A marriage of manifolds and algebra: the mathematical work of Peter Landweber

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It is no accident that Peter Landweber's career closely matches the striking unification of algebra and topology provided by the theory of bordism of manifolds, especially complex bordism: his work has been at the heart of that interaction. Here I will briefly describe some of Landweber's main contributions to this story.

The starting point was Thom's work [**59**], from 1954, in which he used transversality to identify bordism classes of closed smooth n-manifolds with what we today call the nth homotopy group of the Thom spectrum MO (though of course it was in part an attempt to express Thom's arguments conveniently that later led to the concept of a spectrum) and then computed the homology of this (by the Thom isomorphism) and the homotopy (and actually the homotopy type). The result was that any mod 2 cohomology class was carried by a "singular manifold," the image of the fundamental class of a smooth closed manifold under a map; and the bordism ring is a polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ with one generator in each positive degree not one less than a power of 2. Thom also considered the oriented bordism ring $\Omega_* = \pi_*(MSO)$.

This was followed in 1960 by the use of the newly minted Adams spectral sequence [**1**], independently by Milnor [**44**] and by Novikov [**51**] to make the analogous computation of the complex bordism ring $\Omega_*^U = \pi_*(MU)$. The result was a polynomial algebra over $\mathbb{Z}$ with one

generator in each positive even degree. The following year Atiyah [**6**] provided the natural relativization of bordism, defining the bordism groups of a space X by considering bordism classes of maps into X. A homology theory and its dual cohomology theory were born.

These theories were integrated into the arsenal of working topologists, to borrow Saunders Mac Lane's phrase, and soon in the book *Differentiable Periodic Maps* (1964) Pierre Conner and Ed Floyd [**13**] were using them to solve problems in group actions. This work focused attention on the bordism of classifying spaces, corresponding to classifying bordism classes of free actions, and in particular they showed that for p odd the map

$$\Omega_*(BC_p) \otimes_{\Omega_*} \Omega_*(X) \to \Omega_*(BC_p \times X)$$

is a monomorphism for any X: the beginning of a Künneth theorem.

In 1962 [**7**] Atiyah provided a geometric approach to the Künneth theorem, and used it to prove the result for K-theory. The combination of this idea with the Connor-Floyd result led to Landweber's thesis, written at Harvard under the direction of Raoul Bott, completed in 1965 and published with the same title, "Künneth formulas for bordism theories" [**21**]. (Landweber recalls that "Frank Peterson really served as my advisor (in the sense of giver of support and advice) when it came to matters such as cobordism theory.") Landweber studied both Ω and complex bordism (which he denoted by U_*), and considered spaces W for which every homology class was carried by a manifold, or, equivalently (Conner and Floyd [**13**] (15.1)), for which the relevant Atiyah-Hirzebruch spectral sequence collapsed. (He worked at an odd prime in the case of oriented bordism, which was subsequently understood to be a summand of complex bordism when localized at an odd prime.) Parity implies that $W = BC_p$ is an example. Under this hypothesis he constructed a two-term resolution of W by spaces with free bordism modules, and so obtained a natural short exact Künneth sequence

$$0 \to U_*(W) \otimes_{U_*} U_*(X) \to U_*(W \times X) \to \mathrm{Tor}_1^{U_*}(U_*(W), U_*(X)) \to 0.$$

Landweber made the explicit observation that what mattered here was the fact that $U_*(W)$ had projective dimension at most one over the bordism ring U_*. (He attributes this remark to Don Anderson.) This story was continued by Conner and Larry Smith in their 1969 study [**15**] of the complex bordism of finite complexes. They observed that surjectivity of $U_*(W) \to H_*(W)$ was in fact equivalent to $U_*(W)$

having projective dimension at most one. They also extended Landweber's exact Künneth sequence to a full spectral sequence, removing the restriction on projective dimension.

Landweber's long-term interest in the module structure of the bordism of a space, especially the complex bordism, was thus born with his thesis. In 1970 [25] he continued his study of the complex bordism of a classifying space of a finite group G, showing that hom $\dim_{MU_*} MU_*(BG) = 1$ is equivalent to the well-studied condition that G have periodic cohomology, or, equivalently, that every abelian subgroup of G is cyclic, or that $H^*(BG; \mathbb{Z})$ is even.

His first job was at the University of Virginia, and it was from there, in September, 1965, that he submitted his paper "Cobordism operations and Hopf algebras" [22]. Conner and Floyd, in work published in their 1966 Springer Lecture Note *The Relation of Cobordism to K-Theories* [14], had observed that a splitting principle held in complex cobordism, and this allowed them to compute the cobordism of BU and construct in it specific polynomial generators written later by Adams [3] as cf_i, the "Conner-Floyd Chern classes." The subalgebra $\mathbb{Z}[cf_1, \ldots]$ they generate admits a diagonal, reflecting the Whitney sum formula. Landweber considered the image of this subalgebra under the Thom isomorphism

$$MU^*(BU) \to MU^*(MU).$$

This isomorphism is not an algebra map, but Landweber discovered that the image is a subalgebra of the full algebra of cobordism operations under composition, and that it forms a Hopf algebra if we transfer the Whitney diagonal. He observed that the dual Hopf algebra is polynomial on classes I will write b_i, and gave an explicit expression for the diagonal in the dual Hopf algebra, reflecting the composition product. Novikov, in his epic 1967 paper "The methods of algebraic topology from the viewpoint of cobordism theory" [52], independently constructed this algebra but did not make the structure explicit. In his 1967 Chicago lectures on Landweber and Novikov's work, Adams wrote Landweber's formula as

$$\Delta b = \sum b^{i+1} \otimes b_i.$$

where b is the formal sum of the b_i and $b_0 = 1$. This formula was understood by Dan Quillen and Jack Morava as asserting that the dual "Landweber-Novikov algebra" S_* (in Novikov's notation) represents the functor sending a ring to the group, under composition, of power series of the form $x + \ldots$.

The full algebra $MU^*(MU)$ is a completed tensor product of S with MU^*, and Landweber recognized that the remaining ingredient, the action of S on the coefficient ring MU^*, was important and extremely complex. The key to understanding it was hinted at by the work of Mike Boardman[1] [**9**] and and Novikov, in their observation that the behavior of cobordism Euler classes of complex line bundles under tensor product is described by a formal group law

$$x +_F y = x + y + \cdots .$$

Novikov has said that he did not know of Lazard's theorem ([**43**], 1955), giving the structure of the ring supporting the universal formal group law, and it was left to Quillen [**54**] to prove, in 1969, that the Novikov formal group over MU^* is universal, MU^* is the Lazard ring. The action of the Landweber-Novikov algebra was then described by conjugation of F by formal power series.

This insight has led to a remarkable incursion of algebra into homotopy theory over the subsequent thirty years, in which Peter Landweber has been a four-star general. A first payoff, carried out by Quillen, was the use of Cartier's theory of p-typical parameters to provide a canonical and computable splitting of $MU_{(p)}$ into indecomposables, namely suspensions of the Brown-Peterson spectrum BP [**12**]. This spectrum had been constructed in 1966 and such a splitting established but without enough control to make computations feasible.

While at Virginia, and later at the IAS and Yale, Landweber wrote a series of papers on the geometric side of his subject. To give one example, with Pierre Conner [**23**] he gave a necessary and sufficient condition for an oriented manifold to be unoriented cobordant to an SU-manifold: all the Pontryagin numbers involving p_1 must vanish mod 2. To give another, in a paper [**24**] in the Annals he considers an equivariant map

$$(\mathbb{R}_-^{n+p} \oplus \mathbb{R}_+^q)^+ \to (\mathbb{R}_-^p \oplus \mathbb{R}_+^q)^+$$

and uses equivariant KO-theory to resolve an ambiguous power of 2 in Glen Bredon's calculation of the degree of the restriction to fixed points

$$(\mathbb{R}_+^q)^+ \to (\mathbb{R}_+^q)^+.$$

In 1970 Landweber moved to Rutgers, and returned to the question of what the operations on complex bordism tell you about the MU_*-module structure. The work of Conner and Smith [**15**] (and of

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[1]In these notes, representing a portion of his thesis, Boardman introduced the formal group for unoriented bordism and used it, as Quillen was to do in the complex case, to give a canonical splitting of MO.

Conner and Floyd earlier on) had focused attention on the structure of annihilator ideals of nonzero elements, especially spherical classes. Sphericity has strong implications on the behavior of operations: all the Landweber-Novikov operations vanish—the class is "primitive."

Landweber published a sequence of three papers developing his program. He wanted to use methods from commutative algebra, and in place of the standard Noetherian condition he used the condition of coherence. This had been introduced into Topology by Novikov [52] (without identifying it by name) and Larry Smith [56], and was advertised by Adams in his Seattle lectures [2]. Recall that an R-module is coherent if it is finitely generated and every finitely generated submodule is finitely presented. A ring is coherent if it is coherent as a module over itself, and over a coherent ring the conditions of finitely presented and coherent coincide. Novikov and Smith proved that the complex bordism of any finite complex is coherent, so the category of comodules over MU_*MU which are coherent as MU_*-modules is a good first approximation to the category of finite complexes. In particular, coherent MU_*-modules have finite projective dimension.

In the first of these papers [26] Landweber showed that if the annihilator ideal of an element in such a comodule is prime then it is invariant, i.e. a sub-comodule of MU_*. This prompted a study of the invariant prime ideals in MU_*, and Landweber made explicit Morava's observation [47] that Lazard's classification [43] of formal groups over separably closed fields by their height leads to a determination of these ideals. The ring MU_* admits generators $x_i \in \pi_{2i}MU$ with the property that the Hurewicz image in $H_{2i}MU$ is divisible by p whenever $i + 1$ is a power of the prime p. The ideal

$$I(p,n) = (x_0, x_{p-1}, x_{p^2-1}, \ldots, x_{p^{n-1}-1})$$

(where $x_0 = p$) is independent of choices, and as p runs over the rational primes and n runs between 1 and ∞ this exhausts the list of nonzero invariant prime ideals in MU_*. This was proven inductively, calculating the module of primitive elements in $MU_*/I(p,n)$ to be $\mathbb{F}_p[x_{p^n-1}]$ by means of a purely algebraic version of a theorem of Stong and Hattori concerning the K-theoretic Hurewicz homomorphism. K-theory made its appearance through the multiplicative formal group over $\mathbb{F}_p$, and Landweber used work of Lazard to produce generalizations of this formal group law, distinguished by the property that

$$[p](x) = x^{p^n}.$$

The fact that once you choose a characteristic the invariant primes are linearly ordered has played an absolutely fundamental role in all the subsequent work in this area. See [20] for a slick proof using BP.

Returning to [26], the invariance of prime annihilator ideals was proven using standard primary decomposition together with the following structural result: any coherent comodule has a finite comodule filtration whose quotients are cyclic as MU_*-modules. This result was strengthened in the second paper [27], in which Landweber showed that maximal elements in the partially ordered set of proper annihilator ideals are annihilators of primitive elements. This allowed him to refine the filtration theorem to its final form, the Landweber Filtration Theorem: any coherent comodule has a finite comodule filtration whose associated quotients are suspensions of the comodules $MU_*/I(p,n)$ for $0 \leq n < \infty$.

In the third paper [30] in this series, Landweber put the filtration theorem to work in Topology. His primary goal was to complete the original task of understanding the projective dimension of the bordism module of a finite complex. By this time definitive results had been obtained by Dave Johnson and Steve Wilson [19], using BP in place of MU. They used the new Sullivan-Baas theory of manifolds with singularities [58, 8] to construct certain BP-module theories $BP\langle n \rangle$. The classes $x_{p^i-1} \in MU_*$ project to polynomial generators of BP_* which I will write v_i, and in terms of these generators the coefficient ring of $BP\langle n \rangle$ is the BP_*-module $BP\langle n \rangle_* = BP_*/(v_{n+1}, v_{n+2} \ldots) = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$. Johnson and Wilson showed that hom $\dim_{BP_*} BP_*(X) \leq n+1$ if and only if $BP_*(X) \to BP\langle n \rangle_*(X)$ is surjective. They also made the intriguing observation that after localization $BP\langle n \rangle_*(X)$ is determined algebraically from $BP_*(X)$ (Remark 5.13):

$$v_n^{-1}BP\langle n \rangle_* \otimes_{BP_*} BP_*(X) \xrightarrow{\cong} v_n^{-1}BP\langle n \rangle_*(X).$$

While their results were for the most part stable, their proofs used the unstable splitting of Wilson's thesis, and one of Landweber's objectives was to use stable operations instead. (For more information the reader may consult Wilson's *Primer and Sampler*, [62].)

Landweber succeeded in this goal. As a technical step he made the following observation. Given any MU_*-module M one can form the functor

(1) $$X \mapsto M \otimes_{MU_*} MU_*(X).$$

If M is flat over MU_* then of course you get a homology theory, represented by a spectrum. The fact is, though, that one need test the

exactness of $M \otimes_{MU_*} -$ against exact sequences of modules of a type which is very tightly restricted by the Filtration Theorem. The result was that (1) is a homology theory provided that for every p the sequence $p, x_{p-1}, \ldots$, acts regularly on M: this is the Landweber Exact Functor Theorem.

To return to the original theorem of Conner and Floyd, we begin by classifying the graded multiplicative formal group by a map $MU_* \to K_* = \mathbb{Z}[v^{\pm 1}]$. For any prime p, p is monic on K_*, and x_{p-1} maps to a nonzero multiple of v^{p-1} in K_*. Since it is a unit, $K_*/(p, x_{p-1}) = 0$ and the whole sequence acts regularly. Thus

$$(2) \qquad\qquad X \mapsto K_* \otimes_{MU_*} MU_*(X)$$

is a homology theory. The Todd genus determines a map from this theory to K-homology, which is an isomorphism for X a sphere. It follows that (2) is none other than K-homology. Landweber's proof differs in an important respect from that of Conner and Floyd: he did not need to know that there was a homology theory there in advance; rather, he showed directly that (2) defines a homology theory. Combined with the Brown Representability Theorem, the Exact Functor Theorem provides a tool of almost magical efficacy for constructing homotopy types from pure algebra.

Landweber worked out the evident analogue over BP as well. As an example the localization of the Johnson-Wilson coefficient ring $v_n^{-1}BP\langle n\rangle_*$ clearly satisfies his conditions, and one obtains a new construction of the localized homology theory (later called $E(n)_*(-)$) and a new proof of the observation of Johnson and Wilson. (Of course the connective spectra, $BP\langle n\rangle$, cannot be constructed in this way.) Jack Morava and Mike Hopkins have taught us that for many purposes it is better to consider a complete theory, associated to the Lubin-Tate universal deformation of a formal group of height n. It also is constructed from the Exact Functor Theorem.

The perspective of describing a structure by the functor it represents was beautifully carried over by Landweber to the case of BP in 1975 [29]. Adams, in his magisterial "Lectures on generalized cohomology" at the Battelle Conference in the summer of 1968 [2], had described the structure on the pair (E_*, E_*E) in case E is a commutative ring spectrum such that E_*E is flat over E_*. The theorems just described assert that (MU_*, MU_*MU) represents the functor sending a ring R to the groupoid of formal groups over R and their strict isomorphisms. Landweber's contribution was to note that (BP_*, BP_*BP) represented the functor sending an $\mathbb{Z}_{(p)}$-algebra to the groupoid of p-typical formal group laws over R and their strict isomorphisms. I

vividly recall the moment of enlightenment when he explained this to me in the common room in Fine Hall, and was led by this to propose the term "Hopf algebroid" for a cogroupoid object in the category of commutative rings.

By 1974 the technical advances in the subject had opened the way to serious computations in the Adams spectral sequence based on MU, or, equivalently when localized at a prime, on BP. Landweber's determination of the invariant prime ideals in BP_* amounted to a computation of

$$\mathrm{Ext}^0(BP_*, BP_*/I_n).$$

Steve Wilson and I [**46**] computed

$$\mathrm{Ext}^1(BP_*, BP_*/I_n)$$

for p odd, and then Wilson and Doug Ravenel and I [**45**] computed

$$\mathrm{Ext}^2(BP_*, BP_*)$$

for p odd. Landweber [**31**] interpreted these computations as classifying certain invariant ideals in BP_* which were not necessarily prime but rather regular. Many of these ideals have the form $(p^{a_0}, v_1^{a_1}, \ldots, v_n^{a_n})$, but not all do. Landweber gave some structural properties of these ideals. For example, an invariant regular sequence of length n generates a primary ideal with radical I_n.

The "chromatic resolution" of [**45**] appeared there as an ad hoc device, but of course Landweber was not content with that, and, with Johnson and Zen-ichi Yosimura, he showed [**34**] that in a certain sense it is an injective resolution. This paper established a characterization of injective coherent comodules dual to the criterion in the Exact Functor Theorem.

During this period Landweber also contributed to the study of complex structures, showing [**28**] that if the cohomology of an open almost complex manifold vanishes above the middle dimension then the almost complex structure is integrable.

The Proceedings volume of a 1978 conference in Waterloo, Ontario, contain back-to-back papers by Landweber, entitled "New applications of commutative algebra to Brown-Peterson homology" [**32**] and "The signature of symplectic and self-conjugate manifolds" [**33**]—a nice encapsulation of his breadth of interest and creativity. The first paper carried on the story just told. In it he proved that the radical of the annihilator of a nonzero element in a coherent BP-comodule is invariant and prime. A result of Johnson and Yoshimura then followed from the linear ordering of invariant primes: If x is v_n-torsion then it is v_{n-1}-torsion. For if $v_n^s x = 0$, then v_n lies in the radical of the annihilator of

x, which must thus contain I_n and hence v_{n-1}. A further result from this paper is that for finitely generated BP-comodules M the following conditions are equivalent: M is coherent; M has finite projective dimension; and $v_n|M$ is monic for some n. Landweber subsequently used this observation to show [35] for example that finite projective dimensionality of BP-homology is preserved by attaching of cells.

The second paper is a scholarly completion of the project, begun by Rochlin's theorem, of determining the image of the signature on various standard bordism groups.

In the early 1980's, influenced by the work [60] of Clarence Wilkerson, Landweber became interested in the more classical question of the structure of a commutative algebra endowed with an unstable action of the Steenrod algebra. This led to a fruitful collaboration with Bob Stong, represented for example by [40]. They were led to a conjecture about the depth of the ring of invariants of a subgroup of $GL(n, \mathbb{F}_p)$ acting on the polynomial algebra $S = \mathbb{F}_p[t_1, \ldots t_n]$, $|t_i| = 2$. The ring of invariants contains the "Dickson invariants" $S^{GL(n,\mathbb{F}_p)}$, which is itself known to be a polynomial algebra $\mathbb{F}_p[c_1, \ldots c_n]$, with $|c_i| = 2(p^n - p^{n-i})$. The Landweber-Stong conjecture is that the depth of S^G is the largest r for which $c_1, \ldots, c_r$ forms a regular sequence in S^G. This conjecture generated a substantial body of research, culminating in its proof by Dorra Bourguiba and Said Zarati [11]. A long survey [57] by Larry Smith takes this development as its centerpiece.

A new impetus was provided in the early 1980's by the work of Ed Witten. Landweber himself has given a beautiful account of this development in [37]. Witten asked if certain equivariant characteristic classes were invariant under a circle action. Landweber's student Lucilia Borsari verified this conjecture for semifree actions in her 1985 thesis [10]. In [42] Landweber and Stong produced several other "rigid characteristic classes." These results served as the starting point for Serge Ochanine's introduction [53] of an "elliptic genus," and his proof that any rigid genus is elliptic. Witten [63] immediately gave an interpretation of this genus (and others) as equivariant indices on the space of free loops, and proposed a physics proof that conversely any elliptic genus is rigid. This was subsequently realized as a mathematical theorem by Cliff Taubes and then, sequentially, by Bott and Taubes, by Kefeng Liu, and by Ioanid Rosu.

During this period, when e-mail was still uncommon and the web nonexistent, Landweber acted as a distribution center for preprints connected with elliptic genera. This preliminary work made all the participants at the Princeton conference of September 15–17, 1986, be

they topologists or number theorists or physicists, feel that they were working on the same thing. He laid the groundwork for one of the most exciting interactions between widely diverse specialists in recent years, and followed up by editing the Springer Lecture Notes volume *Elliptic Curves and Modular Forms in Algebraic Topology* [**36**]. According to Landweber, "Something else I was very happy about was getting Hirzebruch interested in what might be called, in imitation of course, the marriage of manifolds and modular forms. I spent a week in Bonn in Spring 1987, talking to Hirzebruch and Zagier." Hirzebruch gave a course on this subject in the Winter Term 1987/88, and notes were written up by Thomas Berger and Rainer Jung. The Preface to [**17**] goes on: "P. Landweber used these notes for a course at Rutgers University in the spring of 1989 and [in characteristic fashion] prepared a translation into English for his students; in addition he proposed numerous corrections and improvements." The result was published as *Manifolds and Modular Forms*, [**17**].

In 1987 Landweber, Ravenel, and Stong [**39**] showed that the elliptic genus, taking values in a ring of modular forms with suitable integrality conditions, was in fact Landweber exact, when regarded as a ring homomorphism into an appropriately localized ring of modular forms. (In "Forms of K-theory" [**50**], dating from the Spring of 1972, Jack Morava consciously avoided considering this case.) The result was the construction of "Elliptic cohomology," $Ell_*(X)$. Jens Frenke [**16**] subsequently put it in a larger framework. This homotopy theoretic construction initiated an intriguing and frustrating search by many of us for a geometric construction of this theory and its many cousins; see Graeme Segal's report [**55**] for an early view of this project.

Perhaps I could say a few words about the questions raised by this work, to indicate that the line of research Landweber initiated and guided is still at center stage. Let us write FGL for the category of Landweber-exact formal groups. There is a functor L from this category to the homotopy category of evenly graded periodic ring spectra. (The functoriality has been a nagging question, and was recently resolved by Mark Hovey and Neil Strickland [**18**], building on [**16**].) Mike Hopkins and I have shown that for various important and natural functors $\pi : M \to FGL$ the composite $L\pi$ lifts to a functor to the strict category of A_∞ rings spectra, and Hopkins and Paul Goerss have improved this in many cases to E_∞. This is a lifting of M into *homotopy theory* rather than merely into the *homotopy category*, and it enables one to form homotopy limits, giving new and interesting spectra such as the spectrum TM of Topological Modular Forms. But in fact it is quite possible that the entire functor L lifts in this way. The resulting

diagram would be "Landweber homology," and, one hopes, would have interesting geometric aspects which are as yet completely obscure.

I'd like to end by recording my own personal appreciation of Peter Landweber. Peter came to Rutgers the same year I entered Princeton graduate school. In the fall of 1973 Steve Wilson, in his second year as an Instructor at Princeton University, initiated a weekly seminar on bordism theory, dubbed the "Bording School." Jack Morava was living in Princeton, Doug Ravenel was at Columbia University, Peter was at Rutgers. It was an exciting moment. Jack was overflowing with ideas and preprints, of which [**47**]–[**50**] are only a fragment. He talked about his geometric perspective, viewing a comodule as an equivariant sheaf over the moduli space of formal group laws; he described Sullivan's theory of bordism with singularities, gave a proof of the classification of invariant prime ideals in this setting, identified the subcategory of "height n" comodules with representations of the "stabilizer group" (the automorphism group of a formal group of height n), and explained that Lazard's work on p-adic Lie groups led to astonishing finiteness results for the cohomology of these comodules. Jack was very exciting, but his ideas were sometimes hard to follow, and for an insecure youngster like me it was essential that there was at least one adult around. Cobordism was the most exciting thing going, and Peter seemed to me to be its éminence grise, though today it's not so hard to compute that he was just a few years older than we were. He came bearing his latest translations from the mysterious Russian school, and always had some new discovery to tell us about. Best of all, he read what we wrote and returned copies to us with meticulous editorial comments, always to be taken seriously since Peter's own writing style is a model of clarity and grace.

Peter Landweber presents us with example of one who pursues an idea tenaciously over a long period of time—the ideas in the LEFT and elliptic cohomology trace right back to his thesis, with its study of the module structure of the complex bordism of a space, and the work immediately after, on operations—while not single-mindly, as we see from his contributions to our understanding of the geometric side of topology as well.

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