is very appealing and intuitive. The option's expected return at a minimum is the risk-free rate and is increased

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[4]It is easy to use the Black-Scholes model to see that the elasticity is not less than 1. Elasticity is defined as $(\partial c/\partial S)(S/c)$. You should recognize this as $N(d_1)S/c$ from the Black-Scholes model. Replacing c with the Black-Scholes formula reveals that elasticity is no less than 1 if $Xe^{-rT}N(d_2)$ is non-negative, which is always true.

by a risk premium, obtained as the product of the risk premium on the stock and the risk of the option relative to the stock. Now let us try to determine the volatility of the option.

Volatilities of Options

Using the expression created above for the return on the option as a function of the return on the stock, we take the variance of the option return:

$$\begin{aligned} \text{Var}\left(\frac{\Delta c}{c}\right) &= \text{Var}\left(r + \left(\left[\frac{\Delta S}{S}\right] - r\right)\Omega\right) \\ &= \text{Var}\left(\frac{\Delta S}{S}\Omega\right) \\ &= \Omega^2 \text{Var}(R_c). \end{aligned}$$

Expressing this result in terms of the standard deviation:

$$\sigma_c = \Omega\sigma_s.$$

We see that the volatility of the option is the volatility of the stock times the elasticity. Thus, the option's risk is the risk of the stock times the risk of the option relative to the stock. This result should seem intuitive.

These results concerning expected returns and volatilities of options apply regardless of how expected returns are determined on the underlying stock. In the special case that the Capital Asset Pricing Model explains expected returns on stocks, we can obtain further insights.

Options and the Capital Asset Pricing Model

Now suppose that the CAPM holds. Recall that the CAPM is a model for the pricing of all risky assets. An option is a risky asset. Therefore, the expected return on the option must also be governed by the CAPM. Hence, we get the equation

$$E(R_c) = r + [E(R_m) - r]\beta_c,$$

where β_c is the beta of the call and is recognized as its risk with respect to the market portfolio. Substituting our CAPM equation for the expected return on the stock into the CAPM for the expected return on the option gives

$$\begin{aligned} E(R_c) &= r + (E(R_s) - r)\Omega \\ &= r + ((r + [E(R_m) - r]\beta_S) - r)\Omega \\ &= r + [E(R_m) - r]\Omega \end{aligned}$$

Hence, the option beta is given as

$$\beta_c = \beta_S \Omega,$$

indicating that the option's beta is the stock beta times the elasticity. Once gain, we see the role that elasticity plays. As we noted earlier, elasticity is a relative measure (the percentage change in the option return divided by the percentage change in the stock return). Because beta is a relative measure, naturally elasticity plays an important part in the relationship of the option's beta to the stock's beta.

Options and the Sharpe Ratio

A widely used measure of investment performance is the Sharpe ratio. For a portfolio with return R_p and volatility σ_p, the Sharpe ratio is

$$\text{Sharpe}_p = \frac{R_p - r}{\sigma_p}.$$

The Sharpe ratio measures the return over and above the risk-free rate expressed relative to the total risk. Using our measures of return and volatility for an option, the Sharpe ratio for an option is

$$\text{Sharpe}_c = \frac{R_c - r}{\sigma_c} = \frac{r + \left[\frac{\Delta S}{S} - r\right] h \frac{S}{c} - r}{h \frac{S}{c} \sigma_s}$$

$$= \frac{\left[\frac{\Delta S}{S} - r\right]}{\sigma_s}.$$

In other words, the Sharpe ratio for an option is the Sharpe ratio for the stock! This result is likely to surprise some, but an explanation is simple. The Sharpe ratio measures whether an investment provided a risk premium in excess of the appropriate risk premium for its level of risk. If the option is correctly priced relative to its stock, it cannot provide a risk premium beyond that already provided by the stock. The option merely levers the risk premium of the stock. And the leverage used by the option, while an advantage in augmenting the return (note the leverage factor in the numerator), is a disadvantage in augmenting the risk (note the leverage factor in the denominator).

Of course if the option is mispriced, then the arbitrage linkage between option and stock that enabled us to obtain the above results is broken. An option would then provide a form of excess return, and the Sharpe ratio of the option would exceed that of the stock.[5]

The Stochastic Process Followed by the Option

When deriving the option pricing model, we assume that the stock follows a stochastic process of the form

$$dS = \mu_s(S,t)dt + \sigma_s(S,t)dW,$$

where dW is a standard Brownian motion. The expected return $\mu_s(S,t)$ normally takes the simple form of $\mu_s(S,t) = \mu_c S$, with the volatility taking the form $\sigma_s(S,t) = \sigma_s S$, and neither being influenced by time. More general versions of the model could have the expected return and volatility be more complex functions of S and t.

To obtain the option pricing model, we require the stochastic process for the stock. We do not specifically require a full analysis of the option's stochastic properties to obtain its price.[6] Studies of the performance of option strategies and of option pricing models, however, invariably make use of probability models and statistical rules. Hence, it may be important for some purposes to know what the stochastic process would look like for the option. In general, we would expect it to be of the form

$$dc = \mu_c(S,t)dt + \sigma_c(S,t)dW,$$

where we note that the option's expected return and volatility are functions of S and t.

We can easily obtain the stochastic process for the option. Recall that in deriving the option pricing formula, we use Itô's Lemma, which expresses the change in the call price as a function of first and second order changes in the stock price and time:

$$dc = \frac{\partial c}{\partial S}dS + \frac{\partial c}{\partial t}dt + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}dS^2 dt.$$

Substituting the stochastic process of the stock for dS and noting that dS^2 is the well-known result $S^2\sigma_s^2 dt$, we obtain

$$dc = \left(\frac{\partial c}{\partial S}S\mu_s + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}S^2\sigma_s{}^2 + \frac{\partial c}{\partial t}\right)dt + \left(\frac{\partial c}{\partial S}S\sigma_s\right)dW.$$

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[5]Using Sharpe ratios for options is problematic, however, because option return distributions are highly non-normal, but the Sharpe ratio characterizes performance exclusively with the expected return and volatility, ignoring higher order moments associated with non-normal distributions.

[6]In other words, the Black-Scholes model is obtained without any reference to the option's expected return or volatility, nor does it directly provide the option's expected return and volatility.

Dividing by c, we find that that the return on the call is

$$\frac{dc}{c} = \left(\frac{\frac{\partial c}{\partial S} S\mu_s + \frac{1}{2}\frac{\partial^2 c}{\partial S^2} S^2 {\sigma_s}^2 + \frac{\partial c}{\partial t}}{c} \right) dt + \left(\frac{\partial c}{\partial S}\frac{S}{c}\sigma_s \right) dW.$$

Now, suppose we wish to obtain the expected return and volatility of the option. We must first recognize the dimension of the parameters of the model. In the option pricing model, the return on the option, dc/c, is measure over an infinitesimal time interval. We can take the expected return but it will be multiplied by dt and will reflect the expectation over this very short time interval. The CAPM reflects returns over a finite interval.[7] To make these values comparable, let us add a dt to the left-hand side in taking both the expected return and volatility of the option. We can then drop the dt's from both sides.

Taking expectations, we obtain[8]

$$\mu_c(S,t)dt = E\left(\frac{dc}{c} \right) = \left(\frac{\frac{\partial c}{\partial S} S\mu_s + \frac{1}{2}\frac{\partial^2 c}{\partial S^2} S^2 {\sigma_s}^2 + \frac{\partial c}{\partial t}}{c} \right) dt.$$

Although this formula for the expected return does not look like the expected return for the option, it can be shown to be the same. All we are required to do is substitute the partial derivatives $\partial c/\partial S$, $\partial^2 c/\partial S^2$, and $\partial c/\partial t$, which are well-known formulas available in most options textbooks. We then obtain

$$\mu_c(S,t) = N(d_1)\frac{S}{c}\mu_s - r\frac{X}{c}e^{-rT}N(d_2).$$

From the Black-Scholes formula, we substitute $SN(d_1) - c$ for $Xe^{-rT}N(d_2)$ and obtain

$$\mu_c(S,t) = r + (\mu_s - r)N(d_1)\frac{S}{c}.$$

Since $N(d_1)$ is $\partial c/\partial S$ and $\mu_s = E(R_s)$, this is the same formula for the option's expected return we obtained previously.

Taking the volatility of the expected return on the option, we obtain[9]

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[7]Alternatively, we could use the continuous time version of the CAPM and adjust that equation so that it would contain a dt term and reflect expected returns over the interval dt.
[8]In taking expectations of this equation, recall that the expectation of dW is zero.

$$\sigma_c(S,t)^2 dt = Var\left(\frac{dc}{c}\right) = \left(\frac{\partial c}{\partial S}\frac{S}{c}\sigma_s\right)^2 dt$$

$$\sigma_c(S,t) = \frac{\partial c}{\partial S}\frac{S}{c}\sigma_s.$$

In that case, this is the same formula we previously obtained for the option's volatility.

Finally, we should likewise recognize that the risk-free rate in the finite interval CAPM and the infinitesimal interval option pricing model need to be expressed on a comparable basis. The CAPM typically uses discrete interest, while the option pricing model uses continuous interest. We would need to be sure that interest is measured in the same manner in both models to make the results comparable.

Final Comments

We see in this note that option pricing is consistent with capital asset pricing. The price obtained from the Black-Scholes model is consistent with the expected return from the CAPM. But even if the CAPM does not hold, the expected return on the option can be related to the expected return on the stock, through the risk-free rate, the risk premium on the stock, and the option's risk relative to the stock. Unless the option is incorrectly priced relative to the stock, the performance of the option as measured by its Sharpe ratio is no different from the performance of the stock.

References

The expected return on an option has been analyzed in

Rubinstein, Mark. "A Simple Formula for the Expected Rate of Return of an Option Over a Finite Holding Period." *The Journal of Finance* 39 (1984), 1503-1509.

An empirical study on option expected returns is

Coval, Joshua and Tyler Shumway. "Expected Option Returns." *The Journal of Finance* 56 (2001), 983-1009.

[9]Remember that the variance of a constant (in this case, ∂c/∂S)(S/c)) times a random variable is the constant squared times the variance of the random variable. The variance of dW is well-known to be dt.