Metastability of the Chafee-Infante Equation with small heavy-tailed Lévy Noise

A Conceptual Climate Model

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von
Herrn Dipl.-Math. Michael Anton Högele
geboren am 16.04.1980 in Cham in der Oberpfalz

Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Dr. h.c. Christoph Markschies

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät II:
Prof. Dr. Peter Frensch

Gutachter:
1. Prof. Dr. Peter Imkeller
2. Prof. Dr. Ilya Pavlyukevich
3. Prof. Dr. Jerzy Zabczyk

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Caminate, son tus huellas
el camino, y nada más;
caminate, no hay camino,
se hace camino al andar.

Al andar se hace camino
y al volver la vista atrás
se ve la senda que nunca
se ha de volver a pisar.

Caminante, no hay camino,
sino estelas en la mar.

A. Machado: Proverbios y Cantares

In Memory of My Mother

Abstract

If equator-to-pole energy transfer by heat diffusion is taken into account, Energy Balance Models turn into reaction-diffusion equations, whose prototype is the (deterministic) Chafee-Infante equation. Its solution has two stable states and several unstable ones on the separating manifold (separatrix) of the stable domains of attraction. We show, that on appropriately reduced domains of attraction of a minimal distance to the separatrix the solution relaxes in time scales increasing only logarithmically in it. Motivated by the statistical evidence from Greenland ice core time series, we consider this partial differential equation perturbed by an infinite-dimensional Hilbert space-valued regularly varying (pure jump) Lévy noise of index alpha and intensity epsilon. A proto-type of this noise is alpha-stable noise in the Hilbert space.

Extending a method developed by Imkeller and Pavlyukevich to the SPDE setting we prove under mild conditions that in contrast to Gaussian perturbations the expected exit and transition times between the domains of attraction increase polynomially in the inverse intensity. In Chapter 6 we introduce an additional natural separatrix hypothesis on the jump measure that implies an upper bound on the exit time of a neighborhood of the separatrix. This allows to obtain an upper bound for the asymptotic exit time uniform for the initial positions inside the entire domain of attraction. It is followed by two localization results. Finally we prove that the solution exhibits metastable behavior. Under the separatrix hypothesis we can extend this to a result that holds uniformly in space.

Zusammenfassung

Wird der Äquator-Pol-Energietransfer als Wärmediffusion berücksichtigt, so gehen Energiebilanzmodelle in Reaktions-Diffusionsgleichungen über, deren Modellfall die (deterministische) Chafee-Infante-Gleichung darstellt. Ihre Lösung besitzt zwei stabile Zustände und mehrere instabile auf der separierenden Mannigfaltigkeit (Separatrix) der stabilen Anziehungsgebiete. Es wird bewiesen, dass die Lösung auf geeignet verkleinerten Anziehungsgebieten mit Minimalabstand zur Separatrix innerhalb von Zeitskalen relaxiert, die höchstens logarithmisch darin anwachsen. Motiviert durch statistische Belege aus grönländischen Zeitreihen wird diese partielle Differentialgleichung unter Störung mit unendlichdimensionalem, Hilbertraumwertigen, regulär variierenden Lévy'schen reinen Sprungrauschen mit index alpha und Intensität epsilon untersucht. Ein kanonisches Beispiel dieses Rauschens ist alpha-stabiles Rauschen im Hilbertraum.

Durch Erweiterung einer Methode von Imkeller und Pavlyukevich auf stochastische partielle Differentialgleichungen wird unter milden Bedingungen bewiesen, dass im Gegensatz zu Gauß'schem Rauschen die erwarteten Austritts- und Übertrittszeiten zwischen Anziehungsgebieten polynomiell mit Ordnung in der inversen Intensität für kleine Rauschintensität anwachsen. In Kapitel 6 wird eine zusätzliche natürliche "Separatrixhypothese" über das Sprungmaß eingeführt, die eine obere Schranke für die Austrittszeiten aus einer Umgebung der Separatrix impliziert. Dies ermöglicht den Nachweis einer oberen Schranke für die Austrittszeiten, welche gleichmäßig für Anfangsbedingungen in dem ganzen Anziehungsgebiet gilt. Es folgen zwei Lokalisierungsergebnisse. Schließlich wird gezeigt, dass die Lösung metastabiles Verhalten aufweist. Unter der "Separatrixhypothese" wird dies auf ein Ergebnis erweitert, welches gleichmäßig im Raum gilt.

Contents

List of frequently used Notation

Important constants

- $\alpha \in (0,2)$, index of the noise, see ν and L
- $\rho \in (\frac{1}{2}, 1)$, see $\frac{1}{\varepsilon^\rho}, \varepsilon > 0$
- $\Gamma > 0$, large geometric constant
- $\gamma > 0$, appropriately small exponent

The spaces

- $(L^2(0,1), |\cdot|)$, Lebesgue space of equivalence classes of square integrable functions on $(0,1)$ with the usual norm
- $H = H_0^1(0,1), (H, \|\cdot\|)$, space of weakly differentiable elements of $L^2(0,1)$ with Dirichlet boundary conditions with $\nabla x \in L^2(0,1)$ for $x \in H$ and with the norm $\|x\|^2 = \int_0^1 (\nabla x(\zeta))^2 \mathrm{d}\zeta, x \in H$.
- $(\mathcal{C}_0(0,1), |\cdot|_\infty)$, space of continuous functions on $[0,1]$ with $x(0) = x(1) = 0$ with the supremum norm.

The deterministic Chafee-Infante equation

- $(S(t))_{t \geqslant 0}$, heat semigroup on H
- $\lambda > \pi^2$, with $\lambda \neq (k\pi)^2$, $k \in \mathbb{N}$, Chafee-Infante parameter
- $u = (u(t;x))_{\substack{t \geqslant 0 \\ x \in H}}$, solution of the deterministic Chafee-Infante equation at time $t \geqslant 0$ with initial value $x \in H$ for fixed parameter λ
- $\phi^\pm$, one of the two stable states $\{\phi^+, \phi^-\}$ of u for fixed λ
- $\mathcal{A}^\lambda$, global attractor of the dynamical system $t \mapsto u(t;\cdot)$ in H

Domains of attraction Let $\delta_i > 0, i = 1,2,3$ and $\varepsilon > 0$.

- $D^\pm$, domain of attraction of $\phi^\pm$ under the flow $t \mapsto u(t;x), x \in H$
- $\mathcal{S} = H \setminus (D^+ \cup D^-)$, smooth manifold separating D^+ and D^-, called separatrix
- $D^\pm(\delta_1)$, positive invariant set of elements $x \in D^\pm$ such that $\mathrm{dist}(u(t;x), \mathcal{S}) \geqslant \delta_1$ for all $t \geqslant 0$

- $D^{\pm}(\delta_1, \delta_2)$, positive invariant set of elements $x \in D^{\pm}(\delta_1)$ with $\mathrm{dist}(u(t;x), \mathcal{S}) \geqslant \delta_1 + \delta_2$ for all $t \geqslant 0$
- $\tilde{D}^{\pm}(\delta_1) = D^{\pm}(\delta_1, \delta_1^2)$
- $D^{\pm}(\delta_1, \delta_2, \delta_3)$, set of elements $x \in D^{\pm}(\delta_1, \delta_2)$ such that $\mathrm{dist}(u(t), \mathcal{S}) \geqslant \delta_1 + \delta_2 + \delta_3$ for all $t \geqslant 0$
- $D^0(\delta_1) = H \setminus (D^+(\delta_1) \cup D^-(\delta_1))$
- $\tilde{D}^0(\delta_1) = H \setminus (\tilde{D}^+(\delta_1) \cup \tilde{D}^-(\delta_1))$
- $\tilde{D}^{\pm 0}(\delta_1) = \tilde{D}^{\pm}(\delta_1) \cup \tilde{D}^0(\delta_1)$
- $T_{rec} + \kappa\gamma|\ln\varepsilon|$, upper bound for $u(t;x)$, $x \in D^{\pm}(\varepsilon^{\gamma})$, to enter $B_{(1/2)\varepsilon^{2\gamma}}(\phi^{\pm})$

Shifted domains of attraction Let $\delta_i > 0, i = 1, 2, 3$.

- $D_0^{\pm} = D^{\pm} - \phi^{\pm}$
- $D_0^{\pm}(\delta_1) = D^{\pm}(\delta_1) - \phi^{\pm}$
- $D_0^{\pm}(\delta_1, \delta_2) = D^{\pm}(\delta_1, \delta_2) - \phi^{\pm}$
- $\tilde{D}_0^{\pm}(\delta_1) = \tilde{D}^{\pm}(\delta_1) - \phi^{\pm}$
- $D_0^{\pm}(\delta_1, \delta_2, \delta_3) = D^{\pm}(\delta_1, \delta_2, \delta_3) - \phi^{\pm}$

The stochastic Chafee-Infante equation

- $\varepsilon > 0$, noise intensity
- ν, symmetric, regularly varying Lévy measure on $\mathcal{B}(H)$ of index $\alpha \in (0, 2)$
- $L = (L(t))_{t \geqslant 0}$, symmetric pure jump Lévy process in H with Lévy measure ν
- $X^{\varepsilon} = (X^{\varepsilon}(t;x))_{t \geqslant 0}$, solution of the stochastic Chafee-Infante equation driven by $\varepsilon \mathrm{d}L$ at time $t \geqslant 0$ with initial value $x \in H$
- $\Delta_t L = L(t) - L(t-)$, jump of L at time $t > 0$
- $\dfrac{1}{\varepsilon^{\rho}}$, for $\varepsilon > 0, \rho \in (0,1)$, critical jump height of L beween "small" and "large" jumps
- $\eta^{\varepsilon} = (\eta^{\varepsilon}(t))_{t \geqslant 0}$, compound Poisson process consisting of all jumps of L with $\|\Delta_t L\| > \frac{1}{\varepsilon^{\rho}}$
- $(T_i)_{i \in \mathbb{N}}$, jump times of η^{ε}
- $t_i = T_i - T_{i-1}, i \in \mathbb{N}$, inter-jump periods between jumps the of η^{ε}
- $W_i = \Delta_{T_i} L, i \in \mathbb{N}$, i-th jump (increment) of η^{ε}

- $\xi^\varepsilon = (\xi^\varepsilon(t))_{t\geqslant 0}$, where $\xi^\varepsilon(t) = L(t) - \eta^\varepsilon(t), t \geqslant 0$, small jump process
- $\xi^* = (\xi^*(t))_{t\geqslant 0}$, where $\xi^*(t) = \int_0^t S(t-s)\mathrm{d}\xi^\varepsilon(s)$, small jumps convolution
- $Y^\varepsilon = (Y^\varepsilon(t;x))_{\substack{t\geqslant 0\\ x\in H}}$, solution of the stochastic Chafee-Infante equation driven by $\varepsilon\mathrm{d}\xi^\varepsilon$ at time $t \geqslant 0$ and initial value $x \in H$

Time scales Let $\varepsilon > 0$, $\rho \in (\frac{1}{2}, 1)$, $\alpha \in (0, 2)$.

- $\lambda^\pm(\varepsilon) = \nu\left(\frac{1}{\varepsilon}(D_0^\pm)^c\right) = \varepsilon^\alpha\ \ell(1/\varepsilon)\ \mu\left((D_0^\pm)^c\right)$, characteristic rate of the first exit time
- $\beta_\varepsilon = \nu\left(\frac{1}{\varepsilon^\rho}\big(B_1(0)\big)^c\right) = \varepsilon^{\alpha\rho}\ \ell(1/\varepsilon^\rho)\ \mu\left(B_1^c(0)\right)$, intensity of η^ε
- $\lambda^0(\varepsilon) = \nu\left(\frac{1}{\varepsilon}B_1(0)^c\right) = \varepsilon^\alpha\ \ell(1/\varepsilon)\ \mu\left(B_1^c(0)\right)$, characteristic rate of the metastability
- $\ell : (0,\infty) \to (0,\infty)$, slowly varying function associated to ν
- μ, limit measure of ν on $\mathcal{B}(H)$

Exit times and transition times Let $\varepsilon > 0$, $\gamma \in (0, 1)$.

- $\tau_x^\pm(\varepsilon)$, first exit time of $X^\varepsilon(\cdot;x), x \in \tilde{D}^\pm(\varepsilon^\gamma)$, from the reduced domain of attraction $D^\pm(\varepsilon^\gamma)$
- $\hat{\tau}_x^\pm(\varepsilon)$, first exit time of $X^\varepsilon(\cdot;x), x \in D^\pm$ from the entire domain of attraction $D^\pm$
- $\tau_x^{\pm 0}(\varepsilon)$, first exit time of $X^\varepsilon(\cdot;x), x \in \tilde{D}^\pm(\varepsilon^\gamma)$ from the enhanced domain of attraction $\tilde{D}^{\pm 0}(\varepsilon^\gamma)$
- $\sigma_x^\pm(\varepsilon)$, first entrance time of $X^\varepsilon(\cdot;x), x \in \tilde{D}^\pm(\varepsilon^\gamma)$ in $B_{\varepsilon^{2\gamma}}(\phi^\mp)$
- $\tau_x^0(\varepsilon)$, first exit time from the neighborhood of the separatrix $\tilde{D}^0(\varepsilon^\gamma)$

1. Introduction

Human mind is limited. Whenever it is confronted with an overwhelmingly complex collection of objects it tries to identify patterns or structures in it and interpret them in concepts concentrated in models. Mathematics develops a language that serves to make those complex collections of objects of reality tractable to our mind by skillfully combining concepts that produce statements and predictions about reality's model counterpart: virtual reality.

On this level, assisted by modern computing power, it has - besides theory and experiment - in recent years created a third column of human acquisition of knowledge: (numerical) simulation, dealing with the outcomes of experiments with virtual reality. Especially if a model is expected to approximate reality very accurately, (simulation) experiments with virtual reality and observation of true reality share one typical feature: they are of comparable levels of complexity. A major example here is the modeling of terrestrial climate.

The variability of global climate patterns for the last decades has received overwhelming interest during the last years. The impact human activities might have on the current terrestrial climate balance underlines the need for reliable climate modeling and simulation. The mathematical models underlying modern simulations are very complex and high dimensional. The closer to reality the resulting virtual pictures are, the closer our understanding of their contents is to our understanding of real climate. This possibly just means that it is equally poor. In addition, climatology is a science without experiments or empirical inference in the usual sense, apart from the reproduction of past climate patterns by statistical inference from paleoclimatic data. The cross-validation of simulation output with these data is usually rather difficult. As a consequence, there certainly is the danger of too much confidence in the simulation output from the models, and the virtual world they create. And it is certainly wrong to consider computer experiments as acceptable compensation for lack of real experiments and empirical data. Therefore a physical or analytical understanding of the phenomena both in the real as in the virtual world of model simulations through conceptual insight is of central importance. It can be provided by considering conceptual, analytically accessible stochastic reductions of the complex models. Accordingly, stochastic model reduction in climate dynamics is of paramount importance.

1.1. A Conceptual Approach to Low-Dimensional Climate Dynamics

One of the main obstacles of climate modeling is the substantial variability on spatial and temporal scales ranging over many orders of magnitude. It reaches from turbulent eddies in the ocean surface due to breaking waves, through mid-latitude cyclonic storms hundreds of kilometers in extent and lasting for days, to millennial scale shifts in ice cover and ocean circulation. The low-lying physical description behind imposes important mutual dependencies of quantities on these highly different time scales, which in general cannot be resolved entirely. This spread poses major challenges for any quantitatively accurate and computationally feasible representation.

To account for this variety of effects on very different scales, the community of climatology developed a big collection of models which are commonly classified into three groups. On the top level of quantitative accuracy are the comprehensive General Circulation Models (GCMs). These are the quantitatively most ambitious models, which attempt to represent the climate system in as much detail as computational resources and conceptional reasoning allow. Earth System Models of Intermediate Complexity (EMICs) instead are models of a more restrained resolution, which attempt to represent some subsystem of Earth's climate in detail, such as the ocean, the land surface or the atmosphere, while the interaction with other subsystems as well as external forcing remains parametrized. At the bottom of the model hierarchy according to Claussen et al. [2002] are low dimensional ones such as for instance energy balance models, that ignore almost all quantities and their interactions, except for a few. They are studied under highly idealized conditions, such that they are hardly of quantitative relevance. Their interest lies in their accessibility for mathematical analysis. Very often they are completely solvable and entirely understandable. They may predict phenomena encountered in more complex models. Their reduced complexity can help to develop conceptual qualitative paradigms capable to interpret and understand simulations obtained on the basis of EMICs or GCMs. Classical examples of this are the prediction of multiple states of the thermohaline circulation by Stommel [1961], of the phenomenon of sensitivity to initial conditions by Lorenz [1963], and of glacial metastability.

In the lower levels of climate modeling it is crucial to decide which processes to represent explicitly, which to parametrize, and how to justify or even construct the parametrization. Following Imkeller and Monahan [2002], in an updated version of the traditional approach an analogy with thermodynamic limit theorems is used: by taking the proportion of scales to an infinite limit, a complete separation of micro and macro scales is obtained. In a first step, averaging of small scale processes produces deterministic dynamics for the large scale processes. In a second step, the fluctuation of the large scale variables around the averaged values of the small scale quantities is expressed by stochastic differential or partial differential equations, in which the large scale variables are driven by random processes representing the small scale components. The mathematically rigorous derivation of such equations by Khasminskii [1966] leads to *linear systems*, however.

1.1.1. Hasselmann's Unfinished Program

There have been serious attempts to derive simple *non-linear* climate models with stochastic forcing from idealized GCMs. This project is labeled "Hasselmann's program" after an article by Arnold [2001], in which the ideas by the climatologist Hasselmann [1976] dating back to the mid-seventies are translated into modern mathematical language. Hasselmann's work is explicitly aimed at increasing the mathematical and physical understanding of more resolved climate models.

We shall briefly sketch the main ideas. In a first step an idealized GCM is considered as a large system of coupled ordinary (or partial) differential equations, in which for $0 < \varepsilon \ll 1$ the climate state $z = (x^\varepsilon, y^\varepsilon)$ can be separated into "slow" $x^\varepsilon(t, y^\varepsilon)$ and "fast" variables $y^\varepsilon(\frac{1}{\varepsilon}t, x^\varepsilon)$. Such a system can be formally described by

$$\begin{aligned} \dot{x}^\varepsilon &= f(x^\varepsilon, y^\varepsilon), \\ \dot{y}^\varepsilon &= \frac{1}{\varepsilon} g(x^\varepsilon, y^\varepsilon). \end{aligned}$$

The scale separation should be described by a small parameter ε corresponding to the "response time“ of the scales of slow and fast variables. Now define in physical jargon $u^\varepsilon(t) := \langle x^\varepsilon(t, \cdot)\rangle, t \geqslant 0$, as an "average" of the slow variables with respect to an invariant measure of the subsystem of the fast ones. This should lead to an averaged ordinary or partial differential equation

$$\dot{u}^\varepsilon = F(u^\varepsilon),$$

where $F(u^\varepsilon) := \langle f(x^\varepsilon, \cdot)\rangle$. The first mathematically rigorous proof of such a procedure was given by Bogolyubov and Mitropolskii [1961], establishing that under appropriate assumptions $\lim_{\varepsilon\to 0+} x^\varepsilon(t) = u^0(t)$.

In a second step, the fluctuation $x^\varepsilon(t) - u^0(t)$ of the solution around the averaged one is studied. Khasminskii [1966] discovered that for $t \in [0, T]$

$$L^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} \left(x^\varepsilon(t) - u^0(t)\right)$$

has a limiting Gaussian law as $\varepsilon \to 0+$. This way, he obtains linear differential equations for the slow variables with a stochastic term replacing the fast ones on finite intervals. In the framework of diffusion limits, deviations from averaged behavior produce non-linear (partial) differential equations with stochastic forcing (see Arnold and Kifer [2001] and Majda et al. [1999]). In this reduction, an assumption is crucial that is usually very hard to rigorously establish: mixing properties of the fast components, which lead to a decay of correlations viewed by an equilibrium measure. Even in simple ocean models studied in Maas [1994] coupled to a Lorenz equation as atmospheric component, different regimes of the fast motion that are only partially chaotic, complicate the mathematical treatment.

Yet many qualitative phenomena could not be captured by these methods, since they happen on ε-dependent time scales, that tend to be large for small ε, i.e. on intervals $[0, T(\varepsilon)]$, where $T(\varepsilon) \to \infty, \varepsilon \to 0+$. Among these are for example the Markovian transitions between stable states of the deterministic system that become metastable by the action of noise.

The systematic mathematical deduction of these stochastically forced equations from deterministic models remains a challenge some 35 years after their heuristic derivation by Hasselmann.

1.1.2. Energy Balance Models perturbed by Noise of Small Intensity

An alternative approach for obtaining relevant conceptual models in climate dynamics short-circuits the derivation according to Hasselmann's program. It consists in the explicit study of given paleoclimatic time series, and the selection of the best fitting dynamical model through statistical inference. Assume that the data in the time series are realized by one of a family of deterministic dynamical systems perturbed by additive stochastic noise. Assume further that the noise is parametrized by a parameter located in a set in Euclidean space. To choose the best fitting one among the dynamical models, one has to develop a statistical test for instance for the noise parameter - often a rather hard task.

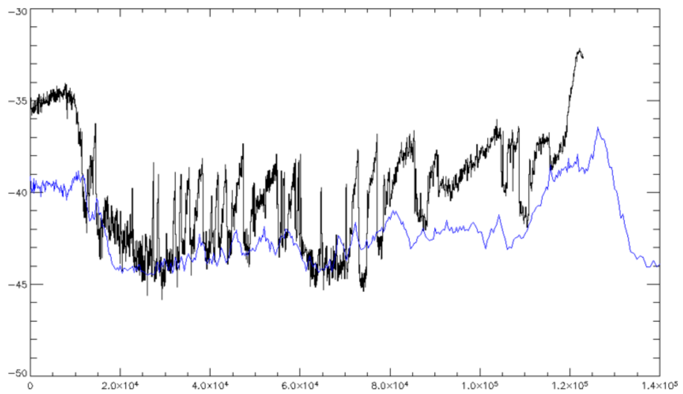

Figure 1.1.: Greenland ice core $\delta^{18}O$ temperature proxies (NGRIP [2004] core data, black line), 50 year average, from 120.000 years before present until now. The higher the values the warmer the average temperature.

For a paleoclimatic time series from the Greenland ice shelf (Figure 1.1) providing proxies for the yearly average temperatures of the last glacial period, climatologists around Ditlevsen [1999] proposed an energy balance model perturbed by heavy-tailed α-stable noise of small intensity. A statistical analysis on a physical level of rigor was used to estimate the best fitting α.

Recently this conclusion has been supported strongly by a mathematical study. In Hein et al. [2009] the model selection problem for the Greenland temperature time series was carried out successfully. The class of models considered is given by a dynamical system driven by a one dimensional additive α-stable process. Based on a path-wise roughness analysis using the power variations of trajectories they establish an estimator

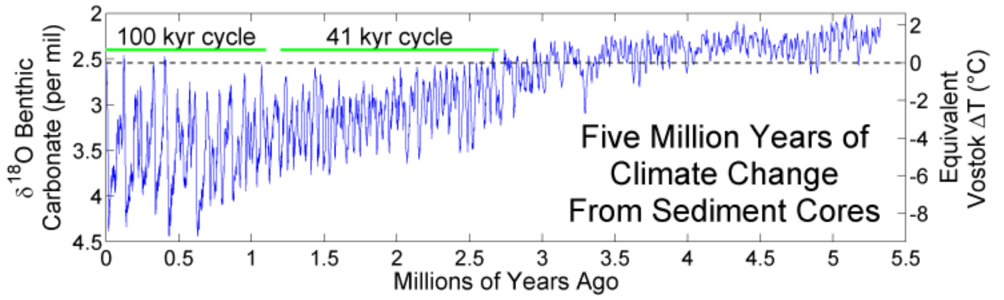

Figure 1.2.: Temperature proxy for the last 5 million years, Lisiecki and Raymo [2005]

for $\alpha \in (0, 2)$. The good convergence of this method to a unique parameter gives at least a good indication that such a signal is observed in the time series. Very recent developments in the thesis work by Gairing [2010] will most probably allow a detailed goodness of fit estimate with confidence intervals for α.

1.1.3. The Motivating Phenomenon: Paleoclimatic Warming Events

In the literature the term “ice age“ has different meanings. In this part we adopt the following convention. *Ice age* denotes a period of lower temperature of Earth’s surface and atmosphere on a scale corresponding roughly to Earth’s age, i.e. on a billion to hundred million year scale. During an ice age, frequent expansions and retreats of continental ice sheets, polar ice caps and alpine glaciers are observed. These episodes of extra cold climate are called *glacial periods*. See IPCC-Report [2010].

Since the estimated formation of Earth about 4.5 billion years ago, five major ice ages are accounted for. The first well-established one, the Huronian Ice Age, happened during the period between 2.4 and 2.1 billion years before present. During the last billion years there is scientific evidence for four distinguishable ice ages. During the Cryogenian Ice Age, considered as the most severe one, around 850-630 million years before present, earth was completely covered by ice (“snowball earth”). It is followed by the minor Andean-Saharan Ice Age, around 460-430 million years before present. The Karroo Ice Age (350-260 million years b.p) is suspected to have been caused by the reduction of CO_2 due to intense vegetation before. Between these periods the land surface seems to have been mostly ice-free. Since 2.58 million years before present polar ice shields appear to reemerge, resulting in the current Quaternary Ice Age, during which around 47 glacial periods have taken place so far (See Figure 1.2).

The eventual causes for the onset of an ice age are not very clear yet. Instead, the succession of glaciation periods at least during the current ice age is closely linked to the periodic behavior of some of Earth’s orbital parameters, the so-called Milankovich cycles.

The theory of climate variability due to the change in planetary orbital parameters goes back to the Serbian civil engineer M. Milankovich (1879-1958). In collaboration with W. Köppen, a German meteorologist, he recognized that the decrease of summer insolation at high latitudes may be responsible for the growth of glaciers. He expresses Earth's incoming solar radiation at a given point on the surface and time as a function of the orbital parameters, but is unsure about the critical latitude to trigger a glaciation period.

If we suppose that Earth's orbit around the sun lies approximately in a plane, it can be decomposed into three major components. The *eccentricity* of the elliptic annual trajectories of Earth around the sun vary regularly over time with periodic components of about 100.000 years.

Earth's axis of rotation has an *inclination* with respect to the normal of the orbital plane, the angle of which varies between 22.1° and 24.5° with an approximate period of 41.000 years. It influences the solar radiation influx at high latitudes, see Hartmann [1994]. A third component is contributed by the periodic *precession* of the equinoxes, i.e. the gyration of Earth's rotation axis around the normal of the orbital plane with major periods of 19.000 and 23.000 years.

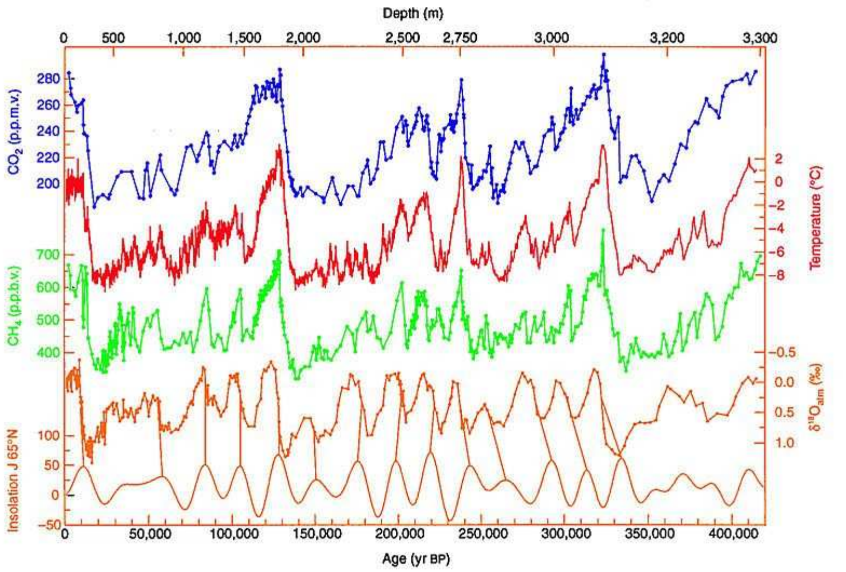

Figure 1.3.: 420.000 years of ice core data from Petit et al. [1997, 1999], Antarctica research station, From Bottom to top: Solar variation at 65° due to Milankovich cycles (connected to ^{18}O), ^{18}O isotope of oxygen, levels of methane, relative temperature

The combined effect of these three components accounts for up to 30% of incoming solar radiation at high latitudes. The diagram of Figure 1.3 exhibits a fairly good correspondence of the summer insolation at 65° North calculated on the basis of this orbital forcing.

In the long-range data plot in Figure 1.2 one recognizes the dominant periodicity of 41.000 years until one million years ago which is replaced by the 100.000 year periodicity since then. For a recent discussion of this phenomenon see Ditlevsen [2009].

The present work is motivated by a phenomenon observed during the last glacial period, about 100.000-10.000 years before present. Temperature proxies in the Greenland ice core indicate that the orbital forcing discussed above does not have a major effect within this period, and temperatures do not stay uniformly low. Instead one can recognize at least 21 major spikes, indicating abrupt extraordinary increases by more than 8° degrees within less than 30 years, followed by a gradual decline during several centuries (see IPCC-Report [2010]). The distribution of the spikes in Figure 1.1 is rather regular over the whole period.

The origin of these patterns is not quite clear. In the literature the spikes are classified into two categories. The first one consists of so-called Heinrich events. They are thought to be caused by ice sheet instabilities with a huge discharge of icebergs, i.e. enormous fresh water influx into the Atlantic. Between three and six rapid warmings are considered to be of Heinrich type. The remaining ones are named Dansgaard-Oeschger events after their discoverers. There is so far no good explanation for their emergence. Some authors, for instance Ganopolski and Rahmstorf [2001], Rahmstorf [2003] and Ditlevsen et al. [2006], suggest a superposition of short periodic signals of solar radiation, leading to temperature evolutions periodic intervals of which determine the Dansgaard-Oeschger and Heinrich events. They are separated by temperature thresholds that may be crossed by random perturbation.

1.2. The Mathematical Model

1.2.1. The Derivation of the Problem

In this work we shall consider a process $X^\varepsilon(t,\zeta)$ that may describe the (annually averaged) temperature evolution in space ζ during a period of time t, subject to spatial-temporal noise of (small) intensity $\varepsilon > 0$. We wish to resolve longitudinally averaged temperatures depending on global latitudes. Therefore the spatial variable ζ takes its values in the interval of latitudes between the poles, normalized to the unit interval $[0,1]$. From a mathematical point of view the underlying description of the evolution of temperature distributions on this interval involves random processes taking their values in sets of functions on compact domains. This leads directly to equations in infinite-dimensional spaces, infinite-dimensional models of noise and eventually from SDEs to SPDEs.

The dynamics of our processes is determined by three components.

1. A *reaction* term f of the evolution equation expresses a deterministic forcing of temperature that can be derived heuristically from simple assumptions on the balance between absorbed and emitted solar radiation energy as a function of time (see Imkeller [2001]). Temperature being a one-dimensional quantity, we may assume that the resulting reaction term is described by the negative gradient of a potential function $f = -U'$ with two local minima, which may be interpreted as a cold "ground" state and a warmer "Dansgaard-Oeschger" state.

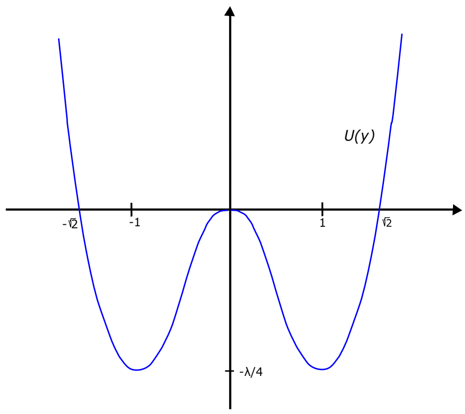

Figure 1.4.: Chafee-Infante potential for $\lambda = 12$

2. A spatial *diffusion term* $\frac{\partial^2}{\partial\zeta^2}X^\varepsilon$ models heat diffusion between equator and poles which is caused by different rates of insolation due to different angles of incidence for solar radiation. The diffusive character of heat transport is a first approximation, but for the time scales under consideration a well-accepted hypothesis. The simplest idealized semi-linear reaction-diffusion equation compatible with our

climate dynamics requirements is the Chafee-Infante equation. Its reaction term is related to a symmetric double-well potential over a bounded interval.

In this work will denote the solution of the deterministic Chafee-Infante equation by $u = X^0$. It satisfies formally

$$\begin{aligned} \frac{\partial}{\partial t}u(t,\zeta) &= \frac{\partial^2}{\partial \zeta^2}u(t,\zeta) + f(u(t,\zeta)) && \zeta \in [0,1],\ t>0, \\ u(t,0) &= u(t,1) = 0, && t>0, \\ u(0,\zeta) &= x(\zeta), && \zeta\in[0,1], \end{aligned} \tag{1.1}$$

where $U(y) = (\lambda/4)y^4 - (\lambda/2)y^2$ for $\lambda > 0$ fixed, and $f = -U'$.

The solution takes values in an infinite-dimensional function space, as for example $L^2(0,1)$, $H_0^1(0,1)$ or $\mathcal{C}(0,1)$, where also the initial state x is taken (see Temam [1992] or Sell and You [2002]). Since its pure reaction term f has two zeros given by the minima of U, apart from singular values of λ, the Chafee-Infante equation possesses two hyperbolic stable states $\phi^+, \phi^- \in \mathcal{C}^\infty(0,1)$. Nevertheless, there may be several unstable saddles, depending on the value of the parameter λ.

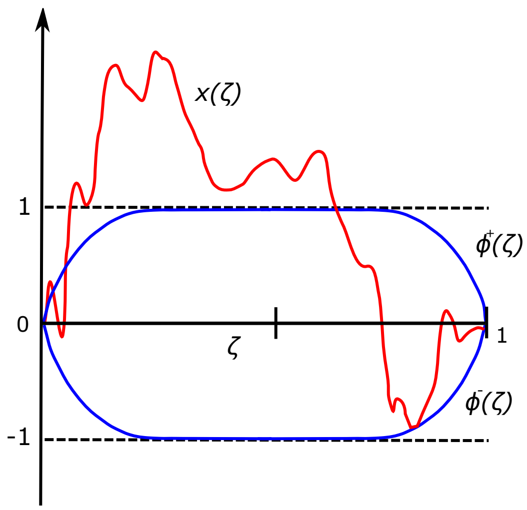

Figure 1.5.: Sketch of a typical element in H and the stable states ϕ^+ and ϕ^-

3. According to our discussion of Hasselmann's approach in Subsection 1.1.1 and the model selection problem in Subsection 1.1.3 for paleoclimatic time series we assume that the two deterministic components of the energy balance determined evolution at the right hand side of equation (1.1) are perturbed by an *additive stochastic process* L of small intensity $\varepsilon > 0$ taking values in the corresponding function space. We follow the suggestion in Hein et al. [2009] according to which the noise is of Lévy type with jump measure tails of polynomial order. The most prominent example is the case of α-stable noise.

Following these heuristic assumptions, for $\varepsilon > 0$ the process is supposed to satisfy the equation

$$\begin{aligned} \frac{\partial}{\partial t} X^\varepsilon(t,\zeta) &= \frac{\partial^2}{\partial \zeta^2} X^\varepsilon(t,\zeta) + f(X^\varepsilon(t,\zeta)) + \varepsilon \dot{L}(t,\zeta) && \zeta \in [0,1],\ t > 0, \\ X^\varepsilon(t,0) &= X^\varepsilon(t,1) = 0, && t > 0, \\ X^\varepsilon(0,\zeta) &= x(\zeta), && \zeta \in [0,1], \end{aligned} \tag{1.2}$$

where $\lambda > 0$ and $f = -U'$. The noise term $\dot{L}$ formally represents the generalized derivative of a pure jump Levy process in the Sobolev space $H = H_0^1(0,1)$ with Dirichlet boundary conditions, regularly varying Lévy measure of index $\alpha \in (0,2)$ and initial value $x \in H$. Since the focus of our mathematical work will be the metastable behavior of X^ε, the periodic orbital forcing effects related to Milankovich cycles are not taken into account in the reaction term at this stage.

For the one-dimensional counterpart of equation (1.2) without diffusion Imkeller and Pavlyukevich investigate the asymptotic behavior of exit and transition times in the small noise limit in Imkeller and Pavlyukevich [2006a], Imkeller and Pavlyukevich [2008] and Imkeller and Pavlyukevich [2006b]. In contrast to the Wiener case, for which exponential growth with respect to the noise intensity is observed (Freidlin and Ventsell [1998]), these models feature exit rates with polynomial growth in the limit of small noise. Accordingly, the critical time scale in which the global metastable behavior of the jump diffusion can be reduced to a finite state Markov chain jumping between the metastable states (see also Bovier et al. [2004]) is equally polynomial in the noise intensity.

To which extent do these results still hold true if a diffusive heat transport from the equator to the poles and infinite-dimensional noise is taken into account?

To find answers to this natural question will be the main objective of this work. We shall show in Theorem 2.18 that the expected exit time from (reduced) domains of attraction of the metastable states ϕ^+, ϕ^- increases polynomially of order $\varepsilon^{-\alpha}$ in the noise intensity ε, and characterize the exit scenarios. We shall also show in Theorem 2.24 that for this time scale of ε the jump diffusion system reduces to a finite state Markov chain with values in the set of stable states $\{\phi^+, \phi^-\}$.

Of course this treatment of the metastability of SPDE with Lévy jump noise can be seen independent of the climate dynamics context in which we embed it following the introductory remarks. So our analysis can be considered as a starting point for studying metastable behavior of dynamical systems induced by reaction-diffusion equations perturbed by Lévy jump noise on a more general basis.

1.2.2. The Basic Idea: Noise Decomposition by the Intensity Parameter

Extending Imkeller and Pavlyukevich [2008] for dimension 1, we next explain the heuristics of the method to determine the expected first exit time for a domain of attraction of the stable states $\phi^{\pm}$ in the asymptotics of small noise intensity. It proceeds along the following lines.

1. For $t \geqslant 0$ and a process Y let us write $\Delta_t Y := Y(t) - Y(t-)$. We fix a certain threshold, say $c > 0$, and consider the sequence of jump times of the driving Lévy noise L in H exceeding c

$$T_{i+1} := \inf\{t > T_i \mid \|\Delta_t L\| > c\}, \qquad T_0 = 0.$$

   If $(S(t))_{t\geq 0}$ is the Markovian semigroup associated with the diffusion operator on $(0,1)$, and we use the mild solution formulation following Peszat and Zabczyk [2007] the jumps of X^ε are just the jumps of L, i.e.

$$\Delta_{T_i} X^\varepsilon = \Delta_{T_i} \int_0^{\cdot} S(\cdot - s)\mathrm{d}L(s) = \Delta_{T_i} L. \tag{1.3}$$

2. The domain of attraction $D^{\pm}$ of the stable solution $\phi^{\pm}$ can be reduced appropriately to $D^{\pm}(\varepsilon^\gamma) \subset D^{\pm}$ such that the solution $u(t;x)$ of the Chafee-Infante equation starting in $x \in D^{\pm}(\varepsilon^\gamma)$ find itself within a small neighborhood $B_{\varepsilon^{2\gamma}}(\phi^{\pm})$ at times t exceeding $T_{rec} + \kappa\gamma|\ln\varepsilon|$, where T_{rec} is a global relaxation time and $\kappa > 0$ a global constant, formally

$$u(t;x) \in B_{\varepsilon^{2\gamma}}(\phi^{\pm}) \qquad \text{for all} \quad t \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon| \quad \text{and} \quad x \in D^{\pm}(\varepsilon^\gamma). \tag{1.4}$$

3. We now let the threshold c depend on ε, and choose $c = c(\varepsilon) = \frac{1}{\varepsilon^\rho}$ for $\rho \in (0,1)$ to split $L(t) = \xi^\varepsilon(t) + \eta^\varepsilon(t)$ into a small jump part ξ^ε, with

$$\varepsilon\|\Delta_t \xi^\varepsilon\| \leqslant \varepsilon\frac{1}{\varepsilon^\rho} \to 0, \quad \varepsilon \to 0+$$

   and a large jump part η^ε, with $\eta^\varepsilon(t) = \sum_{i:T_i \leqslant t} \Delta_{T_i} L,\ t \geq 0$. Between two large jump times T_i and T_{i+1}, the strong Markov property allows us to consider X^ε as being driven by the small jump component $\varepsilon\xi^\varepsilon$ alone. Denote this process by Y^ε. In finite dimensions Y^ε is directly seen to deviate negligibly from the deterministic solution u uniformly in time intervals of the order of its inter-jump times $t_{i+1} = T_{i+1} - T_i$, formally

$$\sup_{x \in D^{\pm}(\varepsilon^\gamma)} \sup_{T_i \leq t \leq T_{i+1}} \|Y^\varepsilon(t) - u(t)\| \to 0 \qquad \text{for} \quad \varepsilon \to 0+ \tag{1.5}$$

   in probability. Since we solve our equation in a mild sense we establish instead that (1.5) is implied by

$$\varepsilon\xi^*(t) \to 0, \quad \varepsilon \to 0+, \text{ for } t \geq 0$$

   where $\xi^*(t) = \int_0^t S(t-s)\ \mathrm{d}\xi^\varepsilon(s)$ the stochastic convolution with respect to ξ^ε.

(see Appendix A.3).

4. The inter-jump times of η^ε are all independent and with exponential law of parameter β_ε,
$$\beta_\varepsilon := \nu\left(\frac{1}{\varepsilon^\rho} B_1^c(0)\right) \approx \varepsilon^{\alpha\rho},$$
where ν is the jump measure of L for which we assume that it varies regularly of index α. They are therefore expected to be of order $\frac{1}{\varepsilon^\alpha}$, which for small ε is much bigger than the relaxation time $T_{rec} + \kappa\gamma|\ln\varepsilon|$ of u to $B_{\varepsilon^{2\gamma}}(\phi^\pm)$. We can now combine (1.3),(1.4) and (1.5). This implies that for small ε exit events start in $B_{\varepsilon^{2\gamma}}(\phi^\pm)$ and are most probably triggered by the large jump part $\varepsilon\eta^\varepsilon$. Hence the first exit time $\tau(\varepsilon)$ from $D^\pm$ is expected to be roughly
$$\tau(\varepsilon) \approx \inf\{T_i = \sum_{j=1}^{i} t_j \mid \phi^\pm + \varepsilon\Delta_{t_i} L \notin D^\pm\}.$$

5. Using the regular variation of the Lévy measure ν of L we obtain for the probability of large jumps high enough to trigger exits
$$\begin{aligned}\mathbb{P}\left(\phi^\pm + \varepsilon\Delta_{t_i} L \notin D^\pm\right) &= \mathbb{P}\left(\Delta_{t_1} L \in \frac{1}{\varepsilon}\left((D^\pm)^c - \phi^\pm\right)\right)\\ &= \frac{\nu\left(\frac{1}{\varepsilon}\left((D^\pm)^c - \phi^\pm\right) \cap \frac{1}{\varepsilon^\rho} B_1^c(0)\right)}{\nu\left(\frac{1}{\varepsilon^\rho} B_1^c(0)\right)} \approx \varepsilon^{\alpha(1-\rho)}.\end{aligned}$$
Therefore
$$\begin{aligned}\mathbb{E}\left[\tau(\varepsilon)\right] &\approx \sum_{i=1}^{\infty} \mathbb{E}\left[T_i\right] \mathbb{P}\left(\phi^\pm + \varepsilon\Delta_{t_i} L \notin D^\pm\right)\\ &\approx \mathbb{E}\left[t_1\right] \mathbb{P}\left(\phi^\pm + \varepsilon\Delta_{t_1} L \notin D^\pm\right) \sum_{i=1}^{\infty} i \,\left(1 - \mathbb{P}\left(\phi^\pm + \varepsilon\Delta_{t_1} L \notin D^\pm\right)\right)^{i-1}\\ &\approx \frac{1}{\varepsilon^{\alpha\rho}}\, \varepsilon^{\alpha(1-\rho)} \,\left(\frac{1}{\varepsilon^{\alpha(1-\rho)}}\right)^2 = \frac{1}{\varepsilon^\alpha}\end{aligned}$$

1.2.3. A Glance at Related Literature

Since to our knowledge the method of this work sketched in Subsection 1.2.2 has not been used in the context of SPDEs so far we shall only give an overview over parts of the literature to which our attention had been drawn on the course of these studies. We do not claim completeness.

The Chafee-Infante equation has been extensively studied, starting with the article by Chafee and Infante [1974]. Its most interesting feature is a bifurcation in the system parameter representing the steepness of the potential, which considerably changes the dynamics in comparison to the finite dimensional case, see for example Carr and Pego

[1989]. Other classical references are the books by Henry [1983] and Hale [1983]. Existence and regularity of its solutions have been investigated, as well as the fine structure of the attractor. We refer to the books Temam [1992], Cazenave and Haraux [1998], Robinson [2001], Chueshov [2002] and references therein.

SPDE with Gaussian noise go back to the seventies with early works by Pardoux [1975], Krylov and Rozovskii [2007] and Walsh [1981], Walsh [1986]. Since then the field has expanded enormously in depth and variety, as is impressively documented recently for example in Khoshnevisan et al. [2008]. More recent treatments can be found among others for instance in the books DaPrato and Zabczyk [1992], Chow [2007], Prevot and Röckner [2007], Kotelenez [2008] and references therein.

The treatment of the asymptotic dynamical behavior for finite dimensional Gaussian diffusions mainly by techniques related to large deviations was developed in Freidlin and Ventsell [1970, 1998]. In Faris and Jona-Lasinio [1982b], the authors use methods based on large deviations in order to analyze the stochastic dynamics for SPDE with Gaussian noise. The *tunneling effects* they discover interpret the phenomenon of metastable behavior of solutions switching between stable equilibria at time scales exponential in the noise intensity. Additionally they show that the transitions asymptotically take place at the saddle points, the number of which varies according to the bifurcation scenarios of the deterministic part. Martinelli et al. [1989] show that suitably renormalized exit times are asymptotically exponential. Brassesco [1991] shows that the process is asymptotically concentrated in balls around the stable states and that the average along trajectories remains close to the stable state before the switching time.

SPDEs with jump noise have been studied since the late eighties, see for example Chojnowska-Michalik [1987] and Kallianpur and Perez-Abreu [1988]. At the end of the nineties the subject is picked up again with a rich series of articles for example by Albeverio et al. [1998], Mueller [1998], Bie [1998], Applebaum and Wu [2000], Fuhrmann and Röckner [2000], Fournier [2000], Fournier [2001], Mytnik [2002], Knoche [2004], Stolze [2005], Hausenblas [2005], Hausenblas [2006], Bo and Wang [2006], Peszat and Zabczyk [2006], Röckner and Zhang [2007], Marinelli et al. [2010], Filipović et al. [2008], Filipović et al. [2010]. We refer to the monograph Peszat and Zabczyk [2007] for a more comprising view on SPDEs with Lévy noise and the bibliography therein.

1.2.4. Organization of the Work

In Section 2.1 we set up of the mathematical framework. We split our driving noise process into "small" and "large" jump components, in dependence on the noise intensity $\varepsilon > 0$. In the sequel we establish properties of the stochastically perturbed Chafee-Infante and characterize the crucial feature of the noise, i.e. its asymptotic polynomial decay of order $\varepsilon^\alpha, \alpha \in (0, 2)$. We discuss the dynamics of the deterministic equation, its attractor and its domains of attraction.

In Section 2.2 we state the main results of this thesis precisely.

Chapter 3 justifies the distinction between "small" and "large" jumps. We show that between two "large" jumps the deterministic system perturbed by only the "small" jumps will deviate only moderately from the deterministic trajectories, hence will cause an exit only with asymptotically vanishing probability.

Chapter 4 is devoted to the main part of the derivation of the asymptotic behavior of the first exit time from a reduced domain of attraction of the deterministic system. In this rather technical part we extend the methods developed in Imkeller and Pavlyukevich [2008] to regularly varying jump measures. To overcome the lack of moments due to the heavy-tailed noise, the crucial tool lies in precise asymptotic estimates of critical events, obtained by using the strong Markov property and the continuous dependence of the solution on the noise.

In Chapter 5 we exploit this result in order to determine the asymptotics of the transition times between small balls around the stable state.

Chapter 6 starts with a detailed discussion of an additional hypothesis, which implies an upper bound for the time to leave neighborhoods of the separating manifold between domains of attraction. In Section 6.2 we prove an upper bound for the asymptotic first exit time of the entire domain of attraction $D^{\pm}$. In Section 6.3 we derive two localization results for the solution on subcritical and critical time scales. Section 6.4 is devoted to the main result of this work, the description of metastable behavior of the stochastic Chafee-Infante equation. It states the convergence on a critical time scale of the solution of the stochastic Chafee-Infante equation to a continuous time Markov chain switching between the stable states $\phi^{\pm}$. Its switching rates are directly related to the mass of the reshifted domain of attraction with respect to the limiting measure of the regularly varying Lévy jump measure ν.

The Appendices cover the material which is needed for the Chapters in the main part. Since many results in the literature are not exactly in a useful form for our purposes we fill this gap here.

Appendix A mainly collects all the properties needed for stochastic Chafee-Infante equation and provides in particular the sketch for the proof of the strong Markov property. It ends with a short Section A.6 containing results about regularly and slowly varying functions, which we shall use useful for the tails of our Lévy noise.

Appendix B concentrates on fine properties of the dynamics of the deterministic Chafee-Infante equation. We start with a consistency result for reduced domains of attraction. In the sequel we show the existence of constants $T_{rec}, \kappa > 0$ such that for any $\gamma > 0$ the deterministic solution of the Chafee-Infante equation is confined to a ball of radius $\varepsilon^{2\gamma}$ around $\phi^{\pm}$ for times after $T_{rec} + \kappa\gamma|\ln\varepsilon|$ initial values x in an appropriate, reduced domain of attraction $D^{\pm}(\varepsilon^{\gamma})$ in the small noise limit $\varepsilon \to 0+$. Due to the bifurcation of the attractor of the Chafee-Infante equation this argument needs some care, and exploits for instance the hyperbolicity of the equilibrium points and the transversality of their respective local stable and unstable manifold. We prove very useful uniform (in x, T) boundedness properties of $\int_0^T |u(t;x)|_\infty^p \, \mathrm{d}t < \infty$ for $p > 1$.

2. The Main Results

2.1. The Mathematical Framework

In this work the natural numbers $\mathbb{N}$ do not contain 0.

The Spaces: For $p \geqslant 1$, the norm on the Banach space $L^p(0,1)$ of equivalence classes of functions on the unit interval Lebesgue integrable in the p-th power will be denoted by $|\cdot|_p$. In the case of the Hilbert space $L^2(0,1)$, we drop the subscript and simply write $|\cdot|$, and denote the corresponding scalar product by $\langle \cdot, \cdot \rangle$. Our processes usually will be supposed to take their values on the separable Hilbert space $H = H_0^1(0,1) := \overline{\mathcal{C}_c^\infty(0,1)}^{\|\cdot\|}$, normed by

$$||u|| := (\int_0^1 (\nabla u(\zeta))^2 d\zeta)^{\frac{1}{2}} = |\nabla u| = \langle \nabla u, \nabla u \rangle^{\frac{1}{2}}, \quad u \in H,$$

where ∇u is written for the derivative of $u \in H$ in the sense of generalized functions. We further use the uniform norm for functions usually in the space $C_0(0,1)$ of continuous functions with Dirichlet boundary conditions on the unit interval, and denote it by $|\cdot|_\infty$. The norms can be compared through Poincaré's inequality $|u| \leqslant ||u||, u \in H$, (see e.g. Brezis [1983]) and $|u|_\infty \leqslant ||u||, u \in H$, which follows from the easiest version of Gauss theorem: For $u \in \mathcal{C}_c^\infty(0,1)$ and $s \in (0,1)$

$$u(s) = u(s) - u(0) = \int_0^s \nabla u(\zeta) \, \mathrm{d}\zeta \leqslant s \, (\int_0^s (\nabla u(\zeta))^2 \, \mathrm{d}\zeta)^{\frac{1}{2}} \leqslant \|u\|.$$

Hence we can take the supremum on the left-hand side. The latter just expresses the one-dimensional Sobolev embedding

$$(H, \|\cdot\|) \cong (H_0^1(0,1), |\cdot|_{H_0^1}) \hookrightarrow (\mathcal{C}_0(0,1), |\cdot|_\infty).$$

The driving Lévy Process, "Small" and "Large" Jumps: Let $(L(t))_{t \geqslant 0}$ be the càdlàg (continue à droite avec limites à gauche, that is right continuous and with limits from the left) version of a pure jump Lévy process with values in H with a symmetric Lévy measure ν on its Borel σ–algebra $\mathcal{B}(H)$ satisfying

$$\int_H (1 \wedge \|y\|^2) \nu(\mathrm{d}y) < \infty.$$

For definitions and properties see Appendix A.1. We denote the jump increment of L at time $t \geqslant 0$ by $\Delta_t L := L(t) - L(t-)$ and decompose the process $L = \eta^\varepsilon + \xi^\varepsilon$ for $\rho \in (0,1)$ and $\varepsilon > 0$ in the following way. Denote by η^ε the compound Poisson process with intensity

$$\beta_\varepsilon := \nu\left(\frac{1}{\varepsilon^\rho} B_1^c(0)\right)$$

and the jump probability measure as ν outside the ball $\frac{1}{\varepsilon^\rho} B_1(0)$ by

$$\nu\left(\cdot \cap \frac{1}{\varepsilon^\rho} B_1^c(0)\right) / \beta_\varepsilon. \tag{2.1}$$

We further define the complimentary process

$$\xi^\varepsilon := L - \eta^\varepsilon. \tag{2.2}$$

The process ξ^ε will be referred to as “small jumps” process, and η^ε as “large jumps” process respectively. Note that for any $\varepsilon > 0$ the processes ξ^ε and η^ε are independent càdlàg Lévy processes with the respective Lévy measures $\nu(\cdot \cap B_{\varepsilon^{-\rho}}(0))$ and $\nu(\cdot \cap B^c_{\varepsilon^{-\rho}}(0))$ but in general of very different properties. ξ^ε is a mean zero martingale in H thanks to the symmetry of ν and possesses finite exponential moments.
Since the process η^ε is a compound Poisson process we can define its jump times. We set recursively

$$T_0 := 0, \qquad T_k := \inf\left\{t > T_{k-1} \;\middle|\; \|\Delta_t L\| > \varepsilon^{-\rho}\right\}, \quad k \geqslant 1,$$

and the periods between successive large jumps of η_t^ε as

$$t_0 = 0, \qquad t_k := T_k - T_{k-1}, \quad k \geqslant 1.$$

These waiting times are exponentially distributed, formally $\mathcal{L}(t_k) = EXP(\beta_\varepsilon)$. We shall denote the k-th large jump by

$$W_0 = 0, \qquad W_k = \Delta_{T_k} L, \quad k \geqslant 1,$$

with the jump distribution (2.1).

Càdlàg Mild Solutions of (1.2): Fix for the moment $\varepsilon > 0$. Consider the formal system (1.2) driven by $(\xi^\varepsilon(t))_{t\geqslant 0}$ instead of L

$$\begin{aligned}
\frac{\partial}{\partial t} Y^\varepsilon(t,\zeta) &= \frac{\partial^2}{\partial \zeta^2} Y^\varepsilon(t,\zeta) + f(Y^\varepsilon(t,\zeta)) + \varepsilon \dot{\xi}^\varepsilon(t,\zeta) && \zeta \in [0,1],\ t > 0, \\
Y^\varepsilon(t,0) &= Y^\varepsilon(t,1) = 0, && t > 0, \\
Y^\varepsilon(0,\zeta) &= x(\zeta), && \zeta \in [0,1].
\end{aligned} \tag{2.3}$$

Definition 2.1. Denote by $(S(t))_{t\geqslant 0}$ the $\mathcal{C}^0$-semigroup generated by the second derivative $\Delta = \frac{\partial^2}{\partial \zeta^2}$ over $(0,1)$ with Dirichlet boundary conditions in H. Then for any time horizon $T > 0$ and $x \in H$ a mild solution of equation (2.3) is a progressively measurable process $(Y^\varepsilon(t))_{t\in[0,T]}$ in H fulfilling for all $t \in [0,T]$ the integral equation

$$Y^\varepsilon(t) = S(t)x + \int_0^t S(t-s)f(Y^\varepsilon(s))\,\mathrm{d}s + \varepsilon \int_0^t S(t-s)\mathrm{d}\xi^\varepsilon(s) \qquad \mathrm{d}\zeta \otimes \mathbb{P}\text{-a.s.} \tag{2.4}$$

Theorem 2.2. *Let the preceding assumptions for $(S(t)_{t\geqslant 0})$ and f be fulfilled. Then for any mean zero $L^2(\mathbb{P};H)$-martingale $(\xi^\varepsilon(t))_{t\geqslant 0}$, $T > 0$ and initial value $x \in H$ there exists a unique càdlàg mild solution $(Y^\varepsilon(t;x))_{t\in[0,T]}$ of equation (2.3). The solution process induces a homogeneous Markov family with the Feller property.*

A proof can be found in Peszat and Zabczyk [2007] Chapter 10. In order to precisely describe solutions of (1.2) we shall need the following localization argument.

Definition 2.3. We consider an increasing exhaustion $(V_k)_{k\geqslant 1}$ of H by symmetric sets $V_k \in \mathcal{B}(H)$, and define a monotone localizing sequence $\tau^1 \leqslant \tau^2 \leqslant \dots$ of stopping times by

$$\tau^0 = 0 \qquad \tau^k := \inf\{t > \tau^{k-1} \mid \Delta_t L \in V_k\}, \quad k \geqslant 1.$$

The mild solution for the original system (1.2) is a progressively measurable process $(X^{\varepsilon,k}(t))_{t\in[0,T]}$ fulfilling for each $k \in \mathbb{N}$ and $t \in [0,\tau^k]$

$$X^{\varepsilon,k}(t) = S(t)x + \int_0^t S(t-s)f(X^{\varepsilon,k}(s))\,\mathrm{d}s + \int_0^t S(t-s)\mathrm{d}\xi^\varepsilon(s) + \int_0^t S(t-s)\mathrm{d}\eta^{\varepsilon,k}(s),$$

if we define $\eta^{\varepsilon,k}$ as the compound Poisson process with intensity

$$\beta_\varepsilon^k := \nu(V_k \setminus B_{\varepsilon^{-\rho}}(0)),$$

and jump probability measure

$$\frac{\nu\left((V_k \setminus B_{\varepsilon^{-\rho}}(0)) \cap \cdot\right)}{\beta_k^\varepsilon}.$$

For $\xi^{\varepsilon,k} := \xi^\varepsilon + \eta^{\varepsilon,k}$ and $t \in [0,\tau^k]$ this is equivalent to

$$X^{\varepsilon,k}(t) = S(t)x - \int_0^t S(t-s)f(X^{\varepsilon,k}(s))\,\mathrm{d}s + \int_0^t S(t-s)\mathrm{d}\xi^{\varepsilon,k}(s).$$

where $\xi^{\varepsilon,k}$ is also a mean zero $L^2(\mathbb{P};H)$-martingale.

We can summarize these facts in the following Corollary.

Corollary 2.4. *For $x \in H$ equation (1.2) has a càdlàg mild solution $(X^\varepsilon(t;x))_{t\geqslant 0}$, which satisfies the strong Markov property.*

The lack of moment regularity for heavy tailed noise L will give us numerous occasions to exploit the strong Markov property. We refer to Appendix A.5 for a sketch of the proof.

The Regularly Varying Jump Measure ν: So far we did not make any assumption on ν besides symmetry. From now on we shall concentrate on ν for which the tail decays asymptotically in the order of $r^{-\alpha}, \alpha \in (0,2)$, for $r \to \infty$. This is a natural generalization of α-stable processes with values in H. In order to describe the asymptotic polynomial increase in ε of the large jumps we introduce the following notions (see Bingham et al. [1987], Hult and Lindskog [2006] and Appendix A.6).

Definition 2.5. A regularly varying function with index $-\beta$ for $\beta > 0$, is a nonnegative, measurable function $v : (0,\infty) \to (0,\infty)$ with the property that for any $y > 0$

$$\lim_{x\to\infty} \frac{v(xy)}{v(x)} = y^{-\beta}.$$

Let us extend the notion of regular variation to measures on a Hilbert space H.

Definition 2.6.
1. Let H be a separable Hilbert space. Denote by $M_0(H)$ the class of all Radon measures $\nu : \mathcal{B}(H) \to [0,\infty)$ with the property

$$\mu(A) < \infty \quad \Leftrightarrow \quad A \in \mathcal{B}(H),\ 0 \notin \bar{A},$$

where $\bar{A}$ stands for the closure of a set A in a topological space.

2. A measure $\nu \in M_0(H)$ is *called regularly varying with index* $-\beta$ if there exists a non-zero measure $\mu \in M_0(H)$ and a regularly varying function v of index $-\beta$ such that
$$\lim_{t\to\infty} v(t)\nu(tA) = \mu(A) \qquad \text{for } A \in \mathcal{B}(H), 0 \notin \bar{A}.$$

For properties of regularly and slowly varying functions we refer to Appendix A.6.

We fix from now on

- a symmetric, regularly varying measure $\nu \in M_0(H)$ with index $\alpha = -\beta \in (0,2)$ and limiting measure $\mu \in M_0(H)$.

By Theorem A.41 this means that there is a slowly varying function $\ell : (0,\infty) \to (0,\infty)$ such that for all $A \in \mathcal{B}(H)$
$$\nu(tA) = t^{-\alpha}\ell(t)\mu(A).$$

In particular
$$\beta_\varepsilon = \nu\left(\frac{1}{\varepsilon^\rho} B_1^c(0)\right) = \varepsilon^{\alpha\rho}\ell\left(\frac{1}{\varepsilon^\rho}\right)\mu(B_1^c(0)). \tag{2.5}$$

Example 2.7. The Lévy measure ν for an α-stable process in a Hilbert space H is introduced via spherical coordinates and has the shape

$$\nu(\mathrm{d}x) = \sigma(\mathrm{d}s)\frac{\mathrm{d}r}{r^{1+\alpha}},$$

where $s = x/\|x\|$ and $r = \|x\|$, and σ is an arbitrary finite measure on the unit sphere (see e.g. Araujo and Giné [1979]). It is regularly varying with index $-\alpha$ with $\alpha \in (0,2)$. For $t > 0$ and $A \in \mathcal{B}(H)$ with $0 \notin \bar{A}$ we may calculate by substitution

$$\nu(tA) = \int\limits_{tA} \nu(\mathrm{d}y) = \int\limits_{tA} \sigma(s)\frac{\mathrm{d}r}{r^{1+\alpha}} = \int\limits_{A} \sigma(s')\frac{t\mathrm{d}r'}{(tr')^{1+\alpha}} = \nu(A)t^{-\alpha}.$$

Hence in this case $\mu = \nu$ and $\ell = 1$.

Crucial Features of the Deterministic System: A main feature of our system is the continuous dependence on individual "large" jumps of the noise. This means that "large" jumps of the noise can be treated as "large" jumps of the solution, while "small" ones do not perturb the deterministic solution by much. This interplay between large jumps and the behavior of the deterministic solution will be seen to provide the key for understanding the dynamics of our system. Existence, uniqueness and regularity results for a large class of deterministic reaction-diffusion equations are well known for a long time. We summarize them quoting Temam [1992], p.84. The deterministic Chafee-Infante equation is given by

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial \zeta^2} + \lambda(u^3 - u) = 0, && t > 0, \zeta \in [0,1], \\ &u(t,0) = u(t,1) = 0, && t > 0, \\ &u(0,\zeta) = x(\zeta), && \zeta \in [0,1]. \end{aligned} \tag{2.6}$$

For its solutions we write $u = (u(t))_{t\geqslant 0}$ resp. $(u(t;x))_{t\geqslant 0}$ if we wish to emphasize the initial state $x \in H$. For convenience of notation, integrating a function $v \in L^1(0,1)$ in $\zeta \in [0,1]$ we often write $\int_0^1 v d\zeta$, omitting the integration parameter ζ.

It is well-known that the solution flow $(t,x) \mapsto u(t;x)$ is continuous in t and x and defines a dynamical system in H. Furthermore the solutions are extremely regular for any positive time, i.e. $u(t) \in \mathcal{C}^\infty(0,1)$ for $t > 0$. The (compact) attractor of the Chafee-Infante equation is explicitly known to be contained in the unit ball with respect to the norm $|\cdot|_\infty$ (see for instance Eden et al. [1994], Chapter 5.6 and Temam [1992]). Since the attractor is absorbing for bounded sets, we obtain the following result.

Proposition 2.8. *For any Chafee-Infante parameter $\lambda > 0$ and any $r > 0$ there is a uniform time $T^r_{rec}(\lambda) > 0$, such that for all $t > T^r_{rec}(\lambda)$*

$$\sup_{x\in H} |u(t;x)|_\infty \leqslant 1 + r. \tag{2.7}$$

In Eden et al. [1994], the weaker boundedness result

$$\sup_{x\in H} |u(t;x)|_\infty < \sqrt{2}$$

for any $t \geqslant T_{rec}(\lambda) > 0$ is proved, see a more detailed discussion in Appendix B.2. The proof of this result will be of particular importance to us, since it will provide an argument to find uniform bounds on $\int_0^t |u(s;x)|_\infty^2 ds$, see Chapter 3 and Appendix B.3. This property implies that the polynomial nonlinearity becomes uniformly Lipschitz in finite time. Let us next give a more precise description of the global attractor $\mathcal{A}^\lambda$ of the Chafee-Infante equation. Its shape depends on the parameter λ and has the following structure

$$\mathcal{A}^\lambda = \mathcal{E}^\lambda \cup \bigcup_{v\in\mathcal{E}^\lambda} \mathcal{M}^u(v), \tag{2.8}$$

where $\mathcal{E}^\lambda$ is the set of fixed points and $\mathcal{M}^u(v)$ the unstable manifold of $v \in \mathcal{E}^\lambda$. We define for $v, w \in \mathcal{E}^\lambda$ the set of complete connecting orbits

$$\begin{aligned} C(v,w) := \{x \in H \mid \exists\ (u(t))_{t\in\mathbb{R}} \text{ solution of equation (2.6) in } H \text{ such that} \\ \exists\, t_0 \in \mathbb{R}:\ x = u(t_0) \text{ and } \lim_{t\to\infty} u(t;x) = w \text{ and } \lim_{t\to-\infty} u(t;x) = v\}, \end{aligned} \tag{2.9}$$

unless it is non-empty. If such an orbit does not exist we set $C(v,w) = \emptyset$. For convenience we introduce the notation for $v, w \in \mathcal{E}^\lambda, v \neq w$

$$v \to w \qquad :\Longleftrightarrow \qquad C(v,w) \neq \emptyset.$$

In addition, for any $v \in \mathcal{E}^\lambda$ we have

$$\mathcal{M}^u(v) = \bigcup_{\substack{w\in\mathcal{E}^\lambda \\ v\to w}} C(v,w).$$

Proposition 2.9. *For any $\lambda > 0$ and initial value $x \in H$ there exists a stationary state $\psi \in \mathcal{E}^\lambda$ of the system (2.6) such that*

$$\lim_{t\to\infty} u(t;x) = \psi.$$

This relies on the fact that there is an energy functional, which may serve as a Lyapunov function for the system. A proof can be found in Faris and Jona-Lasinio [1982b] and Henry [1983].

Proposition 2.10 (Morse-Smale property of fixed points)**.** *For the Chafee-Infante equation with $\pi^2 < \lambda \neq (k\pi)^2, k \in \mathbb{N}$, all fixed points in $\mathcal{E}^\lambda$ are hyperbolic, and the stable and the unstable manifolds of any unstable fixed point $\psi \in \mathcal{E}^\lambda$ intersect transversally.*

A proof is given in Henry [1985]. See also Appendix B.2.1.

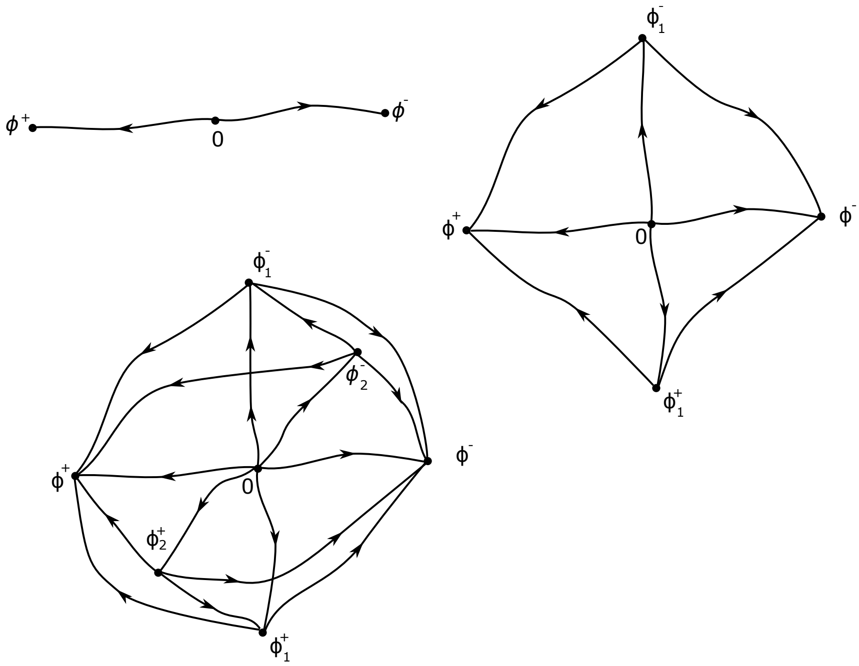

Figure 2.1.: Sketch of $\mathcal{A}^\lambda$ for $\lambda \in (\pi^2, (2\pi)^2)$, $\lambda \in ((2\pi)^2, (3\pi)^2)$, $\lambda \in ((3\pi)^2, (4\pi)^2)$

Definition 2.11. For $\lambda > \pi^2$ the solution of system (2.6) has two stable stationary solutions, which we shall denote throughout by ϕ^+ and ϕ^-. The full domains of attraction are denoted by

$$D^{\pm} := \{x \in H \mid \lim_{t\to\infty} u(t;x) = \phi^{\pm}\},$$

and the separatrix by

$$\mathcal{S} := H \setminus \left(D^+ \cup D^-\right).$$

We use the symbol $\pm$ whenever we can choose simultaneously for all those symbols either $+$ or $-$. In this sense we define the reshifted domains by

$$D_0^{\pm} := D^{\pm} - \phi^{\pm}.$$

Due to the Morse-Smale property the separatrix is a closed $\mathcal{C}^1$-manifold without boundary in H of dimension 1 separating D^+ from D^-. All unstable fixed points lie in $\mathcal{S}$. See Raugel [2002].

We fix from now on

- the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ in equation (2.6).

Definition 2.12. Writing $B_\delta(x)$ for the ball of radius $\delta > 0$ in H with respect to the

$|\cdot|_\infty$–norm centered at x, denote for $\delta_1, \delta_2, \delta_3 \in (0,1)$

$$\begin{aligned} D^{\pm}(\delta_1) :=& \{x \in D^{\pm} \mid \cup_{t\geqslant 0} B_{\delta_1}(u(t;x)) \subset D^{\pm}\}, \\ D^{\pm}(\delta_1,\delta_2) :=& \{x \in D^{\pm} \mid \cup_{t\geqslant 0} B_{\delta_2}(u(t;x)) \subset D^{\pm}(\delta_1)\}, \\ D^{\pm}(\delta_1,\delta_2,\delta_3) :=& \{x \in D^{\pm} \mid \cup_{t\geqslant 0} B_{\delta_3}(u(t;x)) \subset D^{\pm}(\delta_1,\delta_2)\}. \end{aligned} \tag{2.10}$$

For $\gamma \in (0,1)$ the sets $\tilde{D}^{\pm}(\varepsilon^\gamma) := D^{\pm}(\varepsilon^\gamma, \varepsilon^{2\gamma})$ and $D^{\pm}(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})$ will be of particular importance to our analysis.
Analogously we define the reshifted domains of attraction

$$\begin{aligned} D_0^{\pm}(\delta_1) :=& D^{\pm}(\delta_1) - \phi^{\pm}, \\ D_0^{\pm}(\delta_1,\delta_2) :=& D^{\pm}(\delta_1,\delta_2) - \phi^{\pm}, \\ D_0^{\pm}(\delta_1,\delta_2,\delta_3) :=& D^{\pm}(\delta_1,\delta_2,\delta_3) - \phi^{\pm}, \end{aligned}$$

with the particularly important $\tilde{D}_0^{\pm}(\varepsilon^\gamma) = D_0^{\pm}(\varepsilon^\gamma, \varepsilon^{2\gamma})$, and $D_0^{\pm}(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})$, and the following neighborhood of the separatrix $\mathcal{S}$

$$\tilde{D}^0(\varepsilon^\gamma) := H \setminus \left(\tilde{D}^{+}(\varepsilon^\gamma) \cup\ \tilde{D}^{-}(\varepsilon^\gamma)\right).$$

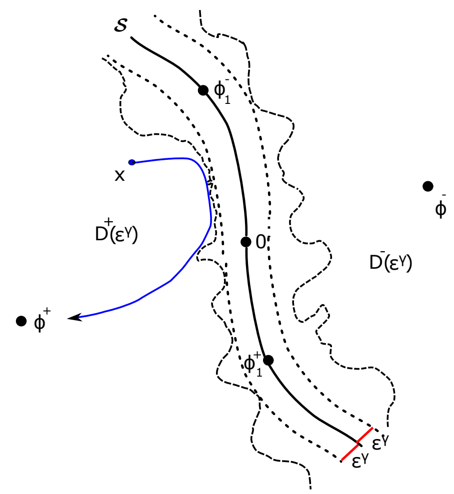

Figure 2.2.: The reduced domains of attraction $D^{\pm}(\varepsilon^\gamma)$

For consistency we need the following elementary but non-trivial lemma.

Lemma 2.13. *For any $\gamma \in (0,1)$ we have*

1. $D^{\pm} = \bigcup_{\varepsilon>0} D^{\pm}(\varepsilon^\gamma)$,
2. $D^{\pm} = \bigcup_{\varepsilon>0} \tilde{D}^{\pm}(\varepsilon^\gamma)$,
3. $D^{\pm} = \bigcup_{\varepsilon>0} D^{\pm}(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})$.

The proof is given in Appendix B.1. The crucial property of $\tilde{D}^{\pm}(\varepsilon^{\gamma})$ is related to the positive invariance under the deterministic solution flow.

Lemma 2.14. *The reduced domains of attraction $D^{\pm}(\varepsilon^{\gamma})$ and $D^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})$ are positively invariant under the deterministic flow, and the following relations are valid with respect to the $|\cdot|_{\infty}$-norm in H:*

$$\tilde{D}^{\pm}(\varepsilon^{\gamma}) + B_{\varepsilon^{2\gamma}}(0) \subset D^{\pm}(\varepsilon^{\gamma}),$$
$$D^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) + B_{\varepsilon^{2\gamma}}(0) \subset \tilde{D}^{\pm}(\varepsilon^{\gamma}).$$

Proof.

1. If $x \in D^{\pm}(\varepsilon^{\gamma})$, by definition $\cup_{t\geqslant 0} B_{\varepsilon^{\gamma}}(u(t,x)) \subset D^{\pm}$. Hence for $s \geqslant 0$

   $$\cup_{t\geqslant 0} B_{\varepsilon^{\gamma}}(u(t,u(s,x)) = \cup_{t\geqslant 0} B_{\varepsilon^{\gamma}}(u(s+t,x)) = \cup_{t\geqslant s} B_{\varepsilon^{\gamma}}(u(t,x)) \subset D^{\pm}.$$

   This proves that $u(s,x) \in D^{\pm}(\varepsilon^{\gamma})$, hence that $D^{\pm}(\varepsilon^{\gamma})$ is positively invariant.

2. If $x \in D^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})$, again by definition $\cup_{t\geqslant 0} B_{\varepsilon^{2\gamma}}(u(t,x)) \subset D^{\pm}(\varepsilon^{\gamma})$. Hence for $s \geqslant 0$

   $$\cup_{t\geqslant 0} B_{\varepsilon^{2\gamma}}(u(t,u(s,x)) = \cup_{t\geqslant 0} B_{\varepsilon^{2\gamma}}(u(s+t,x)) = \cup_{t\geqslant s} B_{\varepsilon^{2\gamma}}(u(t,x)) \subset D^{\pm}(\varepsilon^{\gamma}).$$

   This proves that $u(s,x) \in D^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})$, and therefore that $D^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})$ is positively invariant as well.

3. If $x \in D^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})$ and $y \in B_{\varepsilon^{2\gamma}}(0)$, then by definition

   $$B_{\varepsilon^{2\gamma}}(x) = B_{\varepsilon^{2\gamma}}(u(0,x)) \subset \cup_{t\geqslant 0} B_{\varepsilon^{2\gamma}}(u(t,x)) \subset D^{\pm}(e^{\gamma}).$$

4. Again by definition $D^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) + B_{\varepsilon^{2\gamma}}(0) \subset D^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})$.

□

The following theorem about the deterministic dynamics on the reduced domain of attraction is fundamental for our purposes.

Proposition 2.15. *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ be given. Then there exists an independent finite time $T_{rec} > 0$ and a constant $\kappa > 0$ such that for each $\gamma > 0$ there is $\varepsilon_0 = \varepsilon_0(\gamma) > 0$, such that for all $0 < \varepsilon \leqslant \varepsilon_0$, $t \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|$ and $x \in D^{\pm}(\varepsilon^{\gamma})$*

$$|u(t;x) - \phi^{\pm}|_{\infty} \leqslant (1/2)\varepsilon^{2\gamma}.$$

This means roughly, that as long as the system does not start in an ε^{γ}-neighborhood of the separatrix, it takes a time of only logarithmic order of magnitude in $\varepsilon > 0$ to reach a very small neighborhood of a stable state. In Appendix B.2.2 we prove a slightly stronger version of this result.

2.2. The Main Results

We shall now define our principal object of study, exit times from domains of attraction of the dynamical system generated by the Chafee-Infante equation of the preceding Section.

Definition 2.16. Let the conventions of the Section 2.1 be valid. For $\gamma \in (0,1)$, $\varepsilon > 0$ and $X^\varepsilon(\cdot;x)$ the càdlàg mild solution of (1.2), with initial position $x \in \tilde{D}^\pm(\varepsilon^\gamma)$ we define the *first exit time from the reduced domain of attraction*

$$\tau_x^\pm(\varepsilon) := \inf\{t > 0 \mid X^\varepsilon(t;x) \notin D^\pm(\varepsilon^\gamma)\}.$$

For $x \in D^\pm$ we define the *first exit time from the entire domain of attraction*

$$\hat{\tau}_x^\pm(\varepsilon) := \inf\{t > 0 \mid X^\varepsilon(t;x) \notin D^\pm\}.$$

For $x \in \tilde{D}^\pm(\varepsilon^\gamma)$ we define the *first entrance time of a neighborhood of the opposite stable state* $\phi^\pm$

$$\sigma_x^\pm(\varepsilon) := \inf\{t > 0 \mid X^\varepsilon(t;x) \in B_{\varepsilon^{2\gamma}}(\phi^\mp)\}.$$

For $\varepsilon > 0$ and $x \in \tilde{D}^0(\varepsilon^\gamma)$ let

$$\tau_x^0(\varepsilon) := \inf\{t > 0 \mid X^\varepsilon(t;x) \notin \tilde{D}^0(\varepsilon^\gamma)\}$$

be the *first exit time from the neighborhood of the separatrix* $\tilde{D}^0(\varepsilon^\gamma)$.

In order to obtain non-trivial and non-degenerate exit and transition behavior of our system we have to impose the following hypothesis on the limiting jump measure μ of the regularly varying Lévy measure ν defined in the previous Section.

Hypotheses: Let the conventions of the Section 2.1 be valid.

(H.1) Non-trivial transitions:

$$\mu\left(\left(D_0^\pm\right)^c\right) > 0.$$

This condition excludes that the system remains in one domain of attraction.

(H.2) Non-degenerate limiting measure: *For $\alpha \in (0,2)$ and $\Gamma > 0$ chosen large enough according to Proposition 3.1 and Remark 4.5 let*

$$0 < \Theta < \frac{2-\alpha}{2\alpha}, \quad \rho \in \left(\frac{1}{2}, \frac{2-\alpha}{2-(1-\Theta)\alpha}\right), \quad 0 < \gamma < \frac{(2-\alpha)(1-\rho) - \Theta\alpha\rho}{2(\Gamma+2)}. \qquad (2.11)$$

For each $k = \pm$ and any $\eta > 0$ we can choose $\varepsilon_0 > 0$ small enough such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$\mu\left(H \setminus \left((D^+(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) \cup D^-(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})) + B_{\varepsilon^{2\gamma}}(0)\right) - \phi^k\right) < \eta. \qquad (2.12)$$

This property expresses that the limiting measure μ centered at the stable points does not have too much of its mass concentrated near the separatrix. Hypothesis **(H.2)** implies a sequence of more sophisticated, but slightly less restrictive estimates of similar type. They can be found in Lemma 4.3.

(H.3) Restriction on large jumps close to the separatrix: *Let γ be given according to (2.11). There is $\gamma/2 < \tilde{\gamma} \leqslant \gamma$ and $r > 0$ such that*

$$\lim_{\varepsilon \to 0+} \sup_{x \in B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)} \frac{\nu\left(\frac{1}{\varepsilon^{1-\tilde{\gamma}}} B_1^c(0) \cap \frac{1}{\varepsilon}\left(\tilde{D}^0(\varepsilon^\gamma) - x\right)\right)}{\nu\left(\frac{1}{\varepsilon^{1-\tilde{\gamma}}} B_1^c(0)\right)} = 0.$$

This condition stipulates that the probability for large jumps from positions inside $\tilde{D}^0(\varepsilon^\gamma)$ bounded by $1+r$ to $\tilde{D}^0(\varepsilon^\gamma)$ tends to zero with ε for some parameter $r > 0$. A detailed discussion of Hypothesis (H.3) with the explanation of the precise choice of $\tilde{\gamma}$ will be given in Section 6.1.

The solution X^ε of equation (1.2) defines a process in the state space H. Before we state our main results on the asymptotic behavior of its exit times and its metastable behavior, let us define the relevant ε–dependent rates.

Definition 2.17. We define the asymptotic weight the tail of the jump measure attributes to the reshifted domains of attraction by

$$\lambda^\pm(\varepsilon) := \nu\left(\frac{1}{\varepsilon}\left(D_0^\pm\right)^c\right), \quad \varepsilon > 0,$$

and the critical time scale for metastable behavior by

$$\lambda^0(\varepsilon) := \nu\left(\frac{1}{\varepsilon} B_1^c(0)\right), \quad \varepsilon > 0.$$

Recall the closely related scale of the intensity of the large jumps

$$\beta_\varepsilon = \nu\left(\frac{1}{\varepsilon^\rho} B_1^c(0)\right), \quad \varepsilon > 0.$$

The scales thus defined increase polynomially in ε in the limit $\varepsilon \to 0+$, since ν is regularly varying. More precisely, according to Theorem A.41 with a slowly varying ℓ for any $\varepsilon > 0$ we have

$$\begin{aligned}
\lambda^\pm(\varepsilon) &= \varepsilon^\alpha\, \ell(1/\varepsilon)\, \mu\left((D_0^\pm)^c\right), \\
\lambda^0(\varepsilon) &= \varepsilon^\alpha\, \ell(1/\varepsilon)\, \mu\left(B_1^c(0)\right), \\
\beta_\varepsilon &= \varepsilon^{\alpha\rho}\, \ell(1/\varepsilon^\rho)\, \mu\left(B_1^c(0)\right).
\end{aligned} \tag{2.13}$$

1. Asymptotic Exit Times: The first group of results describes the asymptotic behavior of the first exit times from reduced and the entire domains of attraction of the stable states of our system.

Theorem 2.18 (Exponential convergence of first exit times from $D^{\pm}(\varepsilon^{\gamma})$)**.** *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ be given and suppose Hypotheses **(H.1)** and **(H.2)** are satisfied. Then there is family of random variables $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ with exponential law of parameter 1 such that for all $\theta < 1$*

$$\lim_{\varepsilon \to 0+} \mathbb{E}\big[\sup_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} |\exp\big(\theta\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon)\big) - \exp\left(\theta\bar{\tau}(\varepsilon)\right)|\big] = 0.$$

This implies that the first exit times are of asymptotic order $1/\lambda^{\pm}(\varepsilon) \approx 1/\varepsilon^{\alpha}$ and therefore increase polynomially in the noise parameter as $\varepsilon \to 0+$. This strongly contrasts the behavior known for the Wiener case from Brassesco [1991] and Faris and Jona-Lasinio [1982b], and extends the results of Imkeller and Pavlyukevich [2006a], Imkeller and Pavlyukevich [2006b] and Imkeller and Pavlyukevich [2008] to the case of infinite dimensional systems. The Theorem will be shown in Chapter 4.

If we assume in addition Hypothesis (H.3), we obtain even uniformity for all initial values in $x \in D^{\pm}$ for the first exit time $\hat{\tau}_x^{\pm}(\varepsilon)$ of the entire domain of attraction $D^{\pm}$. This result does not refer to the reduced domains of attraction $D^{\pm}(\varepsilon^{\gamma})$ or $\tilde{D}^{\pm}(\varepsilon^{\gamma})$.

Theorem 2.19 (Asymptotic estimate for the first exit time from $D^{\pm}$)**.** *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ be given and assume that Hypotheses **(H.1)**, **(H.2)** and **(H.3)** are satisfied. Then there is a family of exponentially distributed random variables $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ such that for all $h > 0$*

$$\lim_{\varepsilon \to 0+} \sup_{x \in D^{+}} \mathbb{P}(\lambda^{\pm}(\varepsilon)\hat{\tau}_x^{\pm}(\varepsilon) \geqslant \bar{\tau}(\varepsilon) + h) = 0.$$

It is natural that we only obtain an inequality, since by Hypothesis (H.3) the system is forced to leave the region around the separatrix and hence the boundary of the domains of attraction at most at a shorter rate, than the exit times of the reduced domain of Theorem 2.18. But it is not clear whether this happens inside or out of the reduced domain of attraction. In the first case the regime of Theorem 2.18 is attained, in the latter case the exit time equals the time to leave the boundary region at to the shorter rate (than $\lambda^{\pm}(\varepsilon)$). The Theorem is shown in Section 6.2.

2. Asymptotic Transition Times: We are next interested in times needed to transit from small neighborhoods of stable states $\phi^{\pm}$ to small neighborhoods of the opposite stable state $\phi^{\mp}$. A consequence of the Theorem 2.18 is that, starting in a small ball around $\phi^{\pm}$, the time needed to enter a small ball around the opposite stable state is of the order of the first exit time. Since the result in Proposition 3.1 contains a statement that holds only in probability, we cannot expect a convergence result for transition times as strong as the result of Theorem 2.18 for exit times. Since asymptotically transitions are caused by "large" ups we only need Hypothesis (H.1) and (H.2) but not (H.3).

Instead, since the result in Proposition 3.1 is only in probability we cannot expect anything stronger.

Theorem 2.20 (Asymptotic transitions between balls around the stable states)**.** *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ be given and assume that Hypotheses (**H.1**) and (**H.2**) are satisfied. Then there is a family of exponentially distributed random variables $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ with parameter 1 and $h_0 > 0$ such that for $0 < h \leqslant h_0$*

$$\lim_{\varepsilon\to 0+} \mathbb{E}\left[\sup_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}\{|\lambda^{\pm}(\varepsilon)\sigma_x^{\pm}(\varepsilon) - \bar{\tau}(\varepsilon)| > h\}\right] = 0.$$

The Theorem is shown in Chapter 5.

3. Localization on Subcritical and Critical Time Scales: If $0 < \delta < \alpha$ and we consider the entire process $(X^{\varepsilon}(t/\varepsilon^{\delta}))_{t\in[0,T]}$ for fixed $T > 0$ it should converge for $\varepsilon \to 0+$ to the process taking the constant value given by the stable state in the domain of attraction where it started. This can be justified, since the relaxation time of order $T_{rec} + \kappa\gamma|\ln\varepsilon|$ of the small jump solution Y^{ε} of (2.3) to the stable state is clearly dominated by $\frac{1}{\varepsilon^{\delta}}$, but the first exit time $\tau_x^{\pm}(\varepsilon)$ of expected order $\frac{1}{\lambda^{\pm}(\varepsilon)} \approx \frac{1}{\lambda^{0}(\varepsilon)} \approx \frac{1}{\varepsilon^{\alpha}}$ is not yet reached. However, this is only true if we avoid initial values close to the separatrix $\mathcal{S}$. This is the infinite-dimensional analogon to a result of Imkeller and Pavlyukevich [2008].

Theorem 2.21 (Localization on subcritical time scales in $\tilde{D}^{\pm}(\varepsilon^{\gamma})$)**.** *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ be given, $T_{rec}, \kappa > 0$ given by Proposition 2.15 and assume that Hypotheses (**H.1**) and (**H.2**) are satisfied. Fix $0 < \delta < \alpha$. Then there is $h_0 > 0$ such that for $0 < h \leqslant h_0$ and for any $T > 0$*

$$\lim_{\varepsilon\to 0+} \mathbb{E}\left[\sup_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} \sup_{t\in[T_{rec}+\kappa\gamma|\ln\varepsilon|, T/\varepsilon^{\delta}]} \mathbf{1}\{|X^{\varepsilon}(t;x) - \phi^{\pm}|_{\infty} > h\}\right] = 0. \tag{2.14}$$

If we assume in addition that Hypothesis (H.3) is fulfilled, the process leaves the separatrix with high probability before times of the order $\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}$. Then we obtain a result of the type of Theorem 2.21 uniformly for all initial values in H and time scales including the critical time scale $\frac{T}{\lambda^{0}(\varepsilon)}$. Close to the separatrix we cannot decide to which domain of attraction the process tends while apart from it the previous reasoning of Theorem 2.21 continues to hold. The result is a uniform localization theorem in space.

Theorem 2.22 (Uniform localization on subcritical and critical time scales in H)**.** *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ be given and assume that Hypotheses (**H.1**), (**H.2**) and (**H.3**) are satisfied. Then there is $h_0 > 0$ such that for all $T > 0$ and $0 < h \leq h_0$*

$$\lim_{\varepsilon\to 0+} \sup_{x\in H} \mathbb{E}\left[\sup_{t\in[\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}, \frac{T}{\lambda^{0}(\varepsilon)}]} \mathbf{1}\{X^{\varepsilon}(t;x) \in B_h(\phi^{+}) \cup B_h(\phi^{-})\}\right] = 1.$$

The Theorems 2.21 and 2.22 will be shown in Section 6.3.

4. Metastable Behavior: We shall exploit Theorem 2.22 in order to obtain the following principal result of this thesis. It claims a convergence of X^ε running in the critical time scale $\frac{1}{\lambda^0(\varepsilon)}$ in terms of finite dimensional distributions to the reduced dynamics of a Markov chain jumping between the stable states ϕ^+ and ϕ^- back and forth.

Definition 2.23. For $T > 0$ we shall denote a (finite) *partition of* $[0, T]$ into $n \in \mathbb{N}$ time points as a finite family $\pi = (t_1, \ldots, t_n)$ of points in $[0, T]$, with $0 < t_1 < \cdots < t_n = T$, and write $|\pi| = n$. We denote by $\Pi[0, T]$ the collection of all finite partitions in $[0, T]$. For convenience we write

$$X^\varepsilon(\pi; \cdot) := (X^\varepsilon(t_1, \cdot), \ldots, X^\varepsilon(t_n; \cdot))$$

for $\pi \in \Pi[0, T]$ and $\varepsilon > 0$, and respectively for a process $(Y(t; \cdot))_{t \geqslant 0}$ defined below

$$Y(\pi; \cdot) = (Y(t_1; \cdot), \ldots, Y(t_n; \cdot)).$$

For $h > 0$ and $\bar{v} = (v_1, \ldots, v_n) \in \{\phi^+, \phi^-\}^{|\pi|}$ let

$$B_h(\bar{v}) = B_h(v_1) \times \cdots \times B_h(v_n).$$

Theorem 2.24 (Metastability)**.** *Let the Chafee-Infante parameter* $\pi^2 < \lambda \neq (k\pi)^2$ *for* $k \in \mathbb{N}$ *be given and denote by* μ *the limiting measure of* ν *according to Definition (2.6). Assume that Hypotheses* ***(H.1)***, ***(H.2)*** *and* ***(H.3)*** *are satisfied. Then there exists a continuous time Markov chain* $(Y(t))_{t \geqslant 0}$ *switching between the elements of* $\{\phi^+, \phi^-\}$ *with generating matrix*

$$Q = \frac{1}{\mu(B_1^c(0))} \begin{pmatrix} -\mu\left(\left(D_0^+\right)^c\right) & \mu\left(\left(D_0^+\right)^c\right) \\ \mu\left(\left(D_0^-\right)^c\right) & -\mu\left(\left(D_0^-\right)^c\right) \end{pmatrix}$$

which satisfies the following. There is $h_0 > 0$ *such that for all* $T > 0$, $\Pi[0, T]$ *and* $0 < h \leqslant h_0$

$$\lim_{\varepsilon \to 0+} \sup_{x \in \tilde{D}^\pm(\varepsilon^\gamma)} \left| \mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)}; x) \in B_h(\bar{v})\big) - \mathbb{P}\big(Y(\pi, x) = \bar{v}\big) \right| = 0.$$

This is an analogous result to Imkeller and Pavlyukevich [2008]. We can extend this slightly for uniform initial values, but we have to pay a price in terms of a "flip" close to the separatrix, which determines asymptotically to which side solutions, that start on the separatrix will tend.

Theorem 2.25 (Uniform Metastability)**.** *Suppose for* $k \in \mathbb{N}$ *the Chafee-Infante parameter* $\pi^2 < \lambda \neq (k\pi)^2$ *being given and denote by* μ *the limiting measure of* ν *according to Definition (2.6). Assume that Hypotheses* ***(H.1)***, ***(H.2)*** *and* ***(H.3)*** *are satisfied. Then there exists a continuous time Markov chain* $(Y(t; x))_{t \geqslant 0}$ *starting in* $\phi^\pm$ *if* $x \in D^\pm$ *and* *switching between the elements of* $\{\phi^+, \phi^-\}$ *with generating matrix*

$$Q = \frac{1}{\mu(B_1^c(0))} \begin{pmatrix} -\mu\left(\left(D_0^+\right)^c\right) & \mu\left(\left(D_0^+\right)^c\right) \\ \mu\left(\left(D_0^-\right)^c\right) & -\mu\left(\left(D_0^-\right)^c\right) \end{pmatrix}$$

and random intial condition

$$\Phi^\varepsilon(x) = \phi^\pm, \begin{cases} \phi^+ & \text{if } x \in D^+ \text{ or } x \in \mathcal{S} \text{ with } X^\varepsilon(\tau_x^0(\varepsilon); x) \in D^+ \\ \phi^- & \text{if } x \in D^- \text{ or } x \in \mathcal{S} \text{ with } X^\varepsilon(\tau_x^0(\varepsilon); x) \in D^- \end{cases}$$

and $h_0 > 0$ *such that for all* $T > 0$, $\pi \in \Pi[0,T]$, $\bar{v} \in \{\phi^+, \phi^-\}^{|\pi|}$ *and* $0 < h \leq h_0$

$$\lim_{\varepsilon \to 0+} \sup_{x \in H} \left| \mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)}; x) \in B_h(\bar{v})\big) - \mathbb{P}\big(Y(\pi, \Phi^\varepsilon(x)) = \bar{v}\big) \right| = 0.$$

The Theorems 2.24 and 2.25 will be proved in Section 6.4.

3. The Small Deviation of the Small Noise Solution

In this Chapter we shall consider the solution Y^ε of the SPDE (2.3), consisting of the deterministic Chafee-Infante equation perturbed by just the small jump part ξ^ε of our Lévy process L. We will show that with probability converging to 1 as $\varepsilon \to 0+$ the maximal deviation of Y^ε from the deterministic solution u on the time interval before the first big jump T_1, given by $|Y^\varepsilon(t) - u(t)|_\infty$ is at most of order ε to some positive power. This result is crucial for determining the asymptotic behavior of the first exit time in Chapter 4, since it basically states that exits can arise only from big jumps.

Proposition 3.1. *There is a constant* $\Gamma > 0$ *such that for*

$$0 < \Theta < \frac{2-\alpha}{\alpha}, \qquad \rho \in (\frac{1}{2}, \frac{2-\alpha}{2-(1-\Theta)\alpha}) \qquad 0 < \gamma < \frac{(2-\alpha)(1-\rho)-\Theta\alpha\rho}{2(\Gamma+2)}$$

there exists $\vartheta = \vartheta(\Theta, \rho, \gamma, \alpha) > \alpha(1-\rho)$, $C_\vartheta > 0$ *and* $\varepsilon_0 > 0$, *such that for all* $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{P}\left(\exists\, x \in D^\pm(\varepsilon^\gamma) : \sup_{s\in[0,T_1]} |Y^\varepsilon(s;x) - u(s;x)|_\infty \geqslant (1/2)\varepsilon^{2\gamma}\right) \leqslant C_\vartheta \varepsilon^\vartheta.$$

The proof is done in two steps. First we prove the result for maximal deviations on a time interval bounded by a finite deterministic time horizon $T > 0$ in a series of Lemmas. In a second step we generalize it to the random first big jump time T_1 replacing T.

3.1. Small Deviation on Deterministic Time Intervals

Our uniform deviation estimate on deterministic time intervals is given by the following technical Proposition.

Proposition 3.2. *There is a constant* $\Gamma > 0$ *such that for* $0 < \alpha < 2$,

$$0 < \Theta < \frac{2-\alpha}{\alpha}, \qquad \rho \in (1/2, \frac{2-\alpha}{2-(1-\Theta)\alpha}) \qquad 0 < \gamma < \frac{(2-\alpha)(1-\rho)-\Theta\alpha\rho}{2(\Gamma+2)}$$

there exist $\varepsilon_0 > 0$ *and* $C > 0$ *such that for any* $T > 0$, $0 < \varepsilon \leqslant \varepsilon_0$ *and* $x \in D^\pm(\varepsilon^\gamma)$

$$\mathbb{P}\left(\sup_{s\in[0,T]} |Y^\varepsilon(s;x) - u(s;x)|_\infty \geqslant (1/2)\varepsilon^{2\gamma}\right) \leqslant C\, T\, \varepsilon^{2-2(\Gamma+2)\gamma-(2-(1-\Theta)\alpha)\rho}. \tag{3.1}$$

The proof will be a consequence of the combination of a number of Lemmas. Recalling our notation for the small jump part we define its stochastic convolution ξ^* by

$$\xi^*(t) = \int_0^t S(t-s)\mathrm{d}\xi^\varepsilon(s), \quad t \geqslant 0. \tag{3.2}$$

For further details consult Appendix A.3.

3.1.1. Small Deviation with Controlled Small Noise Convolution

In this Subsection we shall show in a series of Lemmas that the deviation of the small noise mild solution from the solution of the deterministic Chafee-Infante equation is small if the convolution of the small noise is uniformly controlled on finite deterministic time intervals.

Lemma 3.3. *For any $T_{rec} > 0, \kappa > 0$ there is a constant $\Gamma = \Gamma(\kappa) > 0$ such that for $\rho \in (1/2, 1)$, $K > 0$ and $\gamma > 0$ there exists $\varepsilon_0 = \varepsilon_0(K, T_{rec}, \gamma, \kappa) > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$, $x \in D^\pm(\varepsilon^\gamma)$, and $0 \leqslant T \leqslant T_{rec} + \kappa\gamma|\ln\varepsilon|$, the remainder process $R^\varepsilon(\cdot; x) := Y^\varepsilon(\cdot; x) - u(\cdot; x) - \varepsilon\xi^*(\cdot)$ satisfies*

$$\sup_{t\in[0,T]} |R^\varepsilon(t; x)|_\infty \leqslant \frac{1}{K}\varepsilon^{2\gamma}$$

on the event $\mathcal{E}_T(\varepsilon^{(\Gamma+2)\gamma}) := \{\sup_{r\in[0,T]} ||\varepsilon\xi^*(r)|| < \varepsilon^{(\Gamma+2)\gamma}\}$.

Proof. Let $x \in D^\pm(\varepsilon^\gamma)$. The process $R^\varepsilon(\cdot; x) := Y^\varepsilon(\cdot; x) - u(\cdot; x) - \varepsilon\xi^*(\cdot)$ for which we note briefly R^ε in the sequel satisfies the equation

$$\frac{\mathrm{d}R^\varepsilon}{\mathrm{d}t} = \Delta R^\varepsilon + f(Y^\varepsilon) - f(u).$$

We first aim at getting an estimate in $L^2(0,1)$. Multiplication with R^ε and integration by part yields [1]

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|R^\varepsilon|^2 + |\nabla R^\varepsilon|^2 = \langle f(Y^\varepsilon) - f(u), R^\varepsilon\rangle.$$

We may assume $\varepsilon_0 \leqslant 1$. Using the scalar identity

$$f(w) - f(z) = \lambda(w^2 + wz + z^2 - 1)(w - z),$$

we obtain for $0 < t < t^\infty := \inf\{t > 0 \mid |R^\varepsilon(t; x)|_\infty > 1\}$ and $\Gamma > 0$ to be specified later

[1] Here and below, computations are done in a formal way. They can be easily justified by Galerkin or Yosida approximations.

on $\mathcal{E}_T(1) \supset \mathcal{E}_T(\varepsilon^{(\Gamma+2)\gamma})$ the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}|R^\varepsilon|^2 \leqslant 2\lambda \int_0^1 \left((Y^\varepsilon)^2 + Y^\varepsilon u + u^2 - 1\right)(R^\varepsilon + \varepsilon\xi^*)R^\varepsilon \,\mathrm{d}\zeta$$

$$= 2\lambda \int_0^1 \left((R^\varepsilon + u + \varepsilon\xi^*)^2 + (R^\varepsilon + u + \varepsilon\xi^*)u + u^2 - 1\right)(R^\varepsilon + \varepsilon\xi^*)R^\varepsilon \,\mathrm{d}\zeta$$

$$\leqslant 12\lambda \int_0^1 \left((R^\varepsilon)^2 + u^2 + (\varepsilon\xi^*)^2 - 1\right)\left((R^\varepsilon)^2 + \varepsilon\xi^* R^\varepsilon\right) \,\mathrm{d}\zeta$$

$$\leqslant 24\lambda \int_0^1 \left((R^\varepsilon)^2 + u^2 + (\varepsilon\xi^*)^2 - 1\right)\left((R^\varepsilon)^2 + (\varepsilon\xi^*)^2\right) \,\mathrm{d}\zeta$$

$$\leqslant 24\lambda \left(|R^\varepsilon|_\infty^2 - 1 + |u|_\infty^2 + |\varepsilon\xi^*|_\infty^2\right) \int_0^1 \left((R^\varepsilon)^2 + (\varepsilon\xi^*)^2\right) \,\mathrm{d}\zeta$$

$$\leqslant 24\lambda \left(|u|_\infty^2 + 1\right)\left(|R^\varepsilon|^2 + \|\varepsilon\xi^*\|^2\right). \quad (3.3)$$

With $C_1 = 24\lambda$, Gronwall's Lemma for $t \in [0, t^\infty]$, Corollary B.11 and the fact that $R^\varepsilon(0) = 0$ lead to

$$|R^\varepsilon(t)|^2 \leqslant C_1 \int_0^t \left(|u(r)|_\infty^2 + 1\right)\|\varepsilon\xi^*(r)\|^2 \exp\left(C_1(t-r) + C_1 \int_r^t |u(\tau)|_\infty^2 \,\mathrm{d}\tau\right) \mathrm{d}r$$

$$\leqslant C_1 \int_0^t \left(|u(r)|_\infty^2 + 1\right)\|\varepsilon\xi^*(r)\|^2 \exp\left(\bar{K} + 3C_1(t-r)\right) \mathrm{d}r$$

$$\leqslant \sup_{r\in[0,t]} \|\varepsilon\xi^*(r)\|^2 \, C_2 \, (\bar{K} + 3t)\exp(3C_1 t), \quad (3.4)$$

where we set $C_2 = C_1 e^{C_1 K}$. In the mild solution representation of R^ε

$$R^\varepsilon(t) = \int_0^t S(t-s)\left(f(Y^\varepsilon(r)) - f(u(r))\right) \,\mathrm{d}r$$

we can use the regularizing effect of $S = (S(t))_{t\geqslant 0}$, the semigroup of the heat equation on $[0,1]$, formally $||S_t h|| \leqslant \frac{C_3}{\sqrt{t}}|h|$, $h \in L^2(0,1), t > 0$, Hölder's inequality and Corollary B.12 for $n = 12$ to obtain with further universal constants $C_3, \cdots, C_6$

$$
\begin{aligned}
&\|R^\varepsilon(t)\| \\
&\leqslant 6\lambda C_3 \int_0^t \frac{1}{(t-s)^{1/2}} \left|(R^\varepsilon)^2(s) + u^2(s) + (\varepsilon\xi^*)^2(s) - 1\right| \left(|R^\varepsilon(s)| + |\varepsilon\xi^*(s)|\right) \,\mathrm{d}s \\
&\leqslant 6\lambda C_3 \int_0^t \frac{1}{(t-s)^{1/2}} \left(2 + |u(s)|_\infty^2\right) \left(|R^\varepsilon(s)| + |\varepsilon\xi^*(s)|\right) \,\mathrm{d}s \\
&\leqslant 6\lambda C_3 \left(\int_0^t \frac{1}{(t-s)^{3/4}} \,\mathrm{d}s\right)^{\frac{2}{3}} \left(\int_0^t (2+|u(s)|_\infty^2)^6 \,\mathrm{d}s\right)^{\frac{1}{6}} \cdot \left(\int_0^t \left(|R^\varepsilon(s)| + |\varepsilon\xi^*(s)|\right)^6 \,\mathrm{d}s\right)^{\frac{1}{6}} \\
&\qquad\qquad \leqslant C_4 \left((\bar{K}_{12} + 64t)^{1/6}\right) \left(\int_0^t \left(|R^\varepsilon(s)| + |\varepsilon\xi^*(s)|\right)^6 \,\mathrm{d}s\right)^{\frac{1}{6}}.
\end{aligned}
$$

Hence we may insert (3.4) in the preceding expression and obtain

$$
\begin{aligned}
&\|R^\varepsilon(t)\| \leqslant C_4 \left((\bar{K}_{12} + 64t)^{1/6}\right) \\
&\cdot \left(\int_0^t \left(\sup_{r\in[0,s]} \|\varepsilon\xi^\varepsilon(r)\|\, C_2^{1/2} (\bar{K} + 3s)^{1/2} \exp\left(C_1 \frac{3}{2} s\right) + \sup_{r\in[0,s]} \|\varepsilon\xi^\varepsilon(r)\|\right)^6 \mathrm{d}s\right)^{1/6} \\
&\quad \leqslant C_4 \left((\bar{K}_{12} + 64t)^{1/6}\right) \left(C_2^{1/2} (\bar{K} + 3t)^{1/2} \exp\left(C_1 \frac{3}{2} t\right) + 1\right) t^{1/6} \sup_{s\in[0,t]} \|\varepsilon\xi^\varepsilon(r)\| \\
&\qquad \leqslant C_5 (1+t)^{1/6} (1+t)^{1/2} (1+t)^{1/6} \exp\left(C_1 \frac{3}{2} t\right) \sup_{s\in[0,t]} \|\varepsilon\xi^\varepsilon(r)\| \\
&\qquad\qquad \leqslant \sup_{r\in[0,t]} \|\varepsilon\xi^*(r)\| C_6 \;(1+t)\; \exp\left(\frac{3}{2} C_1 t\right).
\end{aligned}
$$

Hence for $\kappa > 0$ and $0 \leqslant T \leqslant t^\infty \wedge (T_{rec} + \kappa\gamma|\ln\varepsilon|)$ we obtain

$$
|R^\varepsilon(T)|_\infty \leqslant \sup_{r\in[0,T]} \|\varepsilon\xi^*(r)\| C_6 \;(T_{rec} + \kappa\gamma|\ln\varepsilon| + 1) \exp\left(\frac{3}{2} C_1 T_{rec}\right) \; \varepsilon^{-\frac{3}{2}\, C_1 \kappa\gamma}.
$$

Let $\Gamma := \left(\frac{3}{2} C_1 \kappa + 1\right)$ and fix $K > 0$. Then there exists $\varepsilon_0 > 0$ sufficiently small, such that for $0 < \varepsilon \leqslant \varepsilon_0$ on the event $\mathcal{E}_T(\varepsilon^{(\Gamma+2)\gamma})$ we have $t^\infty > T$ and

$$
\sup_{t\in[0,T]} |R^\varepsilon(t)|_\infty \leqslant \frac{1}{K} \varepsilon^{2\gamma}.
$$

□

Note that the result stays true for any $\tilde{\Gamma} \geqslant \Gamma$.

For a stable fixed point by $v \in \{\phi^+, \phi^-\}$ denote the exponent of stability $\Lambda > 0$ by the minimal number $\tilde{\Lambda}$ such that for all $w \in H$

$$\langle(\Delta + f'(v))w, w\rangle \leqslant -\tilde{\Lambda}|w|^2. \tag{3.5}$$

There is a universal $\delta_0 > 0$ such that for $v \in \{\phi^+, \phi^-\}$ we have $B_\delta(v)$ is positively invariant in $L^\infty(0,1)$ for all $\delta < \delta_0$ under the flow of the Chafee-Infante equation, see Lemma B.3 in Appendix B.2.3, or with a different method Matano [1979]. In the following Lemma we shall see that the remainder process R^ε can be controlled by the small noise part uniformly on finite deterministic time intervals, if the initial value is chosen inside these positive invariant sets.

Lemma 3.4. *Let Λ be the exponent of stability of the Chafee-Infante equation with parameter λ, $v \in \{\phi^+, \phi^-\}$, and $0 < \delta < \delta_0 \wedge \frac{1}{3}\left(\sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty\right)$. Then there is a constant $C > 0$ such that for all $\rho \in (1/2, 1)$, $x \in B_\delta(v)$, $0 < \varepsilon \leqslant 1$ and $0 \leqslant T$ we obtain*

$$\sup_{t\in[0,T]} |R^\varepsilon(t;x)|_\infty \leqslant C \sup_{r\in[0,T]} \|\varepsilon\xi^*(r)\|$$

on $\mathcal{E}_T\left(\frac{1}{3C}\left(\sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty\right)\right)$.

Proof. The proof has three parts.

1. Fix $\eta > 0$ and $x \in B_\delta(v)$ and denote $t_\eta^* := \inf\{t > 0 \mid |R^\varepsilon(t;x)|_\infty > \eta\}$. In this first part of the proof we shall show that there is a constant C_2 depending only on the geometric parameters $\Lambda, \lambda, |v|_\infty$ such that for $\eta \leq \frac{1}{3}\left(\sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty\right)$ and $t \leqslant t_\eta^*$ we have

$$\sup_{s\in[0,t]} |R^\varepsilon(s)| \leqslant C_2 \sup_{s\in[0,t]} \|\varepsilon\xi^*(s)\| \tag{3.6}$$

on $\mathcal{E}_t(\eta)$. In fact, using the mean value theorem in Hilbert spaces we calculate

$$\begin{aligned}
\frac{\mathrm{d}R^\varepsilon}{\mathrm{d}t} &= \Delta R^\varepsilon + (f(Y^\varepsilon) - f(u)) \\
&= \Delta R^\varepsilon + \left(\int_0^1 f'(u + \theta_1(R^\varepsilon + \varepsilon\xi^*))\,\mathrm{d}\theta_1\right)(R^\varepsilon + \varepsilon\xi^*) \\
&= \Delta R^\varepsilon + f'(v)R^\varepsilon + \left(\int_0^1 f'(u + \theta_1(R^\varepsilon + \varepsilon\xi^*)) - f'(v)\,\mathrm{d}\theta_1\right)R^\varepsilon \\
&\qquad + \left(\int_0^1 f'(u + \theta_1(R^\varepsilon + \varepsilon\xi^*))\,\mathrm{d}\theta_1\right)\varepsilon\xi^*.
\end{aligned} \tag{3.7}$$

We can continue using it a second time

$$\begin{aligned}\frac{\mathrm{d}R^\varepsilon}{\mathrm{d}t} &= \Delta R^\varepsilon + f'(v)R^\varepsilon \\ &\quad + \left(\iint_0^1 f''(v+\theta_2 u+\theta_2\theta_1(R^\varepsilon+\varepsilon\xi^*))\,(u-v+\theta_1 R^\varepsilon+\varepsilon\xi^*)\ \mathrm{d}\theta_2\mathrm{d}\theta_1\right)R^\varepsilon \\ &\quad + \left(\int_0^1 f'(u+\theta_1(R^\varepsilon+\varepsilon\xi^*))\ \mathrm{d}\theta_1\right)\varepsilon\xi^*. \qquad (3.8)\end{aligned}$$

Multiplying by R^ε and integrating in ζ we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|R^\varepsilon|^2 + \Lambda|R^\varepsilon|^2 &\leqslant \left(\iint_0^1 |f''(v+\theta_2 u+\theta_2\theta_1(R^\varepsilon+\varepsilon\xi^*))|_\infty\ \mathrm{d}\theta_2\ \mathrm{d}\theta_1\right) \\ &\quad \cdot \int_0^1 (|u-v|_\infty + |R^\varepsilon|_\infty + \|\varepsilon\xi^*\|)\,(R^\varepsilon)^2\ \mathrm{d}\zeta \\ &\quad + \left(\int_0^1 |f'(u+\theta_1(R^\varepsilon+\varepsilon\xi^*))|_\infty\ \mathrm{d}\theta_1\right)\int_0^1 |\varepsilon\xi^*|_{\mathbb{R}}|R^\varepsilon|_{\mathbb{R}}\ \mathrm{d}\zeta.\end{aligned}$$

Using now the stability of v which ensures that $\langle \Delta w + f'(v)w, w\rangle \leqslant -\Lambda|w|^2$ for $w \in L^2(0,1)$, we obtain for $0 \leqslant t \leqslant t^*_\eta$ and $a > 0$ to be chosen below

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|R^\varepsilon|^2 + \Lambda|R^\varepsilon|^2 \\ &\leqslant 6\lambda\Big(|v|_\infty + |u|_\infty + |R^\varepsilon|_\infty + \sup_{r\in[0,t]}\|\varepsilon\xi^*\|\Big)\Big(\delta+\eta+\sup_{r\in[0,t]}\|\varepsilon\xi^*(r)\|\Big)|R^\varepsilon|^2 \\ &\quad + 3\lambda\left(\Big(|u|_\infty + |R^\varepsilon|_\infty + \sup_{r\in[0,t]}\|\varepsilon\xi^*\|\Big)^2 + 1\right)\left(\frac{1}{a}\sup_{r\in[0,t]}\|\varepsilon\xi^*\|^2 + a|R^\varepsilon|^2\right).\end{aligned}$$

For the second inequality above we employ the positive invariance of $B_\delta(v)$. Further using this property with respect to the L^∞-norm we can write

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|R^\varepsilon|^2 + \Lambda|R^\varepsilon|^2 \\
&\quad \leqslant 6\lambda\Big((2|v|_\infty + \delta + \eta + \sup_{r\in[0,t]}\|\varepsilon\xi^*\|\Big)\Big(\delta + \eta + \sup_{r\in[0,t]}\|\varepsilon\xi^*(r)\|\Big)|R^\varepsilon|^2 \\
&\qquad + 18\lambda\Big((|v|_\infty + \delta)^2 + \eta^2 + \sup_{r\in[0,t]}\|\varepsilon\xi^*\|^2 + 1\Big)a|R^\varepsilon|^2 \\
&\qquad\quad + 18\lambda\Big((|v|_\infty + \delta)^2 + \eta^2 + \sup_{r\in[0,t]}\|\varepsilon\xi^*\|^2 + 1\Big)\frac{1}{a}\sup_{r\in[0,t]}\|\varepsilon\xi^*\|^2.
\end{aligned}$$

By the choice of δ and η it follows

$$\delta + \eta + \sup_{r\in[0,t]}||\varepsilon\xi^*(r)|| \leqslant \sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty.$$

If we choose in addition

$$a = \frac{\Lambda}{2}\frac{1}{36\lambda\Big((|v|_\infty + \delta)^2 + \eta^2 + \sup_{r\in[0,t]}\|\varepsilon\xi^*\|^2 + 1\Big)},$$

we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|R^\varepsilon|^2 \\
&\leqslant -2\Lambda|R^\varepsilon|^2 + \frac{1}{2}\Lambda|R^\varepsilon|^2 + \frac{1}{2}\Lambda|R^\varepsilon|^2 \\
&\qquad + \frac{72\lambda}{\Lambda}\Big((|v|_\infty + \delta)^2 + \eta^2 + \sup_{r\in[0,t]}\|\varepsilon\xi^*\|^2 + 1\Big)^2\sup_{r\in[0,t]}\|\varepsilon\xi^*\|^2 \\
&\qquad\qquad \leqslant -\Lambda|R^\varepsilon|^2 + C_1\sup_{r\in[0,t]}\|\varepsilon\xi^*\|^2.
\end{aligned}$$

Here $C_1 = C_1(|v|_\infty, \Lambda, \lambda)$ can be chosen as a constant depending on $|v|_\infty, \Lambda$, and λ if also both

$$\delta \vee \eta \leqslant \frac{1}{3}\left(\sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty\right) \tag{3.9}$$

$$\sup_{r\in[0,t]}||\varepsilon\xi^*(r)|| \leqslant \frac{1}{3}\left(\sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty\right) \tag{3.10}$$

as is the case on $\mathcal{E}_t(\frac{1}{3}(\sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty))$.

Gronwall's Lemma and $R^\varepsilon(0) = 0$ imply under these conditions for times $t \leqslant t_\eta^*$, with $\eta \leq \frac{1}{3}(\sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty)$ that

$$\sup_{s\in[0,t]} |R^\varepsilon(s;x)|^2 \leqslant \int_0^t C_1 \sup_{\tau\in[0,r]} \|\varepsilon\xi^*(\tau)\|^2 e^{-\Lambda(t-r)}\ \mathrm{d}r \leqslant C_2 \sup_{s\in[0,t]} \|\varepsilon\xi^*(s)\|^2$$

on $\mathcal{E}_t(\frac{1}{3}(\sqrt{|v|_\infty^2+\frac{\Lambda}{24\lambda}}-|v|_\infty))$, with $C_2 = \frac{C_1}{\Lambda}$.

2. We next sharpen the estimate obtained in the first part to an estimate in the $\|\cdot\|$–norm. To this end, we again use the regularizing effect of the semigroup of the heat equation on $[0,1]$, this time taking into account its smallest (positive) eigenvalue c_0. Similarly to Step 1 above we get estimates for $t \leqslant t^*_\eta$ with some constants C_3, C_4 depending only on $\Lambda, \lambda, |v|_\infty$. For convenience we drop the dependence on $x \in B_\delta(v)$.

$$\begin{aligned}
&\|R^\varepsilon(t)\| \\
&\leqslant C_3 \int_0^t \frac{e^{-(c_0/2)(t-r)}}{(t-r)^{1/2}} |f(Y^\varepsilon(r)) - f(u(r))|\ \mathrm{d}r \\
&\leqslant C_3 \int_0^t \frac{e^{-(c_0/2)(t-r)}}{(t-r)^{1/2}} \int_0^1 |f'(u(r)+\theta(R^\varepsilon(r)+\varepsilon\xi^*(r)))(R^\varepsilon(r)+\varepsilon\xi^*(r))|\ \mathrm{d}\theta\ \mathrm{d}r \\
&\leqslant C_3 \int_0^t \frac{e^{-(c_0/2)(t-r)}}{(t-r)^{1/2}} 3\lambda\Big((|v|_\infty+\delta+\eta+\sup_{\tau\in[0,r]}\|\varepsilon\xi^*(\tau)\|)^2+1\Big)\Big(|R^\varepsilon|+|\varepsilon\xi^*|\Big)\ \mathrm{d}r \\
&\leqslant C_3 3\lambda\Big(|v|_\infty^2+\frac{\Lambda}{24\lambda}+1\Big)\Big(\int_0^t \frac{e^{-(c_0/2)(t-r)}}{(t-r)^{1/2}}\ \mathrm{d}r\Big)\Big(\sup_{r\in[0,t]}|R^\varepsilon(r)|+\sup_{r\in[0,t]}\|\varepsilon\xi^*(r)\|\Big) \\
&\qquad\qquad \leqslant C_4 \sup_{r\in[0,t]}\|\varepsilon\xi^*(r)\|.
\end{aligned}$$

For the last inequality in the preceding chain we make use of Part 1 (3.6) of the proof. Hence we also have

$$\sup_{r\in[0,t]}\|R^\varepsilon(r;x)\| \leqslant C_4 \sup_{r\in[0,t]}\|\varepsilon\xi^*(r)\|, \quad t\leqslant t^*_\eta, x\in B_\delta(v),$$

on $\mathcal{E}_t(\frac{1}{3}(\sqrt{|v|_\infty^2+\frac{\Lambda}{24\lambda}}-|v|_\infty))$, with $\eta \leq \frac{1}{3}(\sqrt{|v|_\infty^2+\frac{\Lambda}{24\lambda}}-|v|_\infty)$.

3. For the last Part of the proof, observe first that by Sobolev embedding we can infer from Part 2 that

$$\sup_{r\in[0,t]}|R^\varepsilon(r)|_\infty \leqslant C_4 \sup_{r\in[0,t]}\|\varepsilon\xi^*(r)\|, \quad t\leqslant t^*_\eta, x\in B_\delta(v),$$

on the event $\mathcal{E}_t(\frac{1}{3}(\sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty))$, with $\eta \leq \frac{1}{3}(\sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty)$. We fix now $\eta = \frac{1}{3}(\sqrt{|v|_\infty^2 + \frac{\Lambda}{24\lambda}} - |v|_\infty)$, and let $T \geqslant 0$ be given. Then on the set $\mathcal{E}_T(\frac{1}{C_4}\eta)$ we will have $\sup_{r\in[0,T]} \|R^\varepsilon(r)\|_\infty \leqslant \eta$, hence $t^*_\eta \geqslant T$. This completes the proof. □

We finally combine the results of the preceding two Lemmas to obtain a uniform estimate for the remainder process R^ε.

Lemma 3.5. *There is a constant $\Gamma > 0$ such that for $\rho \in (1/2, 1), \gamma > 0$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$, $T > 0$, $x \in D^\pm(\varepsilon^\gamma)$ on the event $\mathcal{E}_T(\varepsilon^{(\Gamma+2)\gamma})$ we have the estimate*

$$\sup_{t\in[0,T]} |R^\varepsilon(t;x)|_\infty \leqslant \frac{1}{4}\varepsilon^{2\gamma}.$$

Proof. Let $\kappa > 0$ be fixed, and $\Gamma = \Gamma(\kappa)$ be given according to Lemma 3.3. Let $\gamma, T > 0$ be given, choose δ according to Lemma 3.4, and let $K > 0$ be the global Lipschitz constant of $x \mapsto u(t,x)$ on $B_\delta(v)$ uniformly in $t \geqslant 0$. We apply Lemma 3.3 with the constant $16(1+K)$ to find ε_0 such that for $0 < \varepsilon \leqslant \varepsilon_0$, $x \in D^\pm(\varepsilon^\gamma)$ on $\mathcal{E}_T(\varepsilon^{(\Gamma+2)\gamma})$ the estimate

$$\begin{aligned}\sup_{t\in[0,T]} |R^\varepsilon(t;x)|_\infty &\leqslant \sup_{t\in[0,T_{rec}+\kappa\gamma|\ln\varepsilon|]} |R^\varepsilon(t;x)|_\infty + \sup_{t\geqslant T_{rec}+\kappa\gamma|\ln\varepsilon|} |R^\varepsilon(t;x)|_\infty \\ &\leqslant \frac{1}{16(K+1)}\,\varepsilon^{2\gamma} + \sup_{t\geqslant T_{rec}+\kappa\gamma|\ln\varepsilon|} |R^\varepsilon(t;x)|_\infty \qquad (3.11)\end{aligned}$$

holds. For $T \geqslant t \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|$ we can write, using the flow property

$$\begin{aligned}&|R^\varepsilon(t;x)|_\infty = |Y^\varepsilon(t;x) - u(t;x) - \varepsilon\xi^*(t)|_\infty \\ &= |Y^\varepsilon\,(t - T_{rec} - \kappa\gamma|\ln\varepsilon|;\; Y^\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|; x)) \\ &\qquad - u\,(t - T_{rec} - \kappa\gamma|\ln\varepsilon|;\; Y^\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|; x)) - \varepsilon\xi^*(t)|_\infty \\ &\qquad\quad + |u\,(t - T_{rec} - \kappa\gamma|\ln\varepsilon|;\; Y^\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|; x)) \\ &\qquad\qquad - u\,(t - T_{rec} - \kappa\gamma|\ln\varepsilon|;\; u(T_{rec} + \kappa\gamma|\ln\varepsilon|; x))|_\infty = I_1 + I_2.\end{aligned}$$

By eventually reducing it, assume that ε_0 is such that $2\varepsilon_0^{2\gamma} \leqslant \delta$ is satisfied. By Lemma 3.3 and Proposition 2.15 we know that for $0 < \varepsilon \leqslant \varepsilon_0$

$$\begin{aligned}&|Y^\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|; x) - v|_\infty \\ &\quad= |R^\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|; x) + (u(T_{rec} + \kappa\gamma|\ln\varepsilon|; x) - v) + \varepsilon\xi^*(T_{rec} + \kappa\gamma|\ln\varepsilon|)|_\infty \\ &\quad\leqslant \frac{1}{16(K+1)}\,\varepsilon^{2\gamma} + (1/2)\varepsilon^{2\gamma} + \varepsilon^{(\Gamma+2)\gamma} \leqslant 2\varepsilon^{2\gamma} \qquad (3.12)\end{aligned}$$

on the event $\mathcal{E}_{T_{rec}+\kappa\gamma|\ln\varepsilon|}(\varepsilon^{(\Gamma+2)\gamma})$. Hence according to Lemma 3.4 there is a constant C depending only on the geometric parameters of the Chafee-Infante equation such that for $T \geqslant t \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|$ on $\mathcal{E}_T(\varepsilon^{(\Gamma+2)\gamma})$ we have

$$\begin{aligned} I_1 &\leqslant \sup_{s\geqslant 0} \sup_{y\in B_\delta(v)} |Y^\varepsilon(s;y) - u(s;y) - \varepsilon\xi^*(s + T_{rec} + \kappa\gamma|\ln\varepsilon|)|_\infty \\ &= \sup_{s\geqslant 0} \sup_{y\in B_\delta(v)} |R^\varepsilon(s;y) + \varepsilon\,\xi^*(s) - \varepsilon\xi^*(s + T_{rec} + \kappa\gamma|\ln\varepsilon|)|_\infty \leqslant (C+2)\varepsilon^{(\Gamma+2)\gamma}. \end{aligned}$$

To estimate I_2, we recall the Lipschitz continuity of $x \mapsto u(t;x)$ uniformly in $t \geqslant 0$ with Lipschitz constant K to get with the constants $\varepsilon_0, \Gamma > 0$ already chosen on the event $\mathcal{E}_T(\varepsilon^{(\Gamma+2)\gamma})$ and Lemma 3.3 with prefactor $\frac{1}{16(K+1)}$

$$\begin{aligned} I_2 &\leqslant K|Y^\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|; y) - u(T_{rec} + \kappa\gamma|\ln\varepsilon|; y)|_\infty \\ &\qquad \leqslant \frac{K}{16(K+1)}\varepsilon^{2\gamma} + \varepsilon^{(\Gamma+2)\gamma} \leqslant \frac{1}{16}\varepsilon^{2\gamma} + \varepsilon^{(\Gamma+2)\gamma}. \end{aligned}$$

Therefore by eventually reducing $\varepsilon_0 > 0$ once again we can get for $0 < \varepsilon \leqslant \varepsilon_0$

$$\sup_{t\geqslant T_{rec}+\kappa\gamma|\ln\varepsilon|} |R^\varepsilon(t;x)|_\infty \leqslant (C+3)\varepsilon^{(\Gamma+2)\gamma} + \frac{1}{16}\varepsilon^{2\gamma} \leqslant \frac{1}{8}\varepsilon^{2\gamma}$$

and finally

$$\sup_{t\in[0,T]} |R^\varepsilon(t;x)|_\infty \leqslant \left(\frac{1}{16(K+1)} + \frac{1}{8}\right)\varepsilon^{2\gamma} \leqslant \frac{1}{4}\varepsilon^{2\gamma}.$$

□

3.1.2. Control of the Small Noise Convolution

In this Subsection we shall deal with estimating the convolution of small noise with the semigroup of the heat equation on the unit interval, uniformly on finite deterministic time intervals. Note that in the statement of the following Lemma neither ρ nor γ are restricted within their ranges.

Lemma 3.6. *For $\rho \in (0,1)$, $p > 0$ and $0 < \Theta < 1$ there are constants $C > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$ and $T \geqslant 0$*

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|\varepsilon\xi_t^*\| \geqslant \varepsilon^p\right) \leqslant C\,T\,\varepsilon^{2-2p-(2-(1-\Theta)\alpha)\rho}.$$

Proof. 1. We first show that there exists $C_1 > 0$ such that for any $\rho \in (0,1)$, $p > 0$, $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|\varepsilon\xi^*(t)\| > \varepsilon^p\right) \leqslant C_1\,\varepsilon^{-2(p-1)}\,T\left(\int\limits_{\{0<\|y\|\leqslant\frac{1}{\varepsilon^\rho}\}} \|y\|^2\;\nu(\mathrm{d}y)\right). \tag{3.13}$$

We start by applying Kolmogorov's inequality, to get

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|\varepsilon\xi^*(t)\| > \varepsilon^p\right) \leqslant (\varepsilon^{p-1})^{-2}\, \mathbb{E}\left[\sup_{t\in[0,T]} \|\xi^*(t)\|^2\right].$$

Now consider the stochastic convolution equation

$$\mathrm{d}\xi^* = \Delta\xi^* \;\mathrm{d}t + \mathrm{d}\xi^\varepsilon, \qquad \xi^*(0) = 0.$$

For $t \geqslant 0$ we denote by $\Delta_t X = X(t) - X(t-)$ the jump of a càdlàg process X at time t, and remark that by definition $\Delta_t\xi^* = \Delta_t\xi^\varepsilon$. By Itô's formula we can write for $T \geqslant 0$

$$\begin{aligned}\|\xi^*(T)\|^2 &= 2\int_0^T \langle\xi^*(s-), \mathrm{d}\xi^*(s)\rangle_H + \sum_{s\leqslant T}\left(\|\xi^*(s)\|^2 - \|\xi^*(s-)\|^2 - 2\langle\xi^*(s-), \Delta_s\xi^\varepsilon\rangle_H\right) \\ &= 2\int_0^T \langle\xi^*(s-), \Delta\xi^*(s-)\rangle_H \;\mathrm{d}s + 2\int_0^T \langle\xi^*(s-), \mathrm{d}\xi^\varepsilon(s)\rangle_H \\ &\qquad + \sum_{s\leqslant T}\left(\|\xi^*(s)\|^2 - \|\xi^*(s-)\|^2 - 2\langle\xi^*(s-), \Delta_s\xi^\varepsilon\rangle_H\right).\end{aligned}$$

By the non-positivity of $\int_0^T \langle\xi^*(s-), \Delta\xi^\varepsilon(s-)\rangle_H \;\mathrm{d}s$ we may continue to estimate

$$\begin{aligned}\|\xi^*(T)\|^2 \leqslant 2\int_0^T \langle\xi^*(s-), \mathrm{d}\xi^*(s)\rangle_H \\ + \sum_{s\leqslant T}\left(\|\xi^*(s)\|^2 - \|\xi^*(s-)\|^2 - 2\langle\xi^*(s-), \Delta_s\xi^\varepsilon\rangle_H\right).\end{aligned}$$

Note that for $s \leqslant T$

$$\|\xi^*(s)\|^2 - \|\xi^*(s-)\|^2 - 2\langle\xi^*(s-), \Delta_s\xi^\varepsilon\rangle_H = \|\Delta_s\xi^*\|^2 = \|\Delta_s\xi^\varepsilon\|^2,$$

and therefore

$$\|\xi^*(T)\|^2 \leqslant \int_0^T \langle\xi^*(s-), \mathrm{d}\xi^*(s)\rangle_H + \sum_{s\leqslant T}\|\Delta_s\xi^\varepsilon\|^2.$$

For $t \geqslant 0$ let us denote by $[[X]]_t$ the quadratic variation of a process X on $[0, t]$. Then Burkholder's inequality yields a universal constant $C_2 > 0$ such that by Young's inequality for any $a > 0$ we have

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[0,T]}\|\xi^*(s)\|^2\Big]\\
&\leqslant 2\,\mathbb{E}\Big[\sup_{s\in[0,T]}\Big|\int_0^s\langle\xi^*(r-),\mathrm{d}\xi^\varepsilon(r)\rangle_H\Big|\Big]+\mathbb{E}\Big[\sum_{r\leqslant T}\|\delta_r\xi^*\|^2\Big]\\
&\leqslant 2C_2\,\mathbb{E}\Big[[[\int_0^{\cdot}\langle\xi^*(r-),\mathrm{d}\xi^\varepsilon(r)\rangle_H]]_T^{1/2}\Big]+T\int_{\{0<\|y\|\leqslant 1/\varepsilon^\rho\}}\|y\|^2\,\mathrm{d}\nu(\mathrm{d}y)\\
&=2C_2\,\mathbb{E}\Big[\Big(\sup_{s\in[0,T]}\|\xi^*(s)\|^2\int_0^T\mathrm{d}[[\xi^*]](s)\Big)^{1/2}\Big]+T\int_{\{0<\|y\|\leqslant 1/\varepsilon^\rho\}}\|y\|^2\,\mathrm{d}\nu(\mathrm{d}y)\\
&\leqslant 2C_2\left(a\,\mathbb{E}\Big[\sup_{s\in[0,T]}\|\xi^*(s)\|^2\Big]+\frac{1}{4a}\mathbb{E}\Big[\int_0^T\mathrm{d}[[\xi^*]](s)\Big]\right)+T\int_{\{0<\|y\|\leqslant 1/\varepsilon^\rho\}}\|y\|^2\,\mathrm{d}\nu(\mathrm{d}y)\\
&\qquad=2aC_2\,\mathbb{E}\Big[\sup_{s\in[0,T]}\|\xi^*(s)\|^2\Big]+\frac{C_2\,T}{2a}\int_{\{0<\|y\|\leqslant\frac{1}{\varepsilon^\rho}\}}\|y\|^2\,\nu(\mathrm{d}y)\\
&\qquad\qquad+T\int_{\{0<\|y\|\leqslant 1/\varepsilon^\rho\}}\|y\|^2\,\mathrm{d}\nu(\mathrm{d}y).
\end{aligned}
$$

Choosing now $a=1/(4C_2)$ we obtain

$$
\mathbb{E}\left[\sup_{s\in[0,T]}\|\xi^*(s)\|^2\right]\leqslant\left(4C_2^2+2\right)T\int_{\{0<\|y\|\leqslant\frac{1}{\varepsilon^\rho}\}}\|y\|^2\,\nu(\mathrm{d}y).
$$

Now take $C_1=4C_2^2+2$ to finish our argument.

2. In the second part of the proof it remains to determine the asymptotic behavior of the last factor for small $0<\varepsilon<1$. We first write

$$
\int_{\{0<\|y\|\leqslant\frac{1}{\varepsilon^\rho}\}}\|y\|^2\,\nu(\mathrm{d}y)\leqslant\int_{\{0<\|y\|\leqslant 1\}}\|y\|^2\,\nu(\mathrm{d}y)+\int_{\{1<\|y\|\leqslant\frac{1}{\varepsilon^\rho}\}}\|y\|^2\,\nu(\mathrm{d}y),
$$

and remark that by part 1 of the proof it remains to estimate the asymptotic behavior of the function $\varepsilon\mapsto\int_{\{1<\|y\|\leqslant\frac{1}{\varepsilon^\rho}\}}\|y\|^2\,\nu(\mathrm{d}y)$ for small $0<\varepsilon<1$. To do this, we use the regular variation of $t\mapsto\nu(tB_1^c(0))=t^{-\alpha}\,\ell(t)\,\mu(B_1^c(0))$ with a slowly varying function ℓ and limiting measure μ (see Definition 2.6). We also use Proposition A.43 which implies that for any slowly varying function ℓ and $1>\Theta>0$ there exists $C_3>0$, such that $\ell(t)\leqslant C_3+t^{-\Theta\alpha}$. This results in the following chain of inequalities

$$\int_{\{1<\|y\|\leqslant \frac{1}{\varepsilon^\rho}\}} \|y\|^2 \ \nu(\mathrm{d}y) \ = \ \int_H \int_0^{\|y\|} \mathbf{1}\{1<\|y\|\leqslant \frac{1}{\varepsilon^\rho}\} 2t \ \mathrm{d}t \ \nu(\mathrm{d}y)$$
$$\leqslant 2 \int_0^{\frac{1}{\varepsilon^\rho}} t \ \nu\left((1\vee t)B_1^c(0)\right) \ \mathrm{d}t \ \leqslant \ 1 + 2\mu(B_1^c(0)) \int_1^{\frac{1}{\varepsilon^\rho}} t^{1-\alpha} \ \left(C_3 + t^{-\Theta\alpha}\right) \ \mathrm{d}t$$
$$= \ C_4 + \frac{2\, C_3 \mu(B_1^c(0))}{2-\alpha} \varepsilon^{-\rho(2-\alpha)} + \frac{2\mu(B_1^c(0))}{2-(1+\Theta)\alpha} \varepsilon^{-\rho(2-(1-\Theta)\alpha)},$$

with another universal constant C_4. Therefore there exists $C_5 > 0$ such that for $\varepsilon > 0$ sufficiently small

$$\int_{\{1<\|y\|\leqslant \frac{1}{\varepsilon^\rho}\}} \|y\|^2 \ \nu(\mathrm{d}y) \leqslant C_5 \varepsilon^{-\rho(2-(1-\Theta)\alpha)}.$$

Inserting this into inequality (3.13) we obtain the desired result. □

3.1.3. The Small Deviation Estimate on Deterministic Time Intervals

In this Subsection we combine the results of the preceding two to complete the proof of Proposition 3.2 for deterministic time intervals $[0,T]$, $T > 0$.

Proof. (of Proposition 3.2)
By Lemma 3.5, we find $\Gamma > 0$ such that given $\rho \in (1/2, 1), \gamma > 0$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0, T > 0$, and $x \in D^\pm(\varepsilon^\gamma)$ we have by definition of $\mathcal{E}_T(\varepsilon^{(\Gamma+2)\gamma})$

$$\begin{aligned}
&\{ \sup_{t\in[0,T]} |Y^\varepsilon(t;x) - u(t;x)|_\infty \geqslant (1/2)\varepsilon^{2\gamma}\} \\
&= \{ \sup_{t\in[0,T]} |R^\varepsilon(t;x) + \varepsilon\xi^*(t)|_\infty \geqslant (1/2)\varepsilon^{2\gamma}\} \\
&\qquad \subseteq \{ \sup_{t\in[0,T]} |R^\varepsilon(t;x)|_\infty \geqslant (1/4)\varepsilon^{2\gamma}\} \cup \{ \sup_{t\in[0,T]} \|\varepsilon\xi^*(t)\| \geqslant (1/4)\varepsilon^{2\gamma}\} \\
&\qquad \subseteq \{ \sup_{t\in[0,T]} \|\varepsilon\xi^*(t)\| \geqslant \varepsilon^{(\Gamma+2)\gamma}\} \cup \{ \sup_{t\in[0,T]} \|\varepsilon\xi^*(t)\| \geqslant (1/4)\varepsilon^{2\gamma}\} \\
&\qquad\qquad \subseteq \{ \sup_{t\in[0,T]} \|\varepsilon\xi^*(t)\| \geqslant \varepsilon^{(\Gamma+2)\gamma}\}. \quad (3.14)
\end{aligned}$$

Therefore we can infer from Lemma 3.6 with $p = (\Gamma+2)\gamma$ and $0 < \Theta < 1$ by eventually reducing ε_0 a bit

$$\mathbb{P}\left(\exists\, x \in D^{\pm}(\varepsilon^{\gamma}) : \sup_{t\in[0,T]} |Y^{\varepsilon}(t;x) - u(t;x)|_{\infty} \geqslant (1/2)\varepsilon^{2\gamma}\right) \leqslant \mathbb{P}\left(\sup_{t\in[0,T]} \|\varepsilon\xi_t^*\| \geqslant \varepsilon^{(\Gamma+2)\gamma}\right) \;\leqslant\; C\, T\, \varepsilon^{2-2(\Gamma+2)\gamma-(2-(1-\Theta)\alpha)\rho}.$$

□

3.2. Small Deviation before the First Large Jump (Proof of Proposition 3.1)

In this Section we apply the results from the previous one to finally obtain the small deviation estimate on the stochastic interval between 0 and the first big jump T_1. Since we know the law of T_1, an integration of the estimate just obtained is necessary. This will complete the proof of Proposition 3.1 for the time interval $[0, T_1]$.

Proof. (of Proposition 3.1). We use the inequality derived in the preceding Subsection, as well as the asymptotic behavior of the large jump rate β_ε given in (2.13). We conclude that we can find $\Gamma > 0$ such that for given $\rho \in (1/2, 1), \gamma > 0$, and $0 < \Theta < 1$ there exist constants C_1, C_2 and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$

$$
\begin{aligned}
&\mathbb{P}\Big(\exists\, x \in D^{\pm}(\varepsilon^{\gamma}) : \sup_{s\in[0,T_1]} |Y^{\varepsilon}(s;x) - u(s;x)|_{\infty} \geqslant (1/2)\varepsilon^{2\gamma}\Big) \\
&\leqslant \int_0^{\infty} \mathbb{P}\Big(\exists\, x \in D^{\pm}(\varepsilon^{\gamma}) : \sup_{s\in[0,t]} |Y^{\varepsilon}(s;x) - u(s;x)|_{\infty} \geqslant (1/2)\varepsilon^{2\gamma}\Big)\beta_\varepsilon e^{-\beta_\varepsilon t}\, \mathrm{d}t \\
&\leqslant C_1\, \varepsilon^{2-2(\Gamma+2)\gamma-(2-(1-\Theta)\alpha)\rho} \int_0^{\infty} t\beta_\varepsilon e^{-\beta_\varepsilon t}\, \mathrm{d}t \\
&\leqslant C_1\, \varepsilon^{2-2(\Gamma+2)\gamma-(2-(1-\Theta)\alpha)\rho} \left(\beta_\varepsilon^{-1}\Gamma(2)\right) \\
&\leqslant C_2\, \varepsilon^{2-2(\Gamma+2)\gamma-(2-(1-\Theta)\alpha)\rho-\alpha\rho},
\end{aligned}
$$

where Γ is the classical Gamma function $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\mathrm{d}t,\ z \geqslant 0$. Let now $\vartheta = 2 - 2(\Gamma + 2)\gamma - (2 - (1 - \Theta)\alpha)\rho - \alpha\rho$. Upon setting $C_\vartheta = C_2$ it remains to check the conditions under which $\vartheta > \alpha(1 - \rho)$. We have

$$
\begin{aligned}
\vartheta - \alpha(1-\rho) &= 2 - 2(\Gamma+2)\gamma - (2-(1-\Theta)\alpha)\rho - \alpha\rho - \alpha(1-\rho) \\
&= 2 - \alpha - 2(\Gamma+2)\gamma - (2-(1-\Theta)\alpha)\rho \\
&= \underbrace{(2-\alpha)(1-\rho)}_{>0} - \Theta\alpha\rho - 2(\Gamma+2)\gamma > 0
\end{aligned}
$$

if and only if

$$
0 < \gamma < \frac{(2-\alpha)(1-\rho) - \Theta\alpha\rho}{2(\Gamma+2)}.
$$

The right-hand side of the last inequality is positive if

$$
(2-\alpha)(1-\rho) - \Theta\alpha\rho > 0 \quad \text{and thus iff} \quad \rho < \frac{2-\alpha}{2-(1-\Theta)\alpha}.
$$

Since $\rho > 1/2$, the last inequality forces us to restrict $\Theta > 0$ to fulfill

$$\frac{1}{2} < \frac{2-\alpha}{2-(1-\Theta)\alpha} \quad \text{which is equivalent to} \quad \Theta < \frac{2-\alpha}{\alpha}.$$

Under these assumptions, identical to the ones formulated in the statement of Proposition 3.1, we have $\vartheta > \alpha(1-\rho)$. This completes the proof. □

In the next Chapter we shall need a slightly modified form of Proposition 3.1, which we derive in the rest of this Chapter.
For $x \in D^{\pm}(\varepsilon^{\gamma})$ define

$$E_x := \{ \sup_{s\in[0,T_1]} |Y^{\varepsilon}(s;x) - u(s;x)|_{\infty} \leqslant (1/2)\varepsilon^{2\gamma} \}.$$

Corollary 3.1. *Under the assumptions of Proposition 3.1 there is $\vartheta = \vartheta(\alpha, \Theta, \gamma, \rho)$ with $\vartheta > \alpha(1-\rho)$, $C_{\vartheta} > 0$, and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$*

$$\mathbb{E}\left[\sup_{x\in D^{\pm}(\varepsilon^{\gamma})} \mathbf{1}(E_x^c) \right] \leqslant C_{\vartheta}\varepsilon^{\vartheta}.$$

With the following Corollary, we prepare an auxiliary statement to be used in Chapter 4 in the estimate for the Laplace transform of the exit time from reduced domains of attraction.

Corollary 3.2. *Let $C > 0$, and let the assumptions of Proposition 3.1 hold. Then there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$, $\theta > -1$*

$$\mathbb{E}\left[e^{-\theta\lambda^{\pm}(\varepsilon)T_1} \sup_{x\in D^{\pm}(\varepsilon^{\gamma})} \mathbf{1}(E_x^c) \right] \leqslant C \left(\frac{\beta_{\varepsilon}}{\beta_{\varepsilon} + \theta\lambda^{\pm}(\varepsilon)} \right) \frac{\lambda^{\pm}(\varepsilon)}{\beta_{\varepsilon}}. \tag{3.15}$$

Proof. 1. First note that for $\vartheta > \alpha(1-\rho)$ and C_{ϑ} according to the preceding Corollary 3.1, by the asymptotic properties of the functions β_{ε} and $\lambda^{\pm}(\varepsilon)$ stated in (2.13) we may conclude that there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$ we have

$$C_{\vartheta}e^{\vartheta} \leqslant C \left(\frac{\beta_{\varepsilon}}{\beta_{\varepsilon} + \theta\lambda^{\pm}(\varepsilon)} \right) \frac{\lambda^{\pm}(\varepsilon)}{\beta_{\varepsilon}},$$

and also

$$\frac{C_{\vartheta}\varepsilon^{\vartheta+\alpha\rho}}{\beta_{\varepsilon} + \theta\lambda^{\pm}(\varepsilon)} \leqslant C \left(\frac{\beta_{\varepsilon}}{\beta_{\varepsilon} + \theta\lambda^{\pm}(\varepsilon)} \right) \frac{\lambda^{\pm}(\varepsilon)}{\beta_{\varepsilon}}.$$

2. For $\theta \geqslant 0$ Corollary 3.1 provides $\vartheta > \alpha(1-\rho)$ such that

$$\mathbb{E}\left[e^{-\theta\lambda^{\pm}(\varepsilon)T_1} \sup_{x\in D^{\pm}(\varepsilon^{\gamma})} \mathbf{1}(E_x^c) \right] \leqslant \mathbb{E}\left[\sup_{x\in D^{\pm}(\varepsilon^{\gamma})} \mathbf{1}(E_x^c) \right] \leqslant c_{\vartheta}\varepsilon^{\vartheta}$$

Now apply the first inequality of Part 1.

For $\theta \in (-1,0)$, recalling from the proof of Theorem 3.1 that the exponent had the shape $\vartheta = 2 - 2(2+\Gamma)\gamma - (2-(1-\Theta)\alpha)\rho > \alpha(1-\rho)$ we get

$$\begin{aligned}
&\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}(E^c(y))\right] \\
&\leqslant \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\mathbf{1}\{\sup_{s\in[0,T]} \|\varepsilon\xi^*(s)\| > \varepsilon^{(\Gamma+2)\gamma}\}\right] \\
&\qquad\qquad \leqslant \int_0^\infty e^{-\theta\lambda(\varepsilon)t}\mathbb{P}\left(\sup_{s\in[0,t]} \|\varepsilon\xi^*(s)\| > \varepsilon^{(\Gamma+2)\gamma}\right)\beta_\varepsilon e^{-\beta_\varepsilon t}\,\mathrm{d}t \\
&\qquad\qquad \leqslant C_\vartheta \varepsilon^{2-2(2+\Gamma)\gamma-(2-(1+\Theta)\alpha)\rho}\int_0^\infty e^{-(\theta\lambda(\varepsilon)+\beta_\varepsilon)t}\beta_\varepsilon t\,\mathrm{d}t \\
&\qquad\qquad\qquad\qquad \leqslant \left(\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\right)^2 \frac{C_\vartheta}{\beta_\varepsilon}\varepsilon^{\vartheta+\alpha\rho}. \qquad (3.16)
\end{aligned}$$

Now apply the second inequality in Part 1 to conclude. □

4. Asymptotic Exit Times

We shall now use the small deviations estimates of Chapter 3 to give a precise account of the exit times of the system described by our Chafee-Infante equation with small Lévy noise in H of the reduced domains of attraction of the stable states $\phi^{\pm}$ defined in Chapter 2. Our main line of reasoning will be based on the splitting of small and large jumps proposed there. In fact, the Chafee-Infante equation perturbed by small jumps being subject to only small deviations from the solution of the deterministic system, uniformly before the first big jump, as shown in Chapter 3, and the time needed for relaxation in a small neighborhood of $\phi^{\pm}$ being only of logarithmic order in ε, exits will happen at times of big jumps that are big enough to leave the reduced domains of attraction. To characterize the asymptotic law of the exit time, we shall compute its Laplace transform. Making these heuristic arguments mathematically rigorous will be the main task of this Chapter.

4.1. Estimates of Exit Events by Large Jump and Perturbation Events

In this Section we shall exploit the Markov property of our process to rigorously define events that are capable of capturing the successive big jumps linked by periods of relaxation during which only small deviations from the deterministic solutions are possible. The strong Markov property allows us to represent X^ε recursively in the following way. Recall the notation used for the big jump compound Poisson part of our Lévy noise process from Section 2.1, and denote the shift by time t on the space of trajectories by $\theta_t, t \geqslant 0$. For any $k \in \mathbb{N}$, $t \in [0, t_k]$, $x \in H$ we have

$$X^\varepsilon(t + T_{k-1}; x) = Y^\varepsilon(t; X^\varepsilon(0; x)) \circ \theta_{T_{k-1}} + \varepsilon W_k \mathbf{1}\{t = t_k\}. \tag{4.1}$$

In the following two lemmas we estimate certain events connecting the behaviour of X^ε in the domains of type $D^{\pm}(\varepsilon^\gamma)$ with the large jumps η^ε in the reshifted domains of type

$D_0^{\pm}(\varepsilon^\gamma)$. We introduce for $\varepsilon > 0$ and $x \in \tilde{D}^{\pm}(\varepsilon^\gamma)$ the major events

$$
\begin{aligned}
A_x :=& \{Y^\varepsilon(s;x) \in D^{\pm}(\varepsilon^\gamma) \text{ for } s \in [0,T_1] \text{ and } Y^\varepsilon(T_1;x) + \varepsilon W_1 \in D^{\pm}(\varepsilon^\gamma)\},\\
B_x :=& \{Y^\varepsilon(s;x) \in D^{\pm}(\varepsilon^\gamma) \text{ for } s \in [0,T_1] \text{ and } Y^\varepsilon(T_1;x) + \varepsilon W_1 \notin D^{\pm}(\varepsilon^\gamma)\},\\
A_x^- :=& \{Y^\varepsilon(s;x) \in D^{\pm}(\varepsilon^\gamma) \text{ for } s \in [0,T_1] \text{ and } Y^\varepsilon(T_1;x) + \varepsilon W_1 \in \tilde{D}^{\pm}(\varepsilon^\gamma)\},\\
C_x :=& \{Y^\varepsilon(s;x) \in D^{\pm}(\varepsilon^\gamma) \text{ f. } s \in [0,T_1] \text{ a. } Y^\varepsilon(T_1;x) + \varepsilon W_1 \in D^{\pm}(\varepsilon^\gamma) \setminus \tilde{D}^{\pm}(\varepsilon^\gamma)\},\\
A^\diamond :=& \{\varepsilon W_1 \in D_0^{\pm}\},\\
B^\diamond :=& \{\varepsilon W_1 \notin D_0^{\pm}\},\\
E_x :=& \{\sup_{s\in[0,T_1]} |Y^\varepsilon(s;x) - u(s;x)|_\infty \leqslant (1/2)\varepsilon^{2\gamma}\}.
\end{aligned}
\tag{4.2}
$$

We can now exploit the precise definitions of the reduced domains of attraction in order to obtain partial estimates of solution path events by events only depending on the driving noise. For simplicity of notation we abbreviate

$$D_0^*(\varepsilon^\gamma) := \big(D_0^{\pm}(\varepsilon) \setminus D_0(\varepsilon^\gamma, \varepsilon^{2\gamma})\big) + B_{\varepsilon^{2\gamma}}(0).$$

The three following lemmas are proved in Section 4.3. The first one is concerned with events on which only small deviations from the deterministic trajectory are possible before the first big jump time.

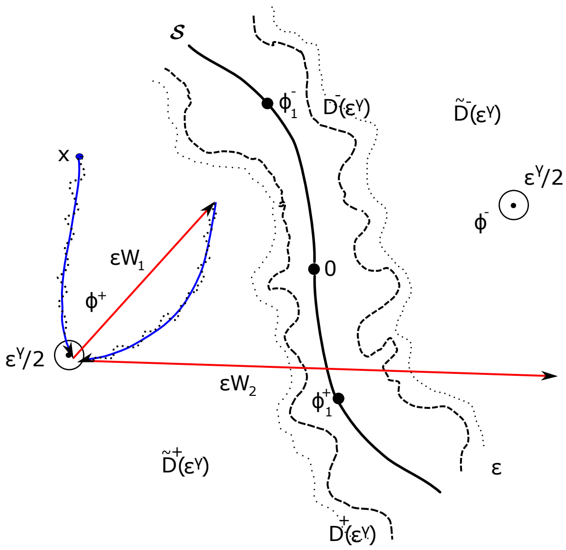

Figure 4.1.: Sketch of a typical first exit event from a reduced domain of attraction

Lemma 4.1 (Partial estimates of the major events).
Let $T_{rec}, \kappa > 0$ given by Proposition 2.15 and assume that Hypotheses (H.1) and (H.2)

are satisfied. For $\rho \in \left(\frac{1}{2}, 1\right)$, $\gamma \in (0, 1-\rho)$ *there exists* $\varepsilon_0 > 0$ *so that the following inequalities hold true for all* $0 < \varepsilon \leqslant \varepsilon_0 > 0$ *and* $x \in D^{\pm}(\varepsilon^{\gamma})$

$$i)\ \mathbf{1}(A_x)\mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \leqslant \mathbf{1}\{\varepsilon W_1 \in D_0^{\pm}\}, \tag{4.3}$$

$$ii)\ \mathbf{1}(B_x)\mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \leqslant \mathbf{1}\{\varepsilon W_1 \notin D_0^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})\}, \tag{4.4}$$

$$iii)\ \mathbf{1}(C_x)\mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \leqslant \mathbf{1}\{\varepsilon W_1 \in D_0^{*}(\varepsilon^{\gamma})\}. \tag{4.5}$$

Additionally, for $x \in D^{\pm}(\varepsilon^{\gamma})$ *we have*

$$iv)\ \mathbf{1}(B_x)\mathbf{1}(E_x)\mathbf{1}\{\|\varepsilon W_1\| \leqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 > T_{rec} + \kappa\gamma|\ln\varepsilon|\} = 0, \tag{4.6}$$

$$v)\ \mathbf{1}(C_x)\mathbf{1}(E_x)\mathbf{1}\{\|\varepsilon W_1\| \leqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} = 0. \tag{4.7}$$

In the opposite sense for $x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})$

$$vi)\ \mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon W_1 \notin D_0^{\pm}\} \leqslant \mathbf{1}(B_x), \tag{4.8}$$

$$vii)\ \mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon W_1 \in D_0^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\} \leqslant \mathbf{1}(A_x^{-}). \tag{4.9}$$

The estimates presented in the preceding Lemma can be readily combined to provide full estimates of the events in terms of the first large jump time T_1, the large jump height W_1 and the perturbation event E_y^c on which deviations obtained from the small jump part are big.

Lemma 4.2 (Full estimates of the major events)**.** *Let* $T_{rec}, \kappa > 0$ *given by Proposition 2.15 and assume that Hypotheses (H.1) and (H.2) are satisfied. Let us denote the shift by time* t *on the path space for our Markov process by* $\theta_t, t \geqslant 0$. *For* $\rho \in \left(\frac{1}{2}, 1\right)$, $\gamma \in (0, 1-\rho)$ *there exists* $\varepsilon_0 > 0$ *such that the following inequalities hold true for all* $0 < \varepsilon \leqslant \varepsilon_0, \kappa > 0$ *and* $x \in D^{\pm}(\varepsilon^{\gamma})$

$$ix)\ \mathbf{1}(A_x) \leqslant \mathbf{1}\{\varepsilon W_1 \in D_0^{\pm}\} + \mathbf{1}\{\|\varepsilon W_1\| > \frac{1}{2}\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} + \mathbf{1}(E_x^c),$$

$$x)\ \mathbf{1}(B_x) \leqslant \mathbf{1}\{\varepsilon W_1 \notin D_0^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})\} + \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} + \mathbf{1}(E_x^c),$$

$$xi)\ \sup_{y \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}\{Y^{\varepsilon}(s; y) \notin D^{\pm}(\varepsilon^{\gamma}) \text{ for some } s \in (0, T_1)\} \leqslant \sup_{y \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}(E_y^c),$$

$$xii)\ \mathbf{1}(A_x)\mathbf{1}\{Y^{\varepsilon}(s; X^{\varepsilon}(0, x)) \circ \theta_{T_1} \notin D^{\pm}(\varepsilon^{\gamma}) \text{ for some } s \in (0, T_1)\}$$
$$\leqslant \mathbf{1}\left\{\varepsilon W_1 \in D_0^{*}(\varepsilon^{\gamma})\right\} + \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} + \sup_{y \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}(E_y^c) \circ \theta_{T_1} + \mathbf{1}(E_x^c).$$

In the opposite sense for $x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})$

$$xiii)\ \mathbf{1}(A_x^{-}) \geqslant \mathbf{1}\{\varepsilon W_1 \in D_0^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\} - \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} - 2\,\mathbf{1}(E_x^c),$$

$$xiv)\ \mathbf{1}(B_x) \geqslant \mathbf{1}\{\varepsilon W_1 \notin D_0^{\pm}\}(1 - \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}) - \mathbf{1}(E_x^c).$$

In particular for $x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})$

xv) $\mathbf{1}(A_x^- \cap A^{\diamond}) \geqslant \mathbf{1}\{\varepsilon W_1 \in D_0^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\} - \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} - 2\,\mathbf{1}(E_x^c)$,

xvi) $\mathbf{1}(B_x \cap B^{\diamond}) \geqslant \mathbf{1}\{\varepsilon W_1 \notin D_0^{\pm}\}(1 - \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}) - \mathbf{1}(E_x^c)$.

In order to make the previous estimates fertile to our analysis we show that Hypthesis (H.2) implies a sequence of similar slightly less restrictive inequalities.

Lemma 4.3. *Assume that Hypthesis (H.2) with (2.11) is true. Then for any* $\eta > 0$ *we can choose* $\varepsilon_0 > 0$ *small enough such that for all* $0 < \varepsilon \leqslant \varepsilon_0$

$$\begin{aligned}
&i)\ \mu\left(\left(D_0^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})\right)^c \setminus \left(D_0^{\pm}\right)^c\right) < \eta,\\
&ii)\ \mu\left(\left(D_0^{\pm}(\varepsilon^{\gamma}) \setminus D_0^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})\right) + B_{\varepsilon^{2\gamma}}(0)\right) < \eta,\\
&iii)\ \mu\left(D_0^{\pm} \setminus D_0^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\right) < \eta,\\
&iv)\ \mu\left((D^{\pm})^c \setminus D^{\mp}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}) - \phi^{\pm}\right) < \eta.
\end{aligned} \tag{4.10}$$

The claims follow from simple set inclusions.

Equipped with estimates of the major events by analytically accessible handy ones containing only information about the time and height of the first big jump and the deviations of the small jump part from the deterministic solution before the first big jump time, we can study their asymptotic behavior. It will turn out that only the large jump event stipulating W_1 to leave $D_0^{\pm}$ or its reduced versions will be asymptotically relevant. This is rigorously stated in the following Lemma.

Lemma 4.4 (Asymptotic behavior of large jump events)**.**
Assume that Hypotheses (H.1) and (H.2) are satisfied and let $1/2 < \rho < 1 - 2\gamma$ *fixed. Then for any* $C > 0$ *there is* $\varepsilon_0 = \varepsilon_0(C) > 0$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_0$

$$\begin{aligned}
&I)\ \left(\frac{\mu\left((D_0^{\pm})^c\right)}{\mu(B_1^c(0))} - C\right)\varepsilon^{\alpha(1-\rho)} \leqslant \frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon} \leqslant \left(\frac{\mu((D_0^{\pm})^c)}{\mu(B_1^c(0))} + C\right)\varepsilon^{\alpha(1-\rho)},\\
&II)\ \mathbb{P}\left(\|\varepsilon W_1\| > (1/2)\varepsilon^{2\gamma}\right) \leqslant 4\varepsilon^{\alpha(1-\rho-2\gamma)},\\
&III)\ \mathbb{P}\left(\varepsilon W_1 \in (\tilde{D}_0^{\pm}(\varepsilon^{\gamma}))^c\right) \leqslant (1 + C)\,\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon},\\
&IV)\ \mathbb{P}\left(\varepsilon W_1 \in D_0^{*}(\varepsilon^{\gamma})\right) \leqslant C\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon},\\
&V)\ \mathbb{P}(\varepsilon W_1 \in D_0^c(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})) \leqslant (1 + C)\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}.
\end{aligned}$$

The proof is given in Section 4.3.

4.2. Asymptotic Exit Times from Reduced Domains of Attraction

In this Section we shall state and prove our main result about the asymptotic behavior of the exit time from the reduced domains of attraction of the equilibria of the Chafee-Infante equation. It will essentially describe the asymptotic behavior of the exit time's Laplace transform. Let us start with a remark concerning the constants appearing in the small deviations estimates in Chapter 3.

Remark 4.5. Recall from Proposition 3.1 that for $\alpha \in (0,2)$ there exists $\Gamma > 0$ such that the constants $\rho \in (1/2, 1), \gamma > 0, 0 < \Theta < 1$ can be chosen according to

$$0 < \Theta < \frac{2-\alpha}{2\alpha}, \qquad \rho \in (\frac{1}{2}, \frac{2-\alpha}{2-(1-\Theta)\alpha}) \qquad 0 < \gamma < \frac{(2-\alpha)(1-\rho)-\Theta\alpha\rho}{2(\Gamma+2)}$$

such that the statement of the Proposition holds true. Also recall that Proposition 3.1 stays true for any constant $\tilde{\Gamma} > \Gamma$ and $\Theta < \frac{2-\alpha}{2\alpha}$. This justifies the choice of constants (2.11) in Hypothesis (H.2).

Claim: We stipulate that additionally to the inequalities stated and the validity of Proposition 3.1 the inequalities

$$2\gamma < \rho < 1 - 2\gamma \tag{4.11}$$

hold true. In fact, the first one is evident since $\rho > 1/2$. For the second inequality we calculate

$$\rho + 2\gamma < \rho + \frac{(2-\alpha)(1-\rho)}{\Gamma+2} \leqslant \rho + 1 - \rho = 1.$$

The following main Theorem states that for all $\theta > -1$ the Laplace transform $\widehat{\lambda(\varepsilon)\tau_x^{\pm}}(\varepsilon)(\theta)$ of the normalized first exit time $\lambda(\varepsilon)\tau_x^{\pm}(\varepsilon)$ from the reduced domain of attraction $D^{\pm}$ converges to $\frac{1}{1+\theta}$ as $\varepsilon \to 0+$. This establishes its convergence in law to an exponentially distributed random variable.

Theorem 4.6 (Asymptotic first exit time law)**.** *Assume that Hypotheses (H.1) and (H.2) are satisfied. Then for all* $\theta > -1$ *and* $C \in (0, 1+\theta)$ *there exists* $\varepsilon_0 = \varepsilon_0(\theta) > 0$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_0$

$$\begin{aligned}\frac{1-C}{1+\theta+C} &\leqslant \mathbb{E}\left[\inf_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} \exp\left(-\theta\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon)\right)\right] \\ &\leqslant \mathbb{E}\left[\sup_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} \exp\left(-\theta\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon)\right)\right] \leqslant \frac{1+C}{1+\theta-C}.\end{aligned}$$

This theorem is proved in Subsection 4.2.1 and 4.2.2. The result of Theorem 4.6 implies a statement about the asymptotic behavior of the expected first exit time.

Corollary 4.7. *Under the assumptions of Theorem 4.6 we have*

$$\lim_{\varepsilon\to 0+} \mathbb{E}\left[\inf_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)\right] = \lim_{\varepsilon\to 0+} \mathbb{E}\left[\sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)\right] = 1. \tag{4.12}$$

Proof. By Theorem 4.6 which holds for $\theta > -1$, we know that $\lambda^\pm(\varepsilon)\tau_x(\varepsilon)$ converges in law to τ as $\varepsilon \to 0$, and τ has an exponential law with parameter 1. In addition, for $\theta < 0$

$$\begin{aligned}
&\mathbb{E}\left[e^{-\theta\left(\inf_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)\right)}\right] \\
&\quad\leqslant \mathbb{E}\left[e^{-\theta\left(\sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)\right)}\right] = \mathbb{E}\left[\sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} e^{-\theta\lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)}\right] \leqslant \frac{1+C}{1+\theta-C} < \infty
\end{aligned}$$

and hence $(\inf_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon))_{0<\varepsilon\leqslant\varepsilon_0}$ and $(\sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon))_{0<\varepsilon\leqslant\varepsilon_0}$ are uniformly integrable. For nonnegative random variables, convergence in law and uniform integrability implies convergence in expectation (see Kallenberg [1997], Lemma 4.11). This implies the formula (4.12). □

In the sequel we construct a family of random variables, $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$, such that in probability $\tau_x^\pm(\varepsilon)\lambda^\pm(\varepsilon) - \bar{\tau}(\varepsilon) \to 0$ for $\varepsilon \to 0+$.

Theorem 4.8 (Asymptotic first exit times in probability)**.** *Assume that Hypotheses (H.1) and (H.2) are satisfied. Then there is a family of random variables* $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ *with exponential law of parameter 1 (on the same probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *as the driving Lévy noise* $(L(t))_{t\geqslant 0}$*) such that in probability*

$$\lim_{\varepsilon\to 0+} \inf_{y\in\tilde{D}^\pm(\varepsilon^\gamma)} |\lambda^\pm(\varepsilon)\tau_y^\pm - \bar{\tau}(\varepsilon)| = \lim_{\varepsilon\to 0+} \inf_{y\in\tilde{D}^\pm(\varepsilon^\gamma)} |\lambda^\pm(\varepsilon)\tau_y^\pm - \bar{\tau}(\varepsilon)| = 0.$$

This theorem is proved in Subsection 4.2.3. Combining these two results, we obtain the announced Theorem 2.18, which we restate for convenience.

Theorem 4.9 (Exponential convergence of first exit times from $D^\pm(\varepsilon^\gamma)$)**.** *Let the Chafee-Infante parameter* $\pi^2 < \lambda \neq (k\pi)^2$ *for* $k \in \mathbb{N}$ *be given and suppose Hypotheses (**H.1**) and (**H.2**) are satisfied. Then there is a family of random variables* $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ *with exponential law of parameter 1 such that for all* $\theta < 1$

$$\lim_{\varepsilon\to 0+} \mathbb{E}\Big[\sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} |\exp\left(\theta\lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)\right) - \exp\left(\theta\bar{\tau}(\varepsilon)\right)|\Big] = 0.$$

Proof. By Theorem 4.6 for each $\theta > -1$ and $C \in (0, 1-\theta)$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$\frac{1}{1+\theta} - C_1 \leqslant \mathbb{E}\left[\inf_{x \in \tilde{D}^\pm(\varepsilon^\gamma)} \exp\left(-\theta\lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)\right)\right]$$
$$\leqslant \mathbb{E}\left[\sup_{x \in \tilde{D}^\pm(\varepsilon^\gamma)} \exp\left(-\theta\lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)\right)\right] \leqslant \frac{1}{1+\theta} + C_2.$$

where $C_1 = \frac{1+C}{1+\theta-C} - \frac{1}{1+\theta} > 0$ and $C_2 = \frac{1}{1}1+\theta - \frac{1-C}{1+\theta+C} > 0$. Clearly $C_1, C_2 \to 0$ for $C \to 0+$. By Theorem 4.8 there is a family of random variables $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ with exponential law of parameter 1 such that for all $\theta < 1$

$$\mathbb{E}\left[e^{-\theta\bar{\tau}(\varepsilon)}\right] = \frac{1}{1+\theta}.$$

Hence for all $\theta < 1$

$$\mathbb{E}\left[\sup_{x \in \tilde{D}^\pm(\varepsilon^\gamma)} \left|\exp\left(\theta\lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)\right) - \exp\left(\theta\bar{\tau}(\varepsilon)\right)\right|\right] \leqslant \max\{C_1, C_2\}.$$

□

4.2.1. The Upper Estimate of the Laplace Transform

In this Subsection we shall establish the upper estimate part of Theorem 2.18.

Proposition 4.10 (The upper estimate)**.** *Assume that Hypotheses (H.1) and (H.2) are satisfied. Then for all* $\theta > -1$ *and* $C \in (0, 1+\theta)$ *there exists* $\varepsilon_0 = \varepsilon_0(\theta) > 0$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{E}\left[\sup_{x \in \tilde{D}^\pm(\varepsilon^\gamma)} \exp\left(-\theta\lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)\right)\right] \leqslant \frac{1+C}{1+\theta-C}.$$

Proof. Fix $\Gamma > 0$ such that Proposition 3.1 and inequality (4.11) are true and let C be given as stated. For convenience we drop the superscript $\pm$. As the jumps of the noise process L exceed any fixed barrier $\mathbb{P}$-a.s., i.e. $\tau_x(\varepsilon)$ is $\mathbb{P}$-a.s. finite, we can rewrite the Laplace transform of $\tau_x(\varepsilon)$ in the following way for $\varepsilon > 0$:

$$\mathbb{E}\left[\sup_{x \in \tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)\tau_x(\varepsilon)}\right] = \sum_{k=1}^{\infty}\Bigg(\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_k} \sup_{x \in \tilde{D}(\varepsilon^\gamma)} \mathbf{1}\{\tau_x(\varepsilon) = T_k\}\right]$$
$$+ \mathbb{E}\left[\sup_{x \in \tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)\tau_x(\varepsilon)}\mathbf{1}\{\tau_x(\varepsilon) \in (T_{k-1}, T_k)\}\right]\Bigg) \quad (4.13)$$

We shall estimate the first and second sums in (4.13) separately. As (4.13) indicates, our arguments will be based on the separation of a large jump compound Poisson part, and a small jump part which does not deviate from the rapidly relaxing deterministic solution trajectories of the Chafee-Infante equation by much.

Estimate of the first sum of (4.13): For $k \in \mathbb{N}$ we can decompose the large jump exit by writing

$$\begin{aligned}
&\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_k} \sup_{\sup_{x\in\tilde{D}(\varepsilon^\gamma)}} \mathbf{1}\{\tau_x(\varepsilon) = T_k\}\right] \\
&= \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_k} \sup_{x\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}\{X^\varepsilon(s;x) \in D(\varepsilon^\gamma) \text{ for } s \in [0,T_k) \text{ and } X^\varepsilon(T_k;x) \notin D(\varepsilon^\gamma)\}\right] \\
&= \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_k} \sup_{x\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}\left(\bigcap_{i=1}^{k-1} A_{X^\varepsilon(0;x)} \circ \theta_{T_{i-1}} \cap B_{X^\varepsilon(0;x)} \circ \theta_{T_{k-1}}\right)\right] \\
&= \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_k} \sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \prod_{i=1}^{k-1} \mathbf{1}\left(A_{X^\varepsilon(0;x)} \circ \theta_{T_{i-1}}\right) \mathbf{1}\left(B_{X^\varepsilon(0;x)} \circ \theta_{T_{k-1}}\right)\right].
\end{aligned}$$

Note that $T_k = T_{k-1} + T_1 \circ \theta_{T_{k-1}}$. We use the strong Markov property, conditioning on the past of T_{k-1}, and then estimate from above by the supremum over all values $X(T_{k-1};x)$ can take. This gives

$$\begin{aligned}
&\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_k} \sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \prod_{i=1}^{k-1} \mathbf{1}\left(A_{X^\varepsilon(0;x)} \circ \theta_{T_{i-1}}\right) \mathbf{1}\left(B_{X^\varepsilon(0;x)} \circ \theta_{T_{k-1}}\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_k} \sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \prod_{i=1}^{k-1} \mathbf{1}\left(A_{X^\varepsilon(0;x)} \circ \theta_{T_{i-1}}\right) \mathbf{1}\left(B_{X^\varepsilon(0;x)} \circ \theta_{T_{k-1}}\right) |\mathcal{F}_{T_{k-1}}\right]\right] \\
&\leqslant \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_{k-1}} \sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \prod_{i=1}^{k-2} \mathbf{1}\left(A_{X^\varepsilon(0;x)} \circ \theta_{T_{i-1}}\right) \mathbf{1}\left(A_{X^\varepsilon(0;x)} \circ \theta_{T_{k-2}}\right)\right] \\
&\qquad \cdot \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)\, T_1} \sup_{y\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}\left(B_y\right)\right].
\end{aligned}$$

By $k-1$-fold iteration of this argument we obtain for $k \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)T_k} \mathbf{1}\{\tau_x(\varepsilon) = T_k\}\right] \\
&\qquad \leqslant \left(\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y\in D(\varepsilon^\gamma)} \mathbf{1}\left(A_y\right)\right]\right)^{k-1} \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y\in D(\varepsilon^\gamma)} \mathbf{1}\left(B_y\right)\right].
\end{aligned}$$

Now we have to estimate the individual terms corresponding to A_y and $B_y, y \in D(\varepsilon^\gamma)$ by exploiting Lemma 4.2.

Claim 1: There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{E}_x \left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y \in D(\varepsilon^\gamma)} \mathbf{1}(A_y) \right] \leqslant \frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)} \left(1 - \frac{\lambda(\varepsilon)}{\beta_\varepsilon}(1 - \frac{C}{5}) \right).$$

In the inequality of Lemma 4.2 *ix*) we can pass to the supremum in $y \in D(\varepsilon^\gamma)$, and integrate to obtain, using the independence of jump times and heights

$$\begin{aligned}
&\mathbb{E} \left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y \in D(\varepsilon^\gamma)} \mathbf{1}(A_y) \right] \\
&\qquad \leqslant \mathbb{E} \left[e^{-\theta\lambda(\varepsilon)T_1} \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} \right] \mathbb{P}\left(\varepsilon\|W_1\| > (1/2)\varepsilon^{2\gamma} \right) \\
&\qquad + \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \right] \mathbb{P}\left(\varepsilon W_1 \in D_0 \right) + \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y \in D(\varepsilon^\gamma)} \mathbf{1}(E_y^c) \right] \\
&\qquad =: K_1K_2 + K_3K_4 + K_5.
\end{aligned}$$

Let us estimate $K_1, \ldots, K_5$ separately, in the order of increasing complexity. We shall see that the asymptotic behavior of the aggregate is dominated by the second summand.

1. Clearly
$$K_3 = \int_0^\infty e^{-\theta\lambda(\varepsilon)s} \beta_\varepsilon e^{-\beta_\varepsilon s} \, \mathrm{d}s = \frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}. \tag{4.14}$$

2. By definition of $\lambda(\varepsilon)$ we know
$$K_4 = \mathbb{P}\left(W_1 \in (1/\varepsilon)D_0 \right) = 1 - \lambda(\varepsilon)/\beta_\varepsilon. \tag{4.15}$$

3. The recurrence time of logarithmic order in ε enters into the calculation of K_1. Remember that κ is fixed along with Γ. We have
$$\begin{aligned}
K_1 &= \int_0^{T_{rec}+\kappa\gamma|\ln\varepsilon|} e^{-\theta\lambda(\varepsilon)s} \beta_\varepsilon e^{-\beta_\varepsilon s} \, \mathrm{d}s \\
&= \frac{\beta_\varepsilon}{\theta\lambda(\varepsilon) + \beta_\varepsilon} \left[1 - \exp\left(-(\theta\lambda(\varepsilon) + \beta_\varepsilon)(T_{rec} + \kappa\gamma|\ln\varepsilon|) \right) \right].
\end{aligned}$$

4. For the estimation of $K_2 = \mathbb{P}\left(\|\varepsilon W_1\| > (1/2)\varepsilon^{2\gamma} \right)$ we use Lemma 4.4 *II*), providing $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$
$$K_2 \leqslant 4\varepsilon^{\alpha(1-2\gamma-\rho)}. \tag{4.16}$$

5. For K_5 we refer to Proposition 3.1 and its Corollary 3.2 ensuring that for ε_0 small

enough we have for $0 < \varepsilon \leqslant \varepsilon_0$

$$K_5 \leqslant \frac{C}{10} \frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)} \frac{\lambda(\varepsilon)}{\beta_\varepsilon}. \tag{4.17}$$

Inserting the estimates we obtained for $K_1, \ldots, K_5$ into our original inequality we can write for ε_0 small enough and $0 < \varepsilon \leq \varepsilon_0$

$$\begin{aligned}
&\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y\in D(\varepsilon^\gamma)} \mathbf{1}(A_y)\right] \\
&\leqslant \frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}\big[1 - \exp\left(-(\theta\lambda(\varepsilon) + \beta_\varepsilon)(T_{rec} + \kappa\gamma|\ln\varepsilon|)\right)\big]4\varepsilon^{\alpha(1-2\gamma-\rho)} + \\
&\qquad \frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}\left(1 - \frac{\lambda(\varepsilon)}{\beta_\varepsilon}\right) + \frac{C}{10}\frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon} \\
&= \frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}\left[1 - \frac{\lambda(\varepsilon)}{\beta_\varepsilon}(1 - \frac{C}{10})\right] \\
&\qquad + \frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}\big[1 - \exp\left(-(\theta\lambda(\varepsilon) + \beta_\varepsilon)(T_{rec} + \kappa\gamma|\ln\varepsilon|)\right)\big]4\varepsilon^{\alpha(1-2\gamma-\rho)} \\
&\qquad\qquad = \frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}\left[1 - \frac{\lambda(\varepsilon)}{\beta_\varepsilon}\left(1 - \frac{C}{10} - K_6\right)\right],
\end{aligned}$$

where

$$K_6 = \frac{\beta_\varepsilon}{\lambda(\varepsilon)}\big[1 - \exp\left(-(\theta\lambda(\varepsilon) + \beta_\varepsilon)(T_{rec} + \kappa\gamma|\ln\varepsilon|)\right)\big]4\varepsilon^{\alpha(1-2\gamma-\rho)}.$$

To estimate K_6, recall that by (2.5) and (2.13) and Lemma 4.4 *I*) there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$

$$\frac{\beta_\varepsilon}{\lambda(\varepsilon)} \leqslant \varepsilon^{-\alpha(1-\rho)}. \tag{4.18}$$

By (4.18) and $2\gamma < \rho$ we know that by eventually choosing ε_0 smaller we may obtain for $0 < \varepsilon \leqslant \varepsilon_0$

$$K_6 \leqslant 4(\theta\lambda(\varepsilon) + \beta_\varepsilon)(T_{rec} + \kappa\gamma|\ln\varepsilon|)\varepsilon^{-2\alpha\gamma} \leqslant \frac{C}{10}.$$

We can summarize our findings in stating that there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{x\in D(\varepsilon^\gamma)} \mathbf{1}\left(A_x\right)\right] \leqslant \frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}\left(1 - \frac{\lambda(\varepsilon)}{\beta_\varepsilon}\left(1 - \frac{C}{5}\right)\right). \tag{4.19}$$

Claim 2: There is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y\in D(\varepsilon^\gamma)} \mathbf{1}\left(B(y)\right)\right] \leqslant (1 + C)\frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon}.$$

In the inequality of Lemma 4.2 *x*) we again pass to the supremum in $y \in D(\varepsilon^\gamma)$, and integrate to obtain, using the independence of jump times and heights

$$\begin{aligned}
&\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\sup_{y\in D(\varepsilon^\gamma)}\mathbf{1}\left(B^1_y\right)\right]\\
&\leqslant \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\right]\mathbb{P}\left(\varepsilon W_1\notin D_0(\varepsilon^\gamma,\varepsilon^{2\gamma})\right)\\
&\qquad+\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\mathbf{1}\{T_1<T_{rec}+\kappa\gamma|\ln\varepsilon|\}\right]+\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\sup_{y\in D(\varepsilon^\gamma)}\mathbf{1}(E^c_y)\right]\\
&\qquad\qquad=: K_3(1-K_8)+K_1+K_5. \qquad (4.20)
\end{aligned}$$

Examining K_1 more closely, we recognize by Lemma 4.4 I) and by $\rho>1/2$ that there exists $\varepsilon_0>0$ such that for $0<\varepsilon\leqslant\varepsilon_0$ we have

$$\begin{aligned}
K_1 &= \frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon}\left[1-\exp\left(-(\theta\lambda(\varepsilon)+\beta_\varepsilon)(T_{rec}+\kappa\gamma|\ln\varepsilon|)\right)\right]\\
&\leqslant \frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon}(\theta\lambda(\varepsilon)+\beta_\varepsilon)(T_{rec}+\kappa\gamma|\ln\varepsilon|)\\
&\leqslant \frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon}\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\left(\frac{\beta_\varepsilon}{\lambda(\varepsilon)}(\theta\lambda(\varepsilon)+\beta_\varepsilon)(T_{rec}+\kappa\gamma|\ln\varepsilon|)\right)\\
&\leqslant \frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon}\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\frac{C}{10}. \qquad (4.21)
\end{aligned}$$

In order to estimate K_8 we use Lemma 4.4 III). It yields that there is $\varepsilon_0>0$ such that for $0<\varepsilon\leqslant\varepsilon_0$

$$K_8\leqslant(1+\frac{C}{5})\frac{\lambda(\varepsilon)}{\beta_\varepsilon}.$$

Recalling the estimates for K_3 and K_5 from the preceding part, we find ε_0 such that for $0<\varepsilon\leqslant\varepsilon_0$

$$\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\sup_{y\in D(\varepsilon^\gamma)}\mathbf{1}\left(B(y)\right)\right]\leqslant(1+C)\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon}.$$

Estimate of the second sum of (4.13): In order to treat the summands

$$\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)}e^{-\theta\lambda(\varepsilon)\tau_x(\varepsilon)}\mathbf{1}\{\tau_x(\varepsilon)\in(T_{k-1},T_k)\}\right]$$

for $k\in\mathbb{N}$ we have to distinguish the cases $\theta\geqslant0$ and $\theta\in(-1,0)$. More precisely, for $k\in\mathbb{N}$ we start with the inequality

$$\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)\tau_x(\varepsilon)}\mathbf{1}\{\tau_x(\varepsilon)\in(T_{k-1},T_k)\}\right]$$
$$\leqslant \begin{cases} \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_{k-1}}\sup_{x\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}\{\tau_x(\varepsilon)\in(T_{k-1},T_k)\}\right], & \text{if } 0\leqslant\theta,\\ \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_{k}}\sup_{x\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}\{\tau_x(\varepsilon)\in(T_{k-1},T_k)\}\right], & \text{if } -1<\theta<0.\end{cases}$$

We have to argue separately for the cases $k=1$ and $k\geqslant 2$.

Claim 3: There is $\varepsilon_0>0$ such that for all $0<\varepsilon\leqslant\varepsilon_0$

$$\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)\tau_x(\varepsilon)}\mathbf{1}\{\tau_x(\varepsilon)\in(0,T_1)\}\right]\leqslant\frac{C}{5}\left(\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\right)\frac{\lambda(\varepsilon)}{\beta_\varepsilon}.$$

1. First consider the case $\theta\geqslant 0$. With $T_0=0$ we see

$$\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)T_0}\mathbf{1}\{\tau_x(\varepsilon)\in(0,T_1)\}\right]$$
$$\leqslant\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}\{X^\varepsilon(s;x)\notin D(\varepsilon^\gamma)\text{ for some } s\in(0,T_1)\}\right]$$
$$=\mathbb{E}\left[\sup_{y\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}\{Y^\varepsilon(s;y)\notin D(\varepsilon^\gamma)\text{ for some } s\in(0,T_1)\}\right].$$

We apply Lemma 4.2 *xi)* for the event under the expectation and the first part of the proof of Corollary 3.1 which guarantees that there is $\varepsilon_0>0$ such that for $0<\varepsilon\leqslant\varepsilon_0$

$$\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)T_0}\mathbf{1}\{\tau_x(\varepsilon)\in(T_0,T_1)\}\right]\leqslant\frac{C}{5}\,\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\,\frac{\lambda(\varepsilon)}{\beta_\varepsilon}.$$

2. In case $\theta\in(-1,0)$ we may write

$$\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)\tau_x(\varepsilon)}\mathbf{1}\{\tau_x(\varepsilon)\in(0,T_1)\}\right]$$
$$\leqslant\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\sup_{x\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}\{X^\varepsilon(s;x)\notin D(\varepsilon^\gamma)\text{ for some } s\in(0,T_1)\}\right]$$
$$=\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\sup_{y\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}\{Y^\varepsilon(s;y)\notin D(\varepsilon^\gamma)\text{ for some } s\in(0,T_1)\}\right].$$

Using Lemma 4.2 *xi*) and Corollary 3.2, we obtain for sufficiently small $\varepsilon > 0$ an analogous inequality

$$\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)T_1}\mathbf{1}\{\tau_x(\varepsilon)\in(0,T_1)\}\right] \leqslant \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\sup_{y\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}(E^c(y))\right] = \left(\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\right)\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\frac{C}{5}. \quad (4.22)$$

We continue for the case $k \geqslant 2$.

Claim 4: There exists $\varepsilon_0 > 0$ such that for any $k \geqslant 2$ and $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)\tau_x(\varepsilon)}\mathbf{1}\{\tau_x(\varepsilon)\in(T_{k-1},T_k)\}\right] \leqslant \left(\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\left(1-\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\left(1-\frac{C}{5}\right)\right)\right)^{k-2}\frac{C}{5}\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon}.$$

1. For $\theta \geqslant 0$ we use the strong Markov property as in the estimate for the first summand to get for $k \geqslant 2$

$$\begin{aligned}&\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)T_{k-1}}\mathbf{1}\{\tau_x(\varepsilon)\in(T_{k-1},T_k)\}\right]\\ &= \left(\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\sup_{y\in D(\varepsilon^\gamma)}\mathbf{1}(A_y)\right]\right)^{k-2}\\ &\cdot\ \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\sup_{y\in D(\varepsilon^\gamma)}\mathbf{1}(A_y)\mathbf{1}\{Y^\varepsilon(s;X^\varepsilon(0;y))\circ\theta_{T_1}\notin D(\varepsilon^\gamma)\text{ f. s. } s\in(0,T_1)\}\right]\end{aligned} \quad (4.23)$$

The event appearing in the last integral is estimated in Lemma 4.2 *xii*). With this in mind we obtain

$$\begin{aligned}&\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\sup_{y\in D(\varepsilon^\gamma)}\mathbf{1}(A_y)\mathbf{1}\{Y^\varepsilon(s,X^\varepsilon(0;x))\circ\theta_{T_1}\notin D(\varepsilon^\gamma)\text{ f. s. } s\in(0,T_1)\}\right]\\ &\leqslant \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\right]\mathbb{P}\left(\varepsilon W_1\in D_0^*(\varepsilon^\gamma)\right)+\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\mathbf{1}\{T_1\leqslant T_{rec}+\kappa\gamma|\ln\varepsilon|\}\right]\\ &\qquad + 2\,\mathbb{E}\left[\sup_{y\in D(\varepsilon^\gamma)}\mathbf{1}(E^c(y))\right] =: K_3K_9+K_1+2K_5.\end{aligned}$$

Lemma 4.4 IV) provides $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$

$$K_9 = \mathbb{P}\left(\varepsilon W_1 \in D_0^*(\varepsilon^\gamma)\right) \leqslant \frac{C}{20}\frac{\lambda(\varepsilon)}{\beta_\varepsilon}.$$

The asymptotic behavior of K_3, K_5 and K_1 as $\varepsilon \to 0$ is known from previous parts of the proof. We may therefore deduce that there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y\in D(\varepsilon^\gamma)} \mathbf{1}(A_y)\mathbf{1}\{Y^\varepsilon(s, X^\varepsilon(0;x)) \circ \theta_{T_1} \notin D(\varepsilon^\gamma) \text{ for s. } s \in (0,T_1)\}\right] \leqslant \frac{C}{5}\frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon}. \qquad (4.24)$$

An estimate for the first factor is known from Claim 1 in (4.19). This completes the proof of the inequality of Claim 4 for $\theta \geqslant 0$.

2. Let us consider the case $\theta \in (-1, 0)$. With arguments as before employing the strong Markov property we obtain this time the estimate

$$\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_{k-1}} \sup_{x\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}\{\tau_x(\varepsilon) \in (T_{k-1}, T_k)\}\right]$$
$$\leqslant \left(\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y\in D(\varepsilon^\gamma)} \mathbf{1}(A_y)\right]\right)^{k-2}$$
$$\cdot\ \mathbb{E}\Bigg[e^{-\theta\lambda(\varepsilon)(T_1+t_2\circ\theta_{T_1})} \sup_{y\in D(\varepsilon^\gamma)} \mathbf{1}(A_y)\mathbf{1}\{Y^\varepsilon(X^\varepsilon(0,y)) \circ \theta_{T_1} \notin D(\varepsilon^\gamma) \text{ for some } s \in (0,T_1)\}\Bigg].$$

To estimate the last factor in the previous expression, we use the strong Markov property once again, and then Claim 1 and the previous result, keeping in mind Lemma 4.2. We conclude that there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)(T_1+t_2\circ\theta_{T_1})} \sup_{y\in D(\varepsilon^\gamma)} \mathbf{1}(A_y)\mathbf{1}\{Y^\varepsilon(X^\varepsilon(0,y)) \circ \theta_{T_1} \notin D(\varepsilon^\gamma) \text{ for s. } s \in (0,T_1)\}\right] \leqslant \frac{C}{5}\frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon}.$$

This provides the same estimate (4.24) as in the case $\theta \geqslant 0$ and completes the proof of Claim 4.

Combined estimate of (4.13): Combining the estimates for the first and second summands in (4.13) by Claims 1-4 we find $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)\tau_x(\varepsilon)}\right] \\
&\leqslant \sum_{k=1}^{\infty}\left(\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\right)^{k-1}\left(1-\frac{\lambda(\varepsilon)}{\beta_\varepsilon}(1-\frac{C}{5})\right)^{k-1}\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon}(1+\frac{C}{5}) \\
&+\frac{(\frac{C}{5})\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon} \\
&\qquad+\sum_{k=2}^{\infty}\left(\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\right)^{k-2}\left(1-\frac{\lambda(\varepsilon)}{\beta_\varepsilon}(1-\frac{C}{5})\right)^{k-2}\frac{(\frac{C}{5})\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon} \\
&\qquad\leqslant (1+(2/5)C)\,\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\sum_{k=0}^{\infty}\left(\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\left(1-\frac{\lambda(\varepsilon)}{\beta_\varepsilon}(1-\frac{C}{5})\right)\right)^{k} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\leqslant \frac{1+C}{\theta+(1-C)}.
\end{aligned}
$$

The sum obviously converges if and only if $C < \theta + 1$. □

4.2.2. The Lower Estimate of the Laplace Transform

The lower estimate is easier to obtain since we can neglect the non-negative second sum in equation (4.13). The tedious reasoning concerning small deviations of the small noise part from the deterministic solution trajectories of the Chafee-Infante equation is not needed.

Proposition 4.11 (The lower estimate)**.** *Assume that Hypothesis (H.1) and (H.2) are satisfied. Then for all $\theta > -1$ and $C \in (0, 1+\theta)$ there is $\varepsilon_0 = \varepsilon_0(\theta) > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$*

$$\mathbb{E}\left[\inf_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \exp\left(-\theta\lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon)\right)\right] \geqslant \frac{1+C}{1+\theta-C}.$$

Proof. Again we omit the superscript $\pm$ and fix $\Gamma > 0$ large enough such that Proposition 3.1 and inequality (4.11) are true. Reducing equation (4.13) in the way indicated, and applying the strong Markov property in the usual way we obtain the estimate

$$
\begin{aligned}
&\mathbb{E}\left[\inf_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)\tau_x(\varepsilon)}\right]\\
&\geqslant \sum_{k=1}^{\infty}\mathbb{E}\left[\inf_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)T_k}\mathbf{1}\{\tau_x(\varepsilon)=T_k\}\right]\\
&= \sum_{k=1}^{\infty}\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_k}\inf_{x\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}\{X^\varepsilon(s;x)\in D(\varepsilon^\gamma)\text{ f. } s\in[0,T_k).X^\varepsilon(T_k;x)\notin D(\varepsilon^\gamma)\}\right]\\
&\quad\geqslant \sum_{k=1}^{\infty}\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_k}\inf_{x\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}\left(\bigcap_{i=1}^{k-1}A^-_{X^\varepsilon(0;x))}\circ\theta_{T_{i-1}}\cap B_{X^\varepsilon(0;x))}\circ\theta_{T_{k-1}}\right)\right]\\
&\quad\quad\geqslant \sum_{k=1}^{\infty}\left(\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\inf_{y\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}(A^-_y)\right]\right)^{k-1}\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\inf_{y\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}(B_y)\right]. \qquad (4.25)
\end{aligned}
$$

Let us treat the terms appearing in (4.25) in a similar way as for the upper estimate.

Claim 1: There is $\varepsilon_0>0$ such that for all $0<\varepsilon\leqslant\varepsilon_0$

$$
\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\inf_{x\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}(A^-_x)\right]\geqslant\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\left(1-(1+C)\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\right).
$$

First we apply Lemma 4.1 *xiii*) and take the infimum over $y\in\tilde{D}(\varepsilon^\gamma)$ and integrate, to obtain

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\inf_{y\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}(A^-_y)\right]\\
&\geqslant \mathbb{E}\left[\varepsilon^{-\theta\lambda(\varepsilon)T_1}\right]\mathbb{P}\left(\varepsilon W_1\in D_0(\varepsilon^\gamma,\varepsilon^{2\gamma},\varepsilon^{2\gamma})\right)-\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\mathbf{1}\{T_1<T_{rec}+\kappa\gamma|\ln\varepsilon|\}\right]\\
&\qquad\qquad\qquad -2\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\sup_{y\in\tilde{D}(\varepsilon^\gamma)}\mathbf{1}(E^c(y))\right]\\
&\qquad\qquad = K_3\left(1-\mathbb{P}(W_1\in(1/\varepsilon)D_0^c(\varepsilon^\gamma,\varepsilon^{2\gamma},\varepsilon^{2\gamma}))\right)-K_1-2K_5.
\end{aligned}
$$

By Lemma 4.4 V) there exists $\varepsilon_0>0$ such that for $0<\varepsilon\leqslant\varepsilon_0$

$$
\mathbb{P}(\varepsilon W_1\in D_0^c(\varepsilon^\gamma,\varepsilon^{2\gamma},\varepsilon^{2\gamma}))\leqslant(1+\frac{C}{5})\frac{\lambda(\varepsilon)}{\beta_\varepsilon}.
$$

Using the estimates for K_1,K_3,K_5 derived in the proof of the upper estimate, we finally find $\varepsilon_0>0$ such that for $0<\varepsilon\leqslant\varepsilon_0$

$$\begin{aligned}
&\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \inf_{x\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}(A_x^-)\right] \\
&\qquad \geqslant \frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\left(1-(1+\frac{C}{5})\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\right) - \frac{C}{5}\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon} \\
&\qquad\quad - \frac{C}{5}\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\frac{\lambda(\varepsilon)}{\beta_\varepsilon} - 2\frac{C}{5}\frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon}\frac{\lambda(\varepsilon)}{\beta_\varepsilon} \\
&\qquad \geqslant \frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\left(1-(1+C)\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\right).
\end{aligned}$$

Claim 2: There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \inf_{y\in\tilde{D}_0(\varepsilon^\gamma)} \mathbf{1}(B_y)\right] \geqslant \frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon}\left((1-C)\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\right).$$

Using Lemma 4.2 *xiv*) we can infer that for sufficiently small ε

$$\begin{aligned}
&\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \inf_{y\in\tilde{D}_0(\varepsilon^\gamma)} \mathbf{1}(B_y)\right] \\
&\qquad \geqslant \mathbb{P}\left(\varepsilon W_1 \notin D_0\right)\left(\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\right] - \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\mathbf{1}\{T_1 < T_{rec}+\kappa\gamma|\ln\varepsilon|\}\right]\right) \\
&\qquad\quad - \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}(E^c(y))\right] \\
&\qquad \geqslant \frac{\lambda(\varepsilon)}{\beta_\varepsilon}\left(\frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon} - \frac{C}{5}\frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon}\frac{\lambda(\varepsilon)}{\beta_\varepsilon} - \frac{C}{5}\frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon}\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\right) \\
&\qquad \geqslant \frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon}\left((1-C)\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\right).
\end{aligned}$$

Combined estimate of (4.25): Combining the estimates just obtained in Claim 1 and Claim 2 we finally get a lower estimate by a geometric series, leading to

$$\begin{aligned}
&\mathbb{E}\left[\inf_{x\in\tilde{D}(\varepsilon^\gamma)} e^{-\theta\lambda(\varepsilon)\tau_x(\varepsilon)}\right] \geq \\
&\qquad \geqslant \sum_{k=1}^{\infty}\left(\frac{\beta_\varepsilon}{\beta_\varepsilon+\theta\lambda(\varepsilon)}\left(1-(1+C)\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\right)\right)^{k-1}\frac{\beta_\varepsilon}{\theta\lambda(\varepsilon)+\beta_\varepsilon}\left((1-C)\frac{\lambda(\varepsilon)}{\beta_\varepsilon}\right) \\
&\qquad = \frac{\lambda(\varepsilon)(1-C)}{\theta\lambda(\varepsilon)-(1+C)\lambda(\varepsilon)} = \frac{1-C}{\theta+1+C}.
\end{aligned}$$

The series converges if and only if $-(1+C) < \theta$. This completes the proof of our main theorem. □

4.2.3. Asymptotic Exit Times in Probability

In this subsection we construct explicit an explicit family $(s^{\pm}(\varepsilon))_{\varepsilon>0}$ of random variables exponential law with parameter $\lambda^{\pm}(\varepsilon)$, to which the first exit times $(\tau_x^{\pm}(\varepsilon))_{\varepsilon>0}$ converge in probability.

Definition 4.12. Recall that $W_k = \Delta_{T_k}\eta^{\varepsilon}, k \in \mathbb{N}$ the k-th "large" jump of $(L(t))_{t\geqslant 0}$ in the sense of Section 2.1. For the event $B_k^{\diamond}(\varepsilon) := \{\varepsilon W_k \notin D_0^{\pm}\}, k \in \mathbb{N}, \varepsilon > 0$ we define the random variable

$$s^{\pm}(\varepsilon) := \sum_{k=1}^{\infty} T_k \prod_{j=1}^{k-1}(1 - \mathbf{1}_{B_j^{\diamond}(\varepsilon)})\mathbf{1}_{B_k^{\diamond}(\varepsilon)}, \quad \varepsilon > 0. \tag{4.26}$$

Lemma 4.13. *(Thinning lemma)*
For $0 < \varepsilon < 1$ the random variable $s^{\pm}(\varepsilon)$ is exponentially distributed with parameter $\lambda^{\pm}(\varepsilon)$, $\lambda^{\pm}(\varepsilon) = \nu\left(\frac{1}{\varepsilon}D_0^{\pm}\right)$, where ν is the Lévy jump measure of the noise process $(L(t))_{t\geqslant 0}$ driving X^{ε}.

Proof. Let $\theta > 0$. We can calculate the Laplace transform of $s^{\pm}(\varepsilon)$ directly

$$\begin{aligned}
\mathbb{E}\left[e^{-\theta s^{\pm}(\varepsilon)}\right] &= \mathbb{E}\left[e^{-\theta\sum_{k=1}^{\infty} T_k \prod_{j=1}^{k-1}(1-\mathbf{1}(B_j^{\diamond}))\mathbf{1}(B_k^{\diamond})}\right] \\
&= \mathbb{E}\left[\prod_{k=1}^{\infty} e^{-\theta T_k \prod_{j=1}^{k-1}(1-\mathbf{1}(B_j^{\diamond}))\mathbf{1}(B_k^{\diamond})}\right] \\
&= \sum_{k=1}^{\infty}\mathbb{E}\left[e^{-\theta T_k}\prod_{j=1}^{k-1}(1-\mathbf{1}(B_j^{\diamond}))\mathbf{1}(B_k^{\diamond})\right] \\
&= \sum_{k=1}^{\infty}\mathbb{E}\left[\prod_{j=1}^{k-1} e^{-\theta t_j}(1-\mathbf{1}(B_j^{\diamond}))e^{-\theta t_k}\mathbf{1}(B_k^{\diamond})\right].
\end{aligned}$$

Exploiting the independence of $(W_k)_{k\in\mathbb{N}}$ and $(T_k)_{k\in\mathbb{N}}$ as well as the stationarity of $(W_k)_{k\in\mathbb{N}}$ each summand takes the form

$$\begin{aligned}
&\mathbb{E}\left[\prod_{j=1}^{k-1} e^{-\theta t_j}(1-\mathbf{1}(B_j^\diamond))e^{-\theta t_k}\mathbf{1}(B_k^\diamond)\right]\\
&=\prod_{j=1}^{k-1}\mathbb{E}\left[e^{-\theta t_j}(1-\mathbf{1}(B_j^\diamond))\right]\mathbb{E}\left[e^{-\theta t_k}\mathbf{1}(B_k^\diamond)\right]=\mathbb{E}\left[e^{-\theta t_1}(1-\mathbf{1}(B_1^\diamond))\right]^{k-1}\mathbb{E}\left[e^{-\theta t_1}\mathbf{1}(B_1^\diamond)\right]\\
&=\left(\mathbb{E}\left[e^{-\theta t_1}\right](1-\mathbb{P}(B_1^\diamond))\right)^{k-1}\mathbb{E}\left[e^{-\theta t_1}\right]\mathbb{P}(B_1^\diamond)\\
&=\left(\frac{\beta_\varepsilon}{\theta+\beta_\varepsilon}(1-\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon})\right)^{k-1}\frac{\beta_\varepsilon}{\theta+\beta_\varepsilon}\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}\left[e^{-\theta\sigma^\pm(\varepsilon)}\right] &= \sum_{k=1}^{\infty}\left(\frac{\beta_\varepsilon}{\theta+\beta_\varepsilon}(1-\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon})\right)^{k-1}\frac{\beta_\varepsilon}{\theta+\beta_\varepsilon}\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon}\\
&=\frac{\beta_\varepsilon}{\theta+\beta_\varepsilon}\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon}\frac{1}{1-\frac{\beta_\varepsilon}{\theta+\beta_\varepsilon}(1-\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon})}=\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon}\frac{1}{\frac{\theta+\beta_\varepsilon}{\beta_\varepsilon}-(1-\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon})}\\
&=\frac{\lambda^\pm(\varepsilon)}{\theta+\lambda^\pm(\varepsilon)}.
\end{aligned}$$

□

Theorem 4.14. *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ be given and suppose Hypotheses (H.1) and (H.2) are satisfied. Then the random variable $s^\pm(\varepsilon)$ satisfies that for any $\theta > 0$ and $C > 0$ there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$*

$$\mathbb{E}\left[\inf_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} e^{-\theta|\tau_x^\pm(\varepsilon)-s^\pm(\varepsilon)|}\right] \geqslant 1-C. \tag{4.27}$$

Corollary 4.15. *Under the assumptions of Theorem 4.15 there is a family of exponentially distributed random variables $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ with parameter 1 on the same probability space $(\Omega,\mathcal{F},\mathbb{P})$ as the driving Lévy noise $(L(t))_{t\geqslant 0}$ such that in probability*

$$\lim_{\varepsilon\to 0+}\inf_{y\in\tilde{D}^\pm(\varepsilon^\gamma)}|\lambda^\pm(\varepsilon)\tau_y^\pm(\varepsilon)-\bar{\tau}(\varepsilon)| = \lim_{\varepsilon\to 0+}\sup_{y\in\tilde{D}^\pm(\varepsilon^\gamma)}|\lambda^\pm(\varepsilon)\tau_y^\pm(\varepsilon)-\bar{\tau}(\varepsilon)| = 0.$$

Proof. By Lemma 4.13 the random variable $s^\pm(\varepsilon), \varepsilon > 0$ defined by (4.26) is an exponentially distributed random variable with parameter $\lambda^\pm(\varepsilon)$. Hence $\bar{\tau}(\varepsilon) := \lambda^\pm(\varepsilon)s^\pm(\varepsilon)$, $\varepsilon > 0$, is exponentially distributed with parameter 1. Theorem 4.14 shows that for any $\theta > 0$ and $C > 0$ there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ and $\theta > 0$ we have

$$1-C \leqslant \mathbb{E}\left[\inf_{y\in\tilde{D}^\pm(\varepsilon^\gamma)} e^{-\theta|\tau_y^\pm(\varepsilon)-\frac{\bar{\tau}(\varepsilon)}{\lambda^\pm(\varepsilon)}|}\right] \leqslant \mathbb{E}\left[\sup_{y\in\tilde{D}^\pm(\varepsilon^\gamma)} e^{-\theta|\tau_y^\pm(\varepsilon)-\frac{\bar{\tau}(\varepsilon)}{\lambda^\pm(\varepsilon)}|}\right] \leqslant 1. \tag{4.28}$$

Since $0 < \lambda^{\pm}(\varepsilon) = \varepsilon^{\alpha}\ell\left(\frac{1}{\varepsilon}\right)\mu\left((D_0^{\pm})^c\right) \searrow 0$ for $\varepsilon \to 0+$, we may fix $\varepsilon_0 > 0$ such that $\lambda^{\pm}(\varepsilon) < 1$ for $0 < \varepsilon \leq \varepsilon_0$. By monotonicity we obtain for this $\varepsilon_0 > 0$ and the same $C > 0$, $\theta > 0$ as before that

$$1 - C \leqslant \mathbb{E}\left[\inf_{y \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} e^{-\theta|\lambda^{\pm}(\varepsilon)\tau_y^{\pm}(\varepsilon) - \bar{\tau}(\varepsilon)|}\right] \leqslant \mathbb{E}\left[\sup_{y \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} e^{-\theta|\lambda^{\pm}(\varepsilon)\tau_y^{\pm}(\varepsilon) - \bar{\tau}(\varepsilon)|}\right] \leqslant 1 \tag{4.29}$$

for $0 < \varepsilon \leq \varepsilon_0$. Hence

$$\lim_{\varepsilon \to 0+} \mathcal{L}(\inf_{y \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} |\lambda^{\pm}(\varepsilon)\tau_y^{\pm}(\varepsilon) - \bar{\tau}(\varepsilon)|) = \lim_{\varepsilon \to 0+} \mathcal{L}(\sup_{y \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} |\lambda^{\pm}(\varepsilon)\tau_y^{\pm}(\varepsilon) - \bar{\tau}(\varepsilon)|) = \delta_0$$

in the weak sense. Due to the equivalence of weak convergence and convergence in probability for constant limits we obtain the desired result. □

Proof. (of Theorem 4.14):
Step 1: Reduction to incremental events
Define for $\varepsilon > 0$ the events

$$A_j^{\diamond} := \left(B_j^{\diamond}\right)^c = \{\varepsilon W_j \in D_0^{\pm}\}, \text{ for } j \in \mathbb{N}.$$

In this step we follow the lines of the first part of the proof of Theorem 2.18. Exploiting the strong Markov property of X^{ε} and Y^{ε} respectively and the Lévy property of L we may estimate the exit first exit time from below by probabilities of exit events A_x^-, B_x, $A_1^{\diamond}$ and $B_1^{\diamond}$ in the following way

$$\begin{aligned}
&\mathbb{E}\left[\inf_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} e^{-\theta|\tau_x^{\pm}(\varepsilon) - s^{\pm}(\varepsilon)|}\right] \\
\geqslant\ & \sum_{k=1}^{\infty} \mathbb{E}\left[\inf_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} e^{-\theta|\tau_x^{\pm}(\varepsilon) - s^{\pm}(\varepsilon)|}\mathbf{1}_{\{\tau_x^{\pm} = T_k\} \cap \bigcap_{j=1}^{k-1} A_{j-1}^{\diamond} \cap B_k^{\diamond}}\right] \\
=\ & \sum_{k=1}^{\infty} \mathbb{E}\left[\inf_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}_{\{\tau_x^{\pm} = T_k\} \cap \bigcap_{j=1}^{k-1} A_{j-1}^{\diamond} \cap B_k^{\diamond}}\right] \\
\geqslant\ & \sum_{k=1}^{\infty} \mathbb{E}\left[\inf_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}_{\bigcap_{j=1}^{k-1} A^-_{X^{\varepsilon}(0;x)} \circ \theta_{T_{j-1}} \cap B_{X^{\varepsilon}(0;x)} \circ \theta_{T_{k-1}} \cap \bigcap_{j=1}^{k-1} A_{j-1}^{\diamond} \cap B_k^{\diamond}}\right] \\
\geqslant\ & \sum_{k=1}^{\infty} \mathbb{E}\left[\inf_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \prod_{j=1}^{k-1}\left(\mathbf{1}_{A^-_{X^{\varepsilon}(0;x)} \circ \theta_{T_{j-1}} \cap A_j^{\diamond}}\right)\mathbf{1}_{B_{X^{\varepsilon}(0;x)} \circ \theta_{T_{k-1}} \cap B_k^{\diamond}}\right].
\end{aligned}$$

With the help of the strong Markov property of X^{ε} we obtain for $k \in \mathbb{N}$ that

$$\begin{aligned}
&\mathbb{E}\left[\inf_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\prod_{j=1}^{k-1}\left(\mathbf{1}_{A^{-}_{X^{\varepsilon}(0;x)}\circ\theta_{T_{j-1}}\cap A^{\diamond}_{j}}\right)\mathbf{1}_{B_{X^{\varepsilon}(0;x)}\circ\theta_{T_{k-1}}\cap B^{\diamond}_{k}}\right]\\
\geqslant\ &\mathbb{E}\left[\mathbb{E}\left[\inf_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\prod_{j=1}^{k-1}\left(\mathbf{1}_{A^{-}_{X^{\varepsilon}(0;x)}\circ\theta_{T_{j-1}}\cap A^{\diamond}_{j}}\right)\mathbf{1}_{B_{X^{\varepsilon}(0;x)}\circ\theta_{T_{k-1}}\cap B^{\diamond}_{k}}\mid\mathcal{F}_{T_{k-1}}\right]\right]\\
\geqslant\ &\mathbb{E}\left[\inf_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\prod_{j=1}^{k-2}\left(\mathbf{1}_{A^{-}_{X^{\varepsilon}(0;x)}\circ\theta_{T_{j-1}}\cap A^{\diamond}_{j}}\right)\mathbf{1}_{A^{-}_{X^{\varepsilon}(0;x)}\circ\theta_{T_{k-2}}\cap A^{\diamond}_{k-1}}\right]\mathbb{E}\left[\inf_{y\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\mathbf{1}_{B_{y}\cap B^{\diamond}_{1}}\right].
\end{aligned}$$

By $k-1$-fold iteration of this argument we obtain for $k\in\mathbb{N}$

$$\begin{aligned}
&\mathbb{E}\left[\inf_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\prod_{j=1}^{k-1}\left(\mathbf{1}_{A^{-}_{X^{\varepsilon}(0;x)}\circ\theta_{T_{j-1}}\cap A^{\diamond}_{j}}\right)\mathbf{1}_{B_{X^{\varepsilon}(0;x)}\circ\theta_{T_{k-1}}\cap B^{\diamond}_{k}}\right]\\
&\qquad\geqslant\ \mathbb{E}\left[\inf_{y\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\mathbf{1}_{A^{-}_{y}\cap A^{\diamond}_{1}}\right]^{k-1}\mathbb{E}\left[\inf_{y\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\mathbf{1}_{B_{y}\cap B^{\diamond}_{1}}\right].
\end{aligned}\tag{4.30}$$

Step 2: Inspection of the incremental events
This step consists in the estimate of the events $A^{-}_{y}\cap A^{\diamond}_{1}$ and $B_y\cap B^{\diamond}_{1}$ by the small deviation event E_x, the relaxation event $\{T_1\geqslant T_{rec}+\kappa\gamma|\ln\varepsilon|\}$ and the pure jump events $\{\varepsilon W_1\in D^{\pm}_{0}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})\}$ and $A^{\diamond}_{1}$. By Lemma 4.2 *xv)* and *xvi)* we know for $x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})$

$$\mathbf{1}(A^{-}_{x}\cap A^{\diamond}_{1})\geqslant\mathbf{1}\{\varepsilon W_1\in D^{\pm}_{0}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})\}-\mathbf{1}\{T_1<T_{rec}+\kappa\gamma|\ln\varepsilon|\}-2\cdot\mathbf{1}(E^{c}_{x}),\tag{4.31}$$

$$\mathbf{1}(B_x\cap B^{\diamond}_{1})\geqslant\mathbf{1}\{\varepsilon W_1\notin D^{\pm}_{0}\}(1-\mathbf{1}\{T_1<T_{rec}+\kappa\gamma|\ln\varepsilon|\})-\mathbf{1}(E^{c}_{x}).\tag{4.32}$$

Step 3: Lower estimate of the first factor
In this step we exploit Lemma 4.4 V) and Corollary 3.2 in order to estimate the first factor in the summands of the right-hand side of (4.13)

$$\begin{aligned}
&\mathbb{E}\left[\inf_{y\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\mathbf{1}_{A^{-}_{y}\cap A^{\diamond}_{1}}\right]\\
&\quad\geqslant\ \mathbb{P}\left(\varepsilon W_1\in D^{\pm}_{0}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})\right)-\mathbb{P}\left(T_1<T_{rec}+\kappa\gamma|\ln\varepsilon|\right)-2\mathbb{E}\left[\inf_{y\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\mathbf{1}(E^{c}_{y})\right].
\end{aligned}$$

By Lemma 4.4 V) for any $C_1>0$ given there is $\varepsilon_1>0$ such that such that for $0<\varepsilon\leq\varepsilon_1$

$$\mathbb{P}(\varepsilon W_1\in D^{c}_{0}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma}))\leqslant(1+C_1)\frac{\lambda^{\pm}(\varepsilon)}{\beta_{\varepsilon}}.$$

Furthermore for any $C_2 > 0$ there is $\varepsilon_2 > 0$ such that for $0 < \varepsilon \leq \varepsilon_2$

$$\mathbb{P}\left(T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\right) = \int\limits_0^{T_{rec}+\kappa\gamma|\ln\varepsilon|} \beta_\varepsilon e^{-\beta_\varepsilon s} \mathrm{d}s = 1 - e^{-\beta_\varepsilon(T_{rec}+\kappa\gamma|\ln\varepsilon|)}$$
$$\leqslant (1 + C_2)\beta_\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|)$$

and there is $\vartheta > \alpha(1-\rho)$ such that for any $C_3 > 0$ there is $\varepsilon_3 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_3$

$$\mathbb{E}\left[\inf_{y\in\tilde{D}^\pm(\varepsilon^\gamma)} \mathbf{1}(E_y^c)\right] \leqslant C_3\varepsilon^\vartheta.$$

This means if $C_i \leq \frac{J_1}{4}$, $i = 1, 2, 3$ that for $0 < \varepsilon \leqslant \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ respectively

$$\mathbb{E}\left[\inf_{y\in\tilde{D}^\pm(\varepsilon^\gamma)} \mathbf{1}_{A_y^- \cap A_1^\diamond}\right]$$
$$\geqslant 1 - (1 + C_1)\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon} - (1 + C_2)\beta_\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|) - 2C_3\varepsilon^\vartheta$$
$$\geqslant 1 - (1 + J_1)\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon}. \quad (4.33)$$

Step 4: Lower estimate of the second factor
Since $\rho > \frac{1}{2}$, we know $\frac{\beta_\varepsilon}{\lambda^\pm(\varepsilon)}\beta_\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|) \to 0+$, $\varepsilon \to 0+$. Hence for given $C_4 > 0$ there is $\varepsilon_4 > 0$ with $\frac{\beta_\varepsilon}{\lambda^\pm(\varepsilon)}\beta_\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|) \leq C_4$ for $0 < \varepsilon \leqslant \varepsilon_4$. Thus for $C_i \leq \frac{J_2}{3}, i = 3, 4$ and $0 < \varepsilon \leqslant \min\{\varepsilon_4, \varepsilon_5\}$

$$\mathbb{E}\left[\inf_{y\in\tilde{D}^\pm(\varepsilon^\gamma)} \mathbf{1}(B_y \cap B_1^\diamond)\right]$$
$$\geqslant \mathbb{P}(\varepsilon W_1 \notin D_0^\pm) - \mathbb{P}\left(T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\right) - \mathbb{E}\left[\inf_{y\in\tilde{D}^\pm(\varepsilon^\gamma)} \mathbf{1}(E_y^c)\right]$$
$$\geqslant \frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon} - (1 + C_4)\beta_\varepsilon(T_{rec} + \kappa\gamma|\ln\varepsilon|) - C_3\varepsilon^\vartheta \geqslant (1 - J_2)\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon}.$$

Step 5: The asymptotic geometric series
We can now combine Step 4 and 5 with estimate (4.13) for $0 < \varepsilon \leqslant \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, such that

$$\sum_{k=1}^\infty \left(1 - (1 + J_1)\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon}\right)^{k-1} (1 - J_2)\frac{\lambda^\pm(\varepsilon)}{\beta_\varepsilon} \geqslant \frac{1 - J_2}{1 + J_1} \geqslant 1 - C$$

if $J_1 > 0$ and $J_2 > 0$ are chosen to satisfy $0 \leqslant J_2 \leqslant C + (1 - C)J_1$. □

4.3. Proofs of the Estimates for the Exit Events

4.3.1. Partial Estimates (Proof of Lemma 4.1)

Proof. 1. On the event $\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}$ we have $u(T_1; x) \in B_{(1/2)\varepsilon^{2\gamma}}(\phi^{\pm})$. Hence on $E_x \cap \{T_1 \geq T_{rec} + \kappa\gamma|\ln\varepsilon|\}$ the relationship $Y^{\varepsilon}(T_1; x) \in B_{\varepsilon^{2\gamma}}(\phi^{\pm})$ holds.

2. On the event A_x, on which $Y^{\varepsilon}(T_1; x) + \varepsilon W_1 \in D^{\pm}(\varepsilon^{\gamma})$, we can infer that necessarily $\varepsilon W_1 \in D_0^{\pm}(\varepsilon^{\gamma}) + B_{\varepsilon^{2\gamma}}(0) \subset D_0^{\pm}$ in order to obtain *i*).

3. On B_x we have
$$Y^{\varepsilon}(T_1; x) + \varepsilon W_1 \notin D^{\pm}(\varepsilon^{\gamma}).$$
Hence by Part 1, on $B_x \cap E_x \cap \{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}$ we have $\varepsilon W_1 \notin D_0^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})$, proving *ii*).

4. On C_x, we have
$$Y^{\varepsilon}(T_1; x) + \varepsilon W_1 \in D^{\pm}(\varepsilon^{\gamma}) \setminus \tilde{D}^{\pm}(\varepsilon^{\gamma}).$$
Therefore on $C_x \cap E_x \cap \{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}$ the relationship
$$\varepsilon W_1 \in \left(D_0^{\pm}(\varepsilon^{\gamma}) \setminus D_0^{\pm}(\varepsilon^{\gamma})\right) + B_{\varepsilon^{2\gamma}}(0)$$
follows, proving *iii*).

5. On B_x, we know that $Y(T_1, x) + \varepsilon W_1 \notin D^{\pm}(\varepsilon^{\gamma})$. But $T_1 > T_{rec} + \kappa\gamma|\ln\varepsilon|$ and $\|\varepsilon W_1\| \leqslant (1/2)\varepsilon^{2\gamma}$ additionally entail that
$$Y(T_1, x) + \varepsilon W_1 \in B_{(3/2)\varepsilon^{2\gamma}}(\phi^{\pm}) \cap \left(D^{\pm}(\varepsilon^{\gamma})\right)^c.$$
This set is empty for sufficiently small $\varepsilon > 0$, proving *iv*).

6. On C_x, $Y(T_1, x) + \varepsilon W_1 \in D^{\pm}(\varepsilon^{\gamma}) \setminus \tilde{D}^{\pm}(\varepsilon^{\gamma})$ holds. Imposing $T_1 > T_{rec} + \kappa\gamma|\ln\varepsilon|$ and $\|\varepsilon W_1\| \leqslant (1/2)\varepsilon^{2\gamma}$ additionally leads to the intersection
$$\left(D^{\pm}(\varepsilon^{\gamma}) \setminus \tilde{D}^{\pm}(\varepsilon^{\gamma})\right) \cap B_{(3/2)\varepsilon^{2\gamma}}(0)$$
which is empty for sufficiently small $\varepsilon > 0$. This proves *v*).

7. By the positive invariance of $\tilde{D}^{\pm}(\varepsilon^{\gamma})$, for $x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})$ we know $u(t; x) \in \tilde{D}^{\pm}(\varepsilon^{\gamma})$ for all $t \geqslant 0$. Hence on the event E_x we have $Y^{\varepsilon}(t; x) \in \tilde{D}^{\pm}(\varepsilon^{\gamma}) + B_{\varepsilon^{2\gamma}}(0)$ for all $t \in [0, T_1]$. Lemma 2.14 guarantees that $\tilde{D}^{\pm}(\varepsilon^{\gamma}) + B_{\varepsilon^{2\gamma}}(0) \subset D^{\pm}(\varepsilon^{\gamma})$. Part 1 of the proof shows $Y^{\varepsilon}(T_1; x) \in B_{\varepsilon^{2\gamma}}(\phi^{\pm})$ on $E_x \cap \{T_1 > T_{rec} + \kappa\gamma|\ln\varepsilon|\}$. Hence on this set, if $Y^{\varepsilon}(T_1; x) + \varepsilon W_1 \in D^{\pm}(\varepsilon^{\gamma})$ then
$$\varepsilon W_1 = Y^{\varepsilon}(T_1; x) + \varepsilon W_1 - Y^{\varepsilon}(T_1; x) \in D_0^{\pm}.$$
This proves *vi*) by contraposition.

8. By Part 1 of the proof, $Y^\varepsilon(T_1;x) \in B_{\varepsilon^{2\gamma}}(\phi^\pm)$ on $E_x \cap \{T_1 > T_{rec} + \kappa\gamma|\ln\varepsilon|\}$. The positive invariance of $\tilde{D}^\pm(\varepsilon^\gamma)$ implies for $x \in \tilde{D}^\pm(\varepsilon^\gamma)$ that $u(t;x) \in \tilde{D}^\pm(\varepsilon^\gamma)$ for $t \in [0,T_1]$. On E_x consequently $Y^\varepsilon(t;x) \in \tilde{D}^\pm(\varepsilon^\gamma) + B_{\varepsilon^{2\gamma}}(\phi^\pm)$ for $t \in [0,T_1]$. Lemma 2.14 guarantees that $\tilde{D}^\pm(\varepsilon^\gamma) + B_{\varepsilon^{2\gamma}}(0) \subset D^\pm(\varepsilon^\gamma)$. Thus on the event $E_x \cap \{T_1 > T_{rec} + \kappa\gamma|\ln\varepsilon|\}$, the condition $\varepsilon W_1 \in D_0^\pm(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})$ implies with the help of Lemma 2.14

$$\begin{aligned} Y^\varepsilon(T_1;x) + \varepsilon W_1 &\in B_{\varepsilon^{2\gamma}}(\phi^\pm) + D_0^\pm(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) \\ &= B_{\varepsilon^{2\gamma}}(0) + D^\pm(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) \subset D^\pm(\varepsilon^\gamma, e^{2\gamma}) = \tilde{D}^\pm(\varepsilon^\gamma). \end{aligned}$$

This proves vii).

□

4.3.2. Full Estimates (Proof of Lemma 4.2)

Proof. We drop the superscript $\pm$ for convenience.

1. After a repartition of the event A_x for $x \in D(\varepsilon^\gamma)$ we exploit Lemma 4.1 i) in the third step of

$$\begin{aligned} \mathbf{1}(A_x) &\leqslant \mathbf{1}(A_x)\mathbf{1}(E_x) + \mathbf{1}(E_x^c) \\ &\leqslant \mathbf{1}(A_x)\mathbf{1}(E_x)\mathbf{1}\{\varepsilon\|W_1\| > (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \\ &\quad + \mathbf{1}(A_x)\mathbf{1}(E_x)\mathbf{1}\{\varepsilon\|W_1\| > (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} \\ &\quad + \mathbf{1}\{\varepsilon\|W_1\| \leqslant (1/2)\varepsilon^{2\gamma}\} + \mathbf{1}(E_x^c) \\ &\leqslant \mathbf{1}\{\varepsilon\|W_1\| > (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{\varepsilon W_1 \in D_0\} \\ &\quad + \mathbf{1}\{\varepsilon\|W_1\| > (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} \\ &\quad + \mathbf{1}\{\varepsilon\|W_1\| \leqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{\varepsilon W_1 \in D_0\} + \mathbf{1}(E_x^c) \\ &= \mathbf{1}\{\varepsilon W_1 \in D_0\} + \mathbf{1}\{\varepsilon\|W_1\| > (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} + \mathbf{1}(E_x^c). \end{aligned}$$

This proves ix).

2. In the same way we decompose B_x for $x \in D(\varepsilon^\gamma)$ and use Lemma 4.1 ii) and iv)

in the second estimate to get

$$\begin{aligned}
&\mathbf{1}(B_x)\\
&\leqslant \mathbf{1}(E_x^c) + \mathbf{1}(B_x)\mathbf{1}(E_x)\mathbf{1}\{\|\varepsilon W_1\| \geqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\qquad + \mathbf{1}(B_x)\mathbf{1}(E_x)\mathbf{1}\{\|\varepsilon W_1\| \geqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\qquad + \mathbf{1}(B_x)\mathbf{1}(E_x)\mathbf{1}\{\|\varepsilon W_1\| < (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\qquad + \mathbf{1}(B_x)\mathbf{1}(E_x)\mathbf{1}\{\|\varepsilon W_1\| < (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\leqslant \mathbf{1}(E_x^c)+\mathbf{1}\{\varepsilon W_1 \notin D_0(\varepsilon^\gamma,\varepsilon^{2\gamma})\}+\mathbf{1}\{\|\varepsilon W_1\| \geqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec}+\kappa\gamma|\ln\varepsilon|\}\\
&\qquad\quad + 0 + \mathbf{1}\{\|\varepsilon W_1\| < (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\qquad = \mathbf{1}(E_x^c) + \mathbf{1}\{\varepsilon W_1 \notin D_0(\varepsilon^\gamma,\varepsilon^{2\gamma})\} + \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}. \qquad (4.34)
\end{aligned}$$

Hence $x)$ is shown.

3. For $y \in \tilde{D}(\varepsilon^\gamma)$ by definition of the small jumps component and $\tilde{D}(\varepsilon^\gamma)$ we have

$$\begin{aligned}
&\mathbf{1}\{Y^\varepsilon(s;y) \notin D(\varepsilon^\gamma) \text{ for some } s \in (0,T_1)\}\\
&\quad\leqslant \mathbf{1}(E_y^c) + \mathbf{1}(E_y)\ \mathbf{1}\{Y^\varepsilon(s;y) \notin D(\varepsilon^\gamma) \text{ for some } s \in (0,T_1)\} = \mathbf{1}(E_y^c). \qquad (4.35)
\end{aligned}$$

This proves $xi)$.

4. Recall the convention $\tilde{D}(\varepsilon^\gamma) = D(\varepsilon^\gamma,\varepsilon^{2\gamma})$. In this tedious estimate we have to take into account the evolution of the solution trajectory over two adjacent big jump intervals. We have by definition

$$\begin{aligned}
&\mathbf{1}(A_x)\mathbf{1}\{Y^{\varepsilon,2}(s;X^\varepsilon(T_1,x)) \notin D(\varepsilon^\gamma) \text{ for some } s \in (0,t_2)\}\\
&= \mathbf{1}\{Y^\varepsilon(s;x) \in D(\varepsilon^\gamma) \text{ for } s \in (0,t_1) \text{ and } X^\varepsilon(T_1;x) \in \tilde{D}(\varepsilon^\gamma)\}\\
&\qquad\qquad \cdot\ \mathbf{1}\{Y^{\varepsilon,2}(s;X^\varepsilon(T_1;x)) \notin D(\varepsilon^\gamma) \text{ for some } s \in (0,t_2)\}\\
&\quad + \mathbf{1}\{Y^\varepsilon(s;x) \in D(\varepsilon^\gamma) \text{ for } s \in (0,t_1) \text{ and } X^\varepsilon(T_1;x) \in \big(D(\varepsilon^\gamma) \setminus \tilde{D}(\varepsilon^\gamma)\big)\}\\
&\qquad\qquad \cdot\ \mathbf{1}\{Y^{\varepsilon,2}(s;X^\varepsilon(T_1;x)) \notin D(\varepsilon^\gamma) \text{ for some } s \in (0,t_2)\}\\
&\leqslant \sup_{z\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}\{Y^{\varepsilon,2}(s;z) \notin D(\varepsilon^\gamma) \text{ for some } s \in (0,t_2)\}\\
&\quad + \mathbf{1}\{Y^\varepsilon(s;x) \in D(\varepsilon^\gamma) \text{ for } s \in (0,t_1) \text{ and } X^\varepsilon(T_1;x) \in \big(D(\varepsilon^\gamma) \setminus \tilde{D}(\varepsilon^\gamma)\big)\}\\
&\qquad = \sup_{z\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}\{Y^\varepsilon(s;z) \notin D(\varepsilon^\gamma) \text{ for some } s \in (0,T_1)\} \circ \theta_{T_1} + \mathbf{1}(C_x). \qquad (4.36)
\end{aligned}$$

Now we repeat the arguments employed for Part 2, replacing $D_0^c(\varepsilon^\gamma,\varepsilon^{2\gamma})$ by

$D_0^*(\varepsilon^\gamma) = \big(D_0(\varepsilon^\gamma) \setminus \tilde{D}_0(\varepsilon^\gamma)\big) + B_{\varepsilon^{2\gamma}}(0)$. We may exploit Lemma 4.1 *iii*) to obtain

$$\begin{aligned}
&\mathbf{1}(C_x)\\
&\leqslant \mathbf{1}(E_x^c) + \mathbf{1}(C_x)\mathbf{1}(E_x)\mathbf{1}\{\|\varepsilon W_1\| \geqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\qquad + \mathbf{1}(C_x)\mathbf{1}(E_x)\mathbf{1}\{\|\varepsilon W_1\| \geqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\qquad + \mathbf{1}(C_x)\mathbf{1}(E_x)\mathbf{1}\{\|\varepsilon W_1\| < (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\qquad + \mathbf{1}(C_x)\mathbf{1}(E_x)\mathbf{1}\{\|\varepsilon W_1\| < (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\leqslant \mathbf{1}(E_x^c) + \mathbf{1}\left\{\varepsilon W_1 \in \big(D_0(\varepsilon^\gamma) \setminus \tilde{D}_0(\varepsilon^\gamma)\big) + B_{\varepsilon^{2\gamma}}(0)\right\}\\
&\qquad\quad + \mathbf{1}\{\|\varepsilon W_1\| \geqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} + 0\\
&\qquad\quad + \mathbf{1}\{\|\varepsilon W_1\| < (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\qquad\quad = \mathbf{1}(E_x^c) + \mathbf{1}\left\{\varepsilon W_1 \in D_0^*(\varepsilon^\gamma)\right\} + \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}. \qquad (4.37)
\end{aligned}$$

Hence collecting the estimates (4.36) and (4.37) and applying *xi*) we obtain

$$\begin{aligned}
&\mathbf{1}(A_x)\mathbf{1}\{Y^\varepsilon(s; X^\varepsilon(0,x)) \circ \theta_{T_1} \notin D(\varepsilon^\gamma) \text{ for some } s \in (0,T_1)\}\\
&\leqslant \mathbf{1}\left\{\varepsilon W_1 \in D_0^*(\varepsilon^\gamma)\right\} + \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\quad + \sup_{z\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}\{Y^\varepsilon(s;z) \circ \theta_{T^1} \notin D(\varepsilon^\gamma) \text{ for some } s \in (0,T_1)\} + \mathbf{1}(E_x^c)\\
&\qquad \leqslant \mathbf{1}\left\{\varepsilon W_1 \in D_0^*(\varepsilon^\gamma)\right\} + \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\qquad\qquad + \sup_{y\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}(E_y^c) \circ \theta_{T_1} + \mathbf{1}(E_x^c).
\end{aligned}$$

This proves *xii*).

5. For $x \in \tilde{D}(\varepsilon^\gamma)$ we use the definition of A_x^- and E_x to get

$$\mathbf{1}(A_x^-) \geqslant \mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon\|W_1\| \leqslant (1/2)\varepsilon^{2\gamma}\}.$$

Due to Lemma 4.1 *vii*) for the second estimate to follow, as well as the elementary inequality $\mathbf{1}(C_1)\mathbf{1}(C_2) = \mathbf{1}(C_1)(1 - \mathbf{1}(C_2^c)) \geqslant \mathbf{1}(C_1) - \mathbf{1}(C_2^c)$ valid for arbitrary events C_1, C_2 we obtain

$$\begin{aligned}
&\mathbf{1}(A_x^-)\\
&\geqslant \mathbf{1}(A_x^-)\mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon\|W_1\| \leqslant \frac{1}{2}\varepsilon^{2\gamma}\}\\
&\qquad + \mathbf{1}(A_x^-)\mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon\|W_1\| > \frac{1}{2}\varepsilon^{2\gamma}\}
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathbf{1}(A_x^-)\\
&\geqslant \mathbf{1}(E_x)\mathbf{1}\{\varepsilon\|W_1\| \leqslant \frac{1}{2}\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&+ \mathbf{1}\{\varepsilon W_1 \in D_0(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\}\mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon\|W_1\| > \frac{1}{2}\varepsilon^{2\gamma}\}\\
&\geqslant \mathbf{1}\{\varepsilon\|W_1\| \leqslant \frac{1}{2}\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} - 2\cdot\mathbf{1}(E_x^c)\\
&\quad + \mathbf{1}\{\varepsilon W_1 \in D_0(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon\|W_1\| > \frac{1}{2}\varepsilon^{2\gamma}\}.
\end{aligned}$$

Using twice that $\mathbf{1}(G \cap E) \geqslant \mathbf{1}(G) - \mathbf{1}(E^c)$ for sets E, G we continue collecting the terms

$$\begin{aligned}
&\mathbf{1}(A_x^-)\\
&\geqslant -2\cdot\mathbf{1}(E_x^c) + \mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\cdot\left(\mathbf{1}\{\varepsilon W_1 \in D_0(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\}\mathbf{1}\{\varepsilon\|W_1\| > (1/2)\varepsilon^{2\gamma}\} + \mathbf{1}\{\varepsilon\|W_1\| \leqslant \frac{1}{2}\varepsilon^{2\gamma}\}\right)\\
&\geqslant \mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon W_1 \in D_0(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\} - 2\cdot\mathbf{1}(E_x^c)\\
&\qquad \geqslant \mathbf{1}\{\varepsilon W_1 \in D_0(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\} - \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} - 2\cdot\mathbf{1}(E_x^c).
\end{aligned}$$

This proves statement *xiii)*.

6. To obtain the last estimate *xiv)* for $x \in \tilde{D}(\varepsilon^\gamma)$, we use Lemma 4.1 *vi)* which yields

$$\begin{aligned}
\mathbf{1}(B_x) &\geqslant \mathbf{1}(B_x)\mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\\
&\geqslant \mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon W_1 \notin D_0\}\\
&\geqslant \mathbf{1}\{\varepsilon W_1 \notin D_0\}(1 - \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}) - \mathbf{1}(E_x^c). \quad (4.38)
\end{aligned}$$

7. Since

$$\mathbf{1}(A_x^-) \geqslant \mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon W_1 \in D_0^\pm(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\}$$

and $\{\varepsilon W_1 \in D_0^\pm(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\} \subseteq \{\varepsilon W_k \in D_0^\pm\} = A_1^\diamond$ we obtain

$$\mathbf{1}(A_x^- \cap A_1^\diamond) \geqslant \mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon W_1 \in D_0^\pm(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\} \quad (4.39)$$

and by the same reasoning as part 5.

$$\mathbf{1}(A_x^- \cap A_1^\diamond) \geqslant \mathbf{1}\{\varepsilon W_1 \in D_0^\pm(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})\} - \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} - 2\cdot\mathbf{1}(E_x^c). \tag{4.40}$$

This shows inequality *xv*).

8. Similarly since

$$\mathbf{1}(B_x) \geqslant \mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}(B_1^\diamond)$$

it follows

$$\mathbf{1}(B_x \cap B_1^\diamond) \geqslant \mathbf{1}(E_x)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}(B_1^\diamond), \tag{4.41}$$

giving the desired estimate *xvi*)

$$\mathbf{1}(B_x \cap B_1^\diamond) \geqslant \mathbf{1}\{\varepsilon W_1 \notin D_0^\pm\}(1 - \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}) - \mathbf{1}(E_x^c). \tag{4.42}$$

□

4.3.3. Asymptotics of Large Jump Events (Proof of Lemma 4.4)

Proof. 1. For convenience we drop the exponent $\pm$. Since ν is of regular variation of with index $-\alpha$, there exists a regularly varying function v of index $-\alpha$ and a non-zero measure $\mu \in M_0(H)$ such that for all $A \in \mathcal{B}(H)$ with $0 \notin \bar{A}$ follows

$$\lim_{t\to\infty} v(t)\nu(tA) = \mu(A).$$

Therefore from (2.13)

$$\lim_{\varepsilon\to 0+} \frac{\lambda(\varepsilon)}{\beta_\varepsilon}\frac{v(\varepsilon^{-\rho})}{v(\varepsilon^{-1})} = \lim_{\varepsilon\to 0+} \frac{\nu\left(\varepsilon^{-1}D_0^c\right)}{\nu\left(\varepsilon^{-\rho}B_1^c(0)\right)}\frac{v(\varepsilon^{-\rho})}{v(\varepsilon^{-1})} = \frac{\mu(D_0^c)}{\mu(B_1^c(0))}.$$

Since

$$\lim_{\varepsilon\to 0+} \frac{v(\varepsilon^{-1})}{v(\varepsilon^{-\rho})}\frac{\ell(\varepsilon^{-\rho})}{\ell(\varepsilon^{-1})}\varepsilon^{-\alpha(1-\rho)} = 1$$

and the quotient of slowly varying functions $\ell(\varepsilon^{-\rho})/\ell(\varepsilon^{-1})$ tends to 1 as $\varepsilon \to 0+$ we can infer that for any $C > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$\left(\frac{\mu(D_0^c)}{\mu(B_1^c(0))} - C\right)\varepsilon^{\alpha(1-\rho)} \leqslant \frac{\lambda(\varepsilon)}{\beta_\varepsilon} \leqslant \left(\frac{\mu(D_0^c)}{\mu(B_1^c(0))} + C\right)\varepsilon^{\alpha(1-\rho)}.$$

This shows *I*).

2. By the choice $1/2 < \rho < 1 - 2\gamma$ for $1 \geq \varepsilon > 0$ small enough

$$\mathbb{P}\left(\varepsilon\|W_1\| > (1/2)\varepsilon^{2\gamma}\right) = \frac{\nu\left(\left((1/2)\varepsilon^{2\gamma-1} \wedge \varepsilon^{-\rho}\right) B_1^c(0)\right)}{\nu\left(\left(\varepsilon^{-\rho}\right) B_1^c(0)\right)}$$
$$= \frac{\nu\left((1/2)\varepsilon^{2\gamma-1} B_1^c(0)\right)}{\nu\left(\left(\varepsilon^{-\rho}\right) B_1^c(0)\right)}.$$

The slowly varying function $\ell : (0,\infty) \to (0,\infty)$ helps us to identify the limits as $\varepsilon \to 0$. In fact, we may write

$$\lim_{\varepsilon\to 0+} \frac{\nu\left(\left(\dfrac{1}{2\varepsilon^{1-2\gamma}}\right) B_1^c(0)\right)}{\nu\left(\dfrac{2\varepsilon^{1-2\gamma}}{\varepsilon^{\rho}} \dfrac{1}{2\varepsilon^{1-2\gamma}} B_1^c(0)\right)} \frac{\ell\left((1/2)\varepsilon^{2\gamma-1)}\right)}{\ell\left(\varepsilon^{-\rho}\right)} \frac{\varepsilon^{\alpha\rho}}{2^{\alpha}\varepsilon^{\alpha(1-2\gamma)}} = 1$$

Again the quotient $\ell((1/2)\varepsilon^{2\gamma-1})/\ell(\varepsilon^{-\rho})$ tends to one as $\varepsilon \to 0+$. Hence we find that for all sufficiently small $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{P}\left(\varepsilon\|W_1\| > (1/2)\varepsilon^{2\gamma}\right) \leqslant 4\varepsilon^{\alpha(1-2\gamma-\rho)},$$

showing II).

3. First note that by Lemma 2.13 we have $\bigcap_{\varepsilon>0} D_0^c(\varepsilon^{\gamma}, \varepsilon^{2\gamma}) = D_0^c$. By Lemma 4.3$i$), for any $\eta > 0$ there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$

$$\lim_{\varepsilon\to 0+} \left(\frac{\nu\left(\frac{1}{\varepsilon} D_0^c(\varepsilon^{\gamma}, \varepsilon^{2\gamma})\right)}{\beta_\varepsilon} - \frac{\lambda(\varepsilon)}{\beta_\varepsilon}\right) \varepsilon^{-\alpha(1-\rho)}$$
$$= \lim_{\varepsilon\to 0+} \frac{\nu\left(\frac{1}{\varepsilon}\left(D_0^c(\varepsilon^{\gamma}, \varepsilon^{2\gamma}) \setminus D_0^c\right)\right)}{\beta_\varepsilon} \varepsilon^{-\alpha(1-\rho)} = \frac{\mu\left(D_0^c(\varepsilon^{\gamma}, \varepsilon^{2\gamma}) \setminus D_0^c\right)}{\mu(B_1^c(0))} < \eta. \quad (4.43)$$

Assume that $C < \frac{\mu(D_0^c)}{\mu(B_1^c(0))}$, and choose $\eta = C\left(\mu(D_0^c)/\mu(B_1^c(0)) - C\right)$. Then by the lower estimate in Part 1 for ε small enough we have

$$\mathbb{P}\left(\varepsilon W_1 \in \tilde{D}_0^c(\varepsilon^{\gamma})\right) = \frac{\nu\left(\frac{1}{\varepsilon} D_0^c(\varepsilon^{\gamma}, \varepsilon^{2\gamma})\right)}{\beta_\varepsilon} \leqslant \frac{\lambda(\varepsilon)}{\beta_\varepsilon} + \eta\, \varepsilon^{\alpha(1-\rho)} \leqslant (1+C)\frac{\lambda(\varepsilon)}{\beta_\varepsilon}.$$

This proves III).

4. Analogously, using (H.2) and Lemma 4.3 ii), we find that for $\eta > 0$ there exists

$\varepsilon_0 > 0$ such that for $0 \leqslant \varepsilon \leqslant \varepsilon_0$ we have

$$\lim_{\varepsilon \to 0+} \mathbb{P}\left(\varepsilon W_1 \in D_0^*(\varepsilon^\gamma)\right) \varepsilon^{-\alpha(1-\rho)} = \lim_{\varepsilon \to 0+} \frac{\nu\left(\varepsilon^{-1} D_0^*(\varepsilon^\gamma)\right)}{\nu\left(\varepsilon^{-\rho} B_1^c(0)\right)} \varepsilon^{-\alpha(1-\rho)}$$
$$\leqslant \lim_{\varepsilon \to 0+} \frac{\nu\left(\varepsilon^{-1} D_0^*(\varepsilon_0^\gamma)\right)}{\nu\left(\varepsilon^{-\rho} B_1^c(0)\right)} \varepsilon^{-\alpha(1-\rho)} = \frac{\mu\left(D_0^*(\varepsilon_0^\gamma)\right)}{\mu(B_1^c(0))} < \eta.$$

Thus again with the help of part 1 for $\eta = C\left(\mu(D_0^c)/\mu(B_1^c(0)) - C\right)$ and by reducing ε_0 eventually we have for $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{P}\left(\varepsilon W_1 \in D_0^*(\varepsilon^\gamma)\right) \leqslant C \frac{\lambda(\varepsilon)}{\beta_\varepsilon}.$$

Hence $IV)$ is proved.

5. The argument for $V)$ is identical to the one for $III)$. We just have to replace $D_0^c(\varepsilon^\gamma, \varepsilon^{2\gamma}) \setminus D_0^c$ by $D_0 \setminus D_0(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})$ and use Lemma 4.3 *iii*). This yields the desired estimate for ε_0 small enough. The universal ε_0 for which all claims of the Lemma hold, will then be the minimum of the individual ones for $I)$ to $V)$.

□

5. Asymptotic Transition Times

The preceding Chapter is concerned with the effect of small Lévy noise in H of intensity ε of triggering exits from the reduced domains of attraction of the stable states $\phi^{\pm}$ of a Chafee-Infante equation. Noise is seen quite generally to make stable states of deterministic systems given by ordinary or partial differential equations metastable. In this Chapter, we shall investigate more closely the dynamics of the stochastic system, in particular the stochastic transition and wandering behavior between the metastable states. We shall ask questions about the reduced dynamics of the system, i.e. the reduction of the jump diffusion equation to a simple Markov chain in the small noise limit $\varepsilon \to 0+$ boiling down the dynamics to a simple switching between the metastable states. It will be seen that this reduction is related to a scaling limit of the jump diffusion in the polynomial scale $\varepsilon^{-\alpha}$ resulting from the asymptotic behavior of first exit times of domains of attraction encountered in the previous Chapter.

5.1. Asymptotic Times to enter different Reduced Domains of Attraction

For $\varepsilon, \gamma > 0$ we recall the complement of the reduced domains of attraction

$$\tilde{D}^0(\varepsilon^\gamma) = H \setminus \left(\tilde{D}^+(\varepsilon^\gamma) \cup \ \tilde{D}^-(\varepsilon^\gamma)\right),$$

with $\tilde{D}^{\pm}(\varepsilon^\gamma) = D^{\pm}(\varepsilon^\gamma, \varepsilon^{2\gamma})$ according to Definition 2.10. Theorem 2.18 describes the asymptotic behavior of the first exit times $\tau_x^{\pm}(\varepsilon)$ from $D^{\pm}(\varepsilon^\gamma)$ for initial values in $x \in \tilde{D}^{\pm}(\varepsilon^\gamma)$. The aim of this and the next Section is to determine the asymptotic behavior of the transition times between small balls centered in the metastable states. In a first step we consider first exit times from

$$\tilde{D}^{\pm 0}(\varepsilon^\gamma) := \tilde{D}^{\pm}(\varepsilon^\gamma) \cup \tilde{D}^0(\varepsilon^\gamma) = H \setminus \tilde{D}^{\mp}(\varepsilon^\gamma).$$

Definition 5.1. Define for $x \in \tilde{D}^{\pm}(\varepsilon^\gamma)$

$$\tau_x^{\pm 0}(\varepsilon) := \inf\{t > 0 \mid X^\varepsilon(t;x) \notin \tilde{D}^{\pm 0}(\varepsilon^\gamma)\}$$

the first exit time of $X^\varepsilon(\cdot;x)$ to leave the enhanced domain of attraction $\tilde{D}^{\pm 0}(\varepsilon^\gamma)$.

We shall show that the slow deterministic dynamics close to the separatrix $\mathcal{S}$ asymptotically has no contribution to the exit rate, that is $\tau^{\pm}(\varepsilon) \approx \tau^{\pm 0}(\varepsilon)$ for $\varepsilon \to 0+$.

Analogously to (4.2) in Chapter 4 we define for $y \in \tilde{D}^{\pm}(\varepsilon^{\gamma})$ the modified exit event

$$\tilde{B}_y := \left\{\omega \in \Omega \mid Y^{\varepsilon}(s;y) \in D^{\pm}(\varepsilon^{\gamma}) \text{ for } s \in [0,T_1] \text{ and } Y^{\varepsilon}(T_1;y) + \varepsilon W_1 \notin \tilde{D}^{\pm,0}(\varepsilon^{\gamma})\right\}. \tag{5.1}$$

We need the following slight modification of Lemma 4.2.

Lemma 5.2. *For $\rho \in (1/2,1), \gamma \in (0, 1-\rho)$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0, \kappa > 0$*

$$\mathbf{1}(\tilde{B}_y) \leqslant \mathbf{1}\{\varepsilon W_1 \notin D_0^{\pm}\} + \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} + \mathbf{1}(E^c(y)), \quad y \in D^{\pm}(\varepsilon^{\gamma}), \tag{5.2}$$

$$\begin{aligned}\mathbf{1}(\tilde{B}_y) \geqslant \mathbf{1}\{\varepsilon W_1 \in D^{\mp}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) - \phi^{\pm}\}(1 - \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}) \\ - \mathbf{1}(E^c(y)), \qquad y \in \tilde{D}^{\pm}(\varepsilon^{\gamma})\end{aligned} \tag{5.3}$$

The elementary proof is given in Subsection 5.1.1.

Theorem 5.3. *Assume that Hypotheses (H.1) and (H.2) are satisfied. Then for all $\theta > -1$ and $C \in (0, 1+\theta)$ there is $\varepsilon_0 = \varepsilon_0(\theta)$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$*

$$\begin{aligned}\frac{1-C}{1+\theta+C} \leqslant \mathbb{E}\left[\inf_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \exp\left(-\theta\lambda^{\pm}(\varepsilon)\tau_x^{\pm 0}(\varepsilon)\right)\right] \\ \leqslant \mathbb{E}\left[\sup_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \exp\left(-\theta\lambda^{\pm}(\varepsilon)\tau_x^{\pm 0}(\varepsilon)\right)\right] \leqslant \frac{1+C}{1+\theta-C}.\end{aligned}$$

Proof. The proof is a slight modification of the one for Theorem 4.6. The reasoning for the small jump solution Y^{ε} carried out in Chapter 4 remains untouched, whereas for the large jumps η^{ε} we have to replace the event B_y defined in (4.2) by $\tilde{B}_y$ defined by (5.1). Hence it is enough to estimate the expressions

$$\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y \in D(\varepsilon^{\gamma})} \mathbf{1}(\tilde{B}_y)\right] \quad \text{and} \quad \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \inf_{y \in \tilde{D}(\varepsilon^{\gamma})} \mathbf{1}(\tilde{B}_y)\right]$$

analogously to Claim 2 in the proofs of Propositions 4.10 and 4.11.

Upper estimate: We use in Lemma 5.2 estimate (5.2) to estimate $\tilde{B}_y$. We pass to the supremum in $y \in D^{\pm}(\varepsilon^{\gamma})$ on both sides of the inequality and integrate to obtain

$$\begin{aligned}&\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y \in D(\varepsilon^{\gamma})} \mathbf{1}(\tilde{B}_y)\right] \\ &\leqslant \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\right] \mathbb{P}\left(\varepsilon W_1 \notin D_0^{\pm}\right) + \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}\right] \\ &\qquad + \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1} \sup_{y \in D(\varepsilon^{\gamma})} \mathbf{1}(E_y^c)\right] =: K_3(1-K_{10}) + K_1 + K_5.\end{aligned} \tag{5.4}$$

K_1, K_3 and K_5 are expressions appearing in Claim 2 of the proof of Proposition 4.10, and their asymptotic behavior is described in (4.21), (4.14) and (4.17). Moreover, by definition for $\varepsilon > 0$

$$\lambda^{\pm}(\varepsilon) = \nu\left(\frac{1}{\varepsilon}(D_0^{\pm})^c\right)$$

and trivially

$$K_{10} = \frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}.$$

Thus we obtain an estimate identical to the one given in Claim 2 in the proof of Proposition 4.10 by (4.20). Hence there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathbb{E}\left[e^{-\theta\lambda^{\pm}(\varepsilon)T_1} \sup_{y\in D^{\pm}(\varepsilon^\gamma)} \mathbf{1}\left(\tilde{B}_y\right)\right] \leqslant (1 + C/5)\frac{\beta_\varepsilon}{\beta_\varepsilon + \theta\lambda^{\pm}(\varepsilon)}\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}.$$

We can conclude as in the proof of Proposition 4.10.

Lower estimate: We now use the lower estimate in Lemma 5.2 inequality (5.3) for $\tilde{B}_y$. Passing to the infimum over $y \in \tilde{D}^{\pm}(\varepsilon^\gamma)$ on the left-hand side, and correspondingly to the supremum on the right hand side, and then integrating we arrive at

$$\begin{aligned}
&\mathbb{E}\left[e^{-\theta\lambda^{\pm}(\varepsilon)T_1} \inf_{y\in\tilde{D}^{\pm}(\varepsilon^\gamma)} \mathbf{1}(\tilde{B}_y)\right] \\
&\geqslant \mathbb{P}\left(\varepsilon W_1 \in D^{\mp}(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) - \phi^{\pm}\right) \\
&\quad\cdot \left(\mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\right] - \mathbb{E}\left[e^{-\theta\lambda(\varepsilon)T_1}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}\right]\right) - \mathbb{E}\left[\sup_{y\in\tilde{D}(\varepsilon^\gamma)} \mathbf{1}(E_y^c)\right].
\end{aligned}$$

By the estimates in the proof of Proposition 4.10 and 4.11 the asymptotic behavior of all terms arising in the preceding inequality except $\mathbb{P}\left(\varepsilon W_1 \in D^{\mp}(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) - \phi^{\pm}\right)$ are well known. To estimate the latter, we exploit the representation

$$D^{\mp}(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) - \phi^{\pm} = \left(D_0^{\pm}\right)^c \setminus \left(\left(D_0^{\pm}\right)^c \setminus \left(D^{\mp}(\varepsilon^\gamma, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) - \phi^{\pm}\right)\right).$$

Note that by the regular variation of ν and Lemma 4.3 *iv*) for each $C > 0$ there is $\varepsilon_0 > 0$ small enough, such that for $0 < \varepsilon \leqslant \varepsilon_0$

$$
\begin{aligned}
&\mathbb{P}\left(\varepsilon W_1 \in D^{\mp}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) - \phi^{\pm}\right)\\
&= \frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon} - \frac{1}{\beta_\varepsilon}\nu\left(\frac{1}{\varepsilon}\left((D_0^{\pm})^c \setminus (D^{\mp}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma}) - \phi^{\pm})\right)\right)\\
&= \frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon} - \frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}\frac{\nu\left(\frac{1}{\varepsilon}\left(((D^{\pm})^c \setminus D^{\mp}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})) - \phi^{\pm})\right)\right)}{\lambda^{\pm}(\varepsilon)}\\
&\qquad \geqslant \frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}\left(1 - \frac{\mu\left(\left((D^{\pm})^c \setminus D^{\mp}(\varepsilon_0^{\gamma}, \varepsilon_0^{2\gamma}, \varepsilon_0^{2\gamma})\right) - \phi^{\pm}\right)\right)}{\mu\left(\left(D_0^{\pm}\right)^c\right)}\right) \geqslant \frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}(1 - C/2)\,.
\end{aligned}
$$

Taking into account the remaining terms with their asymptotic behavior according to the previous Chapter we find $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\theta\lambda^{\pm}(\varepsilon)T_1}\inf_{y\in\tilde{D}(\varepsilon^{\gamma})}\mathbf{1}(\tilde{B}_y)\right]\\
&\qquad \geqslant \frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}(1 - C/2)\left(\frac{\beta_\varepsilon}{\theta\lambda^{\pm}(\varepsilon)+\beta_\varepsilon} - C/2\frac{\beta_\varepsilon}{\theta\lambda^{\pm}(\varepsilon)+\beta_\varepsilon}\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}\right)\\
&\qquad\qquad \geqslant (1-C)\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}\frac{\beta_\varepsilon}{\theta\lambda^{\pm}(\varepsilon)+\beta_\varepsilon}.
\end{aligned}
$$

Apart from these modifications the lower bound may be obtained as in the proof of Proposition 4.11. □

Theorem 5.4. *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ be given and suppose Hypotheses (H.1) and (H.2) are satisfied. Then the family of random variables $(s^{\pm}(\varepsilon))_{\varepsilon>0}$ defined by (4.26) satisfies that for any $\theta > 0$ and $0 < C < 1$ there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$*

$$
\mathbb{E}\left[\inf_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} e^{-\theta|\tau_x^{\pm 0}(\varepsilon) - s^{\pm}(\varepsilon)|}\right] \geqslant 1 - C. \tag{5.5}
$$

Proof. (of Theorem 5.4):
Step 1: Reduction to incremental events
Recall for $\varepsilon > 0$ the events

$$
A_j^{\diamond} := \left(B_j^{\diamond}\right)^c = \{\varepsilon W_j \in D_0^{\pm}\}, \text{ for } j \in \mathbb{N}.
$$

In this step we follow the lines of the first part proof of Theorem 4.14. Exploiting the strong Markov property of X^ε and Y^ε respectively and the Lévy property of L we may estimate the exit first exit time from below by probabilities of exit events A_x^-, $\tilde{B}_x$, $A_1^{\diamond}$

and $B_1^\diamond$ such that

$$\begin{aligned}
&\mathbb{E}\left[\inf_{x\in\tilde{D}^{\pm}(\varepsilon^\gamma)} e^{-\theta|\tau_x^{\pm 0}(\varepsilon)-s^{\pm}(\varepsilon)|}\right] \\
= \quad & \sum_{k=1}^{\infty}\mathbb{E}\left[\inf_{x\in\tilde{D}^{\pm}(\varepsilon^\gamma)} \mathbf{1}_{\{\tau_x^{\pm 0}(\varepsilon)=T_k\}\cap\bigcap_{j=1}^{k-1}A_{j-1}^\diamond\cap B_k^\diamond}\right] \\
\geqslant \quad & \sum_{k=1}^{\infty}\mathbb{E}\left[\inf_{x\in\tilde{D}^{\pm}(\varepsilon^\gamma)} \prod_{j=1}^{k-1}\left(\mathbf{1}_{A^-_{X^\varepsilon(0;x)}\circ\theta_{T_{j-1}}\cap A_j^\diamond}\right)\mathbf{1}_{\tilde{B}_{X^\varepsilon(0;x)}\circ\theta_{T_{k-1}}\cap B_k^\diamond}\right].
\end{aligned}$$

By $k-1$-fold iterated application of the strong Markov property of X^ε as for example in the proof of Theorem 4.14 we obtain for $k\in\mathbb{N}$ that

$$\begin{aligned}
&\mathbb{E}\left[\inf_{x\in\tilde{D}^{\pm}(\varepsilon^\gamma)} \prod_{j=1}^{k-1}\left(\mathbf{1}_{A^-_{X^\varepsilon(0;x)}\circ\theta_{T_{j-1}}\cap A_j^\diamond}\right)\mathbf{1}_{\tilde{B}_{X^\varepsilon(0;x)}\circ\theta_{T_{k-1}}\cap B_k^\diamond}\right] \\
&\qquad\geqslant \quad \mathbb{E}\left[\inf_{y\in\tilde{D}^{\pm}(\varepsilon^\gamma)}\mathbf{1}_{A_y^-\cap A_1^\diamond}\right]^{k-1}\mathbb{E}\left[\inf_{y\in\tilde{D}^{\pm}(\varepsilon^\gamma)}\mathbf{1}_{\tilde{B}_y\cap B_1^\diamond}\right]. \qquad (5.6)
\end{aligned}$$

The lower estimate of the first factor is given by the asymptotic estimate (4.33) in Step 3 of the proof for Theorem 4.14, for which we keep the notation. Hence it only remains to estimate the second factor.

Step 2: Lower estimate of the second factor
By Lemma 5.2 estimate (5.3) we know for $y\in\tilde{D}^{\pm}(\varepsilon^\gamma)$

$$\mathbf{1}(\tilde{B}_y)\geqslant\mathbf{1}\{\varepsilon W_1\in D^{\mp}(\varepsilon^\gamma,\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm}\}(1-\mathbf{1}\{T_1<T_{rec}+\kappa\gamma|\ln\varepsilon|\})-\mathbf{1}(E^c(y)).$$

Since $\{\varepsilon W_1\in D^{\mp}(\varepsilon^\gamma,\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm}\}\subseteq\{\varepsilon W_1\in(D_0^{\pm})^c\}=B^\diamond$ we may infer that

$$\begin{aligned}
\mathbf{1}(B^\diamond\cap\tilde{B}_y) &\geqslant \mathbf{1}\{\varepsilon W_1\in D^{\mp}(\varepsilon^\gamma,\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm}\}(1-\mathbf{1}\{T_1<T_{rec}+\kappa\gamma|\ln\varepsilon|\}) \\
&\quad -\mathbf{1}(E^c(y)\cap B^\diamond) \\
&\geqslant \mathbf{1}\{\varepsilon W_1\in D^{\mp}(\varepsilon^\gamma,\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm}\}\mathbf{1}\{T_1\geqslant T_{rec}+\kappa\gamma|\ln\varepsilon|\}-\mathbf{1}(E^c(y)).
\end{aligned} \qquad (5.7)$$

Passing to the infimum over $y\in\tilde{D}^{\pm}(\varepsilon^\gamma)$ on the left-hand side, and correspondingly to the supremum on the right hand side, and then integrating we arrive at

$$\mathbb{E}\left[\inf_{y\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\mathbf{1}(\tilde{B}_y\cap B^{\diamond})\right]$$
$$\geqslant \mathbb{P}\left(\varepsilon W_1\in D^{\mp}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm}\right)\ \mathbb{P}\left(T_1\geqslant T_{rec}+\kappa\gamma|\ln\varepsilon|\}\right)-\mathbb{E}\Big[\sup_{y\in\tilde{D}(\varepsilon^{\gamma})}\mathbf{1}(E_y^c)\Big].$$

By the estimates in the proof of Proposition 4.14, Proposition 4.10 and Proposition 4.11 the asymptotic behavior of all terms arising in the preceding inequality with exception of $\mathbb{P}\left(\varepsilon W_1\in D^{\mp}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm}\right)$ are well known. To estimate the latter, we exploit the representation

$$D^{\mp}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm}=\left(D_0^{\pm}\right)^c\setminus\left(\left(D_0^{\pm}\right)^c\setminus\left(D^{\mp}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm}\right)\right).$$

Note that by the regular variation of ν and Lemma 4.3 *iv*) for each $0<C_5<1$ there is $\varepsilon_5>0$ small enough, such that for $0<\varepsilon\leqslant\varepsilon_5$

$$\begin{aligned}
&\mathbb{P}\left(\varepsilon W_1\in D^{\mp}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm}\right)\\
&=\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}-\frac{1}{\beta_\varepsilon}\nu\left(\frac{1}{\varepsilon}\left((D_0^{\pm})^c\setminus(D^{\mp}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm})\right)\right)\\
&=\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}-\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}\frac{\nu\left(\frac{1}{\varepsilon}\left(\left((D^{\pm})^c\setminus D^{\mp}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})\right)-\phi^{\pm}\right)\right)}{\lambda^{\pm}(\varepsilon)}\\
&\qquad\geqslant\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}\left(1-\frac{\mu\left(\left((D^{\pm})^c\setminus D^{\mp}(\varepsilon_0^{\gamma},\varepsilon_0^{2\gamma},\varepsilon_0^{2\gamma})\right)-\phi^{\pm}\right)}{\mu\left(\left(D_0^{\pm}\right)^c\right)}\right)\geqslant\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}(1-C_5).
\end{aligned}$$

Thus keeping all the notation from the proof of Theorem 4.14 we obtain for $C_i\leq\frac{J_2}{3}\leq\frac{1}{3}$, $i=3,4,5$ and $0<\varepsilon\leqslant\min\{\varepsilon_3,\varepsilon_4,\varepsilon_5\}$

$$\begin{aligned}
&\mathbb{E}\left[\inf_{y\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\mathbf{1}(\tilde{B}_y\cap B^{\diamond})\right]\\
&\geqslant \mathbb{P}\left(\varepsilon W_1\in D^{\mp}(\varepsilon^{\gamma},\varepsilon^{2\gamma},\varepsilon^{2\gamma})-\phi^{\pm}\right)\ \mathbb{P}\left(T_1\geqslant T_{rec}+\kappa\gamma|\ln\varepsilon|\}\right)-\mathbb{E}\Big[\sup_{y\in\tilde{D}(\varepsilon^{\gamma})}\mathbf{1}(E_y^c)\Big]\\
&\qquad\geqslant\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}(1-C_5)(1-C_4)-C_3\varepsilon^{\vartheta}\geqslant\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}((1-C_5)(1-C_4)-C_3)\\
&\qquad\qquad\geqslant\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}(1-C_3-C_4-C_5)\geqslant\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}(1-J_2).
\end{aligned}$$

Step 3: The asymptotic geometric series

We can now estimate (5.6) for $0 < \varepsilon \leqslant \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}$, such that

$$\sum_{k=1}^{\infty} \left(1 - (1 + J_1)\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon}\right)^{k-1} (1 - J_2)\frac{\lambda^{\pm}(\varepsilon)}{\beta_\varepsilon} \geqslant \frac{1 - J_2}{1 + J_1} \geqslant 1 - C$$

if $J_1 > 0$ and $J_2 > 0$ are chosen to satisfy $0 \leqslant J_2 \leqslant C + (1 - C)J_1$. □

With an analogous proof as for Corollary 4.15 we obtain the following statement.

Corollary 5.5. *Under the assumptions of Theorem 5.4 there is a family random variables* $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ *with exponential distribution of parameter* 1 *on the same probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *as the driving Lévy noise* $(L(t))_{t\geqslant 0}$ *such that in probability*

$$\lim_{\varepsilon\to 0+} \inf_{y\in\tilde{D}^{\pm}(\varepsilon^\gamma)} |\lambda^{\pm}(\varepsilon)\tau_y^{\pm 0}(\varepsilon) - \bar{\tau}(\varepsilon)| = \lim_{\varepsilon\to 0+} \sup_{y\in\tilde{D}^{\pm}(\varepsilon^\gamma)} |\lambda^{\pm}(\varepsilon)\tau_y^{\pm 0}(\varepsilon) - \bar{\tau}(\varepsilon)| = 0.$$

Now combining the Theorem 5.3 and Corollary 4.15 we obtain with the identical proof as for Corollary 4.9 replacing $\tau_x^{\pm}(\varepsilon)$ by $\tau_x^{\pm 0}(\varepsilon)$

Theorem 5.6 (Exponential convergence of exit times to another reduced domain of attraction)**.** *Let the Chafee-Infante parameter* $\pi^2 < \lambda \neq (k\pi)^2$ *for* $k \in \mathbb{N}$ *be given and suppose Hypotheses (**H.1**) and (**H.2**) are satisfied. Then there is a family of random variables* $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ *with exponential law of parameter 1 such that for all* $\theta < 1$

$$\lim_{\varepsilon\to 0+} \mathbb{E}\Big[\sup_{x\in\tilde{D}^{\pm}(\varepsilon^\gamma)} |\exp\big(\theta\lambda^{\pm}(\varepsilon)\tau_x^{\pm 0}(\varepsilon)\big) - \exp\left(\theta\bar{\tau}(\varepsilon)\right)|\Big] = 0.$$

5.1.1. Estimates of Transition Events (Proof of Lemma 5.2)

Proof. In order to estimate $\mathbf{1}(\tilde{B}_y)$ for $y \in D(\varepsilon^\gamma)$ we need the following estimates analogous to statements *ii*), *iv*) and *vi*) of Lemma 4.1, namely

xvii) $$\mathbf{1}(\tilde{B}_y)\mathbf{1}(E_y)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \leqslant \mathbf{1}\{\varepsilon W_1 \notin D_0^{\pm}\},$$

xviii) $$\mathbf{1}(\tilde{B}_y)\mathbf{1}(E_y)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\|\varepsilon W_1\| \leqslant (1/2)\varepsilon^{2\gamma}\} = 0,$$

xix) $$\mathbf{1}(E_y)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}\mathbf{1}\{\varepsilon W_1 \in \tilde{D}^{\mp}(\varepsilon^\gamma) - \phi^{\pm}\} \leqslant \mathbf{1}(\tilde{B}_y).$$

Proof of *xvii*)**:** Proposition 2.15 states that on the event $\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\}$ we have $u(T_1; y) \in B_{(1/2)\varepsilon^{2\gamma}}(\phi^{\pm})$. Hence on $\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \cap E_y$ the relationship $Y^\varepsilon(T_1; y) \in B_{\varepsilon^{2\gamma}}(\phi^{\pm})$ holds, exactly as in step 1 of the proof of Lemma 4.1. By definition of $\tilde{B}_y$, $Y^\varepsilon(T_1; y) + \varepsilon W_1 \notin D^{\pm,0}(\varepsilon^\gamma)$. For our system this is equivalent to $Y^\varepsilon(T_1; y) + \varepsilon W_1 \in D^{\mp}(\varepsilon^\gamma)$. By definition we have $D^{\pm}(\varepsilon^\gamma) + B_{(1/2)\varepsilon^{2\gamma}}(0) \subset D^{\pm}$ for $\varepsilon < 1$, such that

$$\varepsilon W_1 \in D^{\mp}(\varepsilon^\gamma) - B_{(1/2)\varepsilon^{2\gamma}}(\phi^{\pm}) \subset D^{\mp} - \phi^{\pm} \subset \left(D_0^{\pm}\right)^c.$$

Proof of ***xviii*)**: We argue again for $\varepsilon < 1$. If in addition $\|\varepsilon W_1\| \leqslant (1/2)\varepsilon^{2\gamma}$ and by $D^{\pm}(\varepsilon^{\gamma}) + B_{\varepsilon^{2\gamma}}(0) \subset D^{\pm}$, we obtain

$$0 \in D^{\mp}(\varepsilon^{\gamma}) + B_{\varepsilon^{2\gamma}}(\phi^{\pm}) \subset D^{\mp} + \phi^{\pm} \subset \left(D_0^{\pm}\right)^c$$

which is absurd, i.e. holds for the empty set.

Proof of ***xix*)**: As mentioned in the proof of *xvii*), $Y^{\varepsilon}(T_1; y) \in B_{\varepsilon^{2\gamma}}(\phi^{\pm})$ on the event $\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \cap E_y$. Hence in order to arrive in $\tilde{B}_y$ we only have to ensure that

$$B_{\varepsilon^{2\gamma}}(\phi^{\pm}) + \tilde{D}^{\mp}(\varepsilon^{\gamma}) - \phi^{+} \subset D^{\mp}(\varepsilon^{\gamma}).$$

The shifts by $\phi^{\pm}$ cancel out, reducing the inclusion to

$$B_{\varepsilon^{2\gamma}}(0) + \tilde{D}^{\mp}(\varepsilon^{\gamma}) \subset D^{\mp}(\varepsilon^{\gamma}),$$

which is a result of Lemma 2.14.

Proof of (5.2): Using *xvii*) and *xviii*), we can obtain *xv*) by the following calculation

$$\begin{aligned}
\mathbf{1}(\tilde{B}_y) \leqslant\; & \mathbf{1}(E_y^c) + \mathbf{1}(\tilde{B}_y)\mathbf{1}(E_y)\mathbf{1}\{\|\varepsilon W_1\| \geqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \\
& + \mathbf{1}(\tilde{B}_y)\mathbf{1}(E_y)\mathbf{1}\{\|\varepsilon W_1\| \geqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} \\
& + \mathbf{1}(\tilde{B}_y)\mathbf{1}(E_y)\mathbf{1}\{\|\varepsilon W_1\| < (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \\
& + \mathbf{1}(\tilde{B}_y)\mathbf{1}(E_y)\mathbf{1}\{\|\varepsilon W_1\| < (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} \\
\leqslant\; & \mathbf{1}(E_y^c) + \mathbf{1}\{\varepsilon W_1 \notin D_0^{\pm}\} + \mathbf{1}\{\|\varepsilon W_1\| \geqslant (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} + 0 \\
& + \mathbf{1}\{\|\varepsilon W_1\| < (1/2)\varepsilon^{2\gamma}\}\mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\} \\
=\; & \mathbf{1}(E_y^c) + \mathbf{1}\{\varepsilon W_1 \notin D_0^{\pm}\} + \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}.
\end{aligned}$$

Proof of (5.3): By *xix*), the estimate *xvi*) follows from the calculation

$$\begin{aligned}
\mathbf{1}(\tilde{B}_y) \geqslant\; & \mathbf{1}(\tilde{B}_y)\mathbf{1}(E_y)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \\
\geqslant\; & \mathbf{1}\{\varepsilon W_1 \in \tilde{D}^{\mp}(\varepsilon^{\gamma}) - \phi^{\pm}\}\mathbf{1}(E_y)\mathbf{1}\{T_1 \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\} \\
\geqslant\; & \mathbf{1}\{\varepsilon W_1 \in \tilde{D}^{\mp}(\varepsilon^{\gamma}) - \phi^{\pm}\}(1 - \mathbf{1}\{T_1 < T_{rec} + \kappa\gamma|\ln\varepsilon|\}) - \mathbf{1}(E_y^c).
\end{aligned}$$

□

5.2. Transition Times between Balls Centered in the Stable States

In this Section we shall investigate the asymptotic behavior in the small noise limit $\varepsilon \to 0$ of the times needed to switch between small neighborhoods of the metastable states. As in the Gaussian case it turns out that they differ only to a negligible extent from the transition or exit times investigated before.

Let $\sigma_x^\pm(\varepsilon)$ describe the time needed for X^ε starting in $t = 0$ at $X^\varepsilon(0) = x$ to enter the ball $B_{\varepsilon^{2\gamma}}(\phi^\mp)$ contained in the *opposite* domain of attraction.

Remark 5.7. Note that by definition $X^\varepsilon(\tau^{\pm,0}(\varepsilon); x) \in D^\mp(\varepsilon^\gamma)$ for all $x \in \tilde{D}^\pm(\varepsilon^\gamma)$, which differs clearly from $\sigma_x^\pm(\varepsilon)$ for systems with more than two stable solutions.

The following Proposition confirms that in our situation the asymptotic transitions between small neighborhoods of the stable states of the deterministic Chafee-Infante equation do not differ essentially from the asymptotic first exit times from reduced domains of attraction.

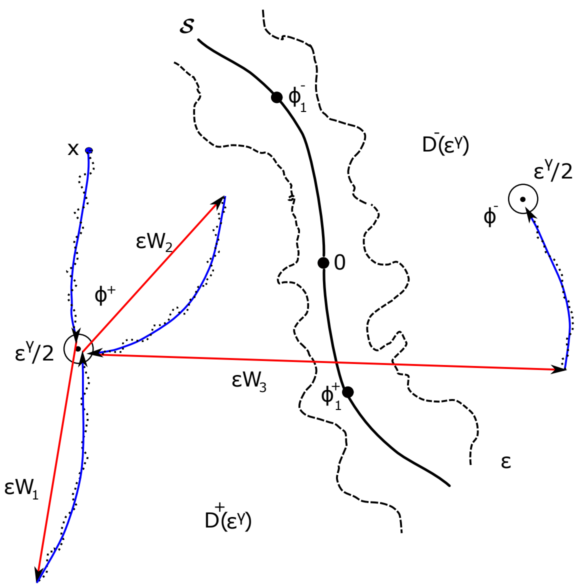

Figure 5.1.: Sketch of a transition event

Theorem 5.8 (Asymptotic transitions between balls around the stable states)**.** *Assume that (H.1) and (H.2) are satisfied. There is a family of random variables $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ with exponential law of parameter 1 and $h_0 > 0$ such that for any $0 < h \leqslant h_0$*

$$\lim_{\varepsilon \to 0+} \mathbb{E}\left[\sup_{x \in \tilde{D}^\pm(\varepsilon^\gamma)} \mathbf{1}\{|\lambda^\pm(\varepsilon)\sigma_x^\pm(\varepsilon) - \bar{\tau}(\varepsilon)| > h\}\right] = 0.$$

Proof. For $x \in D^{\pm}$ and $\varepsilon > 0$ sufficiently small we obviously have

$$\tau_x^{\pm}(\varepsilon) \leqslant \tau_x^{\pm 0}(\varepsilon) \leqslant \sigma_x^{\pm}(\varepsilon). \qquad \mathbb{P}-\text{a.s.} \tag{5.8}$$

Inequality (5.8) can in fact be rewritten as

$$\begin{aligned}
&\inf\{t > 0 \mid X^{\varepsilon}(t;x) \in \big(D^{\pm}(\varepsilon)\big)^c\} \\
&\leqslant \inf\{t > 0 \mid X^{\varepsilon}(t;x) \in \big(D^{\pm}(\varepsilon) \cup D^0(\varepsilon)\big)^c\} \\
&\qquad\qquad \leqslant \inf\{t > 0 \mid X^{\varepsilon}(t;x) \in \big(D^{\pm}(\varepsilon) \cup D^0(\varepsilon) \cup (D^{\mp}(\varepsilon) \setminus B_{\varepsilon^{2\gamma}}(\phi^{\mp}))\big)^c\}.
\end{aligned}$$

Claim: *For $\kappa > 0$ we have*

$$\lim_{\varepsilon \to 0+} \mathbb{E}\left[\sup_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}\{\sigma_x^{\pm}(\varepsilon) < \tau_x^{\pm 0}(\varepsilon) + T_{rec} + \kappa\gamma|\ln\varepsilon|\}\right] = 1. \tag{5.9}$$

To prove the claim, fix $\varepsilon, \gamma, \kappa > 0$. By Proposition 2.15 and Remark 5.7

$$\begin{aligned}
&\mathbb{E}\left[\sup_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}\{\sigma_x^{\pm}(\varepsilon) \geqslant \tau_x^{\pm 0}(\varepsilon) + T_{rec} + \kappa\gamma|\ln\varepsilon|\}\right] \\
&\leqslant \mathbb{E}\left[\sup_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}\{|X^{\varepsilon}(\tau_x^{\pm 0}(\varepsilon) + T_{rec} + \kappa\gamma|\ln\varepsilon|; x) - \phi^{\mp}|_{\infty} \geqslant \varepsilon^{2\gamma}\}\right] \\
&\leqslant \mathbb{E}\Bigg[\sup_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}\{|X^{\varepsilon}(T_{rec} + \kappa\gamma|\ln\varepsilon|; X^{\varepsilon}(\tau^{\pm 0}; x)) \\
&\qquad\qquad -u(T_{rec} + \kappa\gamma|\ln\varepsilon|; X^{\varepsilon}(\tau^{\pm 0}; x))|_{\infty} \geqslant \frac{1}{2}\varepsilon^{2\gamma}\}\Bigg] \\
&\qquad + \mathbb{E}\left[\sup_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}\{|u(T_{rec} + \kappa\gamma|\ln\varepsilon|; X^{\varepsilon}(\tau^{\pm 0}; x)) - \phi^{\pm}|_{\infty} \geqslant \frac{1}{2}\varepsilon^{2\gamma}\}\right] \\
&\leqslant \mathbb{E}\left[\mathbf{1}\{\sup_{x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})} \sup_{t \in [0, T_{rec} + \kappa\gamma|\ln\varepsilon|]} |X^{\varepsilon}(t; X^{\varepsilon}(\tau^{\pm 0}; x)) - u(t; X^{\varepsilon}(\tau^{\pm 0}; x))|_{\infty} \geqslant \frac{1}{2}\varepsilon^{2\gamma}\}\right] \\
&\qquad \leqslant \mathbb{E}\left[\sup_{y \in D^{\mp}(\varepsilon^{\gamma})} \mathbf{1}\{\sup_{t \in [0, T_{rec} + \kappa\gamma|\ln\varepsilon|]} |X^{\varepsilon}(t;y) - u(t;y)|_{\infty} \geqslant \frac{1}{2}\varepsilon^{2\gamma}\}\right].
\end{aligned} \tag{5.10}$$

By estimate (3.14), we find $\Gamma = \Gamma(\kappa) > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$ and $x \in \tilde{D}^{\pm}(\varepsilon^{\gamma})$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{y \in D^{\mp}(\varepsilon^{\gamma})} \mathbf{1}\{\sup_{t \in [0, T_{rec} + \kappa\gamma|\ln\varepsilon|]} |X^{\varepsilon}(t;y) - u(t;y)|_{\infty} \geqslant \frac{1}{2}\varepsilon^{2\gamma}\}\right] \\
&\qquad\qquad \leqslant \mathbb{P}(\tilde{\mathcal{E}}^c),
\end{aligned} \tag{5.11}$$

where

$$\tilde{\mathcal{E}}(\varepsilon) := \{ \sup_{t\in[0,T_{rec}+\kappa\gamma|\ln\varepsilon|} \|\varepsilon\xi^*(t)\| < \varepsilon^{(\Gamma+2)\gamma}\} \cap \{T_1 > T_{rec} + \kappa\gamma|\ln\varepsilon|\}. \tag{5.12}$$

Moreover, by Lemma 3.6 for ε_0 small enough and a constant $C > 0$

$$\begin{aligned}
\mathbb{P}(\tilde{\mathcal{E}}^c(\varepsilon)) &\leqslant \mathbb{P}\left(\left(\{ \sup_{t\in[0,T_{rec}+\kappa\gamma|\ln\varepsilon|]} \|\varepsilon\xi^*(t)\| < \varepsilon^{(\Gamma+2)\gamma}\} \cap \{T_1 > T_{rec} + \kappa\gamma|\ln\varepsilon|\}\right)^c\right) \\
&\leqslant \mathbb{P}\left(\sup_{t\in[0,T_{rec}+\kappa\gamma|\ln\varepsilon|]} \|\varepsilon\xi^*(t)\| \geqslant \varepsilon^{(\Gamma+2)\gamma}\right) + \mathbb{P}\left(T_1 \leqslant T_{rec} + \kappa\gamma|\ln\varepsilon|\right) \\
&\leqslant C\left(T_{rec} + \kappa\gamma|\ln\varepsilon|\right)\varepsilon^{2-2(\Gamma+2)\gamma-(2-(1-\Theta)\alpha)\rho} + 1 - e^{-(T_{rec}+\kappa\gamma|\ln\varepsilon|)\beta_\varepsilon}, \qquad (5.13)
\end{aligned}$$

an estimate that converges to 0 as $\varepsilon \to 0+$. This establishes

$$\lim_{\varepsilon\to 0+} \mathbb{E}\left[\sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \mathbf{1}\{\sigma_x^\pm(\varepsilon) \geqslant \tau_x^{\pm 0}(\varepsilon) + T_{rec} + \kappa\gamma|\ln\varepsilon|\}\right] = 0, \tag{5.14}$$

and therefore in combination with the inequalities (5.8) and (5.14)

$$\mathbb{E}\left[\sup_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \mathbf{1}\{\lambda^\pm(\varepsilon)\tau^{\pm 0}(\varepsilon) \leqslant \lambda^\pm(\varepsilon)\sigma^\pm(\varepsilon) \leqslant \lambda^\pm(\varepsilon)\left(\tau^{\pm 0}(\varepsilon) + T_{rec} + \kappa\gamma|\ln\varepsilon|\right)\}\right] \to 1, \qquad \varepsilon \to 0+ \tag{5.15}$$

as well as

$$\mathbb{E}\left[\inf_{x\in\tilde{D}^\pm(\varepsilon^\gamma)} \mathbf{1}\{\lambda^\pm(\varepsilon)\tau^{\pm 0}(\varepsilon) \leqslant \lambda^\pm(\varepsilon)\sigma^\pm(\varepsilon) \leqslant \lambda^\pm(\varepsilon)\left(\tau^{\pm 0}(\varepsilon) + T_{rec} + \kappa\gamma|\ln\varepsilon|\right)\}\right] \to 1, \qquad \varepsilon \to 0+ . \tag{5.16}$$

By Corollary 5.5 there is a family of random variables $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ with exponential law of parameter 1 such that

$$\begin{aligned}
&\lim_{\varepsilon\to 0+} \sup_{x\in\tilde{D}^\pm(\varepsilon)} |\lambda^\pm(\varepsilon)\tau_x^{\pm 0}(\varepsilon) - \bar{\tau}(\varepsilon)| \\
&\qquad = \lim_{\varepsilon\to 0+} \sup_{x\in\tilde{D}^\pm(\varepsilon)} |\lambda^\pm(\varepsilon)\left(\tau^{\pm 0}(\varepsilon) + T_{rec} + \kappa\gamma|\ln\varepsilon|\right) - \bar{\tau}(\varepsilon)| = 0
\end{aligned}$$

in probability. This implies for given $h > 0$

$$\lim_{\varepsilon \to 0+} \mathbb{E}\left[\sup_{x \in \tilde{D}^{\pm}(\varepsilon)} \mathbf{1}\{|\lambda^{\pm}(\varepsilon)\sigma_x^{\pm 0}(\varepsilon) - \bar{\tau}(\varepsilon)| > h\}\right] \leq \lim_{\varepsilon \to 0+} \mathbb{P}\left(\sup_{x \in \tilde{D}^{\pm}(\varepsilon)} |\lambda^{\pm}(\varepsilon)\sigma_x^{\pm 0}(\varepsilon) - \bar{\tau}(\varepsilon)| > h\right) = 0.$$

This finishes the proof. □

6. Localization and Metastability

In this Chapter, equipped with our previously obtained knowledge of exit and transition times in the limit of small noise amplitude $\varepsilon \to 0$, we shall investigate the global asymptotic behavior of our jump diffusion process in the time scale in which transitions occur, i.e. in the scale given by $\lambda^0(\varepsilon) = \nu(\frac{1}{\varepsilon}B_1^c(0)), \varepsilon > 0$. It turns out that in this time scale, the switching of the diffusion between neighborhoods of the stable solutions $\phi^\pm$ can be well described by a Markov chain jumping back and forth between two states with a characteristic Q-matrix determined by the quantities $\frac{\mu((D_0^\pm)^c)}{\mu(B_1^c(0))}$ as jumping rates. To show this, we need to prove that X^ε is localized around the stable fixed points also on the *critical* time scale $T/\lambda^0(\varepsilon)$. This boils down to the control of the exit behavior from the complement of the reduced domains of attraction $\tilde{D}^0(\varepsilon^\gamma)$. Roughly, rates for large jumps between positions inside this set have to converge to 0 with $\varepsilon \to 0$. This condition is made precise in Hypothesis (H.3), that plays an important role in the subsequent study of metastability.

6.1. Hypothesis (H.3) prevents Trapping close to the Separatrix

In this Section we shall justify Hypothesis (H.3) and prove crucial implications.

Definition 6.1. We assume Hypotheses (H.1), (H.2) and (H.3) to hold. For the constant $\gamma/2 < \tilde{\gamma} \leqslant \gamma$ from (H.3), $\varepsilon > 0$, we set $\tilde{\beta}_\varepsilon := \nu\left(\frac{1}{\varepsilon^{1-\tilde{\gamma}}}B_1^c(0)\right)$, and recursively for $i \in \mathbb{N}$

$$\tilde{T}_0 := 0, \qquad \tilde{T}_{i+1} := \inf\{t > T_i \mid \varepsilon\|\Delta_t L\| > \varepsilon^{\tilde{\gamma}}\}, \quad \text{and} \quad \tilde{W}_i := \Delta_{\tilde{T}_i} L. \quad (6.1)$$

By definition, $\tilde{T}_1$ then has an exponential law with parameter $\tilde{\beta}_\varepsilon$. In analogy to Section 2.1 to we shall split $L = \tilde{\eta}^\varepsilon + \tilde{\xi}^\varepsilon$, where

$$\tilde{\eta}^\varepsilon(t) = \sum_{T_i \leqslant t} \tilde{W}_i, \qquad t \geqslant 0,$$

is a compound Poisson process with jump probability measure $\frac{1}{\tilde{\beta}_\varepsilon}\nu(\cdot \cap \frac{1}{\varepsilon^{1-\tilde{\gamma}}}B_1^c(0))$. For further use we denote by

$$\tilde{\xi}^*(t) := \int_0^t S(t-s)\, \mathrm{d}\tilde{\xi}^\varepsilon(s), \qquad t \geqslant 0,$$

and $\tilde{Y}^\varepsilon$ the solution of equation (2.3) driven by $\varepsilon\tilde{\xi}^\varepsilon$ instead of $\varepsilon\xi^\varepsilon$. We recall the notation

$$\lambda^0(\varepsilon) = \nu\left(\frac{1}{\varepsilon}B_1^c(0)\right) = \varepsilon^\alpha \ell(1/\varepsilon)\mu(B_1^c(0))$$

with the slowly varying function ℓ.

Remark 6.2.

1. Let us argue why we introduce the additional parameter $\tilde{\gamma} \in (\gamma/2, \gamma]$. The upper and the lower bounds are derived from two properties important in the sequel.

1.1 In Lemma 6.3 we shall compare $\tau_x^0(\varepsilon)$ with a deterministic time scale $\frac{1}{\varepsilon^g}$, where $g > 0$. In the proof this reduces to the comparison of $\tilde{T}_1$ with $\frac{1}{\varepsilon^g}$. We aim at showing that that $\lim_{\varepsilon\to 0+} \mathbb{P}(\tilde{T}_1 > \frac{1}{\varepsilon^g}) = 0$. For this purpose we may calculate

$$\mathbb{P}(\tilde{T}_1 > \frac{1}{\varepsilon^g}) = \int_0^{\frac{1}{\varepsilon^g}} \tilde{\beta}_\varepsilon \exp\left(-\tilde{\beta}_\varepsilon s\right) \, \mathrm{d}s = 1 - \exp\left(-\frac{\tilde{\beta}_\varepsilon}{\varepsilon^g}\right)$$
$$= 1 - \exp\left(-\frac{\varepsilon^{\alpha(1-\tilde{\gamma})}\ell\left(\frac{1}{\varepsilon}\right)\mu(B_1^c(0))}{\varepsilon^g}\right).$$

In order that the last term tends to zero it is sufficient that $g > \alpha(1-\tilde{\gamma})$. This follows for $g = \alpha(1-\gamma/2)$, if $\tilde{\gamma} > \gamma/2$.

1.2 In Lemma 6.4 we shall compare $\tau_x^0(\varepsilon)$ with $\tilde{T}_1$ in the sense that we have to prove $\lim_{\varepsilon\to 0+} \mathbb{P}(\tau_x^0(\varepsilon) > \tilde{T}_1) = 0$. By definition $\mathrm{dist}(\tilde{D}^\pm(\varepsilon^\gamma), \mathcal{S}) \geqslant \varepsilon^\gamma + \varepsilon^{2\gamma}$. Clearly the choice of $\tilde{\gamma}$ must not inhibit that $\Delta_{\tilde{T}_i} X^\varepsilon = \varepsilon \Delta_{\tilde{T}_i} L = \varepsilon \tilde{W}_i$ for $i \in \mathbb{N}$ triggers the exit of X^ε from $\tilde{D}^0(\varepsilon^\gamma)$ with a reasonable chance. Hence for any reasonable choice of $\tilde{\gamma}$ there must be a constant $C > 0$ such that at least for small $\varepsilon > 0$

$$\mathbb{P}(\|\varepsilon\tilde{W}_i\| > \varepsilon^\gamma + \varepsilon^{2\gamma}) > C.$$

Assume $\tilde{\gamma} > \gamma$ and $y \in \mathcal{S}$. Then

$$\mathbb{P}(y + \varepsilon\tilde{W}_i \in \tilde{D}^+(\varepsilon^\gamma) \cup \tilde{D}^-(\varepsilon^\gamma)) = \frac{\nu\left(\frac{1}{\varepsilon^{1-\tilde{\gamma}}}\left(B_1^c(0)\right) \cap \frac{1}{\varepsilon}\left(\tilde{D}^0(\varepsilon^\gamma)\right)^c\right)}{\nu\left(\frac{1}{\varepsilon^{1-\tilde{\gamma}}}B_1^c(0)\right)}$$
$$\leqslant \frac{\nu\left(\frac{1}{\varepsilon^{1-\tilde{\gamma}}}B_1^c(0) \cap \frac{1}{\varepsilon}B_1^c(0)\right)}{\nu\left(\frac{1}{\varepsilon^{1-\tilde{\gamma}}}B_1^c(0)\right)} = \frac{\ell(\frac{1}{\varepsilon^{1-\gamma}})}{\ell(\frac{1}{\varepsilon^{1-\tilde{\gamma}}})}\varepsilon^{\tilde{\gamma}-\gamma} \to 0,$$

as $\varepsilon \to 0+$. In other words $\varepsilon\tilde{W}_i$ is of an asymptotically too small scale to trigger an exit from $\tilde{D}^0(\varepsilon^\gamma)$. Hence $\tilde{\gamma} \leqslant \gamma$ is a necessary condition. Modulo the appearence of $r > 0$ in Hypothesis (H.3) on ν appears such that now quite natural, since it can be paraphrased as stating that there is $\gamma/2 < \tilde{\gamma} \leqslant \gamma$

$$\sup_{y \in B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(y + \varepsilon \tilde{W}_1 \in \tilde{D}^+(\varepsilon^\gamma) \cup \tilde{D}^-(\varepsilon^\gamma)\right)$$
$$= 1 - \sup_{y \in B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(y + \varepsilon \tilde{W}_1 \in \tilde{D}^0(\varepsilon^\gamma)\right) \to 1, \text{ as } \varepsilon \to 0+ .$$

This means that large jumps $\varepsilon \tilde{W}_i$ of size height $\varepsilon^{\tilde{\gamma}}$ should have a non-negligible chance to leave the set $\tilde{D}^0(\varepsilon^\gamma)$.

2. Could this Hypothesis (H.3) be slightly strengthened, to improve the metastability results of Theorem 6.12 and 6.13? For instance, could the supremum over $x \in H$ appearing in Lemma 6.10 or the one over $x \in D^\pm$ in Theorem 6.12 be taken under the expectation sign, by just requiring a slightly stronger hypothesis? Let us argue that the hypothesis

$$\lim_{\varepsilon \to 0+} \mathbb{E}\left[\sup_{y \in B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)} \mathbf{1}\{y + \varepsilon \tilde{W}_1 \in \tilde{D}^+(\varepsilon^\gamma) \cup \tilde{D}^-(\varepsilon^\gamma)\}\right]$$
$$= \lim_{\varepsilon \to 0+} \frac{\nu\left(\frac{1}{\varepsilon^{1-\tilde{\gamma}}} B_1^c(0) \cap \frac{1}{\varepsilon}\left(\tilde{D}^0(\varepsilon^\gamma) - \left(B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)\right)\right)\right)}{\tilde{\beta}_\varepsilon} = 0,$$

slightly stronger than Hypothesis (H.3), is in general too strong for purely geometric reasons. Recalling Lemma 2.13 the second claim of which states

$$\bigcap_{\varepsilon > 0} \tilde{D}^0(\varepsilon^\gamma) = \mathcal{S},$$

we can say that apart from very special cases (for example in case $\mathcal{S}$ is contained in a subspace of codimension 1), the separatrix $\mathcal{S}$ will not be contained in

$$\bigcap_{\varepsilon > 0} \tilde{D}^0(\varepsilon^\gamma) - \left(B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)\right) = \mathcal{S} - (B_{1+r}(0) \cap \mathcal{S}) .$$

In this case there generically exists a small ball $B_h(0) \subset \mathcal{S} - (B_{1+r}(0) \cap \mathcal{S})$. But then

$$\begin{aligned}
&\frac{\nu\left(\frac{1}{\varepsilon^{1-\tilde{\gamma}}} B_1^c(0) \cap \frac{1}{\varepsilon}\left(\tilde{D}^0(\varepsilon^\gamma) - \left(B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)\right)\right)\right)}{\tilde{\beta}_\varepsilon} \\
&= \frac{\mu\left(B_1^c(0) \cap \frac{1}{\varepsilon^{\tilde{\gamma}}}\left(\tilde{D}^0(\varepsilon^\gamma) - \left(B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)\right)\right)\right)}{\mu\left(B_1^c(0)\right)} \\
&\geqslant \frac{\mu\left(B_1^c(0) \cap \frac{1}{\varepsilon^{\tilde{\gamma}}}\left(\mathcal{S} - (B_{1+r}(0) \cap \mathcal{S})\right)\right)}{\mu\left(B_1^c(0)\right)} \geqslant \frac{\mu\left(B_1^c(0) \cap \frac{1}{\varepsilon^{\tilde{\gamma}}}\left(B_h(0)\right)\right)}{\mu\left(B_1^c(0)\right)} \xrightarrow{\varepsilon \to 0+} 1.
\end{aligned} \tag{6.2}$$

It is therefore difficult to find a stronger hypothesis enhancing the quality of convergence in our metastability results.

The main consequence of Hypothesis (H.3) that will be exploited vastly in the sequel is contained the following lemma.

Lemma 6.3. *Suppose the Hypotheses (H.1), (H.2) and (H.3) are satisfied. Then for any $T > 0$*

$$\lim_{\varepsilon\to 0+} \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(\tau_x^0(\varepsilon) > \frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\right) = 0$$

holds true.

Proof. In fact, for $\varepsilon > 0$ and the first jump $\tilde{T}_1$ of the compound Poisson process $\tilde{\eta}^\varepsilon$ we have

$$\begin{aligned}
&\sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(\tau_x^0(\varepsilon) > \frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\right) \\
&\quad\leqslant \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(\{\tau_x^0(\varepsilon) > \frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\} \cap \{\tau_x^0(\varepsilon) \leqslant \tilde{T}_1\}\right) \\
&\qquad + \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(\tau_x^0(\varepsilon) > \frac{T}{\varepsilon^{\alpha(1-\gamma/2)}} \mid \tau_x^0(\varepsilon) > \tilde{T}_1\right) \mathbb{P}\left(\tau_x^0(\varepsilon) > \tilde{T}_1\right) \\
&\quad\leqslant \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(\tilde{T}_1 > \frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\right) + \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(\tau_x^0(\varepsilon) > \tilde{T}_1\right) \\
&\quad\leqslant \exp\left(-\frac{T\,\tilde{\beta}_\varepsilon}{\varepsilon^{\alpha(1-\gamma/2)}}\right) + \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(\tau_x^0(\varepsilon) > \tilde{T}_1\right).
\end{aligned}$$

Since $\tilde{\gamma} > \gamma/2$, we obtain $\lim_{\varepsilon\to 0+} \frac{\tilde{\beta}_\varepsilon}{\varepsilon^{\alpha(1-\gamma/2)}} \to \infty$, such that the first term in the last line of the preceding estimate tends to zero. The second term

$$p_1(\varepsilon) := \sup_{y\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(\tau_y^0(\varepsilon) > \tilde{T}_1\right), \varepsilon > 0,$$

will be estimated in the subsequent Lemma 6.4. □

Before stating this Lemma, recall the uniform boundedness parameter $T_{rec}^r = T_{rec}^r(\lambda)$ from Proposition 2.8 existing for any $r > 0$, i.e. for all $t \geqslant T_{rec}^r$ we have

$$\sup_{x\in H} |u(t;x)|_\infty \leqslant 1 + r.$$

Lemma 6.4. *Under the assumptions of Lemma 6.3 the relationship*

$$\lim_{\varepsilon\to 0+} \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(\tau_x^0(\varepsilon) > \tilde{T}_1\right) = 0$$

holds.

Proof. For $r > 0$, $\varepsilon > 0$ and $x \in \tilde{D}^0(\varepsilon^\gamma)$ we may write

$$
\begin{aligned}
&\mathbb{P}\left(\tau_x^0(\varepsilon) > \tilde{T}_1\right) \\
&= \mathbb{P}\left(\tau_x^0(\varepsilon) > \tilde{T}_1 \text{ and } X^\varepsilon(\tilde{T}_1-;x) \in B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)\right) \\
&\qquad + \mathbb{P}\left(\tau_x^0(\varepsilon) > \tilde{T}_1 \text{ and } X^\varepsilon(\tilde{T}_1-;x) \notin B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)\right) \\
&\leqslant \mathbb{P}\left(\tilde{Y}^\varepsilon(\tilde{T}_1;x) \in B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma) \text{ and } \tilde{Y}^\varepsilon(\tilde{T}_1,x) + \varepsilon\tilde{W}_1 \in \tilde{D}^0(\varepsilon^\gamma)\right) \\
&\qquad + \mathbb{P}\left(\tilde{Y}^\varepsilon(\tilde{T}_1;x) \notin B_{1+r}(0)\right) \\
&\leqslant \sup_{y \in B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(y + \varepsilon\tilde{W}_1 \in \tilde{D}^0(\varepsilon^\gamma)\right) \\
&\qquad + \mathbb{P}\left(\tilde{Y}^\varepsilon(\tilde{T}_1;x) \notin B_{1+r}(0) \;\middle|\; \tilde{T}_1 > T^r_{rec}\right) + \mathbb{P}\left(\tilde{T}_1 \leqslant T^r_{rec}\right) \\
&\leqslant \sup_{y \in B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(y + \varepsilon\tilde{W}_1 \in \tilde{D}^0(\varepsilon^\gamma)\right) + p_2(\varepsilon) + \left(1 - e^{-\tilde{\beta}_\varepsilon T^r_{rec}}\right), \qquad (6.3)
\end{aligned}
$$

where

$$
p_2(\varepsilon) := \mathbb{P}\left(\sup_{x \in H} |X^\varepsilon(\tilde{T}_1-;x)|_\infty > 1 + r \mid \tilde{T}_1 \geqslant T^r_{rec}\right), \quad \varepsilon > 0.
$$

The third term in the last line of the preceding inequality clearly tends to zero as $\varepsilon \to 0$. By Hypothesis (H.3) we may treat the first term by noting

$$
\begin{aligned}
&\sup_{y \in B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(y + \varepsilon\tilde{W}_1 \in \tilde{D}^0(\varepsilon^\gamma)\right) \\
&\qquad = \sup_{y \in B_{1+r}(0) \cap \tilde{D}^0(\varepsilon^\gamma)} \frac{\nu\left(\frac{1}{\varepsilon^{1-\tilde{\gamma}}} B_1^c(0) \cap \frac{1}{\varepsilon}\left(\tilde{D}^0(\varepsilon^\gamma) - y\right)\right)}{\tilde{\beta}_\varepsilon} \overset{\varepsilon \to 0+}{\longrightarrow} 0. \qquad (6.4)
\end{aligned}
$$

It remains to show the convergence of $p_2(\varepsilon)$ as $\varepsilon \to 0$ for which we shall state Lemma 6.5. □

Lemma 6.5. *Under the assumptions of Lemma 6.3, and with $T^r_{rec} > 0$ chosen in (2.7) for $r > 0$ we have*

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{x \in H} |X^\varepsilon(\tilde{T}_1-;x)|_\infty \geqslant 1 + r \;\middle|\; \tilde{T}_1 \geqslant T^{r/2}_{rec}\right) \\
&\qquad \leqslant \mathbb{P}\left(\sup_{x \in H} \sup_{t \in [0, T^{r/2}_{rec}]} |X^\varepsilon(t;x) - u(t;x)|_\infty \geqslant r/2 \;\middle|\; \tilde{T}_1 \geqslant T^{r/2}_{rec}\right). \qquad (6.5)
\end{aligned}
$$

Proof. If $\tilde{T}_1 \geqslant T_{rec}^r$ we obtain by Proposition (2.8) that

$$\begin{aligned}
&|X^\varepsilon(\tilde{T}_1-;x)| \\
&\leqslant |u(T_{rec}^{r/2}; X^\varepsilon(\tilde{T}_1 - T_{rec}^{r/2};x))|_\infty \\
&\qquad + |\tilde{Y}^\varepsilon(T_{rec}^{r/2}; X^\varepsilon(\tilde{T}_1 - T_{rec}^{r/2};x)) - u(T_{rec}^{r/2}; X^\varepsilon(\tilde{T}_1 - T_{rec}^{r/2};x)|_\infty \\
&\qquad\qquad \leqslant 1 + r/2 \, + \sup_{y\in H} |\tilde{Y}^\varepsilon(T_{rec}^{r/2};y) - u(T_{rec}^{r/2};y)|_\infty. \quad (6.6)
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{P}\left(\sup_{x\in H} |X^\varepsilon(\tilde{T}_1-;x)|_\infty \geqslant 1 + r \;\Big|\; \tilde{T}_1 \geqslant T_{rec}^{r/2}\right) \\
&\qquad \leqslant \mathbb{P}\left(\sup_{x\in H} \sup_{t\in[0,T_{rec}^{r/2}]} |X^\varepsilon(t;x) - u(t;x)|_\infty \geqslant r/2 \;\Big|\; \tilde{T}_1 \geqslant T_{rec}^{r/2}\right). \quad (6.7)
\end{aligned}$$

□

To continue our estimate, we shall next bound the right-hand side of (6.7). For this purpose, we prove a slightly stronger result. As opposed to Chapter 3 we only need the small deviation estimate for the finite time horizon $T_{rec}^{r/2}$. The estimate is uniform in $x \in H$.

Lemma 6.6. *Under the assumptions of Lemma 6.3 there exist $p > 0$, $q > 0$ and $\varepsilon_0 > 0$ such that for all $T > 0$ and $0 < \varepsilon \leqslant \varepsilon_0$*

$$\mathbb{P}\left(\sup_{x\in H} \sup_{t\in[0,T]} |X^\varepsilon(t;x) - u(t;x)|_\infty \geqslant \varepsilon^p \;\Big|\; \tilde{T}_1 \geqslant T\right) \leqslant T\varepsilon^q.$$

Proof. With the notation from Definition 6.1 we shall argue for the remainder term $\tilde{R}^\varepsilon(\cdot;x) := \tilde{Y}^\varepsilon(\cdot;x) - u(\cdot;x) - \varepsilon\tilde{\xi}^*(\cdot)$ and $\varepsilon > 0, x \in H$, in the same manner as in the proof of Lemma 3.3.

Claim 1: There is $0 < \varepsilon_0 = \varepsilon_0(\tilde{\gamma}) \leqslant 1$ such that for all $T > 0$ there is $C = C(T) > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$ we have

$$\sup_{x\in H} \sup_{t\in[0,T]} |\tilde{R}^\varepsilon(t;x)|_\infty \leqslant C \sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\|.$$

on the event $\tilde{\mathcal{E}}_T(1) = \{\sup_{r\in[0,T]} \|\varepsilon\tilde{\xi}^*(r)\| < \frac{1}{C}\}$. As in the proof of Lemma 3.3 $\tilde{R}^\varepsilon$ satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{R}^\varepsilon|^2 + |\nabla\tilde{R}^\varepsilon|^2 = \langle f(\tilde{Y}^\varepsilon) - f(u), \tilde{R}^\varepsilon\rangle.$$

As before, by using the structure of f and applying Gronwall's Lemma, given $T > 0$, we find a constant $C_1 = C_1(T) > 0$ such that for $0 < t < t^\infty$ with time horizon $t^\infty := \inf\{t > 0 : |\tilde{R}^\varepsilon(t)|_\infty > 1\}$ and on $\tilde{\mathcal{E}}_T(1)$

$$|\tilde{R}^\varepsilon(t)|^2 \leqslant C_1 \sup_{r\in[0,T]} \|\varepsilon\tilde{\xi}^*(r)\|^2.$$

We next pass to the mild solution representation of $\tilde{R}^\varepsilon$, given by

$$\tilde{R}^\varepsilon(t) = \int_0^t S(t-s)\left(f(\tilde{Y}^\varepsilon(r)) - f(u(r))\right)\, \mathrm{d}r, \quad t \geqslant 0,$$

use the regularizing effect of S, Hölder's inequality and Corollary B.12 for $n = 12$ to find another constant $C_2 = C_2(T) > 0$ depending otherwise only on the Chafee-Infante parameter λ such that

$$\sup_{x\in H, 0\leqslant t\leqslant t^\infty\wedge T} \|\tilde{R}^\varepsilon(t;x)\| \leqslant C_2 \sup_{r\in[0,T]} \|\varepsilon\tilde{\xi}^*(r)\|.$$

On the event $\mathcal{E}_T(1/C_2) = \{\sup_{r\in[0,T]} \|\varepsilon\tilde{\xi}^*(r)\| < \frac{1}{C_2}\}$ we obtain $t^\infty > T$. This yields the claimed inequality, and we may choose $C = C_2 \vee 1$.

Claim 2: *There is a constant $\varepsilon_0 > 0$, and for all $T > 0$ there is a constant $C = C(T) > 0$ such that for all $p > 0$, $0 < \varepsilon \leqslant \varepsilon_0$ we obtain*

$$\mathbb{P}\left(\sup_{t\in[0,T]}\sup_{x\in H} |X^\varepsilon(t;x) - u(t;x)|_\infty \geqslant \varepsilon^p \;\middle|\; \tilde{T}_1 \geqslant T\right) \leqslant \mathbb{P}\left(\sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \geqslant C\varepsilon^p\right).$$

Let $p > 0$. Then we estimate

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]}\sup_{x\in H} |X^\varepsilon(t;x) - u(t;x)|_\infty \geqslant \varepsilon^p \;\middle|\; \tilde{T}_1 > T\right)\\
&\qquad\leqslant \mathbb{P}\left(\sup_{t\in[0,T]}\sup_{x\in H} |\tilde{Y}^\varepsilon(t;x) - u(t;x)|_\infty \geqslant \varepsilon^p\right)\\
&\qquad\qquad= \mathbb{P}\left(\sup_{t\in[0,T]}\sup_{x\in H} |\tilde{R}^\varepsilon(t;x) + \varepsilon\tilde{\xi}^*(t)|_\infty \geqslant \varepsilon^p\right). \qquad (6.8)
\end{aligned}$$

Let $\tilde{C} \geqslant 1$ be according to Claim 1. Analogously to (3.14) we can calculate

$$
\begin{aligned}
&\{ \sup_{t\in[0,T]} \sup_{x\in H} |\tilde{Y}^\varepsilon(t;x) - u(t;x)|_\infty \geqslant \varepsilon^p\} \\
&\subseteq \{ \sup_{t\in[0,T]} \sup_{x\in H} |\tilde{R}^\varepsilon(t;x) + \varepsilon\tilde{\xi}^*(t)|_\infty \geqslant \varepsilon^p\} \\
&\subseteq \{ \sup_{t\in[0,T]} \sup_{x\in H} |\tilde{R}^\varepsilon(t;x)|_\infty + \sup_{t\in[0,T]} |\varepsilon\tilde{\xi}^*(t)|_\infty \geqslant \varepsilon^p\} \\
&\qquad \subseteq \{ \sup_{t\in[0,T]} \sup_{x\in H} |\tilde{R}^\varepsilon(t;x)|_\infty \geqslant (1/2)\varepsilon^p\} \cup \{ \sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \geq (1/2)\varepsilon^p\} \\
&\qquad\quad \subseteq \{ \sup_{t\in[0,T]} \sup_{x\in H} |\tilde{R}^\varepsilon(t;x)|_\infty \geqslant (1/2)\varepsilon^p\} \cap \{ \sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \geqslant \frac{1}{\tilde{C}}\} \\
&\qquad\quad \cup \{ \sup_{t\in[0,T]} \sup_{x\in H} |\tilde{R}^\varepsilon(t;x)|_\infty \geqslant (1/2)\varepsilon^p\} \cap \{ \sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \leqslant \frac{1}{\tilde{C}}\} \\
&\qquad\qquad\qquad\qquad \cup \{ \sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \geq (1/2)\varepsilon^p\}. \qquad (6.9)
\end{aligned}
$$

By Claim 1 we have

$$
\{ \sup_{t\in[0,T]} \sup_{x\in H} |\tilde{R}^\varepsilon(t;x)|_\infty \geqslant (1/2)\varepsilon^p\} \cap \{ \sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \leqslant \frac{1}{\tilde{C}}\} \subset \{\sup_{[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \geqslant \frac{1}{2\,\tilde{C}}\varepsilon^p\}.
$$

Therefore, since $\tilde{C} \geqslant 1$ and $\varepsilon_0 \leqslant 1$

$$
\begin{aligned}
&\{ \sup_{t\in[0,T]} \sup_{x\in H} |\tilde{Y}^\varepsilon(t;x) - u(t;x)|_\infty \geqslant \varepsilon^p\} \\
&\subseteq \{ \sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \geqslant \frac{1}{\tilde{C}}\} \cup \{ \sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \geqslant \frac{1}{2\,\tilde{C}}\varepsilon^p\} \cup \{\sup_{[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \geqslant (1/2)\varepsilon^p\} \\
&\qquad\qquad\qquad\qquad = \{ \sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \geqslant \frac{1}{2\,\tilde{C}}\varepsilon^p\} \qquad (6.10)
\end{aligned}
$$

Hence we conclude

$$
\mathbb{P}\left(\sup_{x\in H} \sup_{t\in[0,T]} |\tilde{Y}^\varepsilon(t;x) - u(t;x)|_\infty \geqslant \varepsilon^p \right) \leqslant \mathbb{P}\left(\sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*(t)\| \geqslant \frac{1}{2\,\tilde{C}}\varepsilon^p \right).
$$

Claim 3: *For any $\rho := 1 - \tilde{\gamma}$, $p > 0$ and $0 < \Theta < 1$ there are constants $C > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$ and $T \geqslant 0$*

$$
\mathbb{P}\left(\sup_{t\in[0,T]} \|\varepsilon\tilde{\xi}^*_t\| \geqslant \varepsilon^p \right) \leqslant C\, T\, \varepsilon^{2-2p-(2-(1-\Theta)\alpha)\rho}.
$$

This is a consequence of Lemma 3.6. We have to verify that for $\rho = 1 - \tilde{\gamma}$ the exponent of ε provided by the Lemma is positive, that is if

$$2 - 2p - (2 - (1 - \Theta)\alpha)\rho = 2\left((1 - p) - (1 - \frac{(1 - \Theta)\alpha}{2})(1 - \tilde{\gamma})\right) > 0.$$

This true for $p = \tilde{\gamma}$. This completes the proof of Lemma 6.6. □

Final step in the proof of Lemma 6.3. We apply Lemma 6.6 for $T = T_{rec}^{r/2}$ to Lemma 6.5 and thus obtain that $\lim_{\varepsilon \to 0+} p_2(\varepsilon) \to 0$ and in Lemmas 6.3 that $\lim_{\varepsilon \to 0+} p_1(\varepsilon) = 0$. This completes the proof.

6.2. Asymptotic Estimates of Exit Times from Entire Domains of Attraction

Based on the previous estimate of $\tau_x^0(\varepsilon)$ under Hypothesis (H.3), we next will prove a result on first exit times. On the one hand it is much weaker than Theorem 2.18, since it only provides convergence in distribution instead of exponential convergence and uniformity of the event in the intial state is lost. On the other hand it holds for the non-reduced domain of attraction $D^{\pm}$. We restate Theorem 6.2 for convenience.

Theorem 6.7 (Asymptotic estimate for the first exit time from $D^{\pm}$)**.** *Assume that Hypothesis (H.1), (H.2) and (H.3) are satisfied. Then there is a family of random variables $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ with exponential law of parameter 1 and $h_0 > 0$ such that for all* $0 < h \leqslant h_0$

$$\lim_{\varepsilon\to 0+} \sup_{x\in D^+} \mathbb{P}(\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon) \geqslant \bar{\tau}(\varepsilon) + h) = 0.$$

Proof. Let $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ be given by Theorem 5.3. Then for any $h > 0$

$$\begin{aligned}
&\sup_{x\in D^{\pm}} \mathbb{P}(\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon) \geqslant \bar{\tau}(\varepsilon) + h) \\
&\leqslant \sup_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbb{P}(\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon) \geqslant \bar{\tau}(\varepsilon) + h) + \sup_{x\in\tilde{D}^{0}(\varepsilon^{\gamma})\cap D^{\pm}} \mathbb{P}(\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon) > \bar{\tau}(\varepsilon)) \\
&\leqslant \mathbb{E}\left[\sup_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}\{|\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon) - \bar{\tau}(\varepsilon)| \geqslant h\}\right] + \sup_{x\in\tilde{D}^{0}(\varepsilon^{\gamma})\cap D^{\pm}} \mathbb{P}(\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon) > \bar{\tau}(\varepsilon)).
\end{aligned} \tag{6.11}$$

The first term tends to 0 for $\varepsilon \to 0+$ by Theorem 2.18. For the second term we estimate

$$\begin{aligned}
&\sup_{x\in\tilde{D}^{0}(\varepsilon^{\gamma})} \mathbb{P}(\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon) > \bar{\tau}(\varepsilon)) \\
&\leqslant \sup_{x\in\tilde{D}^{0}(\varepsilon^{\gamma})} \mathbb{P}(\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon)\bar{\tau}(\varepsilon) \text{ and } X^{\varepsilon}(\tau_x^0(\varepsilon); x) \notin D^+) \\
&\quad + \sup_{x\in\tilde{D}^{0}(\varepsilon^{\gamma})} \mathbb{P}(\lambda^{\pm}(\varepsilon)\tau_x^{\pm}(\varepsilon) > \bar{\tau}(\varepsilon) \text{ and } X^{\varepsilon}(\tau_x^0(\varepsilon); x) \in \tilde{D}^{\pm}(\varepsilon^{\gamma})) =: I_1 + I_2.
\end{aligned} \tag{6.12}$$

For I_2, we use the strong Markov property and obtain

$$\begin{aligned}
I_2 &\leqslant \sup_{x\in\tilde{D}^{0}(\varepsilon^{\gamma})\cap D^+} \mathbb{P}(\lambda^{\pm}(\varepsilon)\tau_y^{\pm}(\varepsilon) > \bar{\tau}(\varepsilon) \text{ and } y = X^{\varepsilon}(\tau_x^0(\varepsilon); x) \in \tilde{D}^{\pm}(\varepsilon^{\gamma})) \\
&\leqslant \sup_{y\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbb{P}(\lambda^{\pm}(\varepsilon)\tau_y^{\pm}(\varepsilon) > \bar{\tau}(\varepsilon)) \leqslant \sup_{y\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbb{P}(\lambda^{\pm}(\varepsilon)\sigma_y^{\pm}(\varepsilon) > \bar{\tau}(\varepsilon)).
\end{aligned} \tag{6.13}$$

The last term also tends to 0 by Theorem 5.3. We continue to estimate I_1, and get

$$\begin{aligned}
I_1 &\leqslant \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)\cap D^+} \mathbb{P}(\lambda^\pm(\varepsilon)\tau_x^\pm(\varepsilon) > \bar{\tau}(\varepsilon) \text{ and } \tau_x^0(\varepsilon) \geqslant \tau_x^\pm(\varepsilon)) \\
&\leqslant \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)\cap D^+} \mathbb{P}(\lambda^\pm(\varepsilon)\tau_x^0(\varepsilon) > \bar{\tau}(\varepsilon)) \\
&\leqslant \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)\cap D^+} \mathbb{P}(\lambda^\pm(\varepsilon)\tau_x^0(\varepsilon) > \bar{\tau}(\varepsilon) \text{ and } \tau_x^0(\varepsilon) \leqslant \frac{1}{\varepsilon^{\alpha(1-\gamma/2)}}) \\
&\quad + \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)\cap D^+} \mathbb{P}(\lambda^\pm(\varepsilon)\tau_x^0(\varepsilon) > \bar{\tau}(\varepsilon) \text{ and } \tau_x^0(\varepsilon) > \frac{1}{\varepsilon^{\alpha(1-\gamma/2)}}) \\
&\leqslant \mathbb{P}(\bar{\tau}(\varepsilon) < \frac{\lambda^\pm(\varepsilon)}{\varepsilon^{\alpha(1-\gamma/2)}}) + \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{P}\left(\tau_x^0(\varepsilon) > \frac{1}{\varepsilon^{\alpha(1-\gamma/2)}}\right).
\end{aligned}$$

Since $\frac{\lambda^\pm(\varepsilon)}{\varepsilon^{\alpha(1-\tilde{\gamma})}} = \frac{\varepsilon^\alpha \ell(\frac{1}{\varepsilon})\mu((D_0^\pm)^c)}{\varepsilon^{\alpha(1-\gamma/2)}} \to 0$ for $\varepsilon \to 0+$ the first term in the preceding line tends to zero. To see that this is also the case for the second term we have to use Lemma 6.3. This completes the proof. □

6.3. Localization on Subcritical and Critical Time Scales

In Chapters 4 and 5 we have seen that exits and transitions between relevant areas in the domains of attaction of the metastable solutions are of the order of $\varepsilon^{-\alpha}$. We shall now consider our system on time scales smaller than this threshold. We shall thereby confirm the reasonable conjecture that on these time scales the solution trajectories converge in probability to the process spending all the time at the local minimum $\phi^{\pm}$ associated to the domain of attraction of the starting value $x \in D^{\pm}$. Our result is even stronger. We prove that after an initial relaxation time of the order of magnitude $T_{rec} + \kappa\gamma|\ln\varepsilon|$, the solution trajectories of the stochastic Chafee-Infante equation converge to the deterministic stationary solutions in $\phi^{\pm}$ uniformly in probability.

Theorem 6.8 (Uniform convergence in probability on subcritical time scales)**.** *Assume that Hypotheses (H.1) and (H.2) are satisfied and $T_{rec}, \kappa > 0$ are given by Proposition 2.15. Fix $0 < \delta < \alpha$. Then there is $h_0 > 0$ such that for $0 < h \leqslant h_0$ and for any $T > 0$*

$$\lim_{\varepsilon\to 0+} \mathbb{E}\left[\sup_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\ \sup_{t\in[T_{rec}+\kappa\gamma|\ln\varepsilon|, T/\varepsilon^{\delta}]} \mathbf{1}\{|X^{\varepsilon}(t;x) - \phi^{\pm}|_{\infty} > h\}\right] = 0. \tag{6.14}$$

Proof. Let $h_0 > 0$ be the supremum of all radii $h > 0$ such that for all $t \geqslant 0$

$$u(t; B_h(\phi^{\pm})) \subset B_h(\phi^{\pm}) \subset D^{\pm}.$$

First note, that the existence of such $h_0 > 0$ follows from Lemma B.3 or alternativle from investigations of sub- and super-solutions for scalar reaction diffusion equations by Matano [1979]. Secondly, since $0 \notin D^{\pm}$, $h_0 \leqslant |\phi^{\pm}|_{\infty}$. With $\Gamma = \Gamma(\kappa) > 0$ according to Theorem 5.3 let us consider the event

$$\tilde{\mathcal{E}}(\varepsilon) := \{\sup_{t\in[0,T_{rec}+\kappa\gamma|\ln\varepsilon|} \|\varepsilon\xi^{*}(t)\| < \varepsilon^{(\Gamma+2)\gamma}\} \cap \{T_1 > T_{rec} + \kappa\gamma|\ln\varepsilon|\},$$

also defined in (5.12). We can now estimate by the supremum over all possible states the process may take after $T_{rec} + \kappa\gamma|\ln\varepsilon|$ time units, and obtain for $0 < h \leqslant h_0$ and ε small enough

$$\begin{aligned}
&\mathbb{E}\left[\sup_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})}\ \sup_{t\in[T_{rec}+\kappa\gamma|\ln\varepsilon|), T/\varepsilon^{\delta}]} \mathbf{1}\{|X^{\varepsilon}(t;x) - \phi^{\pm}|_{\infty} \geqslant h\}\right] \\
&= \mathbb{E}\Bigg[\sup_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}\{\sup_{t\in[0,T/\varepsilon^{\delta}-T_{rec}-\kappa\gamma|\ln\varepsilon|]} |X^{\varepsilon}(t; X^{\varepsilon}(0;x)) \circ \theta_{T_{rec}+\kappa\gamma|\ln\varepsilon|} - \phi^{\pm}|_{\infty} \geqslant h\} \\
&\qquad\qquad \cdot \big(\mathbf{1}(\tilde{\mathcal{E}}(\varepsilon)) + \mathbf{1}(\tilde{\mathcal{E}}^{c}(\varepsilon))\big)\Bigg] \\
&\leqslant \mathbb{E}\left[\sup_{y\in B_{\varepsilon^{2\gamma}}(\phi^{\pm})} \mathbf{1}\{\sup_{t\in[0,T/\varepsilon^{\delta}-T_{rec}-\kappa\gamma|\ln\varepsilon|} |X^{\varepsilon}(t;y) - \phi^{\pm}|_{\infty} \geqslant h\}\right] + \mathbb{P}\left(\tilde{\mathcal{E}}^{c}(\varepsilon)\right) =: I_1 + I_2
\end{aligned}$$

While $I_2 = I_2(\varepsilon) \to 0$ for $\varepsilon \to 0+$ by Lemma 6.3, for I_1, we get the estimate

$$I_1 \leqslant \mathbb{E}\left[\sup_{y \in B_{\varepsilon^{2\gamma}}(\phi^\pm)} \mathbf{1}\{\tau_y^\pm(\varepsilon) < T/\varepsilon^\delta\}\right]$$

$$\leqslant \mathbb{E}\left[\sup_{y \in B_{\varepsilon^{2\gamma}}(\phi^\pm)} \mathbf{1}\{\lambda^\pm(\varepsilon)\tau_y^\pm(\varepsilon) < T\lambda^\pm(\varepsilon)/\varepsilon^\delta\}\right],$$

which converges to 0 as $\varepsilon \to 0+$. For the second inequality we use (3.14) and the definition of h_0. □

Corollary 6.9 (Convergence on subcritical time scales)**.** *Assume that Hypotheses (H.1) and (H.2) are satisfied. Fix* $0 < \delta < \alpha$. *Then for all* $h > 0$ *and* $T > 0$

$$\lim_{\varepsilon \to 0+} \mathbb{E}\left[\sup_{x \in \tilde{D}^\pm(\varepsilon^\gamma)} \mathbf{1}\{|X^\varepsilon(T/\varepsilon^\delta; x) - \phi^\pm|_\infty > h\}\right] = 0.$$

If we include Hypothesis (H.3) into our reasoning in the situation of Theorem 6.8 we obtain a similar result. But time uniformy in the basic estimates is addressed differently. As opposed to Theorem 6.8, where the estimation is achieved uniformly on time intervals of the order $[T_{rec} + \kappa\gamma|\ln\varepsilon|, T/\varepsilon^\delta]$ with $\delta \in (0, \alpha)$, we are now able to treat intervals of the shape $[T/\varepsilon^{\alpha(1-\gamma/2)} + T_{rec} + \kappa\gamma|\ln\varepsilon|, T/\lambda^0(\varepsilon)]$. In addition, by (H.3) we only control the exit times $\tau_x^0(\varepsilon)$ of the separatrix in terms of ε but we do not know into which direction this exit leads. We have no information whether $X^\varepsilon(\tau_x^0(\varepsilon); x) \in \tilde{D}^+(\varepsilon^\gamma)$ or $X^\varepsilon(\tau_x^0(\varepsilon); x) \in \tilde{D}^-(\varepsilon^\gamma)$. This is natural, since the deterministic dynamics close to the separatrix is too slow to predetermine to which domain X^ε tends.

Theorem 6.10. *Assume that Hypotheses (H.1), (H.2) and (H.3) are satisfied and* $T_{rec}, \kappa > 0$ *given by Proposition 2.15. Set for* $T > 0$, γ *due to (H.2), and* $\varepsilon > 0$

$$s(\varepsilon) := \frac{T}{\varepsilon^{\alpha(1-\gamma/2)}} + T_{rec} + \kappa\gamma|\ln(\varepsilon)|.$$

Then there is $h_0 > 0$ *such that for all* $T > 0$ *and* $0 < h \leq h_0$

$$\lim_{\varepsilon \to 0+} \sup_{x \in H} \mathbb{E}\left[\sup_{t \in [s(\varepsilon), \frac{T}{\lambda^0(\varepsilon)}]} \mathbf{1}\{X^\varepsilon(t; x) \notin B_h(\phi^+) \cup B_h(\phi^-)\}\right] = 0.$$

Proof. The proof is divided in two steps.

1. Since with a slowly varying function ℓ we have $\lambda^0(\varepsilon) = \varepsilon^\alpha\ \ell(1/\varepsilon)\ \mu(B_1^c(0))$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$ we have $\frac{T}{\lambda^0(\varepsilon)} > \frac{T}{\varepsilon^{\alpha(1-\gamma/2)}} + T_{rec} + \kappa\gamma|\ln\varepsilon| =: s(\varepsilon)$. In this case the flow property of X^ε ensures

$$\mathbb{E}\left[\sup_{t\in[s(\varepsilon),\frac{T}{\lambda^0(\varepsilon)}]} \mathbf{1}\{X^\varepsilon(t;x)\notin B_h(\phi^+)\cup B_h(\phi^-)\right]$$
$$= \mathbb{E}\Bigg[\sup_{t\in[s(\varepsilon),\frac{T}{\lambda^0(\varepsilon)}]} \mathbb{E}\Bigg[\mathbf{1}\{X^\varepsilon(s(\varepsilon);x)\circ\theta_{t-s(\varepsilon)}\notin B_h(\phi^+)\cup B_h(\phi^-)\}$$
$$\leqslant \sup_{y\in H}\mathbb{E}\left[\mathbf{1}\{X^\varepsilon(s(\varepsilon);y)\notin B_h(\phi^+)\cup B_h(\phi^-)\}\right].$$

2. We treat the cases $y\in\tilde{D}^\pm(\varepsilon^\gamma)$ and $y\in\tilde{D}^0(\varepsilon^\gamma)$ separately. The first case is already treated in Theorem 6.8 implying for $\delta = 1-\gamma/4$

$$\lim_{\varepsilon\to 0+}\mathbb{E}\left[\sup_{y\in\tilde{D}^\pm(\varepsilon^\gamma)}\sup_{t\in[T_{rec}+\kappa\gamma|\ln\varepsilon|,T/\varepsilon^\delta]}\mathbf{1}\left\{X^\varepsilon(t;y)\notin B_h(\phi^+)\cup B_h(\phi^-)\right\}\right]=0. \quad (6.15)$$

The second case $y\in\tilde{D}^0(\varepsilon^\gamma)$ can be dealt with by the estimate

$$\sup_{y\in\tilde{D}^0(\varepsilon^\gamma)}\mathbb{E}\left[\mathbf{1}\left\{X^\varepsilon(s(\varepsilon);y)\notin B_h(\phi^+)\cup B_h(\phi^-)\right\}\right]$$
$$\leqslant \sup_{y\in\tilde{D}^0(\varepsilon^\gamma)}\mathbb{E}\left[\mathbf{1}\left\{X^\varepsilon(s(\varepsilon);y)\notin B_h(\phi^+)\cup B_h(\phi^-)\right\}\cap\left\{\tau_y^0(\varepsilon)\leqslant\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\right\}\right]$$
$$+\sup_{y\in\tilde{D}^0(\varepsilon^\gamma)}\mathbb{P}\left(\tau_y^0(\varepsilon)>\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\right)=:I_1+I_2.$$

By Lemma 6.3, $I_2=I_2(\varepsilon)$ tends to 0 as $\varepsilon\to 0+$. To estimate the term I_1, we use the strong Markov property of X^ε to obtain

$$I_1=\sup_{y\in\tilde{D}^0(\varepsilon^\gamma)}\mathbb{E}\Bigg[\mathbf{1}\left\{\tau_y^0(\varepsilon)\leqslant\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\right\}$$
$$\cdot\ \mathbf{1}\left\{X^\varepsilon\left(s(\varepsilon)-\tau_y^0(\varepsilon);y\right)\circ\theta_{\tau_y^0(\varepsilon)}\notin B_h(\phi^+)\cup B_h(\phi^-)\right\}\Bigg] \quad (6.16)$$

and consequently

$$
\begin{aligned}
I_1 \leqslant & \sup_{y\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{E}\Big[\mathbf{1}\{\tau_y^0(\varepsilon) \leqslant \frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\} \\
& \qquad \cdot \mathbf{1}\left\{X^\varepsilon\left(s(\varepsilon)-\tau_y^0(\varepsilon);y\right)\circ\theta_{\tau_y^0(\varepsilon)} \notin B_h(\phi^+)\cup B_h(\phi^-)\right\}\Big] \\
\leqslant & \sup_{y\in\tilde{D}^0(\varepsilon^\gamma)} \mathbb{E}\Big[\mathbf{1}\{\tau_y^0(\varepsilon) \leqslant \frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\} \\
& \qquad \cdot \sup_{z\in\tilde{D}^+(\varepsilon^\gamma)\cup\tilde{D}^-(\varepsilon^\gamma)} \sup_{t\in[T_{rec}+\kappa\gamma|\ln\varepsilon|,s(\varepsilon)]} \mathbf{1}\left\{X^\varepsilon(t;z)\notin B_h(\phi^+)\cup B_h(\phi^-)\right\}\Big] \\
& \qquad \leqslant \sum_{k=\pm}\mathbb{E}\Big[\sup_{z\in\tilde{D}^k(\varepsilon^\gamma)} \mathbf{1}\Big\{\sup_{t\in[T_{rec}+\kappa\gamma|\ln\varepsilon|,s(\varepsilon)]}|X^\varepsilon(t;z)-\phi^k|_\infty > h\Big\}\Big].
\end{aligned}
$$

Now choose $\delta=\alpha(1-\gamma/4)$ as for inequality (6.15). Hence Theorem 6.8 ensures that I_1 tends to zero as $\varepsilon\to 0+$. □

Corollary 6.11. *Assume that Hypotheses (H.1), (H.2) and (H.3) are satisfied. Fix $\delta\in(\alpha(1-\gamma/2),\alpha)$. Then there is $h_0>0$ such that for all $T>0$ and $0<h\le h_0$*

$$
\lim_{\varepsilon\to 0+}\sup_{x\in H}\mathbb{E}\left[\sup_{t\in[\frac{T}{\varepsilon^\delta},\frac{T}{\lambda^0(\varepsilon)}]}\mathbf{1}\{X^\varepsilon(t;x)\notin B_h(\phi^+)\cup B_h(\phi^-)\}\right]=0.
$$

The proof if obvious since there is $\varepsilon_0>0$ such that for $0<\varepsilon\le\varepsilon_0$ we have

$$
\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}+T_{rec}+\kappa\gamma|\ln\varepsilon|<\frac{T}{\varepsilon^\delta}.
$$

6.4. Metastable Behavior

In this Section, equipped with our previously obtained knowledge of exit and transition times in the limit of small noise amplitude $\varepsilon \to 0$, we shall investigate the global asymptotic behavior of our jump diffusion process in the time scale in which transitions occur, i.e. in the scale given by $\lambda^0(\varepsilon) = \nu(\frac{1}{\varepsilon}B_1^c(0)), \varepsilon > 0$. It turns out that in this time scale, the switching of the diffusion between neighborhoods of the stable solutions $\phi^\pm$ can be well described by a Markov chain jumping back and forth between two states with a characteristic Q-matrix determined by the quantities $\mu((D_0^\pm)^c)/\mu(B_1^c(0))$ as jumping rates. We shall obtain convergence results for the finite-dimensional distributions and the initial values $x \in \tilde{D}^\pm(\varepsilon^\gamma)$. This convergence result pertains even for all $x \in H$ if the Markov chain has random initial conditions, which contain the information if for initial conditions $x \in \tilde{D}^0(\varepsilon^\gamma)$ the process $X^\varepsilon(\cdot; x)$ exits this set to $\tilde{D}^+(\varepsilon^\gamma)$ or $\tilde{D}^-(\varepsilon^\gamma)$.

Recall the notation of Definition 2.23. For $T > 0$ we consider (finite) *partitions of* $[0, T]$ as finite families of points in $[0, T]$, with $0 < t_1 < \ldots t_n = T$, and write $|\pi| = n$. We shall denote by $\Pi[0, T]$ the collection of all finite partitions in $[0, T]$. For convenience we shall write for $h > 0$, $\pi = (t_1, \ldots, t_n) \in \Pi[0, T]$ and $\bar{v} = (v_1, \ldots, v_n) \in \{\phi^+, \phi^-\}^{|\pi|}$ and $\varepsilon > 0$

$$X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)}; \cdot) := (X^\varepsilon(\frac{t_1}{\lambda^0(\varepsilon)}; \cdot), \ldots, X^\varepsilon(\frac{t_{|\pi|}}{\lambda^0(\varepsilon)}; \cdot))$$

and

$$B_h(\bar{v}) = B_h(v_1) \times \cdots \times B_h(v_n).$$

For convenience we restate Theorem 2.24.

Theorem 6.12 (Metastability). *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ be given and denote by μ the limiting measure of ν according to (2.6). Suppose Hypotheses (H.1), (H.2) and (H.3) are satisfied. Then there exists $h_0 > 0$ and a continuous time Markov chain $(Y(t))_{t\geqslant 0}$ switching between the elements of $\{\phi^+, \phi^-\}$ with generating matrix*

$$Q = \frac{1}{\mu(B_1^c(0))} \begin{pmatrix} -\mu\left(\left(D_0^+\right)^c\right) & \mu\left(\left(D_0^+\right)^c\right) \\ \mu\left(\left(D_0^-\right)^c\right) & -\mu\left(\left(D_0^-\right)^c\right) \end{pmatrix}.$$

which satisfy: For all $T > 0$, $\pi = (t_1, \ldots, t_n) \in \Pi[0, T]$, $\bar{v} \in (v_1, \ldots, v_{|\pi|}) \in \{\phi^+, \phi^-\}^{|\pi|}$ and $0 < h \leqslant h_0$ we have

$$\lim_{\varepsilon \to 0+} \sup_{x \in \tilde{D}^\pm(\varepsilon^\gamma)} \left|\mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)}; x) \in B_h(\bar{v})\big) - \mathbb{P}\big(Y(\pi, x) = \bar{v}\big)\right| = 0.$$

Proof. The proof is given in four Parts.

1. Construction of an auxiliary process $\hat{Y}^\varepsilon$: We fix $h_0 > 0$ such that for $0 < h \leqslant h_0$ Theorem 5.8 and Corollary 6.10 are true and define a two state auxiliary process $\hat{Y}^\varepsilon$ by the sequence of stopping times and stable states marking the transitions between $B_h(\phi^+)$ and $B_h(\phi^-)$. Note that $\tau_x^0(\varepsilon) = 0$ if $x \in \tilde{D}^\pm(\varepsilon^\gamma)$. For $x \in D^\pm$, $\varepsilon > 0$ and $k \geqslant 1$

let

$$\begin{aligned}
\sigma^0(\varepsilon;x) &:= 0,\\
m^0(\varepsilon,x) &:= \phi^\pm,\\
m^k(\varepsilon;x) &:= \phi^+\mathbf{1}\{X^\varepsilon(\sigma^k(\varepsilon;x);x)\in B_h(\phi^+)\} + \phi^-\mathbf{1}\{X^\varepsilon(\sigma^k(\varepsilon;x);x)\in B_h(\phi^-)\},\\
\sigma^k(\varepsilon;x) &:= \inf\{t>\sigma^{k-1}(\varepsilon;x) \mid X^\varepsilon(t;x)\in B_h(\phi^+)\cup B_h(\phi^-)\setminus B_h(m^{k-1}(\varepsilon;x))\}.
\end{aligned}$$

Based on these quantities we define a Markovian finite state process on the critical time scale $T/\lambda^0(\varepsilon)$ by setting

$$\hat{Y}^\varepsilon(T;x) := \sum_{k=0}^{\infty} m^k(\varepsilon;x)\,\mathbf{1}\{\sigma^k(\varepsilon;x)\leqslant T/\lambda^0(\varepsilon)<\sigma^{k+1}(\varepsilon;x)\}.$$

By construction, the jump times of $\hat{Y}^\varepsilon$ are $\lambda^0(\varepsilon)\sigma^k(\varepsilon;x)$ and since there are only two stable states we obtain for any $k\geqslant 2$, $x\in B_h(\phi^\pm)$ and $\varepsilon>0$

$$\begin{aligned}
&\mathbb{P}\left(m^{k+1}(\varepsilon;x)=\phi^\mp \mid m^k(\varepsilon;x)=\phi^\pm\right)\\
&\qquad = \mathbb{P}\left(\hat{Y}^\varepsilon(\lambda^0(\varepsilon)\sigma^{k+1}(\varepsilon);x)=\phi^\mp \mid \hat{Y}^\varepsilon(\lambda^0(\varepsilon)\sigma^k(\varepsilon);x)=\phi^\pm\right)=1. \qquad (6.17)
\end{aligned}$$

Hence $m^k(x,\varepsilon)=y_k$ $\mathbb{P}$-almost surely. This implies that $\hat{Y}^\varepsilon$ is a continuous time Markov chain.

2. Construction of the limiting Markov chain: Define a continuous time Markov process $(Y(t))_{t\geqslant 0}$ taking values in $\{\phi^+,\phi^-\}$ by its generating matrix

$$Q=\begin{pmatrix} -q_+ & q_+\\ q_- & -q_-\end{pmatrix} = \frac{1}{\mu(B_1^c(0))}\begin{pmatrix} -\mu\left((D_0^+)^c\right) & \mu\left((D_0^+)^c\right)\\ \mu\left((D_0^-)^c\right) & -\mu\left((D_0^-)^c\right)\end{pmatrix}.$$

Denote its jump times and states $(\sigma_k, y_k=Y_{\sigma_k}), k\in\mathbb{N}$, where inter jump times are conditionally independent and exponentially distributed with

$$\begin{aligned}
&\mathcal{L}\left(\sigma_{k+1}-\sigma_k|y_k=\phi^\pm\right)=EXP(1/q_\pm)\\
&\mathbb{P}\left(y_{k+1}=\phi^\pm|y_k=\phi^\mp\right)=1.
\end{aligned}$$

For $x \in D^{\pm}$, a partition $\pi \in \Pi[0,T]$ and $\bar{v} \in \{\phi^+, \phi^-\}^{|\pi|}$ we may estimate

$$\begin{aligned}
&\Big|\mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)};x) \in B_h(\bar{v})\big) - \mathbb{P}_{\phi^\pm}\big(Y(\pi) = \bar{v}\big)\Big| \\
&\qquad \leqslant \Big|\mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)};x) \in B_h(\bar{v})\big) - \mathbb{P}_{\phi^\pm}\big(\hat{Y}(\pi) = \bar{v}\big)\Big| \\
&\qquad\qquad + \Big|\mathbb{P}_{\phi^\pm}\big(\hat{Y}(\pi) = \bar{v}\big) - \mathbb{P}_{\phi^\pm}\big(Y(\pi) = \bar{v}\big)\Big|. \qquad (6.18)
\end{aligned}$$

3. Convergence of the rescaled process to the auxiliary process: Let us estimate the first term on the right-hand side of (6.18). For this purpose we cut the events of both summands into comparable pieces. For $\varepsilon > 0$, $\pi \in \Pi[0,T]$, $\bar{v} \in \{\phi^+, \phi^-\}^{|\pi|}$ and $x \in D^{\pm}$ we have

$$\begin{aligned}
&\mathbb{P}\left(X^\varepsilon(\pi/\lambda^0(\varepsilon);x) \in B_h(\bar{v})\right) - \mathbb{P}\left(\hat{Y}^\varepsilon(\pi;x) = \bar{v}\right) \\
&= \mathbb{P}\left(X^\varepsilon(\pi/\lambda^0(\varepsilon);x) \in B_h(\bar{v}) \text{ and } \hat{Y}^\varepsilon(\pi;x) = \bar{v}\right) \\
&\quad + \mathbb{P}\big(X^\varepsilon(\pi/\lambda^0(\varepsilon);x) \in B_h(\bar{v}) \text{ and } \hat{Y}^\varepsilon(\pi;x) \in \{\phi^+, \phi^-\}^{|\pi|} \setminus \{v\}\big) \\
&\quad - \mathbb{P}\big(\hat{Y}^\varepsilon(\pi;x) = \bar{v} \text{ and } X^\varepsilon(\pi/\lambda^0(\varepsilon);x) \in B_h(\bar{v})\big) \\
&\quad - \mathbb{P}\big(\hat{Y}^\varepsilon(\pi;x) = \bar{v} \text{ and } X^\varepsilon(\pi/\lambda^0(\varepsilon);x) \in \bigcup_{w \in \{\phi^+,\phi^-\}^{|\pi|} \setminus \{v\}} B_h(w)\big) \\
&\quad - \mathbb{P}\big(\hat{Y}^\varepsilon(T;x) = \phi^{\pm} \text{ and } X^\varepsilon(\pi/\lambda^0(\varepsilon);x) \notin \bigcup_{w \in \{\phi^+,\phi^-\}^{|\pi|}} B_h(w)\big).
\end{aligned}$$

Note that the first and the third term of the right-hand side cancel and the second and and the fourth term vanish by definition of $\hat{Y}^\varepsilon$. Therefore we are left with the last term, which we may estimate for $\delta > \alpha(1-\gamma/2)$ by

$$\begin{aligned}
&|\mathbb{P}(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)};x) \in B_h(\bar{v}) - \mathbb{P}(\hat{Y}^\varepsilon(t;x) = \bar{v})| \\
&\qquad \leqslant \mathbb{P}\big(X^\varepsilon(\pi/\lambda^0(\varepsilon);x) \notin \bigcup_{w \in \{\phi^+,\phi^-\}^{|\pi|}} B_h(w)\big) \\
&\qquad \leqslant \sup_{y \in H} \mathbb{E}\left[\sup_{t \in [\frac{T}{\varepsilon^\delta}, \frac{T}{\lambda^0(\varepsilon)}]} X^\varepsilon(t;y) \notin B_h(\phi^+) \cup B_h(\phi^-)\right].
\end{aligned}$$

The expression in the last line of the preceding estimate does not depend on $v \in \{\phi^+, \phi^-\}^{|\pi|}$ or $x \in D^+$, such that we can take the supremum on the left-hand side. Hence we have obtained

$$\sup_{x \in D^{\pm}} \left|\mathbb{P}\left(X^\varepsilon(\pi/\lambda^0(\varepsilon);x) \in B_h(\bar{v})\right) - \mathbb{P}\left(\hat{Y}^\varepsilon(\pi;x) = \bar{v}\right)\right| \leq p_1(\varepsilon).$$

The result $\lim_{\varepsilon\to 0+} p_1(\varepsilon) = 0$ is implied by Theorem 6.10.

4. Convergence of the $\hat{Y}^\varepsilon$ to the Markov chain: Let us now treat the second term on the right-hand side of (6.18). Its convergence to zero is a consequence of the weak convergence of $\hat{Y}^\varepsilon$ to Y, and therefore implied by the joint weak convergence of jump times and jump increments (see for example Xia [1992])

$$\mathcal{L}\left(\left((\lambda^0(\varepsilon)\sigma^k(\varepsilon;x), m^k(\varepsilon;x)\right)_{k\geqslant 0})\right) \to \mathcal{L}\left(\left((\sigma_k, y_k)\right)_{k\geqslant 0}\right), \quad \varepsilon \to 0+\,.$$

This in turn follows from the joint weak convergence of the inter jump times and jump increments

$$\mathcal{L}\left(\left((\lambda^0(\varepsilon)(\sigma^{k+1}(\varepsilon;x) - \sigma^k(\varepsilon;x)), m^k(\varepsilon;x)\right)_{k\geqslant 0})\right) \to \mathcal{L}\left(\left((\sigma_{k+1} - \sigma_k, y_k)\right)_{k\geqslant 0}\right). \tag{6.19}$$

Since $\hat{Y}^\varepsilon$ is a Markov chain, to verify (6.19) it is sufficient to show individual convergence for the infinitely many components. To prove the latter we shall treat the case of $k \geqslant 2$ with $x \in B_h(\phi^\pm)$ and $k = 1$ with $x \in \tilde{D}^\pm(\varepsilon^\gamma)$ and $x \in \tilde{D}^0(\varepsilon^\gamma)$ separately. Note that for $k \geq 2$ we only have to treat initial values $x \in B_h(\phi^\pm)$, since $X^\varepsilon(\sigma^1(\varepsilon;x);x) \in B_h(\phi^\pm)$. Together with the fact that the elements of $\left(\sigma^{k+1}(\varepsilon;x) - \sigma^k(\varepsilon;x)\right)_{k\geqslant 1}$ are independent, condition (6.19) boils down to the convergence

$$\begin{aligned}\mathcal{L}(\lambda^0(\varepsilon)(\sigma^{k+1}(\varepsilon;x) - \sigma^k(\varepsilon;x)) \mid m^k(\varepsilon;x) = \phi^\pm)&\\ \xrightarrow{d} EXP(1/q^\pm) = EXP(\frac{\mu(B_1^c(0))}{\mu((D_0^\pm)^c)}) \quad &\text{as } \varepsilon \to 0+\,. \end{aligned}\tag{6.20}$$

To prove this, note that the strong Markov property of X^ε implies for the law of increments of transition times for $x \in B_h(\phi^\pm)$, $\varepsilon > 0$

$$\begin{aligned}&\mathcal{L}\big(\lambda^0(\varepsilon)\left(\sigma^{k+1}(\varepsilon;x) - \sigma^k(\varepsilon;x)\right) \mid m^k(\varepsilon;x) = \phi^\pm\big)\\ &\quad = \mathcal{L}\left(\lambda^0(\varepsilon)\left(\sigma^{k+1}(\varepsilon;x) - \sigma^k(\varepsilon;x)\right) \mid X^\varepsilon(\sigma^k(\varepsilon;x);x) \in B_h(\phi^\pm)\right)\\ &\quad = \mathcal{L}\left(\lambda^0(\varepsilon)\sigma^1(\varepsilon; X^\varepsilon(\sigma^k(\varepsilon;x);x))\right) = \mathcal{L}\left(\lambda^0(\varepsilon)\sigma^\pm(\varepsilon; X^\varepsilon(\sigma^k(\varepsilon;x);x))\right).\end{aligned}\tag{6.21}$$

In addition, the regular variation with index $-\alpha$ of $\lambda^\pm$ and λ^0 implies

$$\lim_{\varepsilon\to 0+} \frac{\lambda^0(\varepsilon)}{\lambda^\pm(\varepsilon)} = \frac{\mu(B_1^c(0)}{\mu\left((D^\pm)^c\right)}. \tag{6.22}$$

Let now $(\bar{\tau}(\varepsilon))_{\varepsilon>0}$ be the family of exponentially distributed random variables with parameter 1 according to Theorem 4.9. It therefore suffices to prove that for $h > 0$

$$\mathbb{E}\left[\mathbf{1}\{|\lambda^0(\varepsilon)\sigma^\pm(\varepsilon;x) - \frac{\mu(B_1^c(0)}{\mu\left((D^\pm)^c\right)}\bar{\tau}(\varepsilon)| > h\}\right] \to 0, \qquad \text{as } \varepsilon \to 0+\,.$$

In fact, we may write

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}\{|\lambda^0(\varepsilon)\sigma^{\pm}(\varepsilon;x) - \frac{\mu(B_1^c(0)}{\mu\left((D^{\pm})^c\right)}\bar{\tau}(\varepsilon)| > h\}\Big] \\
&\qquad\leqslant\ \mathbb{E}\Big[\sup_{x\in\tilde{D}^{\pm}(\varepsilon^{\gamma})} \mathbf{1}\{|\frac{\lambda^0(\varepsilon)}{\lambda^{\pm}(\varepsilon)}\lambda^{\pm}(\varepsilon)\sigma^{\pm}(\varepsilon;x) - \frac{\lambda^0(\varepsilon)}{\lambda^{\pm}(\varepsilon)}\bar{\tau}(\varepsilon)| > h/2\}\Big] \\
&\qquad\qquad + \ \mathbb{P}\Big(\underbrace{|\frac{\lambda^0(\varepsilon)}{\lambda^{\pm}(\varepsilon)} - \frac{\mu(B_1^c(0)}{\mu\left((D_0^{\pm})^c\right)}|}_{\to 0,\ \varepsilon\to 0+}\ \bar{\tau}(\varepsilon) > h/2\Big) \qquad (6.23)
\end{aligned}$$

By (6.22), the second term of the last estimate tends to zero. For the first one we use Theorem 2.18 to conclude. □

We restate Theorem 2.25.

Theorem 6.13 (Uniform Metastability)**.** *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (k\pi)^2$ for $k \in \mathbb{N}$ be given and denote by μ the limiting measure of ν according to (2.6). Suppose that Hypotheses (H.1), (H.2) and (H.3) are satisfied. Then there exists $h_0 > 0$ and a continuous time Markov chain $(Y(t;x))_{t\geqslant 0}$ starting in $\phi^{\pm}$ if $x \in D^{\pm}$ and switching between the elements of $\{\phi^+, \phi^-\}$ with generating matrix*

$$Q = \frac{1}{\mu(B_1^c(0))}\begin{pmatrix} -\mu\left(\left(D_0^+\right)^c\right) & \mu\left(\left(D_0^+\right)^c\right) \\ \mu\left(\left(D_0^-\right)^c\right) & -\mu\left(\left(D_0^-\right)^c\right) \end{pmatrix}$$

and random initial condition

$$\Phi^{\varepsilon}(x) = \phi^{\pm}, \begin{cases} \text{if } x \in D^{\pm} \\ \text{if } x \in \mathcal{S} \text{ and } X^{\varepsilon}(\tau_x^0(\varepsilon); x) \in D^{\pm}, \end{cases}$$

which satisfy the following. For all $T > 0$, $\pi \in \Pi[0,T]$, $\bar{v} \in \{\phi^+, \phi^-\}^{|\pi|}$ and $0 < h \leqslant h_0$ we have

$$\lim_{\varepsilon\to 0+}\sup_{x\in H}\left|\mathbb{P}\big(X^{\varepsilon}(\frac{\pi}{\lambda^0(\varepsilon)}; x) \in B_h(\bar{v})\big) - \mathbb{P}\big(Y(\pi, \Phi^{\varepsilon}(x)) = \bar{v}\big)\right| = 0.$$

Proof. We proceed in similar steps as in the previous proof. For fixed $\pi \in \Pi[0,T]$ we write $\bar{1} := (1, \ldots, 1) \in \{1\}^{|\pi|}$.

1. Construction of an auxiliary process $\hat{Y}^{\varepsilon}$: Fix h_0 and $0 < h \leqslant h_0$ as in the proof of Theorem 6.12. We define again

$$\hat{Y}^{\varepsilon}(T;x) := \sum_{k=0}^{\infty} m^k(\varepsilon;x)\ \mathbf{1}\{\sigma^k(\varepsilon;x) \leqslant T/\lambda^0(\varepsilon) < \sigma^{k+1}(\varepsilon;x)\},$$

with slighly modified transition times and random states

$$\begin{aligned}
\sigma^0(\varepsilon;x) &:= 0,\\
m^0(\varepsilon,x) &:= \Phi^\varepsilon(x)\\
\sigma^1(\varepsilon,x) &:= \tau_x^0(\varepsilon) + \sigma^\pm(\varepsilon; X^\varepsilon(\tau_x^0(\varepsilon);x)),\\
\sigma^k(\varepsilon;x) &:= \inf\{t > \sigma^{k-1}(\varepsilon;x) \mid X^\varepsilon(t;x) \in B_h(\phi^+) \cup B_h(\phi^-) \setminus B_h(m^{k-1}(\varepsilon;x))\}\\
m^k(\varepsilon;x) &:= \phi^+\mathbf{1}\{X^\varepsilon(\sigma^k(\varepsilon;x);x) \in B_h(\phi^+)\} + \phi^-\mathbf{1}\{X^\varepsilon(\sigma^k(\varepsilon;x);x) \in B_h(\phi^-)\}.
\end{aligned}$$

2. Construction of the limiting Markov chain: The continuous time Markov process $(Y(t;x))_{t\geqslant 0}$ with the states $\{\phi^+,\phi^-\}$ is defined identically by its infinitesimally generating matrix

$$Q = \begin{pmatrix} -q_+ & q_+ \\ q_- & -q_- \end{pmatrix} = \frac{1}{\mu(B_1^c(0))} \begin{pmatrix} -\mu\left(\left(D_0^+\right)^c\right) & \mu\left(\left(D_0^+\right)^c\right) \\ \mu\left(\left(D_0^-\right)^c\right) & -\mu\left(\left(D_0^-\right)^c\right). \end{pmatrix}.$$

Denote its jump times and states $(\sigma_k, y_k = Y_{\sigma_k}), k \in \mathbb{N}$, where inter jump times are conditionally independent and exponentially distributed with

$$\begin{aligned}
\mathcal{L}\left(\sigma_{k+1} - \sigma_k | y_k = \phi^\pm\right) &= EXP(1/q_\pm)\\
\mathbb{P}\left(y_{k+1} = \phi^\pm | y_k = \phi^\mp\right) &= 1.
\end{aligned}$$

For $\varepsilon > 0$, $x \in D^\pm$, a partition $\pi \in \Pi[0,T]$ and $\bar v \in \{\phi^+,\phi^-\}^{|\pi|}$ we may estimate

$$\begin{aligned}
&\Big|\mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)};x) \in B_h(\bar v)\big) - \mathbb{P}\big(Y(\pi;\Phi^\varepsilon(x)) = \bar v\big)\Big|\\
&\leqslant \sup_{x\in\tilde D^\pm(\varepsilon^\gamma)} \Big|\mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)};x) \in B_h(\bar v)\big) - \mathbb{P}\big(Y(\pi;\Phi^\varepsilon(x)) = \bar v\big)\Big|\\
&+ \sup_{x\in\tilde D^0(\varepsilon^\gamma)} \Big|\mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)};x) \in B_h(\bar v)\big) - \mathbb{P}\big(\hat Y(\pi;X^\varepsilon(\tau_x^0(\varepsilon);x)) = \bar v\big)\Big|\\
&+ \sup_{x\in\tilde D^0(\varepsilon^\gamma)} \Big|\mathbb{P}\big(\hat Y(\pi;X^\varepsilon(\tau_x^\pm(\varepsilon);x)) = \bar v\big) - \mathbb{P}\big(Y(\pi;\Phi^\varepsilon(x)) = \bar v\big)\Big| = I_1 + I_2 + I_3 \quad (6.24)
\end{aligned}$$

The convergence of I_1 to 0 as $\varepsilon \to 0+$ is covered by Theorem 6.12.

3. Convergence of the rescaled process to the auxiliary process: To show the convergence of I_2 for $\pi = (t_1,\ldots,t_{|\pi|})$ we first choose $\varepsilon > 0$ small enough to have

$\frac{t_1}{\lambda^0(\varepsilon)} > \frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}$ and then use the strong Markov property at $\tau_x^0(\varepsilon)$ to obtain

$$\begin{aligned}
I_2 &\leqslant \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \big|\mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)};x)\in B_h(\bar{v})\big) - \mathbb{P}\big(\hat{Y}(\pi;X^\varepsilon(\tau_x^0(\varepsilon);x))=\bar{v}\big)\big| \\
&\leqslant \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)} \big|\mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)}-\tau_x^0(\varepsilon)\bar{1};\ X^\varepsilon(\tau_x^0(\varepsilon)\bar{1};x))\in B_h(\bar{v}),\ \tau_x^0(\varepsilon)\leqslant \frac{T}{\varepsilon^{\alpha(1-\gamma/2}}\big) \\
&\qquad - \mathbb{P}\big(\hat{Y}(\pi;X^\varepsilon(\tau_x^0(\varepsilon);x))=\bar{v}\big)\big| + \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)}\mathbb{P}(\tau_x^0(\varepsilon)>\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}) \\
&\leqslant \sup_{y\in\tilde{D}^+(\varepsilon^\gamma)\cup\tilde{D}^-(\varepsilon^\gamma)} \big|\mathbb{P}\big(X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)}-\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\bar{1};y)\in B_h(\bar{v})\big) - \mathbb{P}\big(\hat{Y}(\pi;y)=\bar{v}\big)\big| \\
&\qquad + \sup_{x\in\tilde{D}^0(\varepsilon^\gamma)}\mathbb{P}(\tau_x^0(\varepsilon)>\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}) = I_4+I_5. \qquad (6.25)
\end{aligned}$$

Now $I_5\to 0$ as $\varepsilon\to 0+$ due to Lemma 6.3 under Hypothesis (H.3). Since

$$|X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)}-\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\bar{1};y)-X^\varepsilon(\frac{\pi}{\lambda^0(\varepsilon)};y)| \xrightarrow{\mathbb{P}} 0 \text{ as } \varepsilon\to 0+,$$

hence the same limit holds in law.

4. Convergence of the auxiliary process to the Markov chain: We argue in the same way as in Part 4 in the proof of Theorem 6.12 remarking that the convergence of I_3 to 0 as $\varepsilon\to 0+$ in (6.24)

$$\mathcal{L}\left(\big((\lambda^0(\varepsilon)(\sigma^{k+1}(\varepsilon;x)-\sigma^k(\varepsilon;x)),m^k(\varepsilon;x)\big)_{k\geqslant 0}\right)\to\mathcal{L}\left(\big((\sigma_{k+1}-\sigma_k,y_k)\big)_{k\geqslant 0}\right). \qquad (6.26)$$

Let now $\bar{\tau}$ be exponentially distributed with parameter 1 according to Theorem 2.18. The case $k\geqslant 2$ is already proved by (6.23). For $k=1$ and $x\in H$ this is a consequence of

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}\{|\lambda^0(\varepsilon)(\sigma^\pm(\varepsilon;x)+\tau_x^0(\varepsilon))-\frac{\mu(B_1^c(0)}{\mu\left((D^\pm)^c\right)}\bar{\tau}|>h\}\right] \\
&\leqslant \sup_{y\in H}\mathbb{E}\left[\mathbf{1}\{|\frac{\lambda^0(\varepsilon)}{\lambda^\pm(\varepsilon)}\lambda^\pm(\varepsilon)(\sigma^\pm(\varepsilon;y)+\tau_y^0(\varepsilon))-\frac{\lambda^0(\varepsilon)}{\lambda^\pm(\varepsilon)}\bar{\tau}|>h/2,\ \tau_y^0(\varepsilon)\leq\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}\}\right] \\
&\qquad + \sup_{y\in H}\mathbb{P}(\tau_y^0(\varepsilon)>\frac{T}{\varepsilon^{\alpha(1-\gamma/2)}}) + \underbrace{\mathbb{P}\Big(\big|\frac{\lambda^0(\varepsilon)}{\lambda^\pm(\varepsilon)}-\frac{\mu(B_1^c(0)}{\mu\left((D_0^\pm)^c\right)}\big|\ \bar{\tau}>h/2\Big)}_{\to 0,\ \varepsilon\to 0+} \qquad (6.27)
\end{aligned}$$

by (6.22), the third term of the last estimate tends to zero. By Lemma 6.3 the second summand also tends to zero. For the first one we use Theorem 5.8 to conclude. □

Appendix

A. The Stochastic Chafee-Infante Equation

A.1. Lévy Processes in Hilbert Space

In this Section we provide a brief introduction to Lévy processes in a separable Hilbert space $(H, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and present crucial properties, which we exploit frequently in the main part, in particular the Lévy-Khinchine formula and the Lévy-Itô-decomposition. We finish this Section with a corollary of the latter stating that symmetric pure jump Lévy processes can be decomposed in the sum of a martingale, which has all moments, and a compound Poisson process.

Definition A.1. Let $(\Omega, \mathcal{F}; \mathbb{P})$ be a probability space and H a separable Hilbert space. A stochastic process $(L(t))_{t\geqslant 0}$ is a *Lévy process* in H, if it satisfies

1. $L(0) = 0$,
2. for any $n \in \mathbb{N}$ and $0 \leqslant t_1 < t_2 < t_3 < \cdots < t_n$ the vector of increments
$$(L(t_n) - L(t_{n-1}), \ldots, L(t_1) - L(t_0))$$
is a family of independent random vectors in H,
3. for $0 \leqslant s < t$
$$\mathcal{L}(L(t) - L(s)) = \mathcal{L}(L(t-s)),$$
where $\mathcal{L}(X)$ denotes the law of a random vector X in H, and
4. it is continuous in probability, i.e. for any $t \geqslant 0$ and $\eta > 0$
$$\lim_{s\to t} \mathbb{P}\left(\|L(t) - L(s)\| > \eta\right) = 0.$$

Remark A.2. In general neither the marginal nor the incremental distributions of a Lévy process is given explicitily. However at the level of marginals Lévy processes can be characterized by their characteristic functions by the so-called Lévy-Khinchine formula. For a proof see Peszat and Zabczyk [2007], Theorem 4.27.

Theorem A.3 (Lévy-Khinchine decomposition)**.** *Let $(L(t))_{t\geqslant 0}$ be a Lévy process in a separable Hilbert space H. Then there exist*

- *a vector $a \in H$,*

- *a nonnegative operator of trace class $Q \in L_1^+(H)$, i.e. for an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of H*

$$\sum_{i=1}^{\infty} \langle Qe_i, e_i\rangle < \infty,$$

- *and a σ-finite measure $\nu : \mathcal{B}(H) \to [0,\infty]$, where $\mathcal{B}(H)$ is the Borel σ-algebra in H, with $\nu(\{0\}) = 0$ satisfying*

$$\int_H \left(1 \wedge \|y\|^2\right) \ \nu(\mathrm{d}y) < \infty,$$

such that

$$\mathbb{E}\left[e^{i\langle h, L(t)\rangle}\right] = \exp\left(t\, \psi(h)\right), \quad h \in H, \quad t \geq 0, \tag{A.1}$$

and

$$\psi(h) := i\langle h, a\rangle - \frac{1}{2}\langle Qh, h\rangle + \int_H \left(e^{i\langle h,y\rangle} - 1 - i\langle h, y\rangle \mathbf{1}_{\{0<\|y\|\leqslant 1\}}\right) \ \nu(\mathrm{d}y).$$

The three components (Q, ν, a) are called the *characteristic triple* of $(L(t))_{t\geqslant 0}$. The vector a is called the *drift vector* of $(L(t))_{t\geqslant 0}$, Q is called the *covariance operator for the Wiener part* and ν is called the *Lévy measure* or *jump measure* of the pure jump part.

Remark A.4. Due to the special form of equation (A.1) one can verify easily that for each characteristic triple (Q, ν, a) of this form there is a corresponding stochastic process $(L(t))_{t\geqslant 0}$ in H of Lévy type. However the components in the formula (A.1) are not unique in general. While for given Lévy process $(L(t))_{t\geqslant 0}$ the operator Q is unique, only the vector (ν, a) is unique.

Definition A.5. 1. A Lévy process with characteristic triple $(0, \nu, 0)$ is called *pure jump Lévy process.*

2. A *compound Poisson process* $(Y(t))_{t\geqslant 0}$ in H is a stochastic process $(L(t))_{t\geqslant 0}$ in H of the following shape. There is a Poisson process $(\pi(t))_{t\geqslant 0}$ with intensity $\lambda > 0$ and an independent sequence of identically distributed random variables $(X_k)_{k\in\mathbb{N}}$ with probability measure μ such that $\mathbb{P}$-almost surely for all $t \geqslant 0$

$$Y(t) = \sum_{k=0}^{\pi(t)} X_k.$$

3. For a compound Poisson process $(Y(t))_{t\geqslant 0}$ let $\int_H \|y\|\mu(\mathrm{d}y) = \mathbb{E}\|X_1\| < \infty$. Then a *compensated compound Poisson process* $(\bar{Y}(t))_{t\geqslant 0}$ in H is a stochastic process of the shape

$$\bar{Y}(t) = Y(t) - t\lambda \int_H y\mu(\mathrm{d}y), \quad t \geqslant 0.$$

In particular it is a martingale with respect to its natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$, $\mathcal{F}_t = \sigma(\{\bar{Y}(s), 0 \leqslant s \leqslant t\})$, $t \geqslant 0$.

Example A.6. A pure jump Lévy process $(L(t))_{t\geqslant 0}$ with jump measure $\nu(H) < \infty$ is a compound Poisson process with intensity $\lambda = \nu(H)$ and the jump measure $\mu(B) = \frac{\nu(B)}{\lambda}$ for $B \in \mathcal{B}(H)$, where $X_k = L(t_k) - L(t_k-)$ and $t_k = \inf\{t > 0 \mid \pi(t) = k\}$ for $k \in \mathbb{N}$.

Proposition A.7 (De Acosta). *A Lévy process $(L(t))_{t\geqslant 0}$ in a separable Hilbert space H, with Lévy measure ν of bounded support in H has all finite moments. In other words, if there is $r > 0$ such that $\nu(B_r^c(0)) = 0$, then*

$$\mathbb{E}\left[\|L(t)\|^k\right] < \infty \qquad \textit{for all } k \geqslant 1,\ t \geqslant 0.$$

For a proof see Peszat and Zabczyk [2007], Theorem 4.4, p. 39. On a path-wise level there exists a decomposition that corresponds to the three parts corresponding to the characteristic triple of the Lévy-Khinchine decomposition: deterministic drift, Wiener part and pure jump part.

In the following we will see that pure jump Lévy processes can be interpreted as the sum of a compound Poisson process and the limit of a sequence of partial sums of independent compensated compound Poisson processes, whose intensity tends to infinity.

Theorem A.8 (Lévy-Itô decomposition). *Let $(L(t))_{t\geqslant 0}$ be a Lévy process in a separable Hilbert space $(H, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and (ν, Q, a) the corresponding characteristic triple from Theorem (A.3). Then there exist*

1. *a Q-Wiener process $(W(t))_{t\geqslant 0}$ in H, that is $W(t)$ is a centered Lévy process in H, i.e. $\mathbb{E}[W(t)] = 0$ for all $t \geqslant 0$ satisfying $\mathbb{E}\left[\|W(t)\|^2\right] < \infty$ and*

$$\mathbb{E}\left[\langle W(t), x\rangle \langle W(s), y\rangle\right] = (t \wedge s)\langle Qx, y\rangle \qquad \forall\, x, y \in H,$$

2. *for any sequence of positive radii $r_n \searrow 0$ and $O_n := \{y \in H \mid r_{n-1} < \|y\| \leqslant r_n\}$ a sequence of independent compensated compound Poisson processes $(L_n(t))_{t\geqslant 0}$ in H with jump measures $\nu_n(B) = \nu(B \cap O_n)$ for $B \in \mathcal{B}(H)$*

which satisfy $\mathbb{P}$-almost surely for all $t \geqslant 0$

$$L(t) = at + W(t) + \sum_{n=1}^{\infty} \bar{L}_n + L_0(t) \tag{A.2}$$

$$\bar{L}_n(t) = \left(L_n(t) - t \int_H y \nu_n(\mathrm{d}y) \right), n \geqslant 1. \tag{A.3}$$

Furthermore $(W(t))_{t\geqslant 0}$ and $\{(L_n(t))_{t\geqslant 0} \mid n \in \mathbb{N}\}$ are independent. The convergence on the right-hand side in equation (A.2) holds $\mathbb{P}$-almost surely on bounded intervals of $[0, \infty)$.

For a proof see Peszat and Zabczyk [2007], Chapter 4.5. In this work we will focus on a certain class of pure jump processes.

Definition A.9. 1. A Lévy process $(L(t))_{t\geqslant 0}$ in H is called *symmetric* if its Lévy measure is symmetric in the sense that

$$\nu(-A) = \nu(A) \quad \text{for } A \in \mathcal{B}(H).$$

2. Fix $\alpha \in (0,2)$. An *α-stable process* $(L(t))_{t\geqslant 0}$ in H is a pure jump Lévy process in H where ν has the specific shape

$$\nu(B) = \int_B \nu(\mathrm{d}y) = \int_B \frac{\mathrm{d}r}{r^{1+\alpha}} \sigma(\mathrm{d}s),$$

where $r = \|y\|$ and $s = y/\|y\|$ and $\sigma : \mathcal{B}(\partial B_1(0)) \to [0,\infty)$ is an arbitrary finite Radon measure on the unit sphere of H.

See for example Araujo and Giné [1979] for local limit theorems to α-stable laws and their domains of attraction in Banach spaces.

Proposition A.10. *Let $(M(t))_{t\geqslant 0}$ be a $(\mathcal{F}_t)_{t\geqslant 0}$-martinale in a separable Hilbert space H on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$. Then $(M(t))_{t\geqslant 0}$ has a càdlàg version which is also a $\mathcal{F}_t$-martingale.*

Remark A.11. A symmetric pure jump Lévy process $(L(t))_{t\geqslant 0}$ in H with finite expectation $\mathbb{E}[\|L(t)\|] < \infty$, for all $t \geqslant 0$ is a martingale with respect to the natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$,

$$\mathcal{F}_t = \sigma(\{L(s) \mid 0 \leqslant s \leqslant t\}), \quad t \geqslant 0.$$

Moreover it is also martingale with respect to the right-continuous completion $(\bar{\mathcal{F}}_t)_{t\geqslant 0}$, with

$$\bar{\mathcal{F}}_t := \bigcap_{s>t} \bar{\mathcal{F}}_s^0, \qquad t \geqslant 0,$$

where $(\bar{\mathcal{F}}_t^0)_{t\geqslant 0}$ is the completion of $(\mathcal{F}_t)_{t\geqslant 0}$ with respect to the null sets of $\mathcal{F}$, i.e. all subsets of measurable sets with probability zero.

With the help of Proposition A.7 and Theorem A.8 we obtain the following decomposition, which we will use frequently throughout the work.

Theorem A.12 (Properties of symmetric pure jump Lévy processes)**.** *Let $(L(t))_{t\geqslant 0}$ be a symmetric pure jump Lévy process in a separable Hilbert space H with Lévy measure ν.*

1. *Then there exist*
   a) *a càdlàg $\bar{\mathcal{F}}_t$-martingale $(\xi(t))_{t\geqslant 0}$ with Lévy measure $\nu_\xi(B) := \nu(B \cap B_1(0))$ for $B \in \mathcal{B}(H)$ and $0 \notin \bar{B}$, which has all moments of finite order*
   b) *and a compound Poisson process $(\eta(t))_{t\geqslant 0}$ with intensity $\lambda = \nu(B_1^c(0))$ and increment distribution $\mu(B) = \frac{\nu(B\cap B_1^c(0))}{\lambda}$ for $B \in \mathcal{B}(H)$*

such that $\mathbb{P}$*-almost surely for all* $t \geqslant 0$

$$L(t) = \xi(t) + \eta(t).$$

2. *There is a positive trace-class operator* $Q_\xi \in L^1_+(H)$ *such that*

$$\langle Q_\xi u, v \rangle = \int_H \langle u, y\rangle \langle v, y\rangle \; \nu_\xi(\mathrm{d}y), \qquad u, v \in H. \tag{A.4}$$

A.2. Stochastic Integration in Hilbert Space

In this Section we will establish stochastic integration in a separable Hilbert space H with respect to a Lévy process.

We fix the following convention. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$ be a filtered probability space, $(L(t))_{t\geqslant 0}$ a symmetric pure jump process in H and $L(t) = \xi(t) + \eta(t)$, $t \geqslant 0$ by in Theorem A.12. We will first introduce the stochastic integral with respect to the martinale $(\xi(t))_{t\geqslant 0}$.

Definition A.13. For operators $Y_i \in L(H) := L(H; H)$, events $A_i \in \mathcal{F}_{t_i}$, and a time discretization $0 = t_0 < t_1 < \cdots < t_n$ we call

$$Y(t, \omega) := \sum_{i=0}^{n-1} \mathbf{1}_{A_i}(\omega)\; \mathbf{1}_{(t_i, t_{i+1}]}(t)\; Y_i, \qquad \text{for } \omega \in \Omega \text{ and } t \geqslant 0 \tag{A.5}$$

a *simple process* in $L(H)$. Denote $\mathcal{S}(H)$ the space of all simple processes in $L(H)$. Then we define the *stochastic integral* of a simple process $(Y(t))_{t\geqslant 0}$ in $\mathcal{S}(H)$ by

$$\int_0^t Y(s,\omega)\; \mathrm{d}\xi(s,\omega) := \sum_{i=0}^{n-1} \mathbf{1}_{A_i}(\omega)\; Y_i \left(\xi(t_{i+1} \wedge t, \omega) - \xi(t_i \wedge t, \omega)\right) \text{ for } t \geqslant 0,\ \omega \in \Omega. \tag{A.6}$$

Lemma A.14 (Itô isometry for simple processes)**.** *For* $Y = (Y(t))_{t\geqslant 0} \in \mathcal{S}(H)$ *let*

$$I^\xi_t(Y) := \int_0^t Y(s)\; \mathrm{d}\xi(s), \qquad t \geqslant 0.$$

Then

$$\mathbb{E}\left[\|I^\xi_t(Y)\|^2\right] = \mathbb{E}\left[\int_0^t |Y(s) Q_\xi^{1/2}|^2_{L_2(H)}\; \mathrm{d}s\right] \qquad \textit{for } t \geqslant 0,$$

where $Q_\xi \in L^+_1(H)$ *is the covariance operator of* ξ *given in Theorem A.12.2. The operator* $Q_\xi^{1/2} \in L_2(H)$ *is then the unique Hilbert-Schmidt operator such that*

$$Q_\xi^{1/2} Q_\xi^{1/2} = Q_\xi.$$

Proof. First of all for an orthonormal basis $(e_l)_{l\in\mathbb{N}}$ of H

$$\mathbb{E}\left[\|\int_0^t Y(s)\ \mathrm{d}\xi(s)\|^2\right] = \mathbb{E}\left[\left\|\sum_{i=0}^{n-1}\mathbf{1}_{A_i}\ Y_i\left(\xi(t_{i+1}\wedge t)-\xi(t_i\wedge t)\right)\right\|^2\right]$$
$$= \mathbb{E}\left[\sum_{i,k=0}^{n-1}\mathbf{1}_{A_i}\mathbf{1}_{A_k}\ \langle Y_i\left(\xi(t_{i+1}\wedge t)-\xi(t_i\wedge t)\right), Y_k\left(\xi(t_{k+1}\wedge t)-\xi(t_k\wedge t)\right)\rangle\right]$$
$$= \mathbb{E}\left[\sum_{i,k=0}^{n-1}\sum_{l=1}^{\infty}\mathbf{1}_{A_i}\mathbf{1}_{A_k}\ J_{ikl}\right],$$

where

$$J_{ikl} := \mathbb{E}\left[\langle Y_i\left(\xi(t_{i+1}\wedge t)-\xi(t_i\wedge t)\right), e_l\rangle\langle Y_k\left(\xi(t_{k+1}\wedge t)-\xi(t_k\wedge t)\right), e_l\rangle \mid \mathcal{F}_{t_k\vee t_i}\right]$$
$$= \mathbb{E}\left[\langle\xi(t_{i+1}\wedge t)-\xi(t_i\wedge t), Y_i^* e_l\rangle\langle\xi(t_{k+1}\wedge t)-\xi(t_k\wedge t), Y_k^* e_l\rangle \mid \mathcal{F}_{t_k\vee t_i}\right]$$
$$= \begin{cases} 0, & i\neq k \\ ((t_{i+1}\wedge t)-(t_i\wedge t))\ \langle Q_\xi Y_i^* e_l, Y_i^* e_l\rangle, & i=k \end{cases}.$$

Hence

$$\mathbb{E}\left[\|\int_0^t Y(s)\ \mathrm{d}\xi(s)\|^2\right] = \sum_{i=0}^{n-1}\mathbb{P}(A_i)\left((t_{i+1}\wedge t)-(t_i\wedge t)\right)\ \sum_{l=1}^{\infty}\langle Q_\xi Y_i^* e_l, Y_i^* e_l\rangle$$
$$= \sum_{i=0}^{n-1}\mathbb{P}(A_i)\left((t_{i+1}\wedge t)-(t_i\wedge t)\right)\ \sum_{l=1}^{\infty}\|Q_\xi^{1/2}Y_i^* e_l\|_H^2$$
$$= \sum_{i=0}^{n-1}\mathbb{P}(A_i)\left((t_{i+1}\wedge t)-(t_i\wedge t)\right)\ |Q_\xi^{1/2}Y_i^*|_{L_2(H)}^2$$
$$= \sum_{i=0}^{n-1}\mathbb{P}(A_i)\left((t_{i+1}\wedge t)-(t_i\wedge t)\right)\ |Q_\xi^{1/2}Y_i|_{L_2(H)}^2 = \mathbb{E}\left[\int_0^t |Q_\xi^{1/2}Y(s)|_{L_2(H)}^2\ \mathrm{d}s\right].$$

□

Construction of the stochastic integral: Fix $T>0$. It can be easily verified that on the space of simple integrands $\mathcal{S}(H)$ the mapping

$$\|Y\|_T^2 := \mathbb{E}\left[\int_0^T |Q_\xi^{1/2}Y(s)|_{L_2(H)}^2\ \mathrm{d}s\right], \qquad Y\in\mathcal{S}(H).$$

is a seminorm on $\mathcal{S}(H)$. We define on $\mathcal{S}(H)$ the equivalence relation

$$Y \sim Z, \quad Y, Z \in \mathcal{S}(H) \quad :\Leftrightarrow \quad \|Y - Z\|_T = 0$$

and consider $\tilde{\mathcal{S}}(H) := \mathcal{S}(H)/\sim$. We now define

$$\mathcal{L}^2_{\xi,T}(H) := \overline{\tilde{\mathcal{S}}(H)}^{\|\cdot\|_T}.$$

We can extend the stochastic integral operator in (A.6) from simple integrands to integrands in $\mathcal{L}^2_{\xi,T}(H)$.

Theorem A.15. *Under the previous notation for any $T > 0$ and $t \in [0, T]$ there is a unique extension of I^ξ_t also denoted as I^ξ_t to a continuous operator*

$$I^\xi_t : (\mathcal{L}_{2,T}(H); \|\cdot\|_T) \to L^2(\Omega, \mathcal{F}, \mathbb{P}; H).$$

For a proof see Peszat and Zabczyk [2007], Theorem 8.7.

Definition A.16. This operator is called the *stochastic integral* with respect to ξ and for $Y \in \mathcal{L}^2_{\xi,T}(H)$ we will denote by

$$\int_0^t Y(s)\mathrm{d}\xi(s) := I^\xi_t(Y), \qquad t \geqslant 0.$$

Example A.17. Let $A := \Delta = \frac{\partial^2}{\partial\zeta^2}$ the second derivative in $H = H^1_0(0, 1)$ and denote by $S := (S(t))_{t\geqslant 0}$ the $\mathcal{C}_0$-semigroup with generator A on H. Then $S\big|_{t\in[0,T]} \in \mathcal{L}_{2,T}(H)$ since for the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of eigenvectors in H with respect to A with $e_n(\zeta) = \sin(\pi n\zeta)$ for $\zeta \in [0, 1]$ and $n \in \mathbb{N}$

$$\begin{aligned}
\int_0^T \|Q^{1/2}_\xi S(s)\|^2_{L_2(H)} \,\mathrm{d}s &= \int_0^T \sum_{n=1}^\infty \|Q^{1/2}_\xi e^{-(\pi n)^2 s} e_n\|^2_H \,\mathrm{d}s \\
&= \sum_{n=1}^\infty \int_0^T e^{-2(\pi n)^2 s} \,\mathrm{d}s \,\|Q^{1/2}_\xi e_n\|^2_H \leqslant \sum_{n=1}^\infty \frac{1}{2}\left(\frac{\|Q^{1/2}_\xi e_n\|_H}{\pi n}\right)^2 \left(1 - e^{-2(\pi n)^2 T}\right) \\
&\leqslant \frac{1}{\pi^2}\,|Q^{1/2}_\xi|^2_{L_2(H)} < \infty.
\end{aligned}$$

A.3. The Stochastic Convolution with Lévy Noise

In this Section we gather the crucial properties of the stochastic convolution, which plays a crucial role for the notion of mild solution of the Chafee-Infante equation in the sequel and a key role in Chapter 3 of the main part of this work. Due to the previous

Section for each $T > 0$ the stochastic integral

$$\int_0^T S(T-s)\ \mathrm{d}\xi(s)$$

is well-defined. In the sequel we will establish that the process $t \mapsto \int_0^t S(t-s)\ \mathrm{d}\xi(s)$ has a càdlàg version. We establish the notion of a variational generator of a semigroup, and obtain in the sequel that they generate semigroups of generalized contractions, for which a càdlàg version exists.

Definition A.18. For a separable Hilbert space H a linear, unbounded, closed operator $A : D(A) \to H$ is called *variational* if

1. there exists a Hilbert space $V \hookrightarrow H$ densely imbedded and a continuous bilinear form $a : V \times V \to \mathbb{R}$ and constants $\alpha > 0$ and $c_0 \geqslant 0$ such that
$$-a(v,v) \geqslant \alpha |v|_V^2 - c_0 \|v\|^2 \qquad \text{for all } v \in V, \tag{A.7}$$
2. $D(A) = \{v \in V \mid a(v,\cdot) \text{ is continuous with respect to the topology in } H\}$
3. $a(u,v) = \langle Au, v\rangle$ for all $u \in D(A)$ and $v \in V$.

For the definition of an analytic semigroup we refer to DaPrato and Zabczyk [1992], Appendix A.4, from which we cite the following proposition.

Proposition A.19. *Let A be a variational generator in H such that inequality (A.7) is satisfied. Then A is the generator of an analytic semigroup $(S(t))_{t\geqslant 0}$ such that*

$$|S(t)|_{L(H)} \leqslant e^{c_0 t}.$$

If A is symmetric, then A is self-adjoint.

We see in the following statement that the existence of a variational generator is sufficient for the existence of a càdlàg version.

Proposition A.20. *Let ξ be a square integrable martingale in the separable Hilbert space H and $(S(t))_{t\geqslant 0}$ a semigroup of generalized contractions of H, that means there is an exponent $c_0 \in \mathbb{R}$ such that*

$$\|S(t)\|_{L(H)} \leqslant e^{c_0 t}, \qquad \textit{for } t \geqslant 0.$$

Then the process

$$\int_0^t S(t-s)\ \mathrm{d}\xi(s)$$

has a càdlàg version in H.

For the proof see Peszat and Zabczyk [2007], p. 158. It is based on the so-called Kotelenez inequality for convolutions of so-called evolution operators. We can finally verify that Δ is variational in H.

Example A.21. We consider $H = H_0^1(0,1)$ with $\|h\|^2 = \int_0^1 (\nabla h)^2(\zeta)\, \mathrm{d}\zeta$ and

$$V = H_0^2(0,1) = \{v \in H_0^1(0,1) \mid \nabla v \in H_0^1(0,1)\}$$

with norm $|v|_V^2 = \int_0^1 (\Delta v)^2(\zeta)\, \mathrm{d}\zeta$ and $A = \Delta$. Then A is a variational generator in H.

1. For

$$a(u,v) := -\int_0^1 (\Delta v)(\zeta)(\Delta u)(\zeta)\, \mathrm{d}\zeta, \quad u, v \in V$$

we obtain by Poincaré's inequality for $v \in H_0^2(0,1)$

$$-a(v,v) = \|\nabla v\|^2 \geqslant \|v\|^2$$

such that in this case inequality (A.7) is fulfilled with $\alpha = 1$ and $c_0 = 0$.

2. The domain $D(A)$ can be indentified with

$$D(A) = H^3(0,1) \cap H_0^2(0,1).$$

Since for $u \in H^3(0,1) \cap H_0^2(0,1)$ and $v \in V$

$$\begin{aligned} a(u,v) &= -\int_0^1 (\Delta u)(\zeta)(\Delta v)(\zeta)\, \mathrm{d}\zeta = \int_0^1 (\nabla\Delta u)(\zeta)(\nabla v)(\zeta)\, \mathrm{d}\zeta \\ &\leqslant \left(\int_0^1 (\nabla \Delta u)^2(\zeta)\, \mathrm{d}\zeta \right) \int_0^1 (\nabla v)^2(\zeta)\, \mathrm{d}\zeta = |u|_{H^3 \cap H_0^2}^2 \|v\|^2. \end{aligned}$$

3. The operator A is defined via the bilinear form by integration by parts by

$$a(u,v) = \int_0^1 (\nabla\Delta u)(\zeta)(\nabla v)(\zeta)\, \mathrm{d}\zeta, \qquad u \in D(A), v \in H.$$

Hence Δ is variational in H.

Example A.22. In particular, for ξ^ε for $\varepsilon > 0$ and $\rho \in (0,1)$ defined in (2.2) in (3.2) the process

$$\xi^*(t) = \int_0^t S(t-s) d\xi^\varepsilon(s), \qquad \varepsilon > 0, t \geqslant 0 \tag{A.8}$$

has a càdlàg version.

A.4. The Stochastic Chafee-Infante Equation with Lévy Noise

In this Section we to show existence and uniqueness for the solution of the stochastic Chafee-Infante equation. For this purpose we need the following properties of the $f(x) = \lambda(x^3 - x)$. For clarity in this Section we will denote by $|\cdot|$ the modulus in $\mathbb{R}$, by $|\cdot|_{L^2}$ the norm in $L^2(0,1)$ and as before by $\|\cdot\|$ the norm in $H = H_0^1(0,1)$.

Lemma A.23 (The polynomial nonlinearity is locally Lipschitz in H)**.** *For each $R > 0$ there are $K_{1,R} > 0$ and $K_R > 0$ such that*

$$|f(t) - f(s)| \leqslant K_{1,R}|t - s|, \qquad t, s \in \mathbb{R}, \text{ with } |t|, |s| \leqslant R, \tag{A.9}$$

$$\|f(u) - f(v)\| \leqslant K_R\|u - v\|, \qquad u, v \in H, \text{ with } \|u\|, \|v\| \leqslant R. \tag{A.10}$$

Proof. The proof is found in Sell and You [2002], Chapter 5.1.1, we provide it for completeness. The proof of (A.9) is obvious. We show (A.10).

Claim 1: f is locally Lipschitz from $L^2(0,1)$ to H. We start with $u, v \in B_R(0) \subset H = H_0^1(0,1) \hookrightarrow L^\infty(0,1)$. Due to $|u|_\infty \leqslant |\nabla u|$ for all $u \in H$, we have $|u|_\infty, |v|_\infty \leqslant R$. In particular for each $\zeta \in (0,1)$

$$\begin{aligned} |f'(u(\zeta) + \theta(v(\zeta) - u(\zeta)))| &\leqslant \sup_{y \in 2R} |f'(y)|_\infty =: K_{1,R} < \infty \\ |f''(u(\zeta) + \theta(v(\zeta) - u(\zeta)))| &\leqslant \sup_{y \in 2R} |f''(y)|_\infty =: K_{2,R} < \infty. \end{aligned} \tag{A.11}$$

Hence due to the mean value theorem

$$\begin{aligned} |f(u) - f(v)|_{L^2}^2 &= \int_0^1 (f(u(\zeta)) - f(v(\zeta)))^2 \, \mathrm{d}\zeta \\ &= \int_0^1 \big|\big(\int_0^1 f'(u(\zeta) + \theta(v(\zeta) - u(\zeta))) \, \mathrm{d}\theta\big)(u(\zeta) - v(\zeta))\big|^2 \, \mathrm{d}\zeta \\ &\leqslant K_{1,R}^2 |u - v|_{L^2}^2 \leqslant K_{1,R}^2 \|u - v\|^2. \end{aligned}$$

Claim 2: f is locally Lipschitz from H to H For $u, v \in B_R(0) \subset H$ we may calculate

$$\begin{aligned}\|f(u) - f(v)\|^2 &= |f'(u)\nabla u - f'(v)\nabla v|_{L^2}^2 \\ &\leqslant 2|f'(u)\nabla u - f'(v)\nabla u|_{L^2}^2 + 2|f'(v)\nabla u - f'(v)\nabla v|_{L^2}^2 \\ &\leqslant 2|f'(u) - f'(v)|^2\|u\|^2 + 2|f'(v)|^2\|u - v\|^2 \\ &\leqslant 2K_{2,R}^2|u - v|^2 + 2K_{1,R}^2\|u - v\|^2 \leqslant \underbrace{2\left(K_1^2 + K_2^2\right)}_{=:K_R^2}\|u - v\|^2.\end{aligned}$$

□

Proposition A.24. *Consider a càdlàg function $\psi : [0, \infty) \to H$, a $\mathcal{C}_0$-semigroup $(S(t))_{t\geqslant 0}$ and a Lipschitz continuous map $g : H \to H$. Then for all $x \in H$, the integral equation*

$$v(t) = S(t)x + \int_0^t S(t-s)g(v(s) + \psi(s-))\,\mathrm{d}s$$

has a unique continuous solution $t \to v(t)$ in H.

This can be proved by standard contraction principles taking into account that the càdlàg function $\psi \in L^1_{loc}(0, \infty; H)$ and using the Lipschitz continuity of g.

Definition A.25. Let $X = (X(t))_{t\geqslant 0}$ be a an adapted càdlàg process in H. For $R > 0$ and $x \in B_R(0) \subset H$ and we define the first exit time

$$\tau_R(x; X) := \inf\{t > 0 \mid \|X(t)\| > 2R\}.$$

Let $(X(t))_{t\geqslant 0}$ be an adapted càdlàg process in $L^\infty(0,1)$. For $R > 0$ and $x \in B_R(0)$ in $L^\infty(0,1)$ we define the first exit time

$$\tilde{\tau}_R(x; X) := \inf\{t > 0 \mid |X(t)|_\infty > 2R\}.$$

Remark A.26. Both hitting times τ_R and $\tilde{\tau}_R$ are stopping times (see for example Kallenberg [1997], Lemma 7.6.). Due to the imbedding $H \subset L^\infty(0,1)$ with $|u|_\infty \leqslant \|u\|$ for $u \in H$ for an adapted càdlàg process in H

$$\tau_R(x, X) \leqslant \tilde{\tau}_R(x, X).$$

Proposition A.27 (Existence of a unique local mild solution). *For any $R > 0$ and $x \in B_R(0)$ and $T > 0$, $\varepsilon > 0$ there is a unique local mild solution of equation (2.3),* *which satisfies for any* $t \in [0, T \wedge \tau_R(x, Y^\varepsilon)]$

$$Y^\varepsilon(t) = S(t)x + \int_0^t S(t-s) f(Y^\varepsilon(s))\, \mathrm{d}s + \varepsilon \int_0^t S(t-s) \mathrm{d}\xi^\varepsilon(s). \tag{A.12}$$

Moreover Y^ε *has a càdlàg version.*

Proof. First note that since $(L(t))_{t\geqslant 0}$ is a symmetric pure jump Lévy process (see Definition A.9) for any $\rho \in (0,1)$ and $\varepsilon > 0$ the process ξ^ε defined in Section 2.1, (2.2) is a mean zero martingale with moments of second order. We verify the assumptions of Theorem 9.29 in Peszat and Zabczyk [2007]. By Lemma A.23 for all $u, v \in B_R(0)$ and $t \geqslant 0$ it follows

$$\|S(t)(f(u) - f(v))\| \leqslant \|S(t)\| \|f(u) - f(v)\| \leqslant e^{-c_0 t} K_R \|u - v\|$$

where c_0 is the largest negative eigenvalue of Δ in H and in particular

$$\|S(t) f(u)\| \leqslant e^{-c_0 t} K_{1,R} \|u\|, \qquad t \geqslant 0, u \in B_R(0),$$

where $K_{1,R}$ is defined by (A.11). For the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of eigenvectors of Δ, the set $(Q_\xi^{1/2}(e_n)_{n\in\mathbb{N}})$ is an orthonormal basis in $Q_\xi^{1/2}(H)$. We may calculate for $t \geqslant 0$

$$\|S(t)\|_{L_2(Q_\xi^{1/2}(H);H)} = \sum_{k=1}^\infty \|S(t) Q_\xi^{-1/2} Q_\xi^{1/2} e_k\| = \sum_{k=1}^\infty e^{-c_0 k^2 t}.$$

Thus

$$\begin{aligned} \int_0^T \|S(t)\|^2_{L_2(Q_\xi^{1/2}(H);H)}\, \mathrm{d}t &= \int_0^T \left(\sum_{n=1}^\infty e^{-c_0 n^2 t} \right)^2 \mathrm{d}t \leqslant \int_0^T \sum_{n=1}^\infty e^{-c_0 2 n^2 t}\, \mathrm{d}t \\ &= \sum_{n=1}^\infty \int_0^T e^{-2c_0 n^2 t}\, \mathrm{d}t \leqslant \sum_{n=1}^\infty \frac{1}{2c_0 n^2} < \infty. \end{aligned}$$

Hence we may apply Theorem 9.29 in Chapter 9 of Peszat and Zabczyk [2007], which states under these assumptions the existence of a local unique weak solution, and Theorem 9.15 therein, which ensures the existence of an equivalent local mild solution of equation (2.3). If $\xi^*(t) = \int_0^t S(t-s)\xi^\varepsilon(s)$, we can rewrite equation (2.3) as

$$v(t) + \xi^*(t) = S(t)x + \int_0^t S(t-s) f(v(s) + \xi^*(s-))\, \mathrm{d}s + \varepsilon \xi^*(s).$$

We apply Proposition A.20 to Example A.21, which implies that $\psi(t) := \varepsilon\xi^*(t)$ has a càdlàg version. By Proposition A.24 there is a continuous solution $v(t)$ in H for $t \geqslant 0$, such that $t \to Y^\varepsilon(t) = v(t) + \varepsilon\xi^*(t)$ inherits the càdlàg property on the stochastic interval $[0, T \wedge \tau_R(x, Y^\varepsilon)]$. □

Proposition A.28. *There exists a unique mild solution $(Y^\varepsilon(t))_{t\geqslant 0}$ of equation (2.3), which has a càdlàg version.*

Proof. We derive an a priori estimate for the local mild solution and derive a contradiction to a blow-up of the solution in finite time. For the solution u of the Chafee-Infante equation (2.6), there is a radius R' and $T' > 0$ such that for all $t > T'$ and all $x \in H$

$$u(t;x) \in B_{R'}(0).$$

See Appendix B. Since $t \to u(t;x)$ is continuous in H, $\sup_{t\in[0,T']} \|u(t;x)\| < \infty$ for all $x \in H$ such that

$$M(x) := \sup_{t\geqslant 0} \|u(t;x)\| < \infty, \qquad x \in H.$$

We denote by $R^\varepsilon(t;x) := Y^\varepsilon(t;x) - u(t;x) - \varepsilon\xi^*(t)$ for $t \geqslant 0, x \in H$, where $(\varepsilon\xi^*(t))_{t\geqslant 0}$ is defined by (A.8). Then

$$\frac{\mathrm{d}}{\mathrm{d}t} R^\varepsilon(t;x) = \Delta R^\varepsilon(t;x) + \left(f(Y^\varepsilon(s;x)) - f(u(t;x))\right).$$

We denote $|\cdot|_{L^2}$ the norm in $L^2(0,1)$, $|\cdot|$ the modulus in $\mathbb{R}$ and $\|v\|^2 = \int_0^t (\nabla v)^2(\zeta)\, \mathrm{d}\zeta$ the norm in $H_0^1(0,1)$. We multiply the last equation with $R^\varepsilon(t;x)$, integrate in ζ and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|R^\varepsilon(t;x)|_{L^2}^2 + \|R^\varepsilon(t;x)\| \leqslant \int_0^1 |f(Y^\varepsilon(t;x)) - f(u(t;x))||R^\varepsilon(t;x)|\, \mathrm{d}\zeta.$$

For any $r > 0$, the local Lipschitz constant $K_{1,r}$ of f on $(-r, r)$ from Lemma A.23 equals obviously the local Lipschitz constant of f in $L^\infty(0,1)$ on $B_r(0) \subset L^\infty(0,1)$. Hence for any $x \in H$, $r > M(x)$ and $t \leqslant \tilde{\tau}_r(x, Y^\varepsilon)$ by Lemma A.23 we may estimate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|R^\varepsilon(t;x)|_{L^2}^2 &\leqslant 2K_{1,2r}\int_0^1 (|R^\varepsilon(t;x)| + |\varepsilon\xi^*(t)|)|R^\varepsilon(t;x)|\, \mathrm{d}\zeta \\
&\leqslant 2K_{1,2r}\int_0^1 (R^\varepsilon(t;x))^2 + (R^\varepsilon(t;x))^2 + (\varepsilon\xi^*(t))^2\, \mathrm{d}\zeta \\
&= 4K_{1,2r}|R^\varepsilon(t;x)|_{L^2}^2 + 2K_{1,r}\sup_{s\in[0,t]} \|\varepsilon\xi^*(s)\|^2.
\end{aligned}$$

By Gronwall's Lemma we obtain the following estimate

$$|R^\varepsilon(t;x)|_{L^2} \leqslant 2K_{1,r}e^{4K_{2r}t}\sup_{s\in[0,t]} \|\varepsilon\xi^*(s)\|. \tag{A.13}$$

Passing to the mild representation of $R^\varepsilon(t;x)$

$$R^\varepsilon(t;x) = \int_0^t S(t-s)(f(Y^\varepsilon(s;x) - f(u(s;x)))\,\mathrm{d}s$$

and exploiting the regularizing effect of the semigroup $(S(t))_{t\geqslant 0}$ we obtain still for $t \leqslant \tilde{\tau}_r(x, Y^\varepsilon)$

$$\begin{aligned}
\|R^\varepsilon(t;x)\| &\leqslant C_1 \int_0^t \frac{e^{-\frac{3c_0}{4}(t-s)}}{\sqrt{t-s}} |f(Y^\varepsilon(s;x)) - f(u(s;x))|_{L^2}\,\mathrm{d}s \\
&\leqslant C_1 \Big(\int_0^t \frac{e^{-\frac{3c_0}{4}(t-s)}}{(t-s)^{3/4}}\,\mathrm{d}s\Big)^{2/3} \Big(K_{1,2r}^3 \int_0^t |R^\varepsilon(s;x) + \varepsilon\xi^*(s)|_{L^2}^3\,\mathrm{d}s\Big)^{1/3} \\
&\qquad\qquad \leqslant C_2 K_{1,2r} 3 \Big(\int_0^t \big(|R^\varepsilon(s;x)|_{L^2}^3 + |\varepsilon\xi^*(s)|_{L^2}^3\big)\,\mathrm{d}s\Big)^{1/3}.
\end{aligned}$$

We apply inequality (A.13) to the right-hand side of the preceding estimate

$$\int_0^t \big(|R^\varepsilon(s;x)|_{L^2}^3 + |\varepsilon\xi^*(s)|_{L^2}^3\big)\,\mathrm{d}s \leqslant \int_0^t \left(e^{2K_{2r}s} \sup_{s'\in[0,s]} \|\varepsilon\xi^*(s')\|^3 + \sup_{s'\in[0,s]} \|\varepsilon\xi^*(s')\|^3\right)\mathrm{d}s$$

and obtain

$$\|R^\varepsilon(t;x)\| \leqslant C_3 K_{1,2r} t \left(e^{2K_{1,2r}t} + 1\right) \sup_{s\in[0,t]} \|\varepsilon\xi^*(s)\|$$

and

$$\|Y^\varepsilon(t;x) - u(t;x)\| \leqslant \left(C_3 K_{1,2r} t \left(e^{2K_{1,2r}t} + 1\right) + 1\right) \sup_{s\in[0,t]} \|\varepsilon\xi^*(s)\|$$

for $t \leqslant \tilde{\tau}_r(x, Y^\varepsilon) =: \tilde{\tau}_r$. Hence we can conclude

$$\begin{aligned}
\|Y^\varepsilon(t;x)\| &\leqslant r + \left(C_3 K_{1,2r} t \left(e^{2K_{1,2r}t} + 1\right) + 1\right) \sup_{s\in[0,t]} \|\varepsilon\xi^*(s)\| \\
&\qquad =: r + \Psi_x(t) \sup_{s\in[0,t]} \|\varepsilon\xi^*(s)\| \leqslant r + \Psi_x(\tilde{\tau}_r) \sup_{s\in[0,\tilde{\tau}_r]} \|\varepsilon\xi^*(s)\|.
\end{aligned}$$

Assume that the local mild solution $t \mapsto Y^\varepsilon(t;x)$ blows up in finite (random) time in H. That means there is an event A with $\mathbb{P}(A) > 0$, a sequence of stopping times $(\tau^n)_{n\in\mathbb{N}}$ with $\tau^i(\omega) \leqslant \tau^{i+1}(\omega)$ on A such that $\tau^\infty(\omega) := \sup_{n\in\mathbb{N}} \tau^n(\omega) < \infty$ for $\omega \in A$ and

$$\lim_{n\to\infty} \|Y^\varepsilon(\tau^n(\omega);x)(\omega)\| = \infty.$$

This means by definition that for all $r > 0$ the hitting times $\tilde{\tau}_r(\omega) < \tau^\infty(\omega)$ for $\omega \in A$. We denote for a sequence of deterministic radii $M(x) \leqslant r < r_1 < r_2 < \ldots \to \infty$ and

$\omega \in A$ by

$$\tau_\infty(\omega) := \inf\{t > 0 \mid t > \sup_{n\in\mathbb{N}} \tilde{\tau}_{r_n}(\omega)\}.$$

The time τ_∞ does not depend on the sequence $(r_i)_{i\in\mathbb{N}}$, since the hitting times are ordered monotonically. Moreover $\tau_\infty(\omega) \leqslant \tau^\infty(\omega)$ for $\omega \in A$. Without loss of generality we consider the case $\tau_\infty(\omega) = \tau^\infty(\omega)$. Then we obtain

$$\begin{aligned}\|Y^\varepsilon(\tilde{\tau}_{r_i}(\omega); x)(\omega)\| &\leqslant r + \Psi_x(\tilde{\tau}_{r_i}(\omega)) \sup_{s\in[0,\tilde{\tau}_{r_i}(\omega)]} \|\varepsilon\xi^*(s)(\omega)\| \\ \leqslant r + \Psi_x(\tau_\infty(\omega)) \sup_{s\in[0,\tau_\infty(\omega)]} \|\varepsilon\xi^*(s)(\omega)\| &= r + \Psi_x(\tau^\infty(\omega)) \sup_{s\in[0,\tau^\infty(\omega)]} \|\varepsilon\xi^*(s)(\omega)\|.\end{aligned}$$

Since $\tau^n(\omega) \nearrow \tau^\infty(\omega)$ for $n \to \infty$ and $\tau_{r_i}(\omega) \nearrow \tau^\infty(\omega)$ for $i \to \infty$ we obtain for each $\tau^n(\omega)$ there is a $r_{i(n)}(\omega)$ such that $\tau^n(\omega) \leqslant \tau_{r_{i(n)}}(\omega)$ and hence for all $n \in \mathbb{N}$

$$\begin{aligned}\|Y^\varepsilon(\tau^n(\omega); x)(\omega)\| &\leqslant \|Y^\varepsilon(\tau_{r_{i(n)}}(\omega); x)(\omega)\| \\ &\leqslant r + \Psi_x(\tau_{r_i(n)}(\omega)) \sup_{s\in[0,\tau_{r_i(n)}(\omega)]} \|\varepsilon\xi^*(s)(\omega)\| \\ &\leqslant r + \Psi_x(\tau^\infty(\omega)) \sup_{s\in[0,\tau^\infty(\omega)]} \|\varepsilon\xi^*(s)(\omega)\| < \infty\end{aligned}$$

which is a contradiction. The càdlàg version is inherited from each of the unique local milds solutions. Hence Y^ε is a global unique mild solution. □

A.5. The Strong Markov Property

In this Section we sketch how to establish the strong Markov property of our solutions Y^ε of equation (2.3) since it is absolutely crucial for our method. We pass mostly along the lines in the books DaPrato and Zabczyk [1992] and Peszat and Zabczyk [2007], but since it is not covered explicitely we prefer to indicate the arguments here.

Definition A.29. Let $(\mathcal{F}_t)_{t\geqslant 0}$ be a complete right-continuous filtration. A $(\mathcal{F}_t)$-adapted process $(X(t))_{t\in[0,T]}$ in a measurable space $(E, \mathcal{E})$ has the *Markov property* if it satisfies for $0 \leqslant s \leqslant t \leqslant T$

$$\mathbb{P}(X(t) \in A \mid \mathcal{F}_s) = \mathbb{P}(X(t) \in A \mid X(s)) \qquad \text{for all } A \in \mathcal{E}. \tag{A.14}$$

For a separable Hilbert space $E = H$ and the Borel-σ-Algebra $\mathcal{E} = \mathcal{B}(H)$ we denote $B_b(H)$ the space of real-valued, bounded Borel functions equipped with the norm

$$|f|_{B_b} = \sup_{x\in H} |f(x)|_H.$$

We denote $Y^\varepsilon(t; s, x)$ the value of the global mild solution of equation (2.3) established in Proposition A.28 at time $t \in [0, T]$ starting at time $0 \leqslant s \leqslant t$ in $x \in H$. For

$\varphi \in B_b(H)$ and $0 \leqslant s \leqslant t \leqslant T$ and $x \in H$ define by

$$(P_{s,t}\varphi)(x) := \mathbb{E}\left[\varphi(Y^\varepsilon(t; s, x))\right]$$

the *transition operator of* $(Y^\varepsilon(t))_{t\in[0,T]}$. Note that in this case for $A \in \mathcal{B}(H)$

$$\mathbb{E}\left[\mathbf{1}_A(Y^\varepsilon(t; s, x))\right] = \mathbb{P}(Y^\varepsilon(t; s, x) \in A).$$

If the transition operators are translation invariant, i.e. $P_{s,t}\varphi = P_{0,t-s}$ for all $0 \leqslant s \leqslant t$, the family $(P_{s,t})$ is called *homogeneous.* If $\varphi \in \mathcal{C}_b(H)$ and $0 \leqslant s \leqslant t \leqslant T$ the map

$$P_{s,t}\varphi : H \to \mathbb{R}, \qquad x \mapsto (P_{s,t}\varphi)(x)$$

is continuous, the family $(P_{s,t})$ satisfies the *Feller property.* We say that the process $(Y^\varepsilon(t))_{t\in[0,T]}$ is homogeneous, has the Markov or the Feller property if its family of transition functions $(P_{s,t})$ has this property.

Lemma A.30. *If the solution $(Y^\varepsilon(t))_{t\in[0,T]}$ of (2.3) has the Markov property, then for all $\varphi \in B_b(H)$ and $0 \leqslant r \leqslant s \leqslant t \leqslant T$ it follows for an arbitrary $\mathcal{F}_s$ measurable random variable $\tilde{X} \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}; H)$ that*

$$\mathbb{E}\left[\varphi(Y^\varepsilon(t; r, \tilde{X})) \mid \mathcal{F}_s\right] = (P_{s,t}\varphi)(Y^\varepsilon(s; r, \tilde{X})) \qquad \mathbb{P}\text{-}a.s. \tag{A.15}$$

Proof. Each bounded measurable function φ can be approximated monotonically by simple functions φ_n. This allows to pass to the limit in equation (A.14). □

Proposition A.31. *The mild solution Y^ε of equation (2.3) satisfies the homogeneous Markov property and the Feller property.*

Proof. In Peszat and Zabczyk [2007], in the Theorems 9.29 and 9.30 and Remark 9.33, the authors prove that mild solutions of SPDEs with time independent Lipschitz coefficients driven by an additive mean zero Lévy martingale with second moments in H possess the homogeneous Markov property and the Feller property. □

So far all properties were at the level of marginals. In order to prove the strong Markov property one has to pass to the perspective on path space. We follow here DaPrato and Zabczyk [1992], Chapter 9.2.

Proposition A.32. *For the solution $(Y^\varepsilon(t; s, \tilde{X}))_{t\geqslant s}$ of equation (2.3) with initial values $(s, \tilde{X})$, for $\varphi_1, \ldots, \varphi_n \in B_b(H)$, $0 \leqslant s \leqslant t$ and $0 \leqslant h_1 \leqslant h_2 \leqslant \ldots \leqslant h_n$ the relation*

$$\mathbb{E}\left[\varphi_1(Y^\varepsilon(t+h_1; s, \tilde{X}))\, \varphi_2(Y^\varepsilon(t+h_2; s, \tilde{X}))\, \ldots\, \varphi_n(Y^\varepsilon(t+h_n; s, \tilde{X})) \mid \mathcal{F}_t\right]$$
$$= Q^{\varphi_1,\ldots,\varphi_n}_{h_1,\ldots,h_n}(t, Y^\varepsilon(t; s, \tilde{X})) \qquad \mathbb{P}\text{-}a.s. \tag{A.16}$$

is valid, where $Q^{\varphi_1,\dots,\varphi_n}_{h_1,\dots,h_n} : [0,T] \times H \to \mathbb{R}$ *is defined by*

$$Q^{\varphi_1,\dots,\varphi_n}_{h_1,\dots,h_n}(s,x) := \int\limits_H p(s,x,;s+h_1,\mathrm{d}y_1)\varphi_1(y_1)\int\limits_H p(s,x,;s+h_2,\mathrm{d}y_2)\varphi_2(y_2)\dots$$

$$\dots\int\limits_H p(s,x,;s+h_n,\mathrm{d}y_n)\varphi_n(y_n). \quad \text{(A.17)}$$

Due to the iterated integral form $Q^{\varphi_1,\dots,\varphi_n}_{h_1,\dots,h_n}$ *is a Borel function.*

This can be proved by induction over n identically to Proposition 9.11 in DaPrato and Zabczyk [1992], p.252.

Since Y^ε has almost surely càdlàg trajectories the theory differs slightly from the case of continuous trajectories in DaPrato and Zabczyk [1992] Chapter 9.2.

Definition A.33. Denote $D := D([0,\infty);H)$ the space of càdlàg curves in H. We denote by $\mathbf{P}^{s,x}$ the distribution of $Y^\varepsilon(s+\cdot\,;s,x)$ on $(D,\mathcal{B}(D))$ with respect to the Skorohod topology. Thus it is defined by

$$\mathbf{P}^{s,x}(\mathcal{A}) = \mathbb{P}(Y^\varepsilon(s+\cdot\,;s,x)\in\mathcal{A}) \qquad \text{for } \mathcal{A}\in\mathcal{B}(D).$$

A *cylindrical set* $\mathcal{Z}$ in D is defined by $0 \leqslant h_1 \leqslant \dots \leqslant h_n$ and $A_1,\dots,A_n \in \mathcal{B}(H)$

$$\mathcal{Z} = \mathcal{Z}(h_1,\dots,h_n;A_1,\dots,A_n) = \{g\in D \mid g(h_1)\in A_1,\dots,g(h_n)\in A_n\}.$$

Remark A.34.

1. The measure $\mathbf{P}^{s,x}$ is uniquely determined by its values on cylindrical sets.

2. For a cylindrical set in D over H it follows by definition

$$\mathbf{P}^{s,x}(\mathcal{Z}) = \mathbb{P}\big(Y^\varepsilon(s+h_1;s,x)\in A_1,\dots,Y^\varepsilon(s+h_n;s,x)\in A_n\big).$$

In particular by the Chapman-Kolmogorov equation, it follows

$$\mathbf{P}^{s,x}(\mathcal{Z}) = Q^{\mathbf{1}_{A_1},\dots,\mathbf{1}_{A_n}}_{h_1,\dots,h_n}(s,x).$$

Hence equation (A.16) has for $0\leqslant s\leqslant t$ the shape

$$\mathbb{P}(Y^\varepsilon(t+\cdot\,;s,\tilde{X})\in\mathcal{Z}\mid\mathcal{F}_t) = \mathbf{P}^{t,Y^\varepsilon(t;s,\tilde{X})}(\mathcal{Z}). \quad \text{(A.18)}$$

Definition A.35. We denote by $\mathbf{E}^{s,x} := \mathbb{E}_{\mathbf{P^{s,x}}}$.

Remark A.36. With this notation equation (A.18) can be rewritten as

$$\mathbb{E}\left[\mathbf{1}\{Y^\varepsilon(t+\cdot\,;s,\tilde{X})\in\mathcal{Z}\}\mid\mathcal{F}_s\right] = \mathbf{E}^{t,Y^\varepsilon(t;s,\tilde{X})}[\mathbf{1}_\mathcal{Z}]. \quad \text{(A.19)}$$

Since each measurable, bounded function $\Psi : D\to\mathbb{R}$ can be approximate monotonically by simple functions, it follows the identity

$$\mathbb{E}\left[\Psi(Y^\varepsilon(t+\cdot\,;s,\tilde{X}))\mid\mathcal{F}_s\right]=\mathbf{E}^{t,Y^\varepsilon(t;s,\tilde{X})}[\Psi]. \tag{A.20}$$

Definition A.37. Let τ be a $(\mathcal{F}_t)_{t\geqslant 0}$-stopping time. We denote by

$$\mathcal{F}_\tau := \sigma\{A\in\mathcal{F}\mid\{\tau\leqslant t\}\cap A\in\mathcal{F}_t\}.$$

We say Y^ε satisfies the strong Markov property if for each $s\geqslant 0$, stopping time $\tau\geqslant s$, $\tilde{X}$ a $\mathcal{F}_s$-measurale random variable and measureable mapping $\Psi:(D,\mathcal{B}(D))\to\mathbb{R}$ it holds

$$\mathbb{E}\left[\Psi(Y^\varepsilon(\tau+\cdot\,;s,\tilde{X}))\mid\mathcal{F}_\tau\right]=\mathbf{E}^{s,Y^\varepsilon(\tau;s,\tilde{X})}[\Psi]\qquad\mathbb{P}(\cdot\mid\tau<\infty)\text{-a.s.}$$

Proposition A.38. *The mild solution Y^ε of equation (2.3) satisfies the strong Markov property.*

Proof. We have to prove that for all nonnegative Borel functions $\Psi:(D,\mathcal{B}(D))\to\mathbb{R}$, $s\geqslant 0$, $\tilde{X}\in L^2(\Omega,\mathcal{F}_s,\mathbb{P};H)$ and all stopping times $\tau\geqslant s$ and $A\in\mathcal{F}_\tau$ holds

$$\mathbb{E}\left[\Psi(Y^\varepsilon(\tau+\cdot\,;s,\tilde{X}))\mathbf{1}_{A\cap\{\tau<\infty\}}\right]=\mathbb{E}\left[\mathbf{E}^{\tau,Y^\varepsilon(\tau;s,\tilde{X})}\left[\Psi\,\mathbf{1}_{A\cap\{\tau<\infty\}}\right]\right].$$

Each stopping time τ has for fixed $2\leqslant q\in\mathbb{N}$ and $N\in\mathbb{N}$ a q-adic approximation $\tau_N := \frac{\lfloor\tau q^N\rfloor+1}{q^N}$ with countably many values. Since $\tau_N\geqslant\tau$ we have $\mathcal{F}_{\tau_N}\supset\mathcal{F}_\tau$ and $\tau^N\searrow\tau$ for $N\to\infty$ $\mathbb{P}$-a.s. For Markov processes in Polish spaces such as H with a countable set of time parameters I, which is closed under summation, the Markov property implies the strong Markov property, see for example Klenke [2005] p. 352. In our case this property is fulfilled for $(Y^\varepsilon(t))_{t\in I_n}$, where $I_n=\{\frac{k}{q^n},k\in\mathbb{N}\}$. Hence under previous assumptions we already have

$$\mathbb{E}\left[\Psi(Y^\varepsilon(\tau_n+\cdot\,;s,\tilde{X}))\mathbf{1}_{A\cap\{\tau<\infty\}}\right]=\mathbb{E}\left[\mathbf{E}^{\tau_n,Y^\varepsilon(\tau_n;s,\tilde{X})}\left[\Psi\,\mathbf{1}_{A\cap\{\tau<\infty\}}\right]\right]. \tag{A.21}$$

The remaining question is now, whether we can pass to the limit $n\to\infty$ in (A.21). We denote by $\mathcal{M}:=\{\Psi:(D,\mathcal{B}(D))\to\mathbb{R}\mid\Psi\text{ measurable }\}$. Clearly for each $h\geqslant 0$ and $\varphi\in\mathcal{C}_b(H)$ the point evaluations

$$\Psi(f):=\varphi(f(h))$$

are a subclass of $\mathcal{M}$. For point evaluations of this type equation (A.21) has the form

$$\mathbb{E}\left[\varphi(Y^\varepsilon(\tau_n+h;s,\tilde{X}))\mathbf{1}_{A\cap\{\tau<\infty\}}\right]=\mathbb{E}\left[(P_{\tau_n,\tau_n+h}\varphi)(Y^\varepsilon(\tau_n;s,\tilde{X}))\mathbf{1}_{A\cap\{\tau<\infty\}}\right]. \tag{A.22}$$

Since Y^ε has the Feller property, for each $\varphi\in\mathcal{C}_b(H)$ and $t,h>0$ the mapping

$$H\to\mathbb{R},\qquad x\mapsto(P_{t,t+h}\varphi)(x)$$

is continuous. It can be shown analoguously to DaPrato and Zabczyk [1992], Theorem 9.1 that for any $t \geqslant 0$ and $\tilde{X} \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; H)$ the mapping $h \mapsto \mathbb{E}[(Y^\varepsilon(t+h; t, x))^2]$ is continuous and hence that for each $\varphi \in \mathcal{C}_b(H)$, $x \in H$ and $t > 0$ the mapping

$$h \mapsto (P_{t,t+h}\varphi)(x)$$

is continuous. In addition, by Theorem A.28, Y^ε has almost surely right-continuous trajectories. Hence we can pass to the limit and obtain

$$\mathbb{E}\left[\varphi(Y^\varepsilon(\tau + h; s, \tilde{X}))\mathbf{1}_{A\cap\{\tau<\infty\}}\right] = \mathbb{E}\left[(P_{\tau,\tau+h}\varphi)(Y^\varepsilon(\tau; s, \tilde{X}))\mathbf{1}_{A\cap\{\tau<\infty\}}\right]. \tag{A.23}$$

This is equivalent to

$$\mathbb{E}\left[\varphi(Y^\varepsilon(\tau + h; s, \tilde{X})) \mid \mathcal{F}_\tau\right] = (P_{\tau,\tau+h}\varphi)(Y^\varepsilon(\tau; s, \tilde{X})) \quad \mathbb{P}(\cdot \mid \tau < \infty)\text{-a.s.} \tag{A.24}$$

Since $\mathbf{1}_B(v)$ for $B \in \mathcal{B}(D)$ can be approximated by continuous functions, and measurable functions can be approximated by simple functions, we obtain even for $\varphi \in B_b(H)$ that

$$\mathbb{E}\left[\varphi(Y^\varepsilon(\tau + h; s, \tilde{X})) \mid \mathcal{F}_\tau\right] = (P_{\tau,\tau+h}\varphi)(Y^\varepsilon(\tau; s, \tilde{X})) \quad \mathbb{P}(\cdot \mid \tau < \infty)\text{-a.s.} \tag{A.25}$$

By induction one can now show the following stopping time analogon of Proposition A.32. Consider the solution $(Y^\varepsilon(t; s, \tilde{X}))_{t\geqslant s}$ of equation (2.3) with initial values $(s, \tilde{X})$. For $\varphi_1, \dots, \varphi_n \in B_b(H)$, $s \geqslant 0$, a stopping time $\tau \geqslant s$ and $0 \leqslant h_1 \leqslant h_2 \leqslant \dots \leqslant h_n$ we have

$$\begin{aligned} \mathbb{E}\left[\varphi_1(Y^\varepsilon(\tau + h_1; s, \tilde{X}))\, \varphi_2(Y^\varepsilon(\tau + h_2; s, \tilde{X}))\, \dots\, \varphi_n(Y^\varepsilon(\tau + h_n; s, \tilde{X})) \mid \mathcal{F}_\tau\right] \\ = Q^{\varphi_1,\dots,\varphi_n}_{h_1,\dots,h_n}(\tau, Y^\varepsilon(\tau; s, \tilde{X})) \qquad \mathbb{P}(\cdot \mid \tau < \infty)\text{-a.s.} \end{aligned} \tag{A.26}$$

where the right-hand side is defined by (A.16). Hence for a cylindrical set $\mathcal{Z}$ we can hence rewrite the last equation as

$$\mathbb{P}(Y^\varepsilon(\tau + \cdot\,; s, \tilde{X}) \in \mathcal{Z} \mid \mathcal{F}_\tau) = \mathbf{P}^{\tau, Y^\varepsilon(\tau; s, \tilde{X})}(\mathcal{Z}). \tag{A.27}$$

By monotone approximation of measurable and bounded $\Psi : D \to \mathbb{R}$ and the convention $\mathbf{E}^{s,x} = \mathbb{E}_{\mathbf{P}^{s,x}}$ we obtain the desired equation

$$\mathbb{E}\left[\Psi(Y^\varepsilon(\tau + \cdot\,; s, \tilde{X})) \mid \mathcal{F}_\tau\right] = \mathbf{E}^{\tau, Y^\varepsilon(\tau; s, \tilde{X})}(\Psi).$$

This completes the proof. □

A.6. Basics on Slowly and Regularly Varying Functions

In this Section we provide basic properties of slowly varying function, with the help of which we can define regularly varying Lévy measures in Section 2.1. We will exploit these properties extensely in Chapter 4.

We cite from Hult and Lindskog [2006] and Bingham et al. [1987].

Definition A.39. A positive measurable function $\ell : (0, \infty) \to (0, \infty)$ satisfaying

$$\lim_{x\to\infty} \frac{\ell(yx)}{\ell(x)} = 1 \qquad \text{for all } \ y > 0 \tag{A.28}$$

is called *slowly varying.*

The next Theorem states already the uniform convergence in $y > 0$ (Bingham et al. [1987], Theorem 1.3.1).

Theorem A.40. *For a slowly varying function ℓ the limit (A.28) is uniformly in the sense that for all compact sets $K \subset (0, \infty)$*

$$\lim_{x\to\infty} \sup_{y\in K} \frac{\ell(yx)}{\ell(x)} = 1.$$

For the proof see Bingham et al. [1987] Theorem 1.2.1.

The connection between regular varying function and slowly varying functions is given by (Bingham et al. [1987], Theorem 1.4.1).

Theorem A.41. *For any regularly varying function $v : (0, \infty) \to (0, \infty)$ with index β there is a slowly varying function $\ell : (0, \infty) \to (0, \infty)$ such that*

$$v(x) = x^\beta \ell(x) \quad \textit{for all } x \in (0, \infty).$$

Example A.42.

1. $\ell_1(x) = \ln(\ln x)$ is a slowly varying function.

2. $\ell_2(x) = \exp(\ln(x)^{(1/3)} \cos(\ln(x)^{(1/3)})$ is a slowly varying function with infinite oscillations i.e.

$$\liminf_{x\to\infty} \ell_2(x) = 0 \quad \text{and} \quad \limsup_{x\to\infty} \ell_2(x) = \infty.$$

   These oscillations are of vanishing order as we can see in the following proposition though.

The next Proposition is most significant for our studies. It tells us that in comparison to polynomials, slowly varying functions behave asymptotically like constants.

Proposition A.43. *A slowly varying function* $\ell : (0, \infty) \to (0, \infty)$ *has the property that for any* $a > 0$

$$\lim_{x \to \infty} x^a \ell(x) = \infty$$
$$\lim_{x \to \infty} x^{-a} \ell(x) = 0.$$

For the proof see Bingham et al. [1987] Theorem 1.2.1.

B. The Fine Dynamics of the Chafee-Infante Equation

B.1. Consistency of Reduced Domains of Attraction (Proof of Lemma 2.13)

We prove Lemma 2.13, which states, that the reduced domains of attraction asymptotically exhaust the unreduced domains of attraction.

Proof. "$\supset$" is true by definition. We show "$\subset$" in all three cases.

1. Take $x \in D^{\pm}$. It is obviously enough to prove the existence of $\delta > 0$ such that $x \in D^{\pm}(\delta)$, i.e. $\cup_{t \geqslant 0} B_\delta(u(t;x)) \subset D^{\pm}$. In fact, according to Lemma B.3 or Matano [1979] and Sattinger [1972] and by closedness of $\mathcal{S}$ there is $\delta_1 > 0$ such that for all $0 < \delta \leqslant \delta_1$ and $t \geqslant 0$

$$\cup_{t \geqslant 0} u(t; B_\delta(\phi^{\pm})) \subset B_\delta(\phi^{\pm}) \subset D^{\pm}. \tag{B.1}$$

Since $\lim_{t \to \infty} u(t;x) = \phi^{\pm}$ by Proposition 2.9, there exists $t_0 > 0$ such that for $t \geqslant t_0$ we have $u(t;x) \in B_{\delta_1/2}(\phi^{\pm})$, hence

$$\cup_{t \geqslant t_0} B_{\delta_1/2}(u(t,x)) \subset B_{\delta_1}(\phi^{\pm}) \subset D^{\pm}.$$

Let $\delta_2 = \min_{0 \leqslant t \leqslant t_0} \operatorname{dist}(u(t;x), \mathcal{S})$. By continuity and compactness, $\delta_2 > 0$. Then also

$$\cup_{0 \leqslant t \leqslant t_0} B_{\delta_2/2}(u(t;x)) \subset D^{\pm}.$$

Altogether with $\delta = \frac{\delta_1 \wedge \delta_2}{2}$ we obtain

$$\cup_{t \geqslant 0} B_\delta(u(t;x)) \subset D^{\pm}.$$

2. Take $x \in D^{\pm}$. It is enough to prove that there exist $0 < \delta_0, \eta_0$ such that for all $0 < \delta \leqslant \delta_0, 0 < \eta \leqslant \eta_0$ we have $x \in D^{\pm}(\delta, \eta)$. In other words, we have to show that for $0 < \delta \leqslant \delta_0, 0 < \eta \leqslant \eta_0$ we have $\cup_{s \geqslant 0} B_\eta(u(s;x)) \subset D^{\pm}(\delta)$, which amounts to

$$\cup_{t \geqslant 0} B_\delta(u(t; \cup_{s \geqslant 0} B_\eta(u(s;x)))) \subset D^{\pm}.$$

First use part 1 of the proof to obtain the existence of $0 < \eta_0$ such that for all $0 < \eta \leqslant \eta_0$

$$\cup_{s \geqslant 0} B_\eta(u(s;x)) \subset D^{\pm}.$$

Now for any $t_0 > 0$ the map $y \mapsto u(t; y)$ is compact, hence transfers bounded sets to compact sets. Since $\lim_{t\to\infty} u(t; x) = \phi^\pm$, we know that $\cup_{s\geqslant 0} B_\eta(u(s; x))$ is a bounded subset of $D^\pm$. Hence for $t_0 > 0$

$$u(t_0; \cup_{s\geqslant 0} B_\eta(u(s; x))) \quad \text{is compact in} \quad D^\pm.$$

Next choose $\delta_1 > 0$ according to the first part of the proof. Elementary arguments using the Lipschitz property of u on compact sets, as well as $\lim_{t\to\infty} u(t; x) = \phi^\pm$ enable us to show that there exists $t_1 \geqslant t_0 > 0$ such that

$$u(t_1; \cup_{s\geqslant 0} B_{\eta_0}(s; x)) = u(t_1 - t_0; u(t_0; \cup_{s\geqslant 0} B_{\eta_0}(s; x))) \subset B_{\delta_1}(\phi^\pm).$$

By choice of δ_1, analogously to the first part of the proof,

$$\cup_{t\geqslant t_1} B_{\delta_1/2}(u(t; \cup_{s\geqslant 0} B_{\eta_0}(u(s; x)))) \subset D^\pm.$$

Finally, uniform continuity of $u(t; \cdot)$ in $t \in [0, t_1]$ implies that with some $\delta_2 > 0$ we arrive at

$$\cup_{0\leqslant t\leqslant t_1} B_{\delta_2}(u(t; \cup_{s\geqslant 0} B_{\eta_0}(u(s; x)))) \subset D^\pm.$$

Now choose $\delta_0 = \frac{\delta_1}{2} \wedge \delta_2$ to conclude.

3. Take $x \in D^\pm$. It is enough to prove that there exist $0 < \delta_0, \eta_0, \zeta_0$ such that for all $0 < \delta \leqslant \delta_0, 0 < \eta \leqslant \eta_0, 0 < \zeta \leqslant \zeta_0$ we have $x \in D^\pm(\delta, \eta, \zeta)$. In other words, we have to show that for $0 < \delta \leqslant \delta_0, 0 < \eta \leqslant \eta_0, 0 < \zeta \leqslant \zeta_0$ we have $\cup_{s\geqslant 0} B_\zeta(u(s; x)) \subset D^\pm(\delta, \eta)$, which amounts to

$$\cup_{r\geqslant 0} B_\delta(u(r; \cup_{t\geqslant 0} B_\eta(u(t; \cup_{s\geqslant 0} B_\zeta(u(s; x)))))) \subset D^\pm.$$

It is clear from this statement, that the arguments of the second part of the proof just have to be repeated with $\cup_{t\geqslant 0} B_\eta(u(t; \cup_{s\geqslant 0} B_\zeta(u(s; x))))$ replacing $\cup_{s\geqslant 0} B_\eta(u(s; x))$. This completes the proof. $\square$

B.2. Logarithmic Bounds on the Relaxation Time in Reduced D.o.A.

B.2.1. The Fine Structure of the Attractor

Since our results in the Chapters 3, 4, 5 and 6 are based on a pathwise analysis we need to further specify the fine structure of the attractor of the Chafee-Infante equation. Note that its shape depends crucially on the bifurcation parameter λ.

It is known from Faris and Jona-Lasinio [1982a] that the solution u of the Chafee-Infante equation has the following set of fixed points. For a detailed exposition of the bifurcation on the elliptic boundary value problem and the analytic representation of the stationary solutions also consult for instance Henry [1983], Hale [1983], Raugel [2002], Chueshov [2002], Wakasa [2006] or Robinson [2001].

Proposition B.1. *For the Chafee-Infante parameter $\lambda \leqslant \pi^2$ there is a unique stable fixed point $v \equiv 0$. For $\lambda > \pi^2$ there are always two stable fixed points $\phi^\pm \in \mathcal{C}^\infty([0,1])$. More precisely, if $(\pi(n-1))^2 < \lambda \leqslant (\pi n)^2, n \in \mathbb{N}$ there are 2 stable and for $n \geqslant 2$ exactly $(2n-3)$ unstable fixed points $\{0, \phi_j^\pm, j = 1, \dots, n-2\}$. In other words*

$$\mathcal{E}^\lambda := \begin{cases} \{0\}, & 0 < \lambda \leqslant \pi^2, \\ \{0, \phi^\pm\}, & \pi^2 < \lambda \leqslant (2\pi)^2, \\ \{0, \phi^\pm, \phi_j^\pm, j = 1, \dots, n-2\}, & (\pi(n-1))^2 < \lambda \leqslant (\pi n)^2, \qquad n \geqslant 2. \end{cases}$$

In the following we are going to exploit the fine structure of the system's attractor $\mathcal{A}^\lambda$. The attractor $\mathcal{A}^\lambda$ consists of all fixed points and all global trajectories $\{u(t), t \in \mathbb{R}\}$. Following Chueshov [2002], in this case of finitely many fixed points the global attractor has the following shape. For any $\lambda > 0$

$$\mathcal{A}^\lambda = \bigcup_{v \in \mathcal{E}^\lambda} \mathcal{W}^u(v), \quad \text{where} \qquad \mathcal{W}^u(v) = \bigcup_{\substack{v \to w \\ w \in \mathcal{E}^\lambda}} C(v,w) \tag{B.2}$$

with the notation established in (2.9). In other words

$$\mathcal{A}^\lambda = \{\phi^+, \phi^-\} \cup \bigcup_{v \in \mathcal{E}^\lambda \setminus \{\phi^+, \phi^-\}} \{v\} \cup \bigcup_{\substack{v, w \in \mathcal{E}^\lambda \\ v \to w}} C(v,w). \tag{B.3}$$

Summarizing the results in the literature we can describe the structure of the global attractor recursively in λ as follows.

Recursive description of the attractor $\mathcal{A}^\lambda$: For $\lambda \in (((n-1)\pi)^2, (\pi n)^2)$ the elements of $\mathcal{E}^\lambda$ as well as $\mathcal{A}^\lambda$ depend continuously on λ. Thus the topological structure of $\mathcal{A}^\lambda$ remains invariant for $\lambda \in (((n-1)\pi)^2, (\pi n)^2)$. In other words, in this interval all $\mathcal{A}^\lambda$ are homotope. If λ passes $(\pi n)^2$ from the left the connection structure of the elements

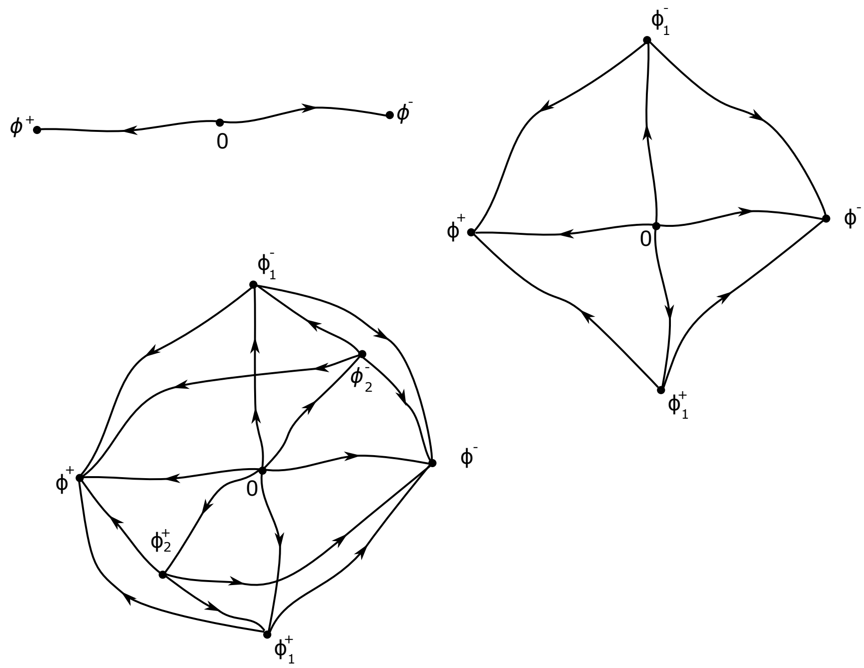

Figure B.1.: Sketch of $\mathcal{A}^\lambda$ for $\lambda \in (\pi^2, (2\pi)^2)$, $\lambda \in ((2\pi)^2, (3\pi)^2)$, $\lambda \in ((3\pi)^2, (4\pi)^2)$

of $\mathcal{E}^\lambda$ for $\lambda \in ((\pi(n-1))^2, (\pi n)^2)$ is retained in $\mathcal{A}^\lambda$ for $\lambda > (n\pi)^2$ as a substructure, but two new unstable fixed points $\phi_n^\pm$ appear in $\mathcal{E}^\lambda$. In addition, exactly $2(n-1)$ new connecting orbits emerge in the attractor: $2(2n-3)$ ones linking the $2n-3$ previously unstable fixed points $\{0, \phi_j^\pm, j = 1, \dots, n-2\}$ with each of the new ones $\{\phi_n^+, \phi_n^-\}$, and 4 trajectories directed from each the latter ones to each of the stable points $\{\phi^+, \phi^-\}$. There is an extensive literature on further properties of attractors for reaction diffusion equations, see instance the survey article by Fiedler and Scheel [1982]. It turns out to be important in the proof of Proposition 2.15 that the longest cascade visits $n-1$ fixed points and any cascade ends in one of the stable points $\phi^\pm$. In particular the number of connecting orbits for $\lambda \in ((\pi(n-1))^2, (\pi n)^2))$ is exactly

$$\sum_{k=1}^{n} 2(2k-1) = 2n^2.$$

B.2.2. Logarithmic Relaxation Times (Proof of Proposition 2.15)

We prove the statement of Proposition 2.15 for the finer topology on H related to the norm $\|\cdot\|$ and then infer the result for the topology related to $|\cdot|_\infty$. Let $\gamma > 0$ be fixed. Denote by $\mathcal{D}^\pm = D^\pm \subset H_0^1(0,1)$ the domain of attraction of $\phi^\pm$ (i.e. ϕ^+ or ϕ^-). We define $\mathcal{D}^\pm(\varepsilon^\gamma), \varepsilon > 0$ as subsets of $\mathcal{D}^\pm$ in the same fashion as $D^\pm(\varepsilon^\gamma)$, with the norm $\|\cdot\|$ replacing $|\cdot|_\infty$. By definition of $D^\pm(\varepsilon^\gamma)$ and by $|\cdot|_\infty \leqslant \|\cdot\|$, we have for $x \in D^\pm(\varepsilon^\gamma)$,

$s \in \mathcal{S}$ and $t \geqslant 0$

$$\|u(t;x) - s\| \geqslant |u(t;x) - s|_\infty > \varepsilon^\gamma,$$

hence $x \in \mathcal{D}^\pm(\varepsilon^\gamma)$, which implies $D^\pm(\varepsilon^\gamma) \subset \mathcal{D}^\pm(\varepsilon^\gamma)$. In addition, by the same argument for $v \in \{\phi^+, \phi^-\}$

$$|u(t;x) - v|_\infty \leqslant \|u(t;x) - v\| \leqslant \frac{1}{2}\varepsilon^{2\gamma}.$$

Therefore it is sufficient to prove the statement of Proposition 2.15 in the topology of $\|\cdot\|$, i.e. for initial values $x \in \mathcal{D}^\pm(\varepsilon^\gamma)$ and the distance $\|\cdot\|$ instead of $|\cdot|_\infty$.

For any set $A \subset H$ and $\sigma > 0$, we define the *σ-neighborhood of A* by

$$U_\sigma(A) := \bigcup_{x\in A} B_\sigma(x).$$

For two fixed points $v, w \in \mathcal{E}^\lambda$ of the Chafee-Infante equation that are connected in $\mathcal{A}^\lambda$ ($C(v,w) \neq \emptyset$) in the previous Subsection B.2.1 we recall the notation $(v \to w) :\Leftrightarrow C(v,w) \neq \emptyset$ and denote for $\eta, \sigma > 0$ by

$$U_\sigma(v,w) := U_\eta(C(v,w))$$

the *σ-tube around the heteroclinic orbit* $C(v,w) \subset \mathcal{A}^\lambda$ and by

$$U^-_{\eta,\sigma}(v,w) := U_\eta(C(v,w)) \setminus (B_\sigma(v) \cup B_\sigma(w))\,.$$

the *η-tube around the heteroclinic orbit $C(v,w) \subset \mathcal{A}^\lambda$ deprived of the σ-balls around the end points v and w.*

For $\eta \leqslant \frac{||v-w||}{3}$ and $\sigma > 0$ it follows for all $v, w \in \mathcal{E}^\lambda$ that

$$U^-_{\eta,\sigma}(v,w) \neq \emptyset \qquad \Longleftrightarrow \qquad v \to w. \tag{B.4}$$

For convenience we write $v \leftarrow w$ equivalently to $w \to v$ ($\Leftrightarrow (C(v,w) \neq \emptyset)$. Define for $\sigma > 0$ the maximal radius $d(\sigma)$ so that the $d(\sigma)$-tubes around the heteroclinic orbits deprived of the σ-balls around the fixed points are all disjoint. More precisely for $\sigma > 0$ we define

$$\begin{aligned} d(\sigma) := \sup\{h > 0 \mid\ & \text{for all } v, w_1, w_2 \in \mathcal{E}^\lambda, \\ & \text{if } w_1 \leftarrow v \to w_2, \text{ then } U_h(v,w_1) \cap U_h(v,w_2) \cap B^c_\sigma(v) = \emptyset, \\ & \text{if } w_1 \to v \leftarrow w_2, \text{ then } U_h(w_1,v) \cap U_h(w_2,v) \cap B^c_\sigma(v) = \emptyset, \\ & \text{if } w_1 \to v \to w_2, \text{ then } U_h(w_1,v) \cap U_h(v,w_2) \cap B^c_\sigma(v) = \emptyset, \\ & \text{if } w_1 \leftarrow v \leftarrow w_2, \text{ then } U_h(v,w_1) \cap U_h(w_2,v) \cap B^c_\sigma(v) = \emptyset\}. \end{aligned} \tag{B.5}$$

For $w_1 \to v \to w_2$, we sketch $d(\sigma)$ in Figure B.2.

It is easy to see that due to the transversality of the fixed points there is $\delta_d > 0$ such that for all $0 < \sigma \leqslant \delta_d$ it follows

$$d(\sigma) > 0. \tag{B.6}$$

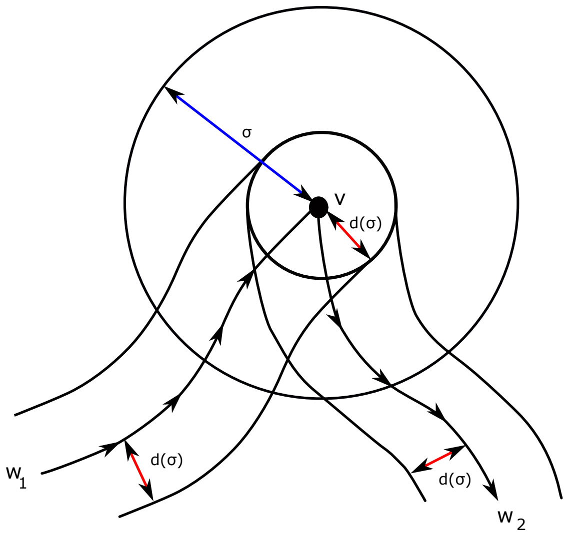

Figure B.2.: Disjoint tubes around an unstable fixed point $w_1 \to v \to w_2$

Note that necessarily $d(\sigma) < \sigma$.

Let us rewrite Proposition 2.15 for the finer topology generated by $\|\cdot\|$.

Proposition B.2. *Let the Chafee-Infante parameter $\pi^2 < \lambda \neq (n\pi)^2$, for $n \in \mathbb{N}$, be given. Then there exists $T_{rec} > 0$ and a constant $\kappa > 0$ such that for each $\gamma > 0$ there exist constants $\varepsilon_0 = \varepsilon_0(\gamma) > 0$, such that for all $0 < \varepsilon \leqslant \varepsilon_0$ and $t \geqslant T_{rec} + \kappa\gamma|\ln\varepsilon|$ and $x \in \mathcal{D}^{\pm}(\varepsilon^{\gamma})$*

$$\|u(t;x) - \phi^{\pm}\| \leqslant (1/2)\varepsilon^{2\gamma}.$$

Proof. The proof is structured into three parts. In Part I we discuss the absorbtion of the trajectories of the Chafee-Infante equation for any initial value $x \in H$ by a neighborhood of the attractor in finite time. This is followed, in Part II, by a detailed discussion of the local behavior of the system when entering different parts of this neighborhood. In other words, using the flow properities we analyze the behaviour of the solution for initial values taking values in the mentioned neighborhood of the attractor. Here we exploit the well-known shape of the attractor and the hyperbolicity of the fixed points. In Part III we can finally use the gradient structure of the system in order to determine the global behavior by the local information gained in Part II.

We fix $\gamma > 0$.

I. The global dynamics absorbed by a neighborhood of the attractor

Claim I.1: There is a universal relaxation time to enter $U_\eta(\mathcal{A}^\lambda)$. *For any $\eta > 0$ there is a time $\tau_1 = \tau_1(\eta) > 0$ such that for all $t \geqslant \tau_1$ and $x \in H$*

$$u(t;x) \in U_\eta(\mathcal{A}^\lambda).$$

Proof. By Temam [1992], Remark 1.4, p. 88 there exist $\rho_1 = \rho_1(\lambda) > 0$ and a uniform upper bound $t_0 = t_0(\lambda) > 0$ such that for all $t \geqslant t_0$ and $x \in H$

$$u(t;x) \in B_{\rho_1}(0).$$

By the definition of a global attractor for each $\eta > 0$ and each bounded set $A \subset H$ there is a time $t_1 = t_1(A, \eta, \lambda) > 0$ such that for all $t \geqslant t_1$ and $x \in A$

$$u(t;x) \in U_\eta(\mathcal{A}^\lambda)$$

holds true. The claim follows for $A = B_{\rho_1}(0)$ and $\tau_1 := t_1 + t_2$. □

Claim I.2: There is a unique last entrance time for open neighborhoods of the attractor. For any open set $\mathcal{O} \supset \mathcal{A}^\lambda$ and $x \notin \mathcal{O}$ there is a unique $\theta_1 = \theta_1(x, \mathcal{O}) > 0$ such that

$$u(\theta_1; x) \in \partial\mathcal{O} \text{ and } u(t;x) \in \mathcal{O} \quad \text{for all } \ t \geqslant \theta_1.$$

For all $x \in H$, $\eta > 0$, open sets $\mathcal{O} \supseteq U_\eta(\mathcal{A}^\lambda)$ and $x \in H \setminus \mathcal{O}$ we have

$$\theta_1(x, \mathcal{O}) \leqslant \tau_1(\eta).$$

Proof. For the first part of the statement it suffices to write

$$\theta_1(x, \mathcal{O}) := \sup\{t > 0 \mid u(t;x) \notin \mathcal{O}\} < \infty,$$

since $\mathcal{A}^\lambda$ is an attracting set. For the second part we use Claim I.1 and obtain

$$\theta_1(x, \mathcal{O}) \leqslant \theta_1(x, U_\eta(\mathcal{A})) \leqslant \tau_1(\eta).$$

□

II. The local behavior in a neighborhood of the attractor in $\mathcal{D}^\pm(\varepsilon^\gamma)$ There exists a universal constant $\delta_b > 0$ such that for $0 < \sigma \leqslant \delta_b$ the balls $B_\sigma(v), v \in \mathcal{E}^\lambda$, are pairwise disjoint, (B.4) and (B.6) are satisfied, and $B_\sigma(\phi^\pm) \subset \mathcal{D}^\pm$. Then there is exists $\varepsilon_b = \varepsilon_b(\sigma) > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_b$, $B_\sigma(\phi^\pm) \subset \mathcal{D}^\pm(\varepsilon^\gamma)$. We shall exploit the segmented structure (B.3) of attractor $\mathcal{A}^\lambda$ which is reflected in the structure of the surface $\partial U_\sigma(\mathcal{A}^\lambda)$. Due to (B.2) and the definition of $U^-_{\sigma,\sigma}(v, w)$ we have the following decomposition in three disjoint sets if $\sigma > 0$ is small enough

$$U_\sigma(\mathcal{A}^\lambda) = B_\sigma(\phi^\pm) \cup \left(\bigcup_{v \in \mathcal{E}^\lambda \setminus \{\phi^+, \phi^-\}} B_\sigma(v) \right) \cup \left(\bigcup_{\substack{v,w \in \mathcal{E}^\lambda \\ v \to w}} U^-_{\sigma,\sigma}(C(v,w)) \right). \qquad \text{(B.7)}$$

By the choice of σ the balls $B_\sigma(v), v \in \mathcal{E}^\lambda$ appearing in (B.7) are pairwise disjoint. However, in general there is no $\sigma > 0$ such that the set

$$\bigcup_{\substack{v,w\in\mathcal{E}^\lambda\\ v\to w}} U^-_{\sigma,\sigma}(C(v,w))$$

becomes a disjoint union. Since we shall use this property we argue for $0 < \sigma \leqslant \delta_b$ and $0 < \eta \leqslant d(\sigma)$ for the following modified neighborhood $U_{\eta,\sigma}(\mathcal{A}^\lambda)$ of $\mathcal{A}^\lambda$ we have

$$U_{\eta,\sigma}(\mathcal{A}^\lambda) = B_\sigma(\phi^\pm) \cup \Bigg(\bigcup_{v\in\mathcal{E}^\lambda\setminus\{\phi^+,\phi^-\}} B_\sigma(v)\Bigg) \cup \Bigg(\bigcup_{\substack{v,w\in\mathcal{E}^\lambda\\ v\to w}} U^-_{\eta,\sigma}(C(v,w))\Bigg). \tag{B.8}$$

Hence for $0 < \varepsilon \leqslant \varepsilon_b$, $0 < \sigma \leqslant \delta_b$ and $0 < \eta \leqslant d(\sigma)$

$$\begin{aligned} &U_{\eta,\sigma}(\mathcal{A}^\lambda) \cap \mathcal{D}^\pm(\varepsilon^\gamma) \\ &= B_\sigma(\phi^\pm) \cup \Bigg(\bigcup_{v\in\mathcal{E}^\lambda\setminus\{\phi^+,\phi^-\}} B_\sigma(v) \cap \mathcal{D}^\pm(\varepsilon^\gamma)\Bigg) \cup \Bigg(\bigcup_{\substack{v,w\in\mathcal{E}^\lambda\\ v\to w}} U^-_{\eta,\sigma}(C(v,w)) \cap \mathcal{D}^\pm(\varepsilon^\gamma)\Bigg), \end{aligned} \tag{B.9}$$

which by definition of $d(\sigma)$ is a union of pairwise disjoint sets. In the sequel we shall further reduce the upper bounds for σ, η and ε appropriately.

The strategy of the proof is the following. For $\sigma, \eta, \varepsilon$ to be determined in the sequel and $x \in \mathcal{D}^\pm(\varepsilon^\gamma)$, we shall use the flow property of the solution $u(\cdot\,; x)$ of the Chafee-Infante equation and treat the local behaviour of $u(t; y)$ for $t \geqslant 0$ after having entered $U_{\eta/2}(\mathcal{A}^\lambda) \subset U_{\eta,\sigma}(\mathcal{A}^\lambda)$, i.e. for initial conditions

$$y = u(\theta_1; x) \qquad \text{for} \quad \theta_1 = \theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda) \cap \mathcal{D}^\pm(\varepsilon^\gamma)).$$

In Part II.A we treat $y \in B_\sigma(\phi^\pm)$, followed by the case $y \in B_\sigma(v) \cap \mathcal{D}^\pm(\varepsilon^\gamma)$ for $v \in \mathcal{E}^\lambda$ in Part II.B, and finally the situation $y \in U^-_{\eta,\sigma}(v, w)$ for $v, w \in \mathcal{E}^\lambda$ with $v \to w$ in Part II.C.

II.A: Local behavior in a ball around a stable state

Claim II.A.1: Close to a stable state there is exponential convergence. For any $v \in \{\phi^+, \phi^-\}$ there are $\delta_s = \delta_s(v) > 0$, $\kappa_s = \kappa_s(v) > 0$ and $\varepsilon_s = \varepsilon_s(v) > 0$ such that for all $0 < \sigma \leqslant \delta_s$, $0 < \varepsilon \leqslant \varepsilon_s$, $y \in B_\sigma(v)$ and

$$\theta_2(y, v, \sigma, \varepsilon) := \{t > 0 \mid u(t; y) \in B_{\varepsilon^\gamma}(v)\}$$

the inequality

$$\theta_2 \leqslant \kappa_s \gamma\, |\ln \varepsilon| \qquad \text{for } \theta_2 = \theta_2(y, v, \sigma, \varepsilon)$$

holds.

This will follow from Lemma B.3 below.

II.B: Local behavior in $\mathcal{D}^{\pm}(\varepsilon^{\gamma})$ in a ball around an unstable state.

Claim II.B.1: Trajectories in $\mathcal{D}^{\pm}(\varepsilon^{\gamma})$ leave balls around unstable states in at most logarithmic time in ε. For each $v \in \mathcal{E}^{\lambda} \setminus \{\phi^+, \phi^-\}$ there exist constants $\delta_u = \delta_u(v) > 0$ and $\kappa_u = \kappa_u(v) > 0$ such that for all $0 < \sigma \leqslant \delta_u$ there is $\varepsilon_u = \varepsilon_u(v, \sigma) > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_u$ we have $\mathcal{D}^{\pm}(\varepsilon^{\gamma}) \cap B_{\sigma}(v) \neq \emptyset$, and for all $y \in \mathcal{D}^{\pm}(\varepsilon^{\gamma}) \cap B_{\sigma}(v)$ and

$$\theta_3(y, v, \sigma, \varepsilon) := \inf\{t > 0 \mid u(t; y) \notin B_{\sigma}(v) \cap \mathcal{D}^{\pm}(\varepsilon^{\gamma})\}$$

the inequality

$$\theta_3 \leqslant k_u \gamma \, |\ln \varepsilon| \quad \text{for} \quad \theta_3 = \theta_3(y, v, \sigma, \varepsilon)$$

holds.

This results from Proposition B.4 below.

II.C: Local behavior in a tube around a connecting orbit. For arbitrary $0 < \eta \leqslant \sigma$ and $v, w \in \mathcal{E}^{\lambda}$ with $v \to w$ and $y \in U^{-}_{\eta,\sigma}(v, w)$ we define the *first exit time from* $U^{-}_{\eta,\sigma}(v, w)$

$$\tau_3(y, \eta, \sigma, v, w) := \inf\{t > 0 \mid u(t; x) \notin U^{-}_{\eta,\sigma}(v, w)\}.$$

Claim II.C.1: Tubes around connecting orbits are left at the edges. For $v, w \in \mathcal{E}^{\lambda}$ with $v \to w$ there is $\delta_1 = \delta_1(v, w) > 0$ such that for any $0 < \sigma \leqslant \delta_1$ there is $\eta_1 = \eta_1(\sigma, v, w) > 0$ such that for all $0 < \eta \leqslant \eta_1$ and $y \in U^{-}_{\eta,\sigma}(v, w)$ we have

$$u(\tau_3; y) \in \partial U^{-}_{\eta,\sigma}(v, w) \cap (\partial B_{\sigma}(v) \cup \partial B_{\sigma}(w)) \quad \text{for } \tau_3 = \tau_3(y, v, w, \eta, \sigma).$$

Proof. Since all trajectories with initial values $x \in H \setminus \mathcal{S}$ finally enter $B_{\sigma}(\phi^+)$ or $B_{\sigma}(\phi^-)$ in finite time they have to leave the connecting tube in finite time. The latter is also true for $x \in \mathcal{S}$ due to the convergence towards a fixed point on $\mathcal{S}$. Hence $\tau_3(y, v, w, \eta, \sigma) < \infty$ for all $v, w \in \mathcal{E}^{\lambda}$, $0 < \eta \leqslant \sigma$, and $y \in U^{-}_{\eta,\sigma}(v, w)$.

Since $\tau_3(y, v, w, \eta, \sigma) > 0$ and by definition of $y = u(\theta_1(x, U_{\eta}(\mathcal{A}^{\lambda})); x)$, the trajectory cannot leave $U^{-}_{\eta,\sigma}(v, w)$ via its outer hull, because this is part of $\partial U_{\eta}(\mathcal{A}^{\lambda})$. But $U_{\eta}(\mathcal{A}^{\lambda})$ is positive invariant for $t \geqslant \theta_1(x, U_{\eta}(\mathcal{A}^{\lambda}))$. Hence the only possible exit locus is

$$u(\tau_3; y) \in \partial U^{-}_{\eta}(v, w) \cap (\partial B_{\sigma}(v) \cup \partial B_{\sigma}(w)) \qquad \text{for} \quad \tau_3 = \tau_3(y, v, w, \eta, \sigma).$$

□

Claim II.C.2: The exit time from connecting tubes is uniformly bounded. For each pair of connected orbits $v, w \in \mathcal{E}^{\lambda}$ with $v \to w$ there is $\delta_h = \delta_h(v, w)$ such that for all $0 < \sigma \leqslant \delta_h$ there exists $\eta_2 = \eta_2(\sigma, v, w) > 0$ such that for $0 < \eta \leqslant \eta_2$ there exists

$\tau_4 = \tau_4(v, w, \sigma, \eta) > 0$ such that for all $y \in U^-_{\eta,\sigma}(v, w)$ we have

$$u(\tau_4; y) \in B_{\sigma/2}(w).$$

In other words for

$$\theta_4(y, v, w, \eta, \sigma) := \inf\{t > 0 \mid u(t; y) \in B_{\sigma/2}(w)\}$$

we have the obtain the estimate

$$\theta_4(y, v, w, \eta, \sigma) \leqslant \tau_4(v, w, \eta, \sigma)$$

for all $y \in U^-_{\eta,\sigma}(v, w)$.

This results from Lemma B.7.

III: Global dynamics by the local behavior and the gradient structure

Claim III.1: Along a cascade of steady states trajectories visit each unstable state at most once.

1. For any "cascade" of length $k \geqslant 2$ in $\mathcal{E}^\lambda$, $v_1 \to \ldots \to v_k$, $v_i \in \mathcal{E}^\lambda$, we have that $v_i \neq v_j$ for $i \neq j$. Hence it has no loops.
2. There is $\delta_g > 0$ such that for $0 < \sigma \leqslant \delta_g$ and any cascade $v_1 \to \ldots \to v_k$ in $\mathcal{E}^\lambda$ the trajectory $t \mapsto u(t; x)$ for $x \in \mathcal{D}^\pm(\varepsilon^\gamma)$ can only move forward along σ-balls centered at fixed points in the cascade.
3. For all $v_1, v_2 \in \mathcal{E}^\lambda$ with $v_1 \to v_2$ there is $\delta_2 = \delta_2(v_1, v_2) > 0$, such that for $0 < \sigma \leqslant \delta_2$ there exists $\eta_3 = \eta_3(v_1, v_2, \sigma) > 0$ and $\varepsilon_0 = \varepsilon_0(v, w, \sigma) > 0$ such that for all $0 < \eta \leqslant \eta_3$, $0 < \varepsilon \leqslant \varepsilon_0$ and $y \in U^-_{\eta,\sigma}(v_1, v_2) \cap \mathcal{D}^\pm(\varepsilon^\gamma)$ there is $v_3 \in \mathcal{E}^\lambda$ with $v_2 \to v_3$ such that for

$$\theta_5(y, v_1, v_2, v_3, \eta, \sigma) := \inf\{t > 0 \mid u(t; y) \in U^-_{\eta,\sigma}(v_2, v_3)\}$$

the inequality

$$\theta_5(y, v_1, v_2, v_3, \eta, \sigma) \leqslant \tau_4(v_1, v_2, \eta, \sigma) + \kappa_u(v_2)\gamma\,|\ln \varepsilon|$$

holds.

Proof. 1. Consider the energy functional

$$\mathfrak{E}(z) = \int_0^1 \left(\frac{1}{2} \int_0^\zeta \left| \frac{\partial z}{\partial \xi}(\xi) \right|^2 \,\mathrm{d}\xi + \lambda \left(z^4(\zeta) - z^2(\zeta) \right) \right) \mathrm{d}\zeta, \quad z \in H.$$

First note that for all steady states $v, w \in \mathcal{E}^\lambda$, $v \to w$ we have $\mathfrak{E}(v) > \mathfrak{E}(w)$. As $t \mapsto \mathfrak{E}(u(t; x))$ is non-increasing, $\mathfrak{E}(v) \geqslant \mathfrak{E}(w)$. The equality $\mathfrak{E}(v) = \mathfrak{E}(w)$ would imply that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}(u(t;x)) = \nabla\mathfrak{E}(u(t;x))\frac{\partial u}{\partial t}(t) = \\ \langle \Delta u(t;x) + f(u(t;x)), \frac{\partial u}{\partial t}(t;x)\rangle = -\left\| \frac{\partial u}{\partial t}(t;x)\right\|^2$$

holds for any $x \in C(v,w)$ and $t \in \mathbb{R}$ implying that $C(v,w) \subset \mathcal{E}^\lambda$, which is absurd. Hence for $v \to w$ we have $\mathfrak{E}(v) > \mathfrak{E}(w)$. Thus for any cascade $v_1 \to \ldots \to v_k$ we obtain

$$\mathfrak{E}(v_1) > \cdots > \mathfrak{E}(v_k).$$

2. Due to the continuity of $H = H_0^1(0,1) \ni z \mapsto \mathfrak{E}(z) \in \mathbb{R}$ there is $\delta_g > 0$ such that for $0 < \sigma \leqslant \delta_g$ and each cascade $v_1 \to \ldots \to v_k$ in $\mathcal{E}^\lambda$ we have

$$\sup_{w\in B_\sigma(v_1)} \mathfrak{E}(w) \geqslant \inf_{w\in B_\sigma(v_1)} \mathfrak{E}(w) > \cdots > \sup_{w\in B_\sigma(v_k)} \mathfrak{E}(w) \geqslant \inf_{w\in B_\sigma(v_k)} \mathfrak{E}(w).$$

   Since $t \to \mathfrak{E}(u(t;x))$ is non-increasing, each trajectory $(u(t;x))_{t\geqslant 0}$ that visits $B_\sigma(v_i)$ in a cascade cannot come back to the previous one $B_\sigma(v_{i-1})$.

3. By Claim II.C.2 for all $v_1, v_2 \in \mathcal{E}^\lambda$ with $v_1 \to v_2$ there exists $\delta_h = \delta_h(v_1, v_1)$ such that for $0 < \sigma \leqslant \delta_h$ there is $\eta_2 = \eta_2(v_1, v_2, \sigma) > 0$ which ensures for $0 < \eta \leqslant \eta_2$ the existence of $\tau_4 = \tau_4(v_1, w_2, \eta, \sigma) > 0$ such that for $y \in U^-_{\eta,\sigma}(v_1, v_2)$ we have

$$u(\tau_4; y) \in B_{\sigma/2}(v_2).$$

   Due to the positive invariance of $U_\eta(\mathcal{A}^\lambda)$ and $\mathcal{D}^\pm(\varepsilon^\gamma)$ for any $\varepsilon > 0$ we obtain

$$u(\tau_4; y) \in B_{\sigma/2}(w) \cap U_{\eta_2}(\mathcal{A}^\lambda) \cap \mathcal{D}^\pm(\varepsilon^\gamma).$$

   By Claim II.B.1 and the positive invariance of $U_\sigma(\mathcal{A}^\lambda)$, for $v_2 \in \mathcal{E}^\lambda \setminus \{\phi^+, \phi^-\}$ there are constants $\kappa_u = \kappa_u(v_2) > 0$ and $\delta_u = \delta_u(v_2) > 0$ such that for $0 < \sigma \leqslant \frac{\delta_u}{2}$ there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$ and $z \in B_\sigma(v_2) \cap \mathcal{D}^\pm(\varepsilon^\gamma)$ we have for $t \geqslant \kappa_u \gamma |\ln \varepsilon|$

$$u(t;z) \in B^c_{2\sigma}(v_2) \cap \mathcal{D}^\pm(\varepsilon^\gamma).$$

   We now fix

$$\delta_2(v_1, v_2) = \frac{1}{2}\left(\delta_b \wedge \delta_h(v_1, v_2) \wedge \delta_1(v_1, v_2) \wedge \delta_u(v_2) \wedge \delta_g(v_1, v_2)\right)$$

   and $0 < \sigma \leqslant \delta_2(v_1, v_2)$. Then there exists

$$\eta_3 = \eta_3(v_1, v_2, \sigma) = \eta_1(v_1, v_2, \sigma) \wedge \eta_2(v_1, v_2, \sigma) \wedge d(\sigma)$$

   with $d(\sigma)$ defined in (B.5), for which we fix $0 < \eta \leq \eta_3$. We denote $N(\lambda) = |\mathcal{E}^\lambda|$.

By Proposition B.1

$$N(\lambda) = 2\lceil \frac{\sqrt{\lambda}}{\pi} \rceil - 1. \tag{B.10}$$

Hence there are finitely many fixed points $w_1, \dots, w_l \in \mathcal{E}^\lambda$, that are connected with v_2, $v_2 \to w_1, \dots, v_2 \to w_l$. Since $0 < \eta \leqslant d(\sigma)$ for $d(\sigma)$ defined in (B.5), the sets

$$U^-_{\eta,\sigma}(v_2, w_k), \quad k = 1, \dots, l_i \tag{B.11}$$

are pairwise disjoint in $U_\eta(\mathcal{A}^\lambda)$, which is positive invariant. Hence, by continuity of $t \mapsto u(t;x)$, leaving $U_\eta(\mathcal{A}^\lambda) \cap B_{2\sigma}(v_2)$ is equivalent to enter one of the connecting tubes (B.11), say $U^-_{\eta,2\sigma}(v_2, w_{k_i})$. By the statement of Part 2 $w_{k_i} \neq v_1$. Call $w_{k_i} = v_3$. In other words for $y \in U_{\eta,\sigma}(v_1, v_2)$

$$\theta_5(y, v_1, v_2, \eta, \sigma, \varepsilon) := \inf\{t > 0 \mid u(t; y) \in U^-_{\eta,\sigma}(v_2, w_3) \cap \mathcal{D}^\pm(\varepsilon^\gamma)\}$$

we have

$$\theta_5(y, v_1, v_2, \eta, \sigma, \varepsilon) = \inf\{t > 0 \mid u(t; y) \in \big(U^-_{\eta,\sigma}(v_1, v_2) \cap B_{2\sigma}(v_2)\big)^c \cap \mathcal{D}^\pm(\varepsilon^\gamma)\}.$$

By the flow property and Claims II.B.1 and II.C.2 we obtain for all $y \in U_{\eta,\sigma}(v_1, v_2)$

$$\begin{aligned} \theta_5(y, v_1, v_2, \eta, \sigma, \varepsilon) &= \theta_4(y, v_1, v_2, \eta, \sigma) + \theta_3(u(\theta_4(y, v_1, v_2, \eta, \sigma), v_2, \sigma, \varepsilon) \\ &\leqslant \tau_4(v_1, v_2, \eta, \sigma) + \kappa_u(v_2)\gamma\,|\ln \varepsilon|. \end{aligned} \tag{B.12}$$

□

We conclude the desired result. We fix

$$\delta_0 := \delta_u(0) \wedge \Big(\min_{\substack{v,w \in \mathcal{E}^\lambda \setminus \{\phi^+,\phi^-\} \\ v \to w}} \delta_2(v, w) \Big) \wedge \delta_c \wedge \Big(\min_{v \in \{\phi^+,\phi^-\}} \delta_s(v) \Big)$$

for

$$\delta_c = \min_{\substack{v \in \mathcal{E}^\lambda \setminus \{\phi^+,\phi^-\} \\ w \in \{\phi^+,\phi^-\} \\ v \to w}} \delta_1(v, w) \wedge \delta_2(v, w).$$

Then for $0 < \sigma \leqslant \delta_0$ define

$$\eta_0 := \Big(\min_{\substack{v,w \in \mathcal{E}^\lambda \setminus \{\phi^+,\phi^-\} \\ v \to w}} \eta_3(v, w, \sigma) \Big) \wedge \Big(\min_{\substack{v \in \mathcal{E}^\lambda \setminus \{\phi^+,\phi^-\} \\ w \in \{\phi^+,\phi^-\} \\ v \to w}} \eta_1(v, w) \wedge \eta_2(v, w) \Big)$$

and fix $0 < \eta \leqslant \eta_0$. Then there exists

$$\varepsilon_0(\sigma) := \Big(\min_{v \in \{\phi^+,\phi^-\}} \varepsilon_s(v, \sigma) \wedge \varepsilon_b(v, \sigma) \Big) \wedge \Big(\min_{v \in \mathcal{E}^\lambda \setminus \{\phi^+,\phi^-\}} \varepsilon_u(v, \sigma) \Big)$$

such that for all $0 < \varepsilon \leqslant \varepsilon_0$ the statements of the Claims I.1, II.A.1, II.B.1, II.C.1, III.C.2 are true simultaneously for all $v \in \mathcal{E}^\lambda$ involved. Define

$$\theta_6(x) := \inf\{t > 0 \mid u(t;x) \in B_{(1/2)\varepsilon^{2\gamma}}(\phi^\pm)\}.$$

For $x \in \mathcal{D}^\pm(\varepsilon^\gamma)$ by Claim I.2

$$u(t + \theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda)); x) \in U_{\eta/2}(\mathcal{A}^\lambda) \subset U_{\eta,\sigma}(\mathcal{A}^\lambda) \tag{B.13}$$

for all $t \geqslant 0$. We have to distinguish three cases according to where $u(\theta(x, U_{\eta/2}(\mathcal{A}^\lambda); x)$ lies. Either

$$u(\theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda); x) \in B_\sigma(\phi^\pm) \tag{B.14}$$

or there is $v \in \mathcal{E}^\lambda \setminus \{\phi^+, \phi^-\}$ such that

$$u(\theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda); x) \in B_\sigma(v) \cap U_{\eta,\sigma}(\mathcal{A}^\lambda) \tag{B.15}$$

or there are $v, w \in \mathcal{E}^\lambda$ such that $v \to w$ and

$$u(\theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda); x) \in U^-_{\eta,\sigma}(v, w). \tag{B.16}$$

Case (B.14) is contained in the two preceding cases. For convenience we first treat case (B.16). By (B.13) and Claim III.1.2 and III.1.3 for $x \in \mathcal{D}^\pm(\varepsilon^\gamma)$ there is $k = k(x) \in \{1, \ldots, N(\lambda)\}$ and a cascade of steady states $v_1 \to \ldots \to v_k \to v_{k+1} = \phi^\pm$ in $\mathcal{E}^\lambda$ of length k such that the trajectory $u(\cdot\,; x)$ visits all balls $B_\sigma(v_i)$ or radius σ once along the cascade. The case $k = 1$ means that there is an unstable state $v \in \mathcal{E}^\lambda$ with $v \to \phi^\pm$ and

$$u(\theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda)); x) \in U^-_{\eta,\sigma}(v, \phi^\pm).$$

Hence

$$\begin{aligned}\theta_6(x) = \theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda)) &+ \theta_4(u(\theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda); x), v, \phi^\pm, \eta, \sigma) \\ &+ \theta_2(u(\theta_4(u(\theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda); x), v, \phi^\pm, \eta, \sigma); x), \phi^\pm, \sigma) \\ &\leqslant \tau_1(\eta/2) + \tau_4(v, \phi^\pm, \eta, \sigma) + \kappa_s(\phi^\pm)\gamma\,|\ln \varepsilon|. \end{aligned} \tag{B.17}$$

This covers the case (B.14). For the cases $k \geqslant 2$ we have to introduce the following global versions of θ_5 form Claim III.1.3

$$\begin{aligned}\sigma_0(x, \eta, \sigma) &= \theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda)) \\ \sigma_i(x, \eta, \sigma) &= \theta_5(u(\sigma_{i-1}(x, \eta, \sigma); x), v_i, v_{i+1}, v_{i+2}, \eta, \sigma), \quad i \in \{1, \ldots, k-1\}.\end{aligned}$$

In this case we may estimate

$$\begin{aligned}\theta_6(x) &= \theta_1(x, U_{\eta/2}(\mathcal{A}^\lambda)) + \sum_{i=1}^{k(x)} \sigma_i(x,\eta,\sigma) + \theta_4(u(\sigma_{k(x)}(x,\eta,\sigma);x), v_k, \phi^\pm, \eta, \sigma) \\ &\qquad + \theta_2(u(\theta_4(u(\sigma_{k(x)}(x,\eta,\sigma);x), v_k, \phi^\pm, \eta, \sigma);x), \phi^\pm, \sigma) \\ &\leqslant \tau_1(\eta/2) + \sum_{i=1}^{k(x)} \tau_4(v_i, v_{i+1}, \eta, \sigma) + \kappa_u(v_{i+1})\gamma|\ln\varepsilon| \\ &\qquad + \tau_4(v_{k(x)}, v_{k(x)+1}, \eta, \sigma) + \kappa_s(\phi^\pm)\gamma\,|\ln\varepsilon| \\ &\leqslant \tau_1(\eta/2) + N(\lambda) \max_{\substack{v,w\in\mathcal{E}^\lambda \\ v\to w}} \tau_4(v,w,\eta,\sigma) \\ &\qquad + N(\lambda)\Big(\max_{v\in\mathcal{E}^\lambda\setminus\{\phi^+,\phi^-|} \kappa_u(v) + \max_{\{v\in\{\phi^+,\phi^-\}} \kappa_s(v) \Big)\gamma|\ln\varepsilon| \\ &\qquad\qquad =: T_{rec} + \kappa\gamma|\ln\varepsilon|.\end{aligned}$$

The expression in the preceding line is also an upper bound for (B.17) covering the case (B.14). Case (B.15) can be obtained analogously with a slightly shifted summation, for which the identical upper bound is attained. This finishes the proof. ☐

B.2.3. Local Convergence to Stable States

In this Subsection we prove Claim II.A.1 in the proof of Proposition B.2.

Lemma B.3 (Local exponential convergence to stable states)**.** *For $v \in \{\phi^+, \phi^-\}$ there are constants $\delta_s = \delta_s(v) > 0$ and $\kappa_s = \kappa_s(v) > 0$ such that for all $\gamma > 0$ there is $\varepsilon_s = \varepsilon_s(\gamma) > 0$ such that for all $0 < \sigma \leqslant \delta_s$, $0 < \varepsilon < \varepsilon_s$, $y \in B_\sigma(v)$ and $t \geqslant \kappa_s\gamma|\ln\varepsilon|$*

$$\|u(t;y) - v\| \leqslant (1/2)\varepsilon^{2\gamma}.$$

In addition, for the balls $B_\sigma(v)$ with respect to $|\cdot|_\infty$ it follows for all $t \geqslant 0$

$$u(t; B_\sigma(v)) \subset B_\sigma(v).$$

Proof. For $t \geqslant 0$, $y \in H$ and $v \in \{\phi^+, \phi^-\}$ denote by $R(\cdot; y) := u(\cdot; y) - v$.

1. We argue similarly as in the proof of Lemma 3.4, equation (3.8), and obtain

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \Delta R + f'(v)R + \left(\iint_0^1 f''(v + \theta_2 u + \theta_2\theta_1 R)\,(u - v + \theta_1 R)\ \mathrm{d}\theta_2 \mathrm{d}\theta_1 \right) R. \quad \text{(B.18)}$$

We fix $\eta > 0$ to be specified later and define for $y \in B_\eta(v)$

$$t^*_\eta := \inf\{t > 0 \mid |R(t;y)|_\infty \geqslant \eta\}.$$

For $y \in B_\eta(v)$ we have $t^*_\eta > 0$. For convenience we drop the arguments of R. Multiplying (B.18) with R and using the stability of v, which ensures that there exists $\Lambda > 0$ and $\delta_1 > 0$ such that for $0 < \sigma \leqslant \delta_1$ and $w \in B_\sigma(v)$ we have $\langle \Delta w + f'(v)w, w\rangle \leqslant -\Lambda |w|^2$. We obtain for $0 \leqslant t \leqslant t^*_\eta$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|R|^2 + \Lambda|R|^2 \\ &\leqslant \left(\iint_0^1 |f''(v+\theta_2 u + \theta_2\theta_1 R)|_\infty \,\mathrm{d}\theta_2\,\mathrm{d}\theta_1\right)\int_0^1 (|u-v|_\infty + |R|_\infty)R^2\,\mathrm{d}\zeta \\ \leqslant \Big(|v|_\infty + |u|_\infty + |R|\Big)\Big(|u-v|_\infty + |R|_\infty\Big)|R|^2 \\ &\leqslant 6\lambda\Big(2|v|_\infty + |u-v|_\infty + |R|_\infty\Big)\Big(|u-v|_\infty + |R|_\infty\Big)|R|^2.\end{aligned}$$

If we choose

$$\delta_2 \leq \frac{|v|_\infty}{2}\left(\sqrt{1+\frac{\Lambda}{12\lambda|v|_\infty}} - 1\right)$$

we obtain for $\eta \leqslant \delta_2$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}|R|^2 \leqslant -\Lambda|R|^2$$

and for $\sigma \leqslant \eta$ and $y \in B_\sigma(v)$ with respect to $|\cdot|_\infty$ by Gronwall's Lemma

$$|R(t)|^2 \leqslant \sigma^2 e^{-\Lambda t} \leqslant \sigma^2 e^{-\tilde{\Lambda} t}. \tag{B.19}$$

where $\tilde{\Lambda} := \min\left(\Lambda, \frac{c_0}{2}\right)$ and $-c_0$ was the largest eigenvalue of the Laplacian.

2. We next sharpen the estimate in the first part to an estimate in the $\|\cdot\|$-norm. To this end, we again use the regularizing effect of the heat semigroup on $[0,1]$ and estimate for $t \geqslant 1$, $\sigma \leqslant \delta_1 \wedge \delta_2$ and $y \in B_\sigma(v)$

$$\begin{aligned}\|R(t;y)\| &\leqslant \frac{e^{-c_0 t}}{t^{1/2}}|R(0;y)| + C_1\int_0^t \frac{e^{-c_0(t-s)}}{(t-s)^{1/2}}|f(u(s;y)) - f(v)|\,\mathrm{d}s \\ &\leqslant e^{-(1/2)\tilde{\Lambda}t}\sigma + C_1\int_0^t \frac{e^{-c_0(t-s)}}{(t-s)^{1/2}}\int_0^1 |f'(v+\theta R(s;y))|_\infty |R(s;y)|\,\mathrm{d}\theta\,\mathrm{d}s \\ &\leqslant e^{-(1/2)\tilde{\Lambda}t}\sigma + 3C_1\lambda\left((|v|_\infty + \eta)^2 + 1\right)\int_0^t \frac{e^{-c_0(t-s)}}{(t-s)^{1/2}}|R(s;y)|\,\mathrm{d}s\end{aligned}$$

Inserting (B.19) we obtain

$$\begin{aligned}\|R(t;y)\| \leqslant & e^{-(1/2)\tilde{\Lambda}t}\sigma + \frac{3}{2}C_1\lambda\left(|v|_\infty^2 + \frac{\Lambda}{9\lambda} + 1\right)\left(\int\limits_0^t \frac{e^{-c_0(t-s)}}{(t-s)^{1/2}}\sigma e^{-(1/2)\tilde{\Lambda}s}\,\mathrm{d}s\right)\\ \leqslant & e^{-(1/2)\tilde{\Lambda}t}\sigma + \underbrace{\frac{3}{2}C_1\lambda\left(|v|_\infty^2 + \frac{\Lambda}{9\lambda} + 1\right)\left(\int\limits_0^t \frac{e^{-(c_0-(1/2)\tilde{\Lambda})(t-s)}}{(t-s)^{1/2}}\,\mathrm{d}s\right)}_{\leqslant C_2}\sigma e^{-(1/2)\tilde{\Lambda}t}.\end{aligned}$$

Hence, for $t \geqslant 1$,

$$\|u(t;y) - v\| \leqslant (1 + C_2)\sigma e^{-(1/2)\tilde{\Lambda}t}.$$

Thus for $\delta_3 < \frac{1}{2(1+C_2)}$ we may choose $\kappa_s := \dfrac{4}{\tilde{\Lambda}}$ and $\delta_s := \min(\delta_1, \delta_2, \delta_3)$. Then for $\sigma \leqslant \eta \leqslant \delta_s$, $\varepsilon_s(\sigma) = \sigma \wedge \exp\left(-\frac{1}{\kappa_s\gamma}\right)$ and $0 < \varepsilon \leqslant \varepsilon_s$, $y \in B_\sigma(v)$ and $t \geqslant \kappa_s\gamma|\ln\varepsilon|$

$$\|u(t;y) - v\| \leqslant (1/2)\varepsilon^{2\gamma}.$$

If we choose additionally

$$\frac{1}{2}\varepsilon_s^{2\gamma} \leqslant \eta$$

we can deduce that $t_\eta^* = \infty$ and the balls $B_\sigma(v)$ with respect to $|\cdot|_\infty$ are positively invariant. This finishes the proof. ☐

B.2.4. Local Repulsion from Unstable States in Reduced D.o.A.

In this Subsection we prove Claim II.B.1 in the proof of Theorem B.2.

Proposition B.4. *For $\pi^2 < \lambda \neq (\pi n)^2, n \in \mathbb{N}$ given and $v \in \mathcal{E}^\lambda \setminus \{\phi^+, \phi^-\}$ there exists $\delta_u = \delta_u(v) > 0$ and $\kappa_u = \kappa_u(v) > 0$ such that for $0 < \sigma \leqslant \delta_u$ there is $\varepsilon_u = \varepsilon_u(v, \sigma) > 0$ ensuring for $0 < \varepsilon \leqslant \varepsilon_u$ that $\mathcal{D}^\pm(\varepsilon^\gamma) \cap B_\sigma(v) \neq \emptyset$, and we have for $x \in \mathcal{D}^\pm(\varepsilon^\gamma) \cap B_\sigma(v)$ and $t \geqslant \kappa_u\gamma|\ln\varepsilon|$ that*

$$u(t;x) \in B_\sigma^c(v) \cap \mathcal{D}^\pm(\varepsilon^\gamma).$$

Let $\gamma > 0$ be fixed throughout this Subsection. We study the local behaviour of the solution $u(\cdot;x)$ of the Chafee-Infante equation starting in $x \in B_\sigma(v)$ for small $\sigma > 0$ and an unstable state $v \in \mathcal{E}^\lambda \setminus \{\phi^+, \phi^-\}$. More precisely we are interested to determine an upper bound in terms of $\varepsilon > 0$ of the time $u(\cdot\,;x)$ starting in $x \in B_\sigma(v) \cap \mathcal{D}^\pm(\varepsilon^\gamma)$ needs to leave $B_\sigma(v) \cap \mathcal{D}^\pm(\varepsilon^\gamma)$ for $\sigma > 0, \varepsilon > 0$ sufficiently small. For convenience we recall that the formal Chafee-Infante equation (2.6) for fixed Chafee-Infante parameter

$\pi^2 < \lambda \neq (\pi n)^2, n \in \mathbb{N}$, is given as

$$\begin{aligned}
\frac{\partial}{\partial t} u(t,\zeta) &= \frac{\partial^2}{\partial \zeta^2} u(t,\zeta) + f(u(t,\zeta)) && \zeta \in [0,1],\ t > 0,\\
u(t,0) &= u(t,1) = 0, && t > 0,\\
u(0,\zeta) &= x(\zeta), && \zeta \in [0,1],
\end{aligned}$$

where $f(y) = -\lambda(y^3 - y)$.

Without loss of generality let $v = 0 \in \mathcal{E}^\lambda \setminus \{\phi^+, \phi^-\}$. Hence $f(v) = 0$. We set

$$\begin{aligned}
B : D(B) \subset H \to H, \qquad Bw &= \frac{\partial^2 w}{\partial \zeta^2} + f'(v)w, \qquad w \in D(B)\\
G : H \to H, \qquad G(w) &= f(w) - f'(v)w, \qquad w \in H.
\end{aligned}$$

Thus $G(v) = G'(v) = 0$. It is well-known that B has a discrete spectrum and a finite number of positive eigenvalues. Denote by ω_0 the smallest positive eigenvalue of B. We denote by $P_u : H \to H^+$ the orthogonal projection onto H^+, the span of the eigenvectors of the positive eigenvalues, and respectively $P_s : H \to H^-$, where H^- is the span of the eigenvectors of the negative eigenvalues. Since for the Chafee-Infante parameter $\pi^2 < \lambda \neq (\pi n)^2, n \in \mathbb{N}$, all steady states in $\mathcal{E}^\lambda$ are hyperbolic, 0 is not an eigenvalue and $H = H^+ \oplus H^-$. We denote by $w_s = P_s w$ and $w_u = P_u w$ for $w \in H$. For $t \geqslant 0$ and $x \in \mathcal{D}^\pm(\varepsilon^\gamma)$ and the solution $u(t;x)$ of equation (2.6) we use in this subsection the notation

$$\begin{aligned}
X(t;x) &:= P_s u(t;x),\\
Y(t;x) &:= P_u u(t;x).
\end{aligned}$$

We write

$$\begin{aligned}
g : H^+ \oplus H^- \to H^-, \quad g(w_u, w_s) &:= P_s G(w),\\
h : H^+ \oplus H^- \to H^+, \quad h(w_u, w_s) &:= P_u G(w)
\end{aligned}$$

and by $(T(t))_{t \geqslant 0}$ the $\mathcal{C}_0$-semigroup by the linearized operator $B = \Delta + f'(v)$. Solving equation (2.6) for $u(t;x)$ is then equivalent to solving the coupled system of projected equations

$$X(t) = T(t)x_s + \int_0^t T(t-\vartheta) g(X(\vartheta), Y(\vartheta))\, \mathrm{d}\vartheta \tag{B.20}$$

$$Y(t) = T(t)x_u + \int_0^t T(t-\vartheta) h(X(\vartheta), Y(\vartheta))\, \mathrm{d}\vartheta \tag{B.21}$$

for $(X(t), Y(t)) = (X(t;x_s), Y(t;x_u))$.

For initial values $x_s \in H^-$ we have for all $t \geqslant 0$

$$|T(t)x_s| \leqslant e^{-\omega_0 t}|x_s|.$$

Moreover $T(t)P_u$ can be extended for $x \in H^+$ to $t \leqslant 0$ with

$$|T(t)x_u| \leqslant e^{\omega_0 t}|x_u|.$$

Before giving the proof we sketch the so-called *Lyapunov-Perron construction of the unstable manifold.* More details can be found in Temam [1992], Chapter IX.

Definition B.5. Let $(\Psi(t))_{t\geqslant 0}$ be a dynamical system on H, i.e. a family of continuous operators $\Psi(t) : H \to H$ satisfying

$$\Psi(t+s) = \Psi(t) \circ \Psi(s), \qquad t, s \geqslant 0.$$

Let $v \in H$ a fixed point, i.e. $\Psi(t)v = v$. The *unstable manifold* $\mathcal{W}^u(v)$ *of* v is defined as

$$\mathcal{W}^u(v) := \{w \in H \mid \lim_{t\to-\infty} \Psi(t; w) = v\}.$$

the *stable manifold* $\mathcal{W}^s(v)$ *of* v by

$$\mathcal{W}^s(v) := \{w \in H \mid \lim_{t\to\infty} \Psi(t; w) = v\}.$$

Sketch of the Lyapunov-Perron Construction of the Unstable Manifold: For a radius $\sigma > 0$ let $B_\sigma(v)$ a ball centered in the unstable solution $v = 0$. First of all we truncate g and h by a function $\psi^\sigma : H \to H$, $\psi^\sigma \in \mathcal{C}^\infty(H; H)$ such that

$$\psi^\sigma(x) = \begin{cases} 1 & \text{if } x \in B_\sigma(v) \\ 0 & \text{if } x \in B^c_{2\sigma}(v). \end{cases}$$

We denote by $L_\sigma > 0$ the common Lipschitz constant of g and h on $B_{2\sigma}(v)$. Clearly

$$L_\sigma \to 0, \text{ if } \sigma \to 0.$$

We want to construct the unstable manifold $\mathcal{W}^u(v)$ as the graph of a bounded function $\Phi_u : H^+ \to H^-$, which is Lipschitz continuous with Lipschitz constant $L_{\Phi_u} > 0$ such that there is $\delta_{\Phi_u} = \delta_{\Phi_u}(v) > 0$ such that for $0 < \sigma \leqslant \delta_{\Phi_u}$

$$\mathcal{W}^u(v) = \{h + \Phi_u(h) \mid h \in B_\sigma(v) \cap H^+\}. \tag{B.22}$$

We assume the existence of Φ_u and that $\Phi_u \in \mathcal{C}^1$ in order to be allowed to apply the calculus. This is done by fixed point arguments, which can be found in Temam [1992], Chapter IX. However, we shall prove that if it exists it provides an exponentially attracting, invariant unstable manifold for the truncated equation on $B_\sigma(v)$.

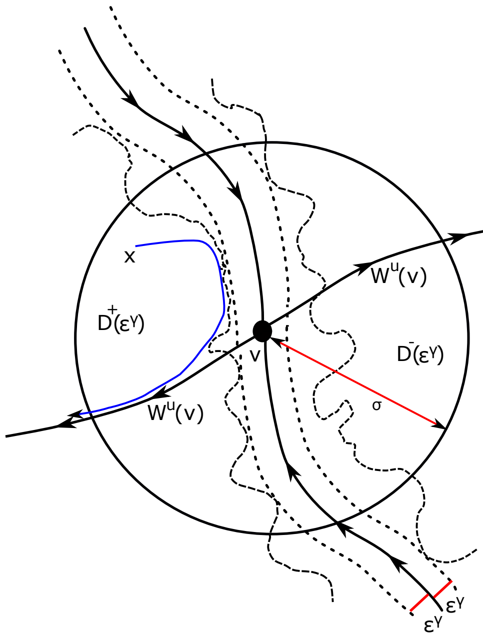

Figure B.3.: Sketch of the exit from a neighborhood an unstable point

Claim 1: *Assume that Φ_u exists, such that (B.22) is fulfilled, then there is $\delta_1 = \delta_1(v) > 0$ such that for $0 < \sigma \leqslant \delta_1$ the projection*

$$R(t;y) := Y(t;y) + \Phi_u(Y(t;y))$$

satisfies (2.6) for all $y \in B_\sigma(v) \cap H^+$ and $t \in \mathbb{R}$ and $\lim_{t\to-\infty} R(t;y) = 0$.

Proof. Let δ_l be the maximal positive number such that for $0 < \sigma \leqslant \delta_l$

$$L_\sigma(1 + L_{\Phi_u}) \leqslant \frac{\omega_0}{2}. \tag{B.23}$$

Set $\delta_1 = \delta_{\Phi_u} \wedge \delta_l$ and fix $0 < \sigma \leqslant \delta_1$. Under the assumption that Φ_u exists and (B.22) holds true the function Φ_u satisfies for $y \in H^+ \cap B_\sigma(0)$

$$\Phi_u(y) = \int_{-\infty}^{0} T(-s)g(\Phi_u(Y(s;y)), Y(s;y))\,\mathrm{d}s, \tag{B.24}$$

$$Y(s;y) = T(s)y + \int_{0}^{s} T(s-r)h(\Phi_u(Y(r;y)), Y(r;y))\,\mathrm{d}r, \quad s \leqslant 0.$$

For details consult Temam [1992], Chapter IX. By the flow property we obtain for $t \in \mathbb{R}$

$$\Phi_u(Y(t;y)) = \int_0^t T(t-s)g(\Phi_u(Y(s;y)), Y(s;y))\ \mathrm{d}s$$

$$Y(t;y) = T(t)y + \int_0^t T(t-s)h(\Phi_u(Y(s;y)), Y(s;y))\ \mathrm{d}s.$$

Hence for $t \in \mathbb{R}$ and $y \in B_\sigma(v) \cap H^+$ the function $R(t;y) = \Phi_u(Y(t;y)) + Y(t;y)$ satisfies

$$R(t;y) = T(t)y + \int_0^t T(t-s)G(R(s;y))\ \mathrm{d}s, \qquad t \leqslant 0.$$

In addition, by the local Lipschitz continuity of Φ_u and G in $B_\sigma(0)$ we obtain for $t \leqslant 0$

$$|Y(t;y)| \leqslant\ e^{\omega_0 t}|y| + \int_t^0 e^{\omega_0(t-\vartheta)} L_\sigma(1 + L_{\Phi_u})|Y(\vartheta;y)|\ \mathrm{d}\vartheta.$$

By Gronwall's Lemma

$$|Y(t;y)| \leqslant\ e^{(\omega_0 - L_\sigma(1+L_{\Phi_u}))t}|y|$$

and due to the choice of σ in (B.23) we obtain

$$|Y(t;y)| \leqslant e^{\frac{\omega_0}{2}t}|y| \to 0, \qquad t \to -\infty.$$

Hence $\Phi_u(Y(t;y)) \to 0, t \to -\infty$. □

Claim 2: *Assume that Φ_u exists and (B.22) is true. Then there is $\delta_2 = \delta_2(v) > 0$ such that for all $0 < \sigma \leqslant \delta_2$ such that for all $y \in B_\sigma(v) \cap H^+$ und $t \geqslant 0$*

$$|u(t;y) - P_u(u(t;y)) - \Phi_u(P_u(u(t;y)))| \leqslant e^{-\frac{\omega_0}{2}t}|P_s(y) - \Phi_u(P_u(y))|.$$

In addition, for all $y \in \mathcal{W}^u(v)$ the global trajectories $(u(t;y))_{t\in\mathbb{R}}$ are of the form $u(t;y) = Y(t;y) + \Phi_u(Y(t;y))$.

Proof. Let $0 < \sigma \leqslant \delta_2$, $\delta_2 = \delta_{\Phi_u} \wedge \delta_l$ such that (B.23) is true and $y \in B_\sigma(v) \cap H^+$. Since $Y(s;Y(t;y)) = Y(t+s;y)$ and by (B.24)

$$\Phi_u(Y(t;y)) = \int_{-\infty}^0 T(-s)g(\Phi_u(Y(s;Y(t;y))), Y(s;Y(t;y)))\ \mathrm{d}s$$

it follows

$$\Phi_u(Y(t;y)) = \int_{-\infty}^{t} T(t-s)g(\Phi_u(Y(s;y));Y(s;y))\ \mathrm{d}s.$$

Thus $\Phi_u(Y(t;y))$ is a mild solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_u(Y(t;y)) = B\Phi_u(Y(t;y)) + g(\Phi_u(Y(t;y)),Y(t;y)) \tag{B.25}$$

with the respective initial condition. Since by the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_u(Y(t;y)) &= (\nabla\Phi_u)(Y(t;y))\frac{\partial}{\partial t}Y(t;y)\\ &= (\nabla\Phi_u)(Y(t;y))(BY(t;y) + h(X(t;y),Y(t;y))\end{aligned} \tag{B.26}$$

we obtain at $t = 0$ that

$$(\nabla\Phi_u)(y)(By + h(\Phi_u(y),y)) - B\Phi_u(y) - G(\Phi_u(y),y) = 0 \qquad \text{for all } y \in H.$$

Let $(X(t;y);Y(t;y))_{t\geqslant 0}$ be the associated solution of $u(t;y)$. In the next calculation we omit the arguments for convenience. By identification of the right-hand side of (B.25) and (B.26) we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(X - \Phi_u(Y)) = BX + g(X,Y) - \nabla\Phi_u(Y)(BY + h(X,Y))\\ &= B(X-\Phi_u(Y)) + g(X,Y) - g(\Phi_u(Y),Y) + \nabla\Phi_u(Y)h(\Phi_u(Y),Y) - \nabla\Phi_u(Y)h(X,Y)\\ &\qquad - \nabla\Phi_u(Y)(BY) - \nabla\Phi_u(Y)h(\Phi_u(Y),Y) + B(\Phi_u(Y)) + g(\Phi_u(Y),Y)\\ &= B(X-\Phi_u(Y)) + g(X,Y) - g(\Phi_u(Y),Y) + \nabla\Phi_u(Y)(h(\Phi_u(Y),Y) - h(X,Y)).\end{aligned}$$

Therefore $Z(t;y) = X(t;y) - \Phi_u(Y(t;y))$ satisfies for $t \geqslant 0$

$$\begin{aligned}Z(t;y) = T(t)Z(0;y) &+ \int_0^t T(t-s)\left(g(X(s;y),Y(s;y)) - g(\Phi_u(Y(s;y)),Y(s;y))\right)\ \mathrm{d}s\\ &+ \int_0^t T(t-s)\nabla\Phi_u(Y(s;y))\left(h(\Phi_u(Y(s;y)),Y(s;y)) - h(X(s;y),Y(s;y))\right)\ \mathrm{d}s.\end{aligned}$$

Taking the L^2-norm we use the Lipschitz continuity of Φ_u and G truncated by ψ^σ we arrive at

$$|Z(t;y)| \leqslant e^{-\omega_0 t}|Z(0;y)| + \int_0^t e^{-\omega_0(t-s)} L_\sigma |Z(s;y)|\, \mathrm{d}s + \int_0^t e^{-\omega_0(t-s)} L_{\Phi_u} L_\sigma |Z(s;y)|\, \mathrm{d}s.$$

By Gronwall's Lemma we obtain

$$|Z(t;y)| \leqslant e^{-(\omega_0 + L_\sigma(1+L_{\Phi_u}))t}|Z(0;y)|.$$

Thus for $0 < \sigma \leqslant \delta_l$ such that (B.23) is true and $y \in B_\sigma(v) \cap H^+$, we have exponential estimate

$$|Z(t;y)| \leqslant |Z(0;y)|e^{-\frac{\omega_0}{2}t} \qquad \text{for } t \geqslant 0.$$

In other words for $y \in B_\sigma(v) \cap H^+$

$$|X(t;y) - \Phi_u(Y(t;y))| \leqslant e^{-\frac{\omega_0}{2}t}|X(0;y) - \Phi_u(Y(0;y))|, \qquad \text{for } t \geqslant 0. \tag{B.27}$$

Hence the graph of Φ_u is exponentially attracting for $t \to \infty$ and invariant. For $y \in B_\sigma(v) \cap H^+$ let $(X(t;y), Y(t;y))$ be a solution defined on $\mathbb{R}^-$ which converges to the unstable state 0 for $t \to -\infty$, it is in particular bounded by a constant, $M > 0$, say

$$|X(t;y)| + |Y(t;y)| \leqslant M, \qquad t \leqslant 0.$$

For $t_0 \leqslant 0$ and all $t \geqslant t_0$ we have

$$|X(t;y) - \Phi_u(Y(t;y))| \leqslant e^{-\frac{\omega_0}{2}(t-t_0)}|X(t_0;y) - \Phi(Y(t_0;y))| \leqslant Me^{-\frac{\omega_0}{2}(t-t_0)}.$$

Since the left-hand side does not depend on t_0 we can pass to the limit $t_0 \to -\infty$ implying

$$X(t;y) = \Phi_u(Y(t;y)) \qquad \text{for } t \in \mathbb{R}, z \in B_\sigma(v) \cap H^+.$$

Hence a global solution $((X(t;y), Y(t;y))_{t\in\mathbb{R}}$ for $y \in B_\sigma(0) \cap H^+$ lives on the unstable manifold given as the local graph of Φ_u in $B_\sigma(v)$. This finishes the proof of Claim 2. □

Remark B.6. The whole construction so far is entirely carried out in the topology of $L^2(0,1)$.

We can now prove Proposition B.4.

Proof. Fix $\gamma > 0$. We show that close to an unstable state $v \in \mathcal{E}^\lambda$ there are times $t \geqslant t_0$ for appropriate t_0 and initial values Z_0 for which we obtain an affine upper bound for the respective unstable projection $Y(t;x) = P_u u(t;x)$ of the solution of (2.6). This will be used for the argument in the second part.

1. Claim: *For $v \in \mathcal{E}^\lambda \setminus \{\phi^+, \phi^-\}$ there is $\delta_3 = \delta_3(v) > 0$ such that for $0 < \sigma \leqslant \delta_3$, there are $\varepsilon_1 = \varepsilon_1(\sigma) > 0$ and $C = C(\sigma) > 0$ which ensures that for all $0 < \varepsilon \leqslant \varepsilon_1$ and $x \in \mathcal{D}^\pm(\varepsilon^\gamma) \cap B_\sigma(v)$ there is $t_0 = t_0(\varepsilon, \sigma) > 0$ exists $Z_0 = Z_0(t_0, x) \in \mathcal{D}^\pm(\varepsilon^\gamma) \cap B_\sigma(v)$ such that for $t_0 \leqslant t \leqslant s$ the inequality*

$$|Y(t; Z_0)| \leqslant e^{-\frac{\omega_0}{2}(s-t)}|Y(s; Z_0)| + C\varepsilon^\gamma.$$

is satisfied.

Without loss of generality, we still consider $v = 0$. Fix $\delta_3 = \delta_3(v) = \delta_1 \wedge \delta_2$ and $0 < \sigma \leqslant \delta_3$ such that Claim 1 and Claim 2 are true. Fix $\varepsilon_1 = \varepsilon_1(\sigma) > 0$ small enough such that for $0 < \varepsilon \leqslant \varepsilon_1$ we have $B_\sigma(v) \cap \mathcal{D}^\pm(\varepsilon^\gamma) \neq \emptyset$ and pick an element $x \in B_\sigma(v) \cap \mathcal{D}^\pm(\varepsilon^\gamma)$. This is justified by Lemma (2.13). Then we can bound the first exit time from $B_\sigma(v) \cap \mathcal{D}^\pm(\varepsilon^\gamma)$ in terms of $0 < \varepsilon \leqslant \varepsilon_1$. Let Φ_u be the generating graph of the unstable manifold as constructed above with Lipschitz constant L_{Φ_u}. Due to estimate (B.27) we obtain

$$|X(t; x) - \Phi_u(Y(t; x))| \leqslant e^{-\frac{\omega_0}{2}t}|P_s(x) - \Phi_u(P_u(x))| \leqslant \frac{1}{8}\varepsilon^\gamma \tag{B.28}$$

for

$$t \geqslant t_0 := -\frac{2\gamma}{\omega_0}\ln(\varepsilon) + \frac{2}{\omega_0}\ln(8\sigma) \tag{B.29}$$

and set $Z_0 = Z_0(x, \varepsilon, \sigma) = (X(t_0; x), Y(t_0; x))$. In other words

$$|P_s(Z_0) - \Phi_u(P_u(Z_0))| \leqslant \frac{1}{8}\varepsilon^\gamma. \tag{B.30}$$

Note that on the finite dimensional subspace H^+ the linearised semigroup $(T(t)\big|_{H^+})$ is defined for all $t \in \mathbb{R}$. For any $t, s \in \mathbb{R}$

$$\begin{aligned} Y(t; Z_0) &= T(t-s)Y(s; Z_0) + \int_s^t T(t-r)h(X(r; Z_0), Y(r; Z_0))\,\mathrm{d}r \\ &= T(t-s)Y(s; Z_0) + \int_s^t T(t-r)h(\Phi_u(Y(r; Z_0)), Y(r; Z_0))\,\mathrm{d}r \\ &\quad + \int_s^t T(t-r)\left(h(X(r; Z_0); Y(r; Z_0)) - h\left(\Phi_u(Y(r; Z_0)), Y(r; Z_0)\right)\right)\mathrm{d}r. \end{aligned}$$

Note that $\Phi_u(v) = 0$ and $T(t)\big|_{H^+}$ is a contraction for $t \leqslant 0$ with operator norm

$$\left|T(t)\big|_{H^+}\right|_{\mathcal{L}(H^+)} = e^{\omega_0 t}.$$

Hence we obtain for $t_0 \leqslant t \leqslant s$ with the help of (B.28)

$$\begin{aligned}|Y(t;Z_0)| &\leqslant e^{\omega_0(t-s)}|Y(s;Z_0)| + \int_t^s e^{\omega_0(t-r)}\ L_\sigma(1+L_{\Phi_u})|Y(r,Z_0)|\ \mathrm{d}r \\ &\quad + \int_t^s e^{\omega_0(t-r)}\ L_\sigma\ |\Phi_u(Y(r;Z_0)) - X(r;Z_0)|\ \mathrm{d}r \\ &\leqslant e^{\omega_0(t-s)}|Y(s;Z_0)| + L_\sigma(1+L_{\Phi_u}) \int_t^s e^{\omega_0(t-r)}\ |Y(r,Z_0)|\ \mathrm{d}r \\ &\quad + \frac{L_\sigma \varepsilon^\gamma}{8} \int_t^s e^{\frac{3\omega_0}{2}(t-r)}\ \mathrm{d}r.\end{aligned}$$

Hence Gronwall's Lemma implies for $\Psi(t) := e^{-\omega_0 t}|Y(t;Z_0)|$ the estimate

$$|Y(t;Z_0)| \leqslant e^{(\omega_0 - L_\sigma(1+L_{\Phi_u}))(t-s)}|Y(s;Z_0)| + \frac{L_\sigma}{8}\frac{2}{3\omega_0}\varepsilon^\gamma \qquad \text{for } t_0 \leqslant t \leqslant s \tag{B.31}$$

Since $0 < \sigma \leqslant \delta_l$ and $L_\sigma(1+L_{\Phi_u}) < \frac{\omega_0}{2}$ we obtain the desired result for $C = \frac{L_\sigma}{8}\frac{2}{3\omega_0}$.

2. Claim: *For $v \in \mathcal{E}^\lambda$ there is $\delta_u = \delta_u(v) > 0$ such that for $0 < \sigma \leqslant \delta_u$ there is $\varepsilon_u = \varepsilon_u(\sigma) > 0$ ensuring that for $0 < \varepsilon \leqslant \varepsilon_u$ we have $\mathcal{D}^\pm(\varepsilon^\gamma) \cap B_\sigma(v) \neq \emptyset$, and for $x \in \mathcal{D}^\pm(\varepsilon^\gamma) \cap B_\sigma(v)$ and $s \geqslant \kappa_u \gamma |\ln \varepsilon|$ that*

$$u(s;x) \notin \mathcal{D}^\pm(\varepsilon^\gamma) \cap B_\sigma(v).$$

Denote by $\Phi_s : H^+ \to H^-$ the function that generates the local stable manifold of v as a graph. More precisely we assume the existence of $\delta_{\Phi_s} > 0$ such that for all $0 < \sigma \leqslant \delta_{\Phi_s}$

$$\mathcal{W}^s(v) = \{h + \Phi_s(h) \mid h \in B_\sigma(v) \cap H^-\}$$

according to an analogue construction as for Φ_u. In the same way as for Φ_u we denote by L_{Φ_s} the Lipschitz constant of Φ_s on $B_\sigma(v)$. Since H^+ is finite dimensional there is $M > 0$ such that $\|y\| \leqslant M|y|$ for all $y \in H^+$. Since the Lipschitz constant L_σ of G on a ball $B_\sigma(v) \cap H^+$ becomes arbitrarily small for $\sigma \to 0$ we may choose $\delta_4 > 0$ such that for $0 < \sigma \leqslant \delta_4$

$$L_\sigma < \frac{4}{M} \wedge \frac{3\omega_0}{M} \wedge 1.$$

Fix $\delta_u = \delta_{\Phi^s} \wedge \delta_3 \wedge \delta_4$ and $0 < \sigma \leqslant \delta_u$. We set $\varepsilon_u := \varepsilon_1 \wedge \exp(-\frac{\ln(256 M\sigma^2)}{3\gamma})$ and $0 < \varepsilon \leqslant \varepsilon_u$. By construction $\mathcal{D}^\pm(\varepsilon^\gamma)$ is invariant for all $\varepsilon > 0$ under the solution flow and such that $(X(t;z), Y(t;z)) \in \mathcal{D}^\pm(\varepsilon^\gamma)$ for all $t \geqslant 0$ and $z \in \mathcal{D}^\pm(\varepsilon^\gamma)$. Hence for all $t \geqslant 0$ and

$z \in \mathcal{D}^{\pm}(\varepsilon^{\gamma})$ by construction $(X(t;z), \Phi_s(X(t;z))) \in \mathcal{S}$. Thus for $z \in \mathcal{D}^{\pm}(\varepsilon^{\gamma}) \cap B_{\sigma}(v)$

$$\|Y(t;z) - \Phi_s(X(t;z))\| \;\geqslant\; \operatorname{dist}((X(t;z), Y(t;z); \mathcal{S}) \geqslant \varepsilon^{\gamma}. \tag{B.32}$$

By definition the functions $Y(\cdot\,;x)$ and $\Phi_s(X(\cdot\,;x))$, $x \in H$, take values in H^+, which is finite dimensional, such that all norms are equivalent there. Hence there is a constant $M > 0$ such that $\|y\| \leqslant M|y|$ for all $y \in H^+$ and

$$\|Y(t;z) - \Phi_s(X(t;z))\| \;\leqslant\; M|Y(t;z) - \Phi_s(X(t;z))|. \tag{B.33}$$

Using $L_{\sigma} \leqslant 1$ and estimate (B.30) in the proof of Claim 1 we obtain for $0 < \varepsilon \leqslant \varepsilon_0$ and $t \geqslant t_0(v, \sigma, \varepsilon)$, $x \in \mathcal{D}^{\pm}(\varepsilon^{\gamma}) \cap B_{\sigma}(v)$ and $z = Z_0 = Z_0(t_0, x) \in \mathcal{D}^{\pm}(\varepsilon^{\gamma}) \cap B_{\sigma}(v)$

$$\begin{aligned} |Y(t;Z_0) - \Phi_s(X(t;Z_0))| \;&\leqslant\; |Y(t;Z_0)| + |\Phi_s(X(t;Z_0)| \leqslant |Y(t;Z_0)| + L_{\sigma}|X(t;Z_0)| \\ &\leqslant |Y(t;Z_0)| + L_{\sigma}|X(t;Z_0) - \Phi_u(Y(t;Z_0))| + |\Phi_u(Y(t;Z_0))| \\ &\leqslant\; (1 + L_{\sigma})|Y(t;Z_0)| + \frac{L_{\sigma}}{8}\varepsilon^{\gamma}. \end{aligned} \tag{B.34}$$

Since in addition $L_{\sigma} < \frac{4}{M}$ by the choice of σ, we can combine (B.34), (B.33) and (B.32) and obtain

$$\frac{\varepsilon^{\gamma}}{2M} \leqslant \frac{1}{1 + L_{\sigma}}\left(\frac{1}{M} - \frac{L_{\sigma}}{8}\right)\varepsilon^{\gamma} \leqslant |Y(t;Z_0)|. \tag{B.35}$$

Applying inequality (B.31) from Claim 1 to the right-hand side of (B.35) we obtain for $s \geqslant t \geqslant t_0$ and Z_0 that

$$\varepsilon^{\gamma}\left(\frac{1}{2M} - \frac{L_{\sigma}}{12\omega_0}\right) \;\leqslant\; e^{\frac{\omega_0}{2}(t-s)}|Y(s;Z_0)|.$$

Since also $L_{\sigma} < \frac{3\omega_0}{M}$ we may estimate for $s \geqslant t \geqslant t_0$ and Z_0

$$\frac{\varepsilon^{\gamma}}{4M} \leqslant e^{\frac{\omega_0}{2}(t-s)}|Y(s;Z_0)|. \tag{B.36}$$

Thus for $0 < \varepsilon \leqslant \varepsilon_u$, $x \in \mathcal{D}^{\pm}(\varepsilon^{\gamma}) \cap B_{\sigma}(v)$ and $s \geqslant t \geqslant t_0$

$$|u(s + t_0; x)| = |u(s;Z_0)| \;\geqslant\; |Y(s;Z_0)| \;\geqslant\; e^{\frac{\omega_0}{2}(s-t)}\frac{\varepsilon^{\gamma}}{4M}$$

Hence

$$s - t \geqslant \frac{2}{\omega_0}(\ln(\sigma 4M) - \gamma\ln(\varepsilon)) \tag{B.37}$$

implies

$$|u(s + t_0; x)| \geqslant \sigma. \tag{B.38}$$

Adding $t \geqslant t_0$ to (B.37) we may eliminate t and obtain by (B.29) that

$$\begin{aligned} s \geqslant \frac{2}{\omega_0}(\ln(\sigma 4M) - \gamma \ln(\varepsilon)) + 2\, t_0 \\ = \frac{2}{\omega_0}(\ln(\sigma 4M) - \gamma \ln(\varepsilon)) + \frac{4}{\omega_0}(-\gamma \ln(\varepsilon) + \ln 8 - \ln \sigma) \\ = \frac{6\gamma}{\omega_0}|\ln(\varepsilon)| + \frac{2}{\omega_0}\ln(256 M\sigma^2) \end{aligned} \tag{B.39}$$

implies the desired result

$$\|u(s;x)\| \;\geqslant\; |u(s;x)| \geqslant \sigma.$$

Since $\varepsilon_u \leqslant \exp(-\frac{\ln(256M\sigma^2)}{3\gamma})$ inequality (B.39) simpifies for $0 < \varepsilon \leqslant \varepsilon_u$ to

$$s \;\geqslant\; \frac{12\gamma}{\omega_0}\gamma|\ln(\varepsilon)|.$$

Set $\kappa_u(v) = \frac{12\gamma}{\omega_0}$. This finishes the proof. □

B.2.5. Uniform Exit from Small Tubes around Heteroclinic Orbits

In this Subsection we prove Claim II.C.2 in the proof of Proposition B.2, stating that the exit times from tubes around heteroclinic orbits can be estimated by a constant.

Lemma B.7 (Uniform exit time from tubes around connecting orbits). *For $v, w \in \mathcal{E}^\lambda$ with $v \to w$ there is $\delta_h = \delta_h(v,w) > 0$ such that for all $0 < \sigma < \delta_h$ there exists $\eta_2 = \eta_2(\sigma, v, w) > 0$ such that for $0 < \eta \leqslant \eta_2$ we obtain $\tau_4 = \tau_4(\eta, \sigma, v, w) > 0$ ensuring for all $y \in U^-_{\eta,\sigma}(v,w)$ that*

$$u(\tau_4; y) \in B_{\sigma/2}(w).$$

Proof. Fix a point $z \in C(v,w)$ such that $\|v - z\| = \sigma$. Since $C(v,w) = \{u(t;z), t \in \mathbb{R}\}$ and $\mathfrak{E}(v) > \mathfrak{E}(w)$ the solution $u(t;z) \to v$ for $t \to -\infty$. Hence

$$t^- := \inf\{t \in \mathbb{R} \mid \|v - u(t;z)\| = \sigma\} > -\infty.$$

Additionally $t \mapsto u(t;z)$ is continuous such that $\|u(t^-;z) - v\| = \sigma$. Denote by $z^- := u(t^-;z)$. It is uniquely determined and depends only on σ. Since $C(v,w) = \{u(t;z^-), t \in \mathbb{R}\}$ and $\mathfrak{E}(v) > \mathfrak{E}(w)$ the solution $u(t;z^-) \to w$ for $t \to +\infty$. Hence there exists a time $\tau_4 = \tau_4(\sigma, v, w) > 0$ such that for $t \geqslant \tau_4$

$$\|u(t;z^-) - w\| \leqslant \frac{\sigma}{4}$$

and thus for all $z \in C_\sigma(v,w) := C(v,w) \cap B_\sigma^c(v) \cap B_\sigma^c(w)$

$$\|u(t;z) - w\| \leqslant \|u(t;z^-) - w\| \leqslant \frac{\sigma}{4}.$$

The map $x \mapsto u(\tau_4;x)$ is uniformly continuous on bounded sets. This implies that there is $\eta_2 = \eta_2(\sigma,v,w) > 0$ such that for $0 < \eta \leqslant \eta_2$ the inequality $\|y_1 - y_2\| \leqslant \eta$ implies

$$\sup_{s\in[0,\tau_4]} \|u(s,y_1) - u(s,y_2)\| \leqslant \frac{\sigma}{4}.$$

We can coose $\eta_2 \leqslant \sigma$. Then for $0 < \eta \leqslant \eta_2$ and $y \in U_{\eta,\sigma}^-(v,w)$ it follows that $\operatorname{dist}(y, C_\sigma(v,w)) \leqslant \eta$ hence there is $z \in C_\sigma(v,w)$ with $\|y - z\| \leqslant \eta$. We conclude that for all $y \in U_{\eta,\sigma}^-(v,w)$

$$\|u(\tau_4;y) - w\| \leqslant \|u(\tau_4;y) - u(\tau_4;z)\| + \|u(\tau_4;z) - w\| \leqslant \frac{\sigma}{2}.$$

□

B.3. An Integrability Result

The structure of the reaction term f in the Chafee-Infante equation guarantees that the solution $u(\cdot;x)$, started in some arbitrary $x \in H$, enters any ball with respect to the $|\cdot|_\infty$-norm around the origin in finite time T_{rec} independent of x. This remarkable fact is exploited in the estimates of Chapter 5, in the form of a bound on $\int_0^t |u(t;x)|_\infty^2 \, ds$ by a value independent of x. For this we need an upper bound $b(t)$ on $|u(t)|_\infty$ which is integrable in time t over finite time intervals. In Eden et al. [1994], p. 99, Lemma 5.6, the authors obtained the following result on the speed of convergence towards the absorbing set $B_{\sqrt{2}}(0)$ in $L^\infty(0,1)$.

Proposition B.8. *For an absorbing set $B \subset H$ of the Chafee-Infante equation which is bounded in $L^\infty(0,1)$, $x \in B$ and $t > \frac{1}{\lambda}$*

$$|u(t;x)|_\infty \leqslant \sqrt{2}.$$

Since for any $x \in H$ it follows that $\sup_{t \geqslant 0} |u(t;x)|_\infty < \infty$, we can get rid of the condition that B should be bounded in $L^\infty(0,1)$. In addition, we can choose $B = H$. With the arguments used to prove this statement the authors actually obtain the following result.

Corollary B.9. *For any $x \in H$ and $t > 0$*

$$|u(t;x)|_\infty \leqslant \max\{\sqrt{2}, \frac{1}{(\lambda t)^{\frac{1}{2}}}\}.$$

We only need to improve this result slightly around $t = 0$ in order to get the integrability of $|u(t;x)|_\infty^2$ in t uniformly in x.

Lemma B.10. *There is a polynomial p such that for any initial value $x \in H$ and any $t \geqslant 0$*

$$\int_0^t |u(s;x)|_\infty^2 \, ds \leqslant p(t).$$

Proof. The proof is structured as follows.

1. In the first part of the proof we use a trick to get rid of the diffusion term in the Chafee-Infante equation. We shall show that there is a polynomial p such that for $t \geqslant 0$ and $x \in H$

$$\int_0^t (t-r)|u(x;r)|_{7/2}^{7/2} dr \leqslant p(t).$$

For $x \neq 0$ the norm $|u(t;x)|_q \neq 0$ for all $q \geqslant 1$ and $t \geqslant 0$. The main idea consists in choosing $\eta > 0$ and multiplying the deterministic equation (2.6) by $u(|u|^2 + \eta)^{-1/4}$, omitting the dependence on $x \in H$ and $\zeta \in [0,1]$, to get for the term on the left hand side

$$\int_0^1 \left(\frac{\partial u}{\partial t}\frac{u}{(u^2+\eta)^{1/4}}\right)\,\mathrm{d}\zeta \;=\; \int_0^1 \frac{\partial}{\partial t}\frac{2}{3}(u^2\;+\;\eta)^{3/4}\;\mathrm{d}\zeta \;=\; \frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u^2\;+\;\eta)^{3/4}\;\mathrm{d}\zeta.$$

The contribution of the diffusion term can be seen to be negative, using integration by parts

$$\int_0^1 (\Delta u)u(u^2+\eta)^{-1/4}\mathrm{d}\zeta \;=\; -\int_0^1 (\nabla u)\nabla\left(u(u^2+\eta)^{-1/4}\right)\,\mathrm{d}\zeta$$

$$= -\int_0^1 (\nabla u)^2\left((u^2+\eta)^{-1/4} - u\frac{1}{4}(u^2+\eta)^{-5/4}2u\right)\,\mathrm{d}\zeta$$

$$= -\int_0^1 (\nabla u)^2\left((u^2+\eta)^{-1/4} - \frac{u^2}{2}(u^2+\eta)^{-5/4}\right)\,\mathrm{d}\zeta =$$

$$-\int_0^1 \underbrace{\frac{(\nabla u)^2}{(u^2+\eta)^{1/4}}}_{\geqslant 0}\underbrace{\left(1-\frac{1}{2}\frac{u^2}{u^2+\eta}\right)}_{\geqslant 1/2}\,\mathrm{d}\zeta.$$

And for the reaction term we can write

$$\int_0^1 (-u^3+u)u(u^2+\eta)^{-1/4}\,\mathrm{d}\zeta \;=\; \int_0^1 \frac{-u^4+u^2}{(u^2+\eta)^{1/4}}\,\mathrm{d}\zeta.$$

Neglecting the negative diffusion term, and integrating in time between s and t for any $\eta > 0$ we obtain the inequality

$$\frac{2}{3}\int_0^1 \left(u(t)^2+\eta\right)^{3/4}\mathrm{d}\zeta - \frac{2}{3}\int_0^1 \left(u(s)^2+\eta\right)^{3/4}\mathrm{d}\zeta \;\leqslant\; \lambda\int_s^t\int_0^1 \frac{-u^4(r)+u^2(r)}{(u(r)^2+\eta)^{1/4}}\,\mathrm{d}\zeta\mathrm{d}r,$$

and hence by monotone convergence for $\eta \to 0$

$$\frac{2}{3}|u(t)|_{3/2}^{3/2} - \frac{2}{3}|u(s)|_{3/2}^{3/2} \;\leqslant\; \lambda\int_s^t -|u(r)|_{7/2}^{7/2} + |u(r)|_{3/2}^{3/2}\,\mathrm{d}r.$$

Dividing by λ we hence get for any $s \leqslant t$

$$\int_s^t |u(r)|_{7/2}^{7/2}\,\mathrm{d}r \;\leqslant\; \int_s^t |u(r)|_{3/2}^{3/2}\,\mathrm{d}r - \frac{2}{3\lambda}|u(t)|_{3/2}^{3/2} + \frac{2}{3\lambda}|u(s)|_{3/2}^{3/2}.$$

Integrating in s from 0 to t and appealing to the preceding Corollary B.9 gives an upper bound independent of an initial value, according to

$$\begin{aligned}\int_0^t (t-r)|u(r)|_{7/2}^{7/2}\,\mathrm{d}r &\leqslant \int_0^t (t-r)|u(r)|_\infty^{3/2}\,\mathrm{d}r + \frac{2}{3\lambda}\int_0^t |u(r)|_\infty^{3/2}\,\mathrm{d}r \\ &\leqslant \int_0^t (t-r)\max\left\{\frac{1}{(\lambda r)^{3/4}}, 2^{3/4}\right\}\mathrm{d}r + \frac{2}{3\lambda}\int_0^t \max\left\{\frac{1}{(\lambda r)^{3/4}}, 2^{3/4}\right\}\mathrm{d}r.\end{aligned}$$

We can obtain an explicit expression for the bound for $t \geqslant 1/(2\lambda)$ by computing

$$\begin{aligned}\int_0^t (t-r)\max\left\{\frac{1}{(\lambda r)^{3/4}}, 2^{3/4}\right\}\mathrm{d}r &= \int_0^{\frac{1}{2\lambda}} (t-r)\frac{1}{(\lambda r)^{3/4}}\,\mathrm{d}r + \int_{\frac{1}{2\lambda}}^t (t-r)2^{3/4}\,\mathrm{d}r \\ &= t\int_0^{\frac{1}{2\lambda}} r^{-3/4}\,\frac{\mathrm{d}r}{\lambda^{3/4}} - \int_0^{\frac{1}{2\lambda}} r^{1/4}\,\frac{\mathrm{d}r}{\lambda^{3/4}} + 2^{3/4}t(t-\frac{1}{2\lambda}) - \frac{2^{3/4}\,t^2}{2} + \frac{2^{3/4}}{(2\lambda)^2 2} \\ &= \frac{4\lambda^{-3/4}}{(2\lambda)^{1/4}}\,t - \frac{4\lambda^{-3/4}}{5(2\lambda)^{5/4}} + 2^{3/4}\,t^2 - \frac{2^{3/4}}{2\lambda}\,t - \frac{2^{3/4}\,t^2}{2} + \frac{2^{3/4}}{(2\lambda)^2 2} \\ &= \frac{t^2}{2^{1/4}} + \left(\frac{2^2}{2^{1/4}} - \frac{1}{2^{1/4}}\right)\frac{t}{\lambda} + \left(\frac{1}{2^{9/4}} - \frac{2^{3/4}}{5}\right)\frac{1}{\lambda^2} = \frac{t^2}{2^{1/4}} + \frac{3}{2^{1/4}}\frac{t}{\lambda} - \frac{3}{2^{9/4}5}\frac{1}{\lambda^2}\end{aligned}$$

and

$$\begin{aligned}\frac{2}{3\lambda}\int_0^t \max\left\{\frac{1}{(\lambda r)^{3/4}}, 2^{3/4}\right\}\mathrm{d}r &= \frac{2}{3\lambda}\frac{1}{\lambda^{3/4}}\int_0^{1/(2\lambda)} r^{-3/4}\,\mathrm{d}r + \frac{2}{3\lambda}2^{3/4}\left(t - \frac{1}{2\lambda}\right) \\ &= \frac{2}{3\lambda^{7/4}}\frac{4}{2^{1/4}\lambda^{1/4}} + \frac{2^{7/4}}{3\lambda}t - \frac{2^{3/4}}{3\lambda^2} = \frac{2^{7/4}}{3}\frac{t}{\lambda} + \frac{4}{3}\frac{1}{\lambda^2}.\end{aligned}$$

Summing up the two preceding estimates yields

$$\begin{aligned}\int_0^t (t-r)|u(r)|_{7/2}^{7/2}\,\mathrm{d}r &\leqslant 2^{-1/4}t^2 + \left(\frac{3}{2^{1/4}} + \frac{2^{7/4}}{3}\right)\frac{t}{\lambda} + \left(\frac{4}{3} - \frac{3}{2^{9/4}5}\right)\frac{1}{\lambda^2} \\ &= 2^{-1/4}t^2 + \frac{13}{3\,2^{1/4}}\frac{t}{\lambda} + \frac{2^{15/4}5 - 9}{3\,5\,2^{9/4}}\frac{1}{\lambda^2} \leqslant 2^{-1/4}\left(t^2 + \frac{13}{2}\frac{t}{\lambda} + \frac{90}{3\,5\,4}\frac{1}{\lambda^2}\right) \\ &\leqslant t^2 + 7\frac{t}{\lambda} + 2\frac{1}{\lambda^2}. \qquad \text{(B.40)}\end{aligned}$$

2. In the second part of the proof we shall show that there is a polynomial q such that for $t \geqslant \frac{1}{2\lambda}, x \in H$, we have

$$\int_0^t ||u(x;s)||^2 \mathrm{d}s \leqslant q(t).$$

Again we omit the dependence of u on x. First note that by Hölder's and Jensen's inequalities for $p = 7/4$ and $p' = 7/3$ and any $t \geqslant 0$

$$\frac{t^{4/7}}{2}\left(\int_0^{t/2} |u(r)|_2^2 \ \mathrm{d}r\right)^{4/7} \leqslant \frac{t}{2}\int_0^{t/2} |u(r)|_2^{7/2} \ \mathrm{d}r \leqslant \frac{t}{2}\int_0^{t/2} |u(r)|_{7/2}^{7/2} \ \mathrm{d}r$$

$$\leqslant \int_0^t (t-r)|u(r)|_{7/2}^{7/2} \ \mathrm{d}r,$$

whence

$$\int_0^{t/2} |u(r)|_2^2 \ \mathrm{d}r \leqslant \frac{4}{t}\left(\int_0^t (t-r)|u(r)|_{7/2}^{7/2} \ \mathrm{d}r\right)^{7/4}. \tag{B.41}$$

Now we return to the representation of $|u|_2^2$ by the Chafee-Infante equation which gives for $s, t \geqslant 0, s \leqslant t$

$$\frac{1}{2}|u(t)|_2^2 - \frac{1}{2}|u(s)|_2^2 + \int_s^t |\nabla u(r,\zeta)|^2 \ \mathrm{d}s + \int_s^t \lambda[|u(r)|_4^4 - |u(r)|_2^2] \ \mathrm{d}r = 0.$$

This immediately implies

$$\int_s^t ||u(r)||^2 \ \mathrm{d}r \leqslant \frac{1}{2}|u(s)|^2 + \lambda\int_s^t |u(r)|^2 \ \mathrm{d}r.$$

Again we integrate in s from 0 to $t/2$ and with the help of (B.40) and (B.41) we arrive at the estimate

$$\frac{t}{4}\int_0^{t/4} ||u(s)||^2 \ \mathrm{d}s \leqslant \int_0^{t/2}\left(\frac{t}{2} - s\right)(||u(s)||^2 \ \mathrm{d}s$$

$$\leqslant \frac{1}{2}\int_0^{t/2} |u(s)|^2 \ \mathrm{d}s + \lambda\int_0^{t/2}\left(\frac{t}{2} - s\right)|u(s)|^2 \ \mathrm{d}s \ \leqslant \ \left(\frac{1}{2} + \frac{\lambda}{2}\,t\right)\int_0^{t/2} |u(s)|^2 \ \mathrm{d}s$$

$$\leqslant 2\left(\frac{1+\lambda\, t}{t}\right)\left(t^2 + \frac{7}{\lambda}\,t + \frac{5}{4\lambda^2}\right)^{7/4}.$$

Dividing both sides of this inequality by t and observing that $t \geqslant \frac{1}{2\lambda}$ we get

$$\int_0^{t/4} ||u(s)||^2 \, \mathrm{d}s \;\leqslant\; 8 \left(\frac{1+\lambda\, t}{t^2}\right) \left(t^2 + \frac{7}{\lambda}\, t + \frac{5}{4\lambda^2}\right)^{7/4}$$
$$\leqslant\; 2\lambda^2 \,(1+\lambda\, t) \left(t^2 + \frac{7}{\lambda}\, t + \frac{5}{4\lambda^2}\right)^{7/4}.$$

Now the claim is obvious for large $t \geqslant 1$, and follows for small t by monotonicity of the integral.

3. To derive the desired estimate from the result of the second part of the proof it remains to refer to the Sobolev embedding implying that $|u|_\infty \leqslant ||u||$ for $u \in H$. □

Corollary B.11. *There is a constant $\bar{K} > 0$ such that for all $t \geqslant 0$ and any initial value $x \in H$*

$$\int_0^t |u(s;x)|_\infty^2 \, \mathrm{d}s \leqslant \bar{K} + 2\, t.$$

Proof. We use Proposition B.8 and Lemma B.10 for the estimate for small t. □

Corollary B.12. *For all $n \in \mathbb{N}$ there is a constant $\bar{K}_n > 0$ such that for all $t \geqslant 0$ and any initial value $x \in H$*

$$\int_0^t |u(s;x)|_\infty^n \, \mathrm{d}s \leqslant \bar{K}_n + 2^{n/2}\, t.$$

Proof. Due to Eden et al. [1994], all we have to show is that $\int_0^t |u(s;x)|_\infty^n \, \mathrm{d}s < \infty$ for $n \in \mathbb{N}$. We proceed by recursion. The case case $n = 1$ is a simple consequence of Corollary B.9 and the integrability of the reciprocal square root. In Lemma B.10 we treat the case $n = 2$. Assume that there is a polynomial p_n such that for all $x \in H$ and $t \geqslant 0$

$$\int_0^t |u(s;x)|_\infty^n \, \mathrm{d}s \leqslant p_n(t).$$

Following the usual procedure we multiply the equation

$$\frac{\partial u}{\partial t} = \Delta u - \lambda(u^3 - u)$$

by $u|u|^{n-2}$ and use the identities

$$\begin{aligned}\langle\frac{\partial u}{\partial t}, u|u|^{n-2}\rangle &= \frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}t}|u|_n^n,\\ \langle\Delta u, u|u|^{n-2}\rangle &= -(n-1)\langle\nabla u)^2, |u|^{n-2}\rangle,\\ \lambda\langle u^3-u, u|u|^{n-2}\rangle &= \lambda(|u|_{n+2}^{n+2}-|u|_n^n).\end{aligned}$$

Note that in the preceding and following equations $|\cdot|$ is used to denote the absolute value of real numbers as well as the L^2 norm in $L^2(0,1)$. We neglect the negative diffusion term, for $s,t \geqslant 0, s \leq t$ integrate from s to t and obtain

$$\frac{1}{n}|u(t)|_n^n - \frac{1}{n}|u(s)|_n^n + \lambda\int_s^t \left(|u(r)|_{n+2}^{n+2} - |u(r)|_n^n\right)\,\mathrm{d}r \leqslant 0.$$

Integrating in s from 0 to t and using $|\cdot| \leqslant |\cdot|_\infty$ we obtain

$$\begin{aligned}\frac{t}{2}\int_0^{t/2} |u(r)|_{n+2}^{n+2}\,\mathrm{d}r \leqslant \int_0^t (t-r)|u(r)|_{n+2}^{n+2}\,\mathrm{d}r&\\ \leqslant \int_0^t (t-r)|u(r)|_\infty^n\,\mathrm{d}r + \frac{1}{\lambda n}\int_0^t |u(r)|_\infty^n\,\mathrm{d}r&\\ \leqslant tp_n(t) + \frac{1}{\lambda n}p_n(t) =: p_{n+2}(t/2).&\end{aligned}$$

For $t \geqslant 2$ this leads to

$$\int_0^t |u(r)|_{n+2}^{n+2}\,\mathrm{d}r \leqslant p_{n+2}(t).$$

For $t \leqslant 2$ the inequality follows again from the monotonicity of the integral, formally by adding a constant to p_{n+2}. This completes the recursion from n to $n+2$. □

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 28.01.2008 Michael Anton Högele