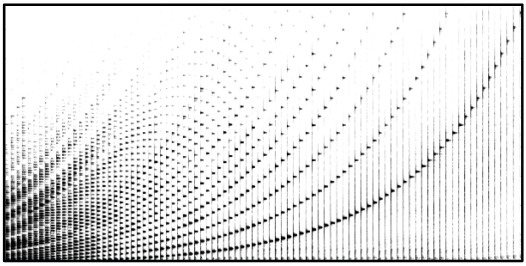
Chapter 2
Fourier Analysis of Signals



As we have seen in the last chapter, music signals are generally complex sound mixtures that consist of a multitude of different sound components. Because of this complexity, the extraction of musically relevant information from a waveform constitutes a difficult problem. A first step in better understanding a given signal is to decompose it into building blocks that are more accessible for the subsequent processing steps. In the case that these building blocks consist of sinusoidal functions, such a process is also called Fourier analysis. Sinusoidal functions are special in the sense that they possess an explicit physical meaning in terms of frequency. As a consequence, the resulting decomposition unfolds the frequency spectrum of the signal—similar to a prism that can be used to break light up into its constituent spectral colors. The Fourier transform converts a signal that depends on time into a representation that depends on frequency. Being one of the most important tools in signal processing, we will encounter the Fourier transform in a variety of music processing tasks.

In Section 2.1, we introduce the main ideas of the Fourier transform and summarize the most important facts that are needed for understanding the subsequent chapters of the book. Furthermore, we introduce the required mathematical notions. A good understanding of Section 2.1 is essential for the various music processing tasks to be discussed. In Section 2.2 to Section 2.5, we cover the Fourier transform in greater mathematical depth. The reader who is mainly interested in the music processing applications may skip these more technical sections on a first reading.

In Section 2.2, we take a closer look at signals and discuss their properties from a more abstract perspective. In particular, we consider two classes of signals: analog signals that give us the right physical interpretation and digital signals that are needed for actual digital processing by computers. The different signal classes lead to different versions of the Fourier transform, which we introduce with math-

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ematical rigor along with intuitive explanations and numerous illustrating examples (Section 2.3). In particular, we explain how the different versions are interrelated and how they can be approximated by means of the discrete Fourier transform (DFT). The DFT can be computed efficiently by means of the fast Fourier transform (FFT), which will be discussed in Section 2.4. Finally, we introduce the short-time Fourier transform (STFT), which is a local variant of the Fourier transform yielding a time–frequency representation of a signal (Section 2.5). By presenting this material from a different perspective as typically encountered in an engineering course, we hope to refine and sharpen the understanding of these important and beautiful concepts.

2.1 The Fourier Transform in a Nutshell

Let us start with an audio signal that represents the sound of some music. For example, let us analyze the sound of a single note played on a piano (see Figure 2.1a). How can we find out which note has actually been played? Recall from Section 1.3.2 that the pitch of a musical tone is closely related to its fundamental frequency, the frequency of the lowest partial of the sound. Therefore, we need to determine the frequency content, the main periodic oscillations of the signal. Let us zoom into the signal considering only a 10-ms section (see Figure 2.1b). The figure shows that the signal behaves in a nearly periodic way within this section. In particular, one can observe three main crests of a sinusoidal-like oscillation (see also Figure 2.1c). Having approximately three oscillation cycles within a 10-ms section means that the signal contains a frequency component of roughly 300 Hz.

The main idea of **Fourier analysis** is to compare the signal with sinusoids of various[1] frequencies $\omega \in \mathbb{R}$ (measured in Hz). Each such sinusoid or pure tone may be thought of as a prototype oscillation. As a result, we obtain for each considered frequency parameter $\omega \in \mathbb{R}$ a magnitude coefficient $d_\omega \in \mathbb{R}_{\geq 0}$ (along with a phase coefficient $\varphi_\omega \in \mathbb{R}$, the role of which is explained later). In the case that the coefficient d_ω is large, there is a high similarity between the signal and the sinusoid of frequency ω, and the signal contains a periodic oscillation at that frequency (see Figure 2.1c). In the case that d_ω is small, the signal does not contain a periodic component at that frequency (see Figure 2.1d).

Let us plot the coefficients d_ω over the various frequency parameters $\omega \in \mathbb{R}$. This yields a graph as shown in Figure 2.1f. In this graph, the highest value is assumed for the frequency parameter $\omega = 262$ Hz. By (1.1), this is roughly the center frequency of the pitch $p = 60$ or the note C4. Indeed, this is exactly the note played in our piano example. Furthermore, as illustrated by Figure 2.1e, one can also observe a

[1] In the following, we also consider *negative frequencies* for mathematical reasons without explaining this concept in more detail. In our musical context, negative frequencies are redundant (having the same interpretation as positive frequencies), but simplify the mathematical formulation of the Fourier transform.

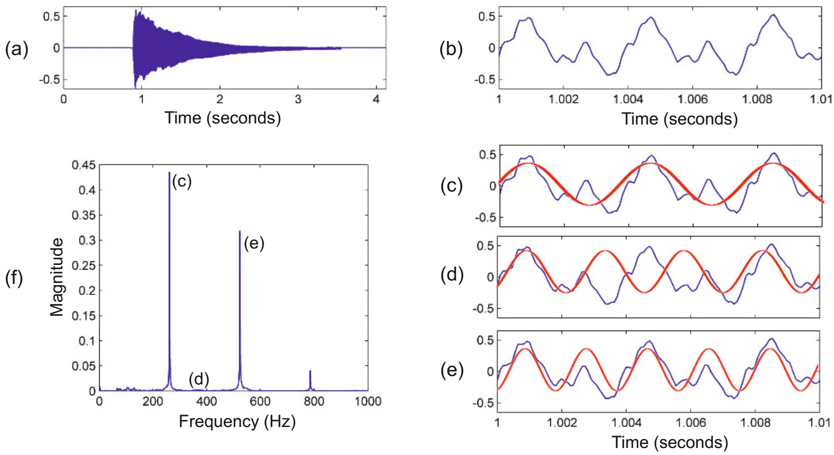

Fig. 2.1 **(a)** Waveform of a note C4 (261.6 Hz) played on a piano. **(b)** Zoom into a 10-ms section starting at time position $t = 1$ sec. **(c–e)** Comparison of the waveform with sinusoids of various frequencies ω. **(f)** Magnitude coefficients d_ω in dependence on the frequency ω.

high similarity between the signal and the sinusoid of frequency $\omega = 523$ Hz. This is roughly the frequency for the second partial of the tone C4.

With this example, we have already seen the main idea behind the **Fourier transform**. The Fourier transform breaks up a signal into its frequency components. For each frequency $\omega \in \mathbb{R}$, the Fourier transforms yields a coefficient d_ω (and a phase φ_ω) that tells us to which extent the given signal matches a sinusoidal prototype oscillation of that frequency.

One important property of the Fourier transform is that the original signal can be reconstructed from the coefficients d_ω (along with the coefficients φ_ω). To this end, one basically superimposes the sinusoids of all possible frequencies, each weighted by the respective coefficient d_ω (and shifted by φ_ω). This weighted superposition is also called the **Fourier representation** of the original signal. The original signal and the Fourier transform contain the same amount of information. This information, however, is represented in different ways. While the signal displays the information across time, the Fourier transform displays the information across frequency. As put by Hubbard [9], the signal tells us when certain notes are played in time, but hides the information about frequencies. In contrast, the Fourier transform of music displays which notes (frequencies) are played, but hides the information about when the notes are played.

In the following sections, we take a more detailed look at the Fourier transform and some of its main properties.

2.1.1 Fourier Transform for Analog Signals

In Section 1.3.1, we saw that a signal or sound wave yields a function that assigns to each point in time the deviation of the air pressure from the average air pressure at a specific location. Let us consider the case of an **analog** signal, where both the time as well as the amplitude (or deviation) are continuous, real-valued parameters. In this case, a signal can be modeled as a function $f\colon \mathbb{R} \to \mathbb{R}$, which assigns to each time point $t \in \mathbb{R}$ an amplitude value $f(t) \in \mathbb{R}$. Plotting the amplitude over time, one obtains a graph of this function that corresponds to the waveform of the signal (see Figure 1.17).

The term **function** may need some explanation. In mathematics, a function yields a relation between a set of input elements and a set of output elements, where each input element is related to exactly one output element. For example, a function can be a polynomial $f\colon \mathbb{R} \to \mathbb{R}$ that assigns for each input element $t \in \mathbb{R}$ an output element $f(t) = t^2 \in \mathbb{R}$. At this point, we want to emphasize that one needs to differentiate between a function f and its output element $f(t)$ (also referred to as the **value**) at a particular input element t (also referred to as the **argument**). In other words, mathematicians think of a function f in an abstract way, where the symbol or physical meaning of the argument does not matter. As opposed to this, engineers often like to emphasize the meaning of the input argument and loosely speak of a function $f(t)$, even though this is strictly speaking an output value. In this book, we assume the viewpoint of a mathematician.

2.1.1.1 The Role of the Phase

After this side note, let us turn towards the spectral analysis of a given analog signal $f\colon \mathbb{R} \to \mathbb{R}$. As explained in our introductory example, we compare the signal f with prototype oscillations that are given in the form of sinusoids. In Section 1.3.2 and Figure 1.19, we have already encountered such sinusoidal signals. Mathematically, a **sinusoid** is a function $g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) := A\sin(2\pi(\omega t - \varphi)) \tag{2.1}$$

for $t \in \mathbb{R}$. The parameter A corresponds to the **amplitude**, the parameter ω to the **frequency** (measured in Hz), and the parameter φ to the **phase** (measured in normalized radians with 1 corresponding to an angle of 360°). In Fourier analysis, we consider prototype oscillations that are normalized with regard to their power (average energy) by setting $A = \sqrt{2}$. Thus for each frequency parameter ω and phase parameter φ we obtain a sinusoid $\mathbf{cos}_{\omega,\varphi} : \mathbb{R} \to \mathbb{R}$ given by

$$\mathbf{cos}_{\omega,\varphi}(t) := \sqrt{2}\cos(2\pi(\omega t - \varphi)) \tag{2.2}$$

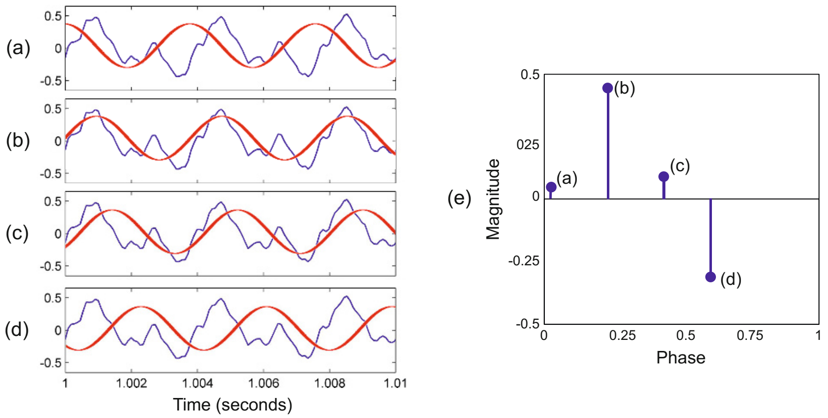

Fig. 2.2 **(a–d)** Waveform and different sinusoids of a fixed frequency $\omega = 262$ Hz but different phases $\varphi \in \{0.05, 0.24, 0.45, 0.6\}$. **(e)** Values that express the degree of similarity between the waveform and the four different sinusoids.

for $t \in \mathbb{R}$. Since the cosine function is periodic, the parameters φ and $\varphi + k$ for integers $k \in \mathbb{Z}$ yield the same function. Therefore, the phase parameter only needs to be considered for $\varphi \in [0,1)$.

When measuring how well the given signal coincides with a sinusoid of frequency ω, we have the freedom of shifting the sinusoid in time. This degree of freedom is expressed by the phase parameter φ. As illustrated by Figure 2.2, the degree of similarity between the signal and the sinusoid of fixed frequency crucially depends on the phase. What have we done with the phase when computing the coefficients d_ω as illustrated by Figure 2.1? The procedure outlined in the introduction was only half the story. When comparing the signal f with a sinusoid $\mathbf{cos}_{\omega,\varphi}$ of frequency ω, we have implicitly used the phase φ_ω that yields the maximal possible similarity. To understand this better, we first need to explain how we actually compare the signal and a sinusoid or, more generally, how we compare two given functions.

2.1.1.2 Computing Similarity with Integrals

Let us assume that we are given two functions of time $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. What does it mean for f and g to be similar? Intuitively, one may agree that f and g are similar if they show a similar behavior over time: if f assumes positive values, then so should g, and if f becomes negative, the same should happen to g. The joint behavior of these functions can be captured by forming the integral of the product of the two functions:

$$\int_{t \in \mathbb{R}} f(t) \cdot g(t) dt. \tag{2.3}$$

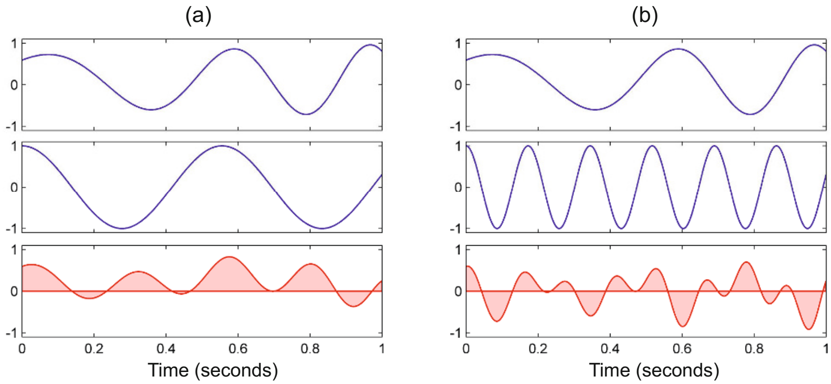

Fig. 2.3 Measuring the similarity of two functions f (top) and g (middle) by computing the integral of the product (bottom). **(a)** Two functions having high similarity. **(b)** Two functions having low similarity.

The integral measures the area delimited by the graph of the product $f \cdot g$, where the negative area (below the horizontal axis) is subtracted from the positive area (above the horizontal axis) (see Figure 2.3). In the case that f and g are either both positive or both negative at most time instances, the product is positive for most of the time and the integral becomes large (see Figure 2.3a). However, if the two functions are dissimilar, then the overall positive and the overall negative areas cancel out, yielding a small overall integral (see Figure 2.3b). Further examples are discussed in Exercise 2.1.

There are many more ways for comparing two given signals. For example, the integral of the absolute difference between the functions also yields a notion of how similar the signals are. In the formulation of the Fourier transform, however, one encounters the measure as considered in (2.3), which generalizes the **inner product** known from linear algebra (see 2.37). We continue this discussion in Section 2.2.3.

2.1.1.3 First Definition of the Fourier Transform

Based on the similarity measure (2.3), we compare the original signal f with sinusoids $g = \mathbf{cos}_{\omega,\varphi}$ as defined in (2.2). For a fixed frequency $\omega \in \mathbb{R}$, we define

$$d_\omega := \max_{\varphi \in [0,1)} \left(\int_{t \in \mathbb{R}} f(t) \mathbf{cos}_{\omega,\varphi}(t) dt \right), \tag{2.4}$$

$$\varphi_\omega := \operatorname*{argmax}_{\varphi \in [0,1)} \left(\int_{t \in \mathbb{R}} f(t) \mathbf{cos}_{\omega,\varphi}(t) dt \right). \tag{2.5}$$

As previously discussed, the magnitude coefficient d_ω expresses the intensity of frequency ω within the signal f. Additionally, the phase coefficient $\varphi_\omega \in [0,1)$ tells

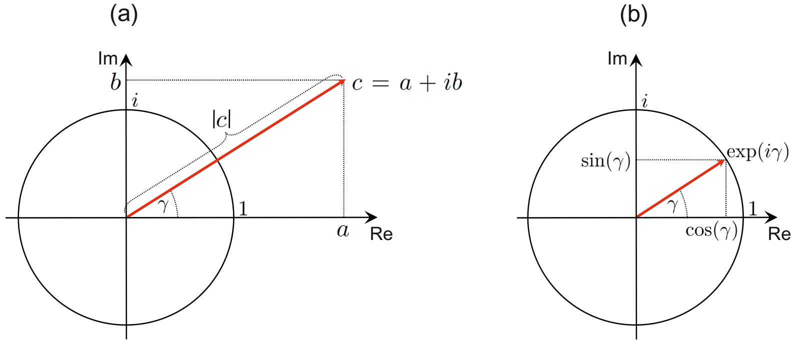

Fig. 2.4 **(a)** Polar coordinate representation of a complex number $c = a + ib$. **(b)** Definition of the exponential function.

us how the sinusoid of frequency ω needs to be displaced in time to best fit the signal f. The **Fourier transform** of a function $f : \mathbb{R} \to \mathbb{R}$ is defined to be the "collection" of all coefficients d_ω and φ_ω for $\omega \in \mathbb{R}$. Shortly, we will state this definition in a more formal way.

The computation of d_ω and φ_ω feels a bit awkward, since it involves an optimization step. The good news is that there is a simple solution to this optimization problem, which results from the existence of certain trigonometric identities that relate phases and amplitudes of certain sinusoidal functions. Using the concept of complex numbers, these trigonometric identities become simple and lead to an elegant formulation of the Fourier transform. We discuss such issues in more detail in Section 2.3. In the following, we introduce the standard complex-valued formulation of the Fourier transform without giving any proofs.

2.1.1.4 Complex Numbers

Let us first review the concept of complex numbers. The complex numbers extend the real numbers by introducing the imaginary number $i := \sqrt{-1}$ with the property $i^2 = -1$. Each complex number can be written as $c = a + ib$, where $a \in \mathbb{R}$ is the real part and $b \in \mathbb{R}$ the imaginary part of c. The set of all complex numbers is written as $\mathbb{C}$, which can be thought of as a two-dimensional plane: the horizontal dimension corresponds to the real part, and the vertical dimension to the imaginary part. In this plane, the number $c = a + ib$ is specified by the Cartesian coordinates (a, b). As illustrated by Figure 2.4a, there is another way of representing a complex number, which is known as the **polar coordinate** representation. In this case, a complex number c is described by its absolute value $|c|$ (distance from the origin) and the angle γ between the positive horizontal axis and the line from the origin and c. The polar coordinates $|c| \in \mathbb{R}_{\geq 0}$ and $\gamma \in [0, 2\pi)$ (given in radians) can be derived from the coordinates (a, b) via the following formulas:

$$|c| := \sqrt{a^2 + b^2}, \tag{2.6}$$

$$\gamma := \text{atan2}(b, a). \tag{2.7}$$

Further details on polar coordinates and the function atan2, which is a variant of the inverse of the tangent function, are explained in Section 2.3.2.2. To regain the complex number c from its polar coordinates, one uses the **exponential function**, which maps an angle $\gamma \in \mathbb{R}$ (given in radians) to a complex number defined by

$$\exp(i\gamma) := \cos(\gamma) + i\sin(\gamma) \tag{2.8}$$

(see also Figure 2.4b). The values of this function turn around the unit circle of the complex plane with a period of 2π (see Section 2.3.2.1). From this, we obtain the following **polar coordinate representation** for a complex number c:

$$c = |c| \cdot \exp(i\gamma). \tag{2.9}$$

2.1.1.5 Complex Definition of the Fourier Transform

What have we gained by bringing complex numbers into play? Recall that we have obtained a positive coefficient $d_\omega \in \mathbb{R}_{\geq 0}$ from (2.4) and a phase coefficient $\varphi_\omega \in [0, 1)$ from (2.5). The basic idea is to use these coefficients as polar coordinates and to encode both coefficients by a single complex number. Because of some technical reasons (a normalization issue that becomes clearer when discussing the mathematical details), one introduces some additional factors and a sign in the phase to yield the complex coefficient

$$c_\omega := \frac{d_\omega}{\sqrt{2}} \cdot \exp(2\pi i(-\varphi_\omega)). \tag{2.10}$$

This complex formulation directly leads us to the Fourier transform of a real-valued function $f : \mathbb{R} \to \mathbb{R}$. For each frequency $\omega \in \mathbb{R}$, we obtain a complex-valued coefficient $c_\omega \in \mathbb{C}$ as defined by (2.4), (2.5), and (2.10). This collection of coefficients can be encoded by a complex-valued function $\hat{f} : \mathbb{R} \to \mathbb{C}$ (called "f hat"), which assigns to each frequency parameter the coefficient c_ω:

$$\hat{f}(\omega) := c_\omega. \tag{2.11}$$

The function $\hat{f}$ is referred to as the **Fourier transform** of f, and its values $\hat{f}(\omega) = c_\omega$ are called the **Fourier coefficients**. One main result in Fourier analysis is that the Fourier transform can be computed via the following compact formula:

$$\hat{f}(\omega) = \int_{t \in \mathbb{R}} f(t) \exp(-2\pi i \omega t) dt \tag{2.12}$$

$$= \int_{t \in \mathbb{R}} f(t) \cos(-2\pi \omega t) dt + i \int_{t \in \mathbb{R}} f(t) \sin(-2\pi \omega t) dt. \tag{2.13}$$

In other words, the real part of the complex coefficient $\hat{f}(\omega)$ is obtained by comparing the original signal f with a cosine function of frequency ω, and the imaginary part is obtained by comparing with a sine function of frequency ω. The absolute value $|\hat{f}(\omega)|$ is also called the **magnitude** of the Fourier coefficient. Similarly, the real-valued function $|\hat{f}| : \mathbb{R} \to \mathbb{R}$, which assigns to each frequency parameter ω the magnitude $|\hat{f}(\omega)|$, is called the **magnitude Fourier transform** of f.

In the standard literature on signal processing, the formula (2.12) is often used to define the Fourier transform $\hat{f}$ and, then, the physical interpretation of the Fourier coefficients is discussed. In particular, the real-valued coefficients d_ω in (2.4) and φ_ω in (2.5) can be derived from $\hat{f}(\omega)$. Using (2.10), one obtains

$$d_\omega = \sqrt{2}|\hat{f}(\omega)|, \tag{2.14}$$

$$\varphi_\omega = -\frac{\gamma_\omega}{2\pi}, \tag{2.15}$$

where $|\hat{f}(\omega)|$ and γ_ω are the polar coordinates of $\hat{f}(\omega)$.

2.1.1.6 Fourier Representation

As mentioned above, the original signal f can be reconstructed from its Fourier transform. In principle, the reconstruction is straightforward: one superimposes the sinusoids of all possible frequency parameters $\omega \in \mathbb{R}$, each weighted by the respective coefficient d_ω and shifted by φ_ω. Both kinds of information are encoded in the complex Fourier coefficient c_ω. In the analog case considered so far, we are dealing with a continuum of frequency parameters, where the superposition becomes an integration over the parameter space. The reconstruction is given by the formulas

$$f(t) = \int_{\omega \in \mathbb{R}_{\geq 0}} d_\omega \sqrt{2} \cos(2\pi(\omega t - \varphi_\omega)) d\omega \tag{2.16}$$

$$= \int_{\omega \in \mathbb{R}} c_\omega \exp(2\pi i \omega t) d\omega, \tag{2.17}$$

first given in the real-valued formulation, and then given in the complex-valued formulation with $c_\omega = \hat{f}(\omega)$. As said before, the representation of a signal in terms of a weighted superposition of sinusoidal prototype oscillations is also called the **Fourier representation** of the signal. Notice that the formula (2.12) for the Fourier transform and the formula (2.17) for the Fourier representation are nearly identical. The main difference is that the roles of the time parameter t and frequency parameter ω are interchanged. The beautiful relationship between these two formulas will be further discussed in later sections of this chapter.

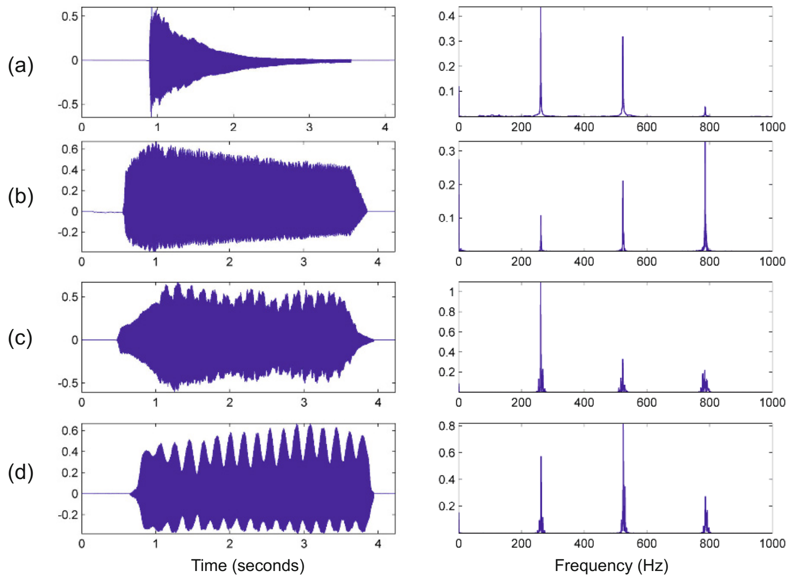

Fig. 2.5 Waveform and magnitude Fourier transform of a tone C4 (261.6 Hz) played by different instruments (see also Figure 1.23). **(a)** Piano. **(b)** Trumpet. **(c)** Violin. **(d)** Flute.

2.1.2 Examples

Let us consider some examples including the one introduced in Figure 2.1. Figure 2.5 shows the waveform and the magnitude Fourier transform for some audio signals, where a single note C4 is played on different instruments: a piano, a trumpet, a violin, and a flute. We have already encountered this example in Figure 1.23 of Section 1.3.4, where we discussed the aspect of timbre. Recall that the existence of certain partials and their relative strengths have a crucial influence on the timbre of a musical tone. In the case of the piano tone (Figure 2.5a), the Fourier transform has a sharp peak at 262 Hz, which reveals that most of the signal's energy is contained in the first partial or the fundamental frequency of the note C4. Further peaks (also beyond the shown frequency range from 0 to 1000 Hz) can be found at integer multiples of the fundamental frequency corresponding to the higher partials.

Figure 2.5b shows that the same note played on a trumpet results in a similar frequency spectrum, where the peaks appear again at integer multiples of the fundamental frequency. However, most of the energy is now contained in the third partial, and the relative heights of the peaks are different compared with the piano. This is one reason why a trumpet sounds different from a piano. For a violin, as shown by Figure 2.5c, most energy is again contained in the first partial. Observe that the peaks are blurred in frequency, which is the result of the vibrato (see also Figure 1.23b). The time-dependent frequency modulations of the vibrato are aver-

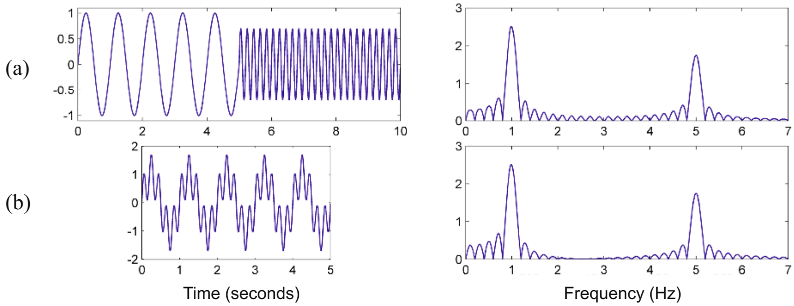

Fig. 2.6 Missing time information of the Fourier transform illustrated by two different signals and their magnitude Fourier transforms. **(a)** Two subsequent sinusoids of frequency 1 Hz and 5 Hz. **(b)** Superposition of the same sinusoids.

aged by the Fourier transform. This yields a single coefficient for each frequency independent of spectro-temporal fluctuations. A similar explanation holds for the flute tone shown in Figure 2.5d.

We have seen that the magnitude of the Fourier transform tells us about the signal's overall frequency content, but it does not tell us at which time the frequency content occurs. Figure 2.6 illustrates this fact, showing the waveform and the magnitude Fourier transform for two signals. The first signal consists of two parts with a sinusoid of $\omega = 1$ Hz and amplitude $A = 1$ in the first part and a sinusoid of $\omega = 5$ Hz and amplitude $A = 0.7$ in the second part. Furthermore, the signal is zero outside the interval $[0, 10]$. In contrast, the second signal is a superposition of these two sinusoids, being zero outside the interval $[0, 5]$. Even though the two signals are different in nature, the resulting magnitude Fourier transforms are more or less the same. This demonstrates the drawbacks of the Fourier transform when analyzing signals with changing characteristics over time. In Section 2.1.4 and Section 2.5 we discuss a short-time version of the Fourier transform, where time information is recovered at least to some degree. Besides the two peaks, one can observe in Figure 2.6 a large number of small "ripples." Such phenomena as well as further properties of the Fourier transform are discussed in Section 2.3.3.

2.1.3 Discrete Fourier Transform

When using digital technology, only a finite number of parameters can be stored and processed. To this end, analog signals need to be converted into finite representations—a process commonly referred to as **digitization**. One step that is often applied in an analog-to-digital conversion is known as **equidistant sampling**. Given an analog signal $f : \mathbb{R} \to \mathbb{R}$ and a positive real number $T > 0$, one defines a function $x : \mathbb{Z} \to \mathbb{R}$ by setting

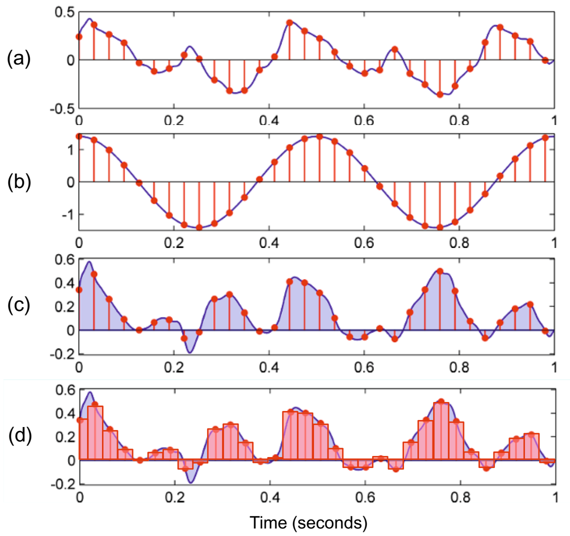

Fig. 2.7 Illustration of the sampling process using a sampling rate of $F_s = 32$. The waveforms of the analog signals are shown as curves and the sampled versions as stem plots. **(a)** Signal f. **(b)** Sinusoid $\mathbf{cos}_{\omega,\varphi}$ with $\omega = 2$ and $\varphi = 0$. **(c)** Product $f \cdot \mathbf{cos}_{\omega,\varphi}$ and its area. **(d)** Approximation of the integral by a Riemann sum obtained from the sampled version.

$$x(n) := f(n \cdot T). \tag{2.18}$$

Since x is only defined on a discrete set of time points, it is also referred to as a **discrete-time** (DT) signal (see Section 2.2.2.1). The value $x(n)$ is called a **sample** taken at time $t = n \cdot T$ of the original analog signal f. This procedure is also known as T**-sampling**, where the number T is referred to as the **sampling period**. The inverse

$$F_s := 1/T \tag{2.19}$$

of the sampling period is also called the **sampling rate** of the process. It specifies the number of samples per second and is measured in Hertz (Hz). Figure 2.7a shows an example of sampling an analog signal using $F_s = 32$ Hz.

In general, one loses information in the sampling process. The famous **sampling theorem** says that the original analog signal f can be reconstructed perfectly from its sampled version x, if f does not contain any frequencies higher than

$$\Omega := F_s/2 = 1/(2T) \text{ Hz}. \tag{2.20}$$

In this case, we also say that f is an Ω**-bandlimited** signal, where the frequency Ω is known as the **Nyquist frequency**. In the case that f contains higher frequencies, sampling may cause artifacts referred to as **aliasing** (see Section 2.2.2 for details). The sampling theorem will be further discussed in Exercise 2.28.

In the following, we assume that the analog signal f satisfies suitable requirements so that the sampled signal x does not contain major artifacts. Now, having a discrete number of samples to represent our signal, how do we calculate the Fourier transform? Recall that the idea of the Fourier transform is to compare the signal with a sinusoidal prototype oscillation by computing the integral over the pointwise product (see (2.12)). Therefore, in the digital domain, it seems reasonable to sample the sinusoidal prototype oscillation in the same fashion as the signal (see Figure 2.7b). By multiplying the two sampled functions in a pointwise fashion, we obtain a sampled product (see Figure 2.7c). Finally, integration in the analog case becomes summation in the discrete case, where the summands need to be weighted by the sampling period T. As a result, one obtains the following approximation:

$$\sum_{n\in\mathbb{Z}} T f(nT)\exp(-2\pi i\omega nT) \approx \hat{f}(\omega). \tag{2.21}$$

In mathematical terms, the sum can be interpreted as the overall area of rectangular shapes that approximates the area corresponding to the integral (see Figure 2.7d). Such an approximation is also known as a **Riemann sum**. As we will show in Section 2.3.4, the quality of the approximation is good for "well-behaved" signals f and "small" frequency parameters ω.

One defines a discrete version of the Fourier transform for a given DT-signal $x:\mathbb{Z}\to\mathbb{R}$ by setting

$$\hat{x}(\omega) := \sum_{n\in\mathbb{Z}} x(n)\exp(-2\pi i\omega n). \tag{2.22}$$

In this definition, where a simple 1-sampling (i.e., T-sampling with $T=1$) of the exponential function is used, one does not assume that one knows the relation between x and the original signal f. If one is interested in recovering the relation to the Fourier transform $\hat{f}$, one needs to know the sampling period T. Based on (2.21), an easy calculation shows that

$$\hat{x}(\omega) \approx \frac{1}{T}\hat{f}\left(\frac{\omega}{T}\right). \tag{2.23}$$

In this approximation, the frequency parameter ω used for $\hat{x}$ corresponds to the frequency ω/T for $\hat{f}$. In particular, $\omega = 1/2$ for $\hat{x}$ corresponds to the Nyquist frequency $\Omega = 1/(2T)$ of the sampling process. Therefore, assuming that f is bandlimited by $\Omega = 1/(2T)$, one needs to consider only the frequencies with $0 \leq \omega \leq 1/2$ for $\hat{x}$. In the digital case, all other frequency parameters are redundant and yield meaningless approximations.

For doing computations on digital machines, we still have some problems. One problem is that the sum in (2.22) involves an infinite number of summands. Another problem is that the frequency parameter ω is a continuous parameter. For both problems, there are some pragmatic solutions. Regarding the first problem, we assume that most of the relevant information of f is limited to a certain duration in time.[2] For example, a music recording of a song hardly lasts for more than ten minutes. Having a finite duration means that the analog signal f is assumed to be zero outside a compact interval. By possibly shifting the signal, we may assume that this interval starts at time $t = 0$. This means that we only need to consider a finite number of

[2] Strictly speaking, this assumption is problematic since it conflicts with the requirement of f being bandlimited. A mathematical fact states that there are no functions that are both limited in frequency (bandlimited) and limited in time (having finite duration).

samples $x(0), x(1), \ldots, x(N-1)$ for some suitable number $N \in \mathbb{N}$. As a result, the sum in (2.22) becomes finite.

Regarding the second problem, one computes the Fourier transform only for a finite number of frequencies. Similar to the sampling of the time axis, one typically samples the frequency axis by considering the frequencies $\omega = k/M$ for some suitable $M \in \mathbb{N}$ and $k \in [0 : M-1]$. In practice, one often couples the number N of samples and the number M that determines the frequency resolution by setting $N = M$. Note that the two numbers N and M refer to different aspects. However, the coupling is convenient. It not only makes the resulting transform invertible, but also leads to a computationally efficient algorithm, as we will see in Section 2.4.3. Setting $X(k) := \hat{x}(k/N)$ and assuming that $x(0), x(1), \ldots, x(N-1)$ are the relevant samples (all others being zero), we obtain from (2.22) the formula

$$X(k) = \hat{x}(k/N) = \sum_{n=0}^{N-1} x(n) \exp(-2\pi i k n/N) \tag{2.24}$$

for integers $k \in [0 : M-1] = [0 : N-1]$. This transform is also known as the **discrete Fourier transform** (DFT), which is covered in Section 2.4.

Next, let us have a look at the frequency information supplied by the Fourier coefficient $X(k)$. By (2.23) the frequency ω of $\hat{x}$ corresponds to ω/T of $\hat{f}$. Therefore, the index k of $X(k)$ corresponds to the physical frequency

$$F_{\mathrm{coef}}(k) := \frac{k}{N \cdot T} = \frac{k \cdot F_{\mathrm{s}}}{N} \tag{2.25}$$

given in Hertz. As we will discuss in Section 2.4.4, the coefficients $X(k)$ need to be taken with care. First, the approximation quality in (2.23) may be rather poor, in particular for frequencies close to the Nyquist frequency. Second, for a real-valued signal x, the Fourier transform fulfills certain symmetry properties (see Exercise 2.24). As a result, the upper half of the Fourier coefficients are redundant, and one only needs to consider the coefficients $X(k)$ for $k \in [0 : \lfloor N/2 \rfloor]$. Note that, in the case of an even number N, the index $k = N/2$ corresponds to $F_{\mathrm{coef}}(k) = F_{\mathrm{s}}/2$, which is the Nyquist frequency of the sampling process.

Finally, we consider some efficiency issues when computing the DFT. To compute a single Fourier coefficient $X(k)$, one requires a number of multiplications and additions linear in N. Therefore, to compute all coefficients $X(k)$ for $k \in [0 : N/2]$ one after another, one requires a number of operations on the order of N^2. Despite being a finite number of operations, such a computational approach is too slow for many practical applications, in particular when N is large.

The number of operations can be reduced drastically by using an efficient algorithm known as the **fast Fourier transform** (FFT). The FFT algorithm, which was discovered by Gauss and Fourier two hundred years ago, has changed whole industries and is now being used in billions of telecommunication and other devices. The FFT exploits redundancies across sinusoids of different frequencies to jointly compute all Fourier coefficients by a recursion. This recursion works particularly well in the case that N is a power of two. As a result, the FFT reduces the overall number of operations from the order of N^2 to the order of $N \log_2 N$. The savings are enormous. For example, using $N = 2^{10} = 1024$, the FFT requires roughly $N \log_2 N = 10240$ instead of $N^2 = 1048576$ operations in the naive approach—a savings factor of about 100. In the case of $N = 2^{20}$, the savings amount to a factor of about 50000 (see Exercise 2.6). In Section 2.4.3, we discuss the algorithmic details of the FFT.

2.1.4 Short-Time Fourier Transform

The Fourier transform yields frequency information that is averaged over the entire time domain. However, the information on *when* these frequencies occur is hidden in the transform. We have already seen this phenomenon in Figure 2.6a, where the change in frequency is not revealed when looking at the magnitude of the Fourier transform. To recover the hidden time information, Dennis Gabor introduced in the year 1946 the **short-time Fourier transform** (STFT). Instead of considering the entire signal, the main idea of the STFT is to consider only a small section of the signal. To this end, one fixes a so-called **window function**, which is a function that is nonzero for only a short period of time (defining the considered section). The original signal is then multiplied with the window function to yield a **windowed signal**. To obtain frequency information at different time instances, one shifts the window function across time and computes a Fourier transform for each of the resulting windowed signals.

This idea is illustrated by Figure 2.8, which continues our example from Figure 2.6a. To obtain local sections of the original signal, one multiplies the signal with suitably shifted rectangular window functions. In Figure 2.8b, the resulting local section only contains frequency content at 1 Hz, which leads to a single main peak in the Fourier transform at $\omega = 1$. Further shifting the time window to the right, the resulting section contains 1 Hz as well as 5 Hz components (see Figure 2.8c). These components are reflected by the two peaks at $\omega = 1$ and $\omega = 5$. Finally, the section shown in Figure 2.8d only contains frequency content at 5 Hz.

Already at this point, we want to emphasize that the STFT reflects not only the properties of the original signal but also those of the window function. First of all, the STFT depends on the length of the window, which determines the size of the section. Then, the STFT is influenced by the shape of the window function. For example, the sharp edges of the rectangular window typically introduce “ripple” artifacts. In Section 2.5.1, we discuss such issues in more detail. In particular, we introduce more suitable, bell-shaped window functions, which typically reduce such artifacts.

In Section 2.5, one finds a detailed treatment of the analog and discrete versions of the STFT and their relationship. In the following, we only consider the discrete case and specify the most important mathematical formulas as needed in practical applications. Let $x : \mathbb{Z} \to \mathbb{R}$ be a real-valued DT-signal obtained by equidistant sampling with respect to a fixed sampling rate F_s given in Hertz. Furthermore, let $w : [0 : N-1] \to \mathbb{R}$ be a sampled window function of length $N \in \mathbb{N}$. For example,

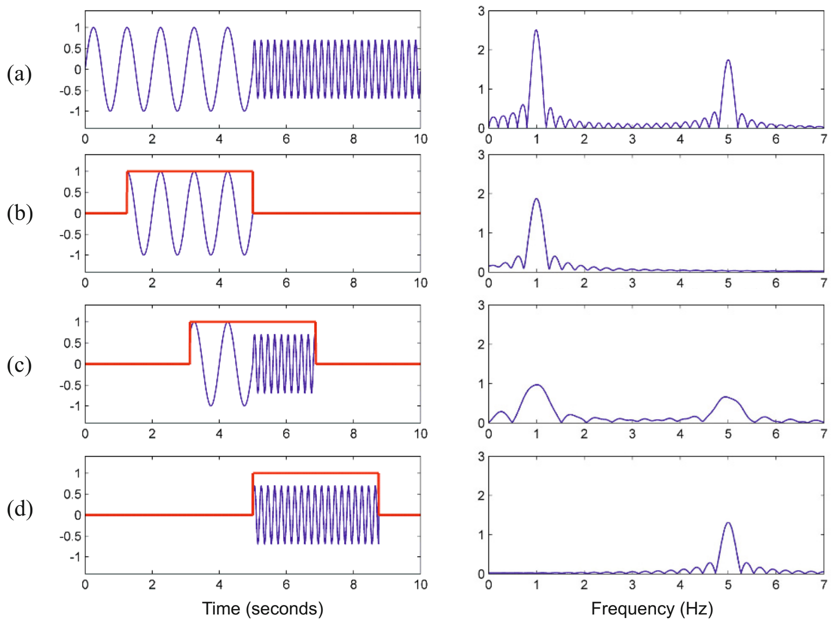

Fig. 2.8 Signal and Fourier transform consisting of two subsequent sinusoids of frequency 1 Hz and 5 Hz (see Figure 2.6a). **(a)** Original signal. **(b)** Windowed signal centered at $t = 3$. **(c)** Windowed signal centered at $t = 5$. **(d)** Windowed signal centered at $t = 7$.

in the case of a rectangular window one has $w(n) = 1$ for $n \in [0 : N-1]$. Implicitly, one assumes that $w(n) = 0$ for all other time parameters $n \in \mathbb{Z} \setminus [0 : N-1]$ outside this window. The length parameter N determines the duration of the considered sections, which amounts to N/F_s seconds. One also introduces an additional parameter $H \in \mathbb{N}$, which is referred to as the **hop size**. The hop size parameter is specified in samples and determines the step size in which the window is to be shifted across the signal.

With regard to these parameters, the **discrete STFT** $\mathcal{X}$ of the signal x is given by

$$\mathcal{X}(m,k) := \sum_{n=0}^{N-1} x(n+mH)w(n)\exp(-2\pi ikn/N) \tag{2.26}$$

with $m \in \mathbb{Z}$ and $k \in [0 : K]$. The number $K = N/2$ (assuming that N is even) is the frequency index corresponding to the Nyquist frequency. The complex number $\mathcal{X}(m,k)$ denotes the k^{th} Fourier coefficient for the m^{th} time frame. Note that for each fixed time frame m, one obtains a **spectral vector** of size $K+1$ given by the coefficients $\mathcal{X}(m,k)$ for $k \in [0 : K]$. The computation of each such spectral vector amounts to a DFT of size N as in (2.24), which can be done efficiently using the FFT.

What have we actually computed in (2.26) in relation to the original analog signal f? As for the temporal dimension, each Fourier coefficient $\mathcal{X}(m,k)$ is associated with the physical time position

$$T_{\mathrm{coef}}(m) := \frac{m \cdot H}{F_{\mathrm{s}}} \tag{2.27}$$

given in seconds. For example, for the smallest possible hop size $H = 1$, one obtains $T_{\mathrm{coef}}(m) = m/F_{\mathrm{s}} = m \cdot T$ sec. In this case, one obtains a spectral vector for each sample of the DT-signal x, which results in a huge increase in data volume. Furthermore, considering sections that are only shifted by one sample generally yields very similar spectral vectors. To reduce this type of redundancy, one typically relates the hop size to the length N of the window. For example, one often chooses $H = N/2$, which constitutes a good trade-off between a reasonable temporal resolution and the data volume comprising all generated spectral coefficients. As for the frequency dimension, we have seen in (2.25) that the index k of $\mathcal{X}(m,k)$ corresponds to the physical frequency

$$F_{\mathrm{coef}}(k) := \frac{k \cdot F_{\mathrm{s}}}{N} \tag{2.28}$$

given in Hertz.

Before we look at some concrete examples, we first introduce the concept of a **spectrogram**, which we denote by $\mathcal{Y}$. The spectrogram is a two-dimensional representation of the squared magnitude of the STFT:

$$\mathcal{Y}(m,k) := |\mathcal{X}(m,k)|^2. \tag{2.29}$$

It can be visualized by means of a two-dimensional image, where the horizontal axis represents time and the vertical axis represents frequency. In this image, the spectrogram value $\mathcal{Y}(m,k)$ is represented by the intensity or color in the image at the coordinate (m,k). Note that in the discrete case, the time axis is indexed by the frame indices m and the frequency axis is indexed by the frequency indices k.

Continuing our running example from Figure 2.8, we now consider a sampled version of the analog signal using a sampling rate of $F_{\mathrm{s}} = 32$ Hz. Having a physical duration of 10 sec, this results in 320 samples (see Figure 2.9a). Using a window length of $N = 64$ samples and a hop size of $H = 8$ samples, we obtain the spectrogram as shown in Figure 2.9b. In the image, the shade of gray encodes the magnitude of a spectral coefficient, where darker colors correspond to larger values. By (2.27), the m^{th} frame corresponds to the physical time $T_{\mathrm{coef}}(m) = m/4$ sec. In other words, the STFT has a time resolution of four frames per second. Furthermore, by (2.28), the k^{th} Fourier coefficient corresponds to the physical frequency $F_{\mathrm{coef}}(k) := k/2$ Hz. In other words, one obtains a frequency resolution of two coefficients per Hertz. The plots of the waveform and the spectrogram with the physically correct time and frequency axes are shown in Figure 2.9c and Figure 2.9d, respectively.

Let us consider some typical settings as encountered when processing music signals. For example, in the case of CD recordings one has a sampling rate of $F_{\mathrm{s}} = 44100$ Hz. Using a window length of $N = 4096$ and a hop size of $H = N/2$,

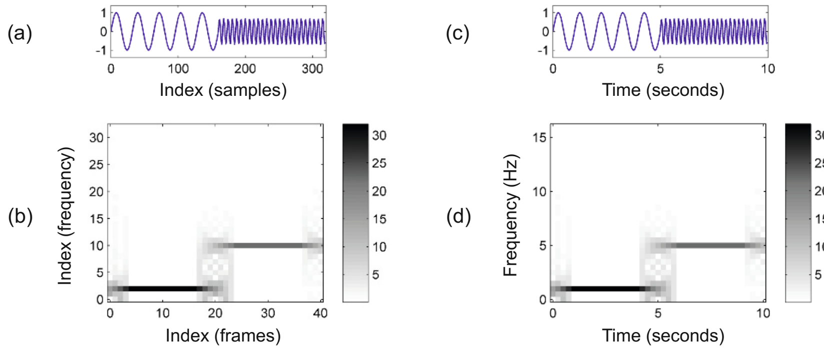

Fig. 2.9 DT-signal sampled with $F_s = 32$ Hz and STFT using a window length of $N = 64$ and a hop size of $H = 8$. **(a)** DT-signal with time axis given in samples. **(b)** STFT with time axis given in frames and frequency axis given in indices. **(c)** DT-signal with time axis given in seconds. **(d)** STFT with time axis given in seconds and frequency axis given in Hertz.

this results in a time resolution of $H/F_s \approx 46.4$ ms by (2.27) and a frequency resolution of $F_s/N \approx 10.8$ Hz by (2.28). To obtain a better frequency resolution, one may increase the window length N. This, however, leads to a poorer localization in time so that the resulting STFT loses its capability of capturing local phenomena in the signal. This kind of trade-off is further discussed in Section 2.5.2 and in the exercises.

We close this section with a further example shown in Figure 2.10, which is a recording of a C-major scale played on a piano. The first note of this scale is C4, which we have already considered in Figure 2.1. In Figure 2.10c, the spectrogram representation of the recording is shown, where the time and frequency axes are labeled in a physically meaningful way. The spectrogram reveals the frequency information of the played notes over time. For each note, one can observe horizontal lines that are stacked on top of each other. As discussed in Section 1.3.4, these equally spaced lines correspond to the partials, the integer multiples of the fundamental frequency of a note. Obviously, the higher partials contain less and less of the signal's energy. Furthermore, the decay of each note over time is reflected by the fading out of the horizontal lines. To enhance small sound components that may still be perceptually relevant, one often uses a logarithmic dB scale (see Section 1.3.3). Figure 2.10d illustrates the effect when applying the dB scale to the values of the spectrogram. Besides an enhancement of the higher partials, one can now observe vertical structures at the notes' onset positions. These structures correspond to the noise-like transients that occur in the attack phase of the piano sound (see Section 1.3.4).

This concludes our "nutshell section" covering the most important definitions and properties of the Fourier transform as needed for the subsequent chapters of this book. In particular, the formula (2.26) of the discrete STFT as well as the physical interpretation of the time parameter (2.27) and the frequency parameter (2.28) are

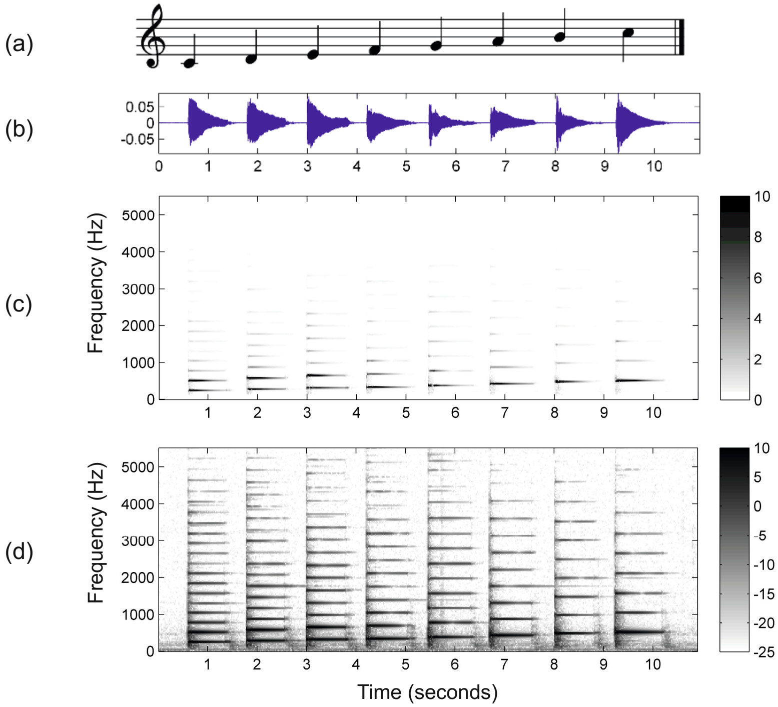

Fig. 2.10 Waveform and spectrogram of a music recording of a C-major scale played on a piano. **(a)** The recording's underlying musical score. **(b)** Waveform. **(c)** Spectrogram. **(d)** Spectrogram with the magnitudes given in dB.

of central importance for most music processing applications to be discussed. As said in the introduction, we provide in the subsequent sections of this chapter some deeper insights into the mathematics underlying the Fourier transform. In particular, we explain in more detail the connection between the various kinds of signals and associated Fourier transforms.

2.2 Signals and Signal Spaces

In technical fields such as engineering or computer science, a **signal** is a function that conveys information about the state or behavior of a physical system. For example, a signal may describe the time-varying sound pressure at some place, the motion of a particle through some space, the distribution of light on a screen representing an image, or the sequence of images as in the case of a video signal. In the following, we consider the case of audio signals as discussed in Section 1.3. We have seen that such a signal can be graphically represented by its waveform, which

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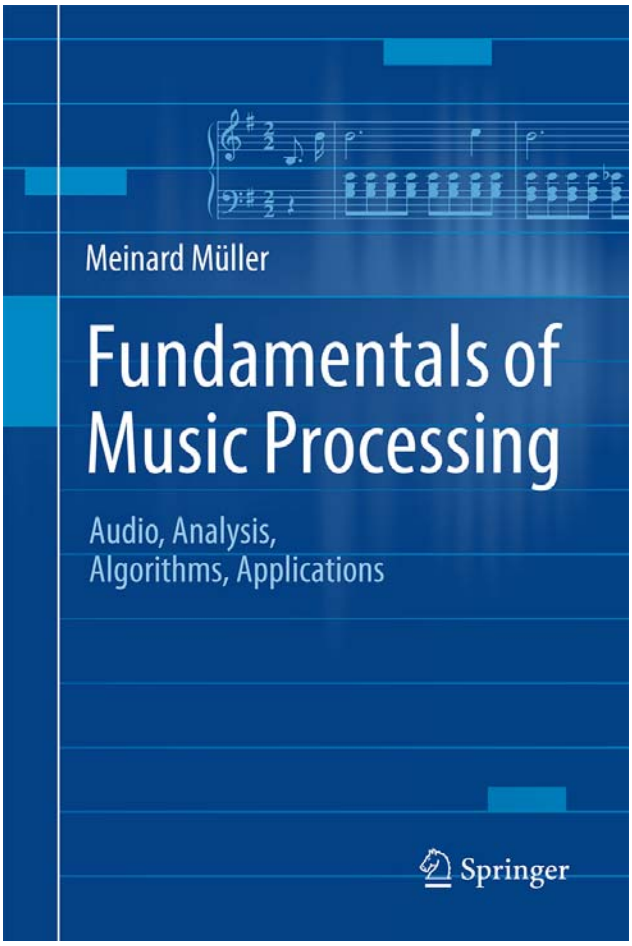

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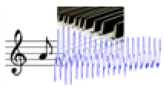
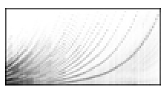
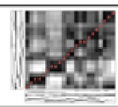
1		Music Representations
2		Fourier Analysis of Signals
3		Music Synchronization
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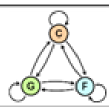
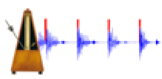
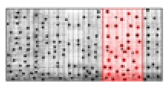
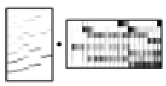
5		Chord Recognition
6		Tempo and Beat Tracking
7		Content-Based Audio Retrieval
8		Musically Informed Audio Decomposition

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