On the Rigid Rotation Concept in n-Dimensional Spaces*

Daniele Mortari†

Department of Aerospace Engineering,
Texas A&M University, College Station, TX 77843-3141

Dedicated to the memory of Carlo Arduini

Abstract

A general mathematical formulation of the $n \times n$ proper Orthogonal matrix, that corresponds to a rigid rotation in n-dimensional real Euclidean space, is given here. It is shown that a rigid rotation depends on an angle (principal angle) and on a set of $(n-2)$ principal axes. The latter, however, can be more conveniently replaced by only 2 Orthogonal directions that identify the plane of rotation. The inverse problem, that is, how to compute these *principal rotation parameters* from the rotation matrix, is also treated. In this paper, the Euler Theorem is extended to rotations in n-dimensional spaces by a constructive proof that establishes the relationship between orientation of the displaced Orthogonal axes in n dimensions and a minimum sequence of rigid rotations. This fundamental relationship, which introduces a new decomposition for proper Orthogonal matrices (those identifying an orientation), can be expressed either by *a product or a sum of the same rotation matrices*. A similar decomposition in terms of the Skew-Symmetric matrices is also given. The extension of the rigid rotation formulation to n-dimensional complex Euclidean spaces, is also provided. Finally, we introduce the *Ortho-Skew* real matrices, which are simultaneously proper Orthogonal and Skew-Symmetric and which exist in even dimensional spaces only, and the *Ortho-Skew-Hermitian* complex matrices which are Orthogonal and Skew-Hermitian. The Ortho-Skew and the Ortho-Skew-Hermitian matrices represent the extension of the scalar imaginary to the matrix field.

---

*Work previously presented in part at the Second International Conference on Non Linear Problems in Aeronautics and Astronautics, "ICNPAA-2000", Daytona Beach, FL, May 10-12, 2000. Work supported by the Italian Space Agency.

†Visiting Associate Professor, Department of Aerospace Engineering, Texas A&M University, College Station, TX 77843-3141, mortari@aero.tamu.edu, and Assistant Professor, Centro di Ricerca Progetto San Marco, Università degli Studi "La Sapienza" di Roma, Via Salaria, 851 - 00138 Roma (Italy), mortari@psm.uniroma1.it, Member AAS.

Introduction

The position of a point in the n-dimensional (n-D) space is defined by an n-long vector whose elements are the coordinates of the point with respect to a reference system of coordinates. Rigid translation is, therefore, simply described by means of a difference between two positions, because rigid translation is a phenomenon which is linear with the position.

The orientation in the n-D space is defined in a non-singular fashion by an $n \times n$ proper Orthogonal matrix C (orientation matrix) containing, as row vectors, the unit-vectors $\hat{\mathbf{c}}_i$ ($i = 1, \ldots, n$) identifying the directions of the reference frame axes. Rigid rotation, however, cannot be described by means of a difference between two orientation matrices, because rigid rotation is not a phenomenon which is linear with the orientation. Orientation, however, can also be described by Skew-Symmetric matrix Q [1] and the Symmetric Cayley Transforms [2, 3] that relate C to Q.

Prior to the detailed developments, let us introduce a heuristic discussion to provide some qualitative insight and motivation for the mathematical discussions that follow. Imagine a 1-D being, living in a 1-D universe, clearly, the rotation concept has no meaning. This being would probably think "Things can go forward or backward only! What does rotation mean?". Therefore, in a hypothetical discussion with a 2-D being, living in a 2-D universe, the latter would probably say "You cannot understand the rotation concept because you need at least two dimensions ... your universe is too small to see that rotation can be performed about a point!". "Rotation about a point?" may ask a 3-D human being living in our 3-D universe, "Rigid rotations are performed about an axis! You live in a too small universe to understand this and, therefore, you can't see that your point of rotation is only the intersection of your flat universe with an axis perpendicular to it, namely the rotation axis!".

Thus, moving from an n-D to an $(n + 1)$-D space, not only are new kind of motions introduced, but also the same physical phenomenon may appear as being ruled by different laws. This short story has the purpose to warn us that, if we proceed with the rigid rotation concept in a four (or higher) dimensional spaces, most likely, we should be prepared for the truth, *the rigid rotational motion is not accomplished by rotating about a single axis* in higher dimensional spaces. As we establish in this paper, the rigid rotation being performed about a single axis is limited only to the 3-D space. In fact, when $n = 2$ the rotation figure is a point, a 0-D figure; when $n = 3$ the figure becomes a 1-D figure (an axis). Therefore, this qualitative thinking suggests that in the n-D space, if the hypothesis of a linear rule holds (*linear hypothesis*), then the figure of rotation would be $(n - 2)$-dimensional. This would mean that in the 4-D space, for instance, the rigid rotation is such that a point would rotate by describing a cone about a plane ... an infinite plane! *What does this mean?* Is it possible that a point in a four dimensional space can move such that it describes a cone about any axis that lies on a plane?

This paper, not only demonstrates that the answer to this question is "yes", but also provides the general mathematical operator that accomplishes such rigid rotations in any dimensional space. In the course of these developments, we introduce the mathematical relationship between orientation and rigid rotation concepts, two concepts that coincide in 2-D and 3-D spaces, only. The general relationship, is nothing but the complete extension to the n-D space of Euler's Theorem. Reference [4] contains the original Eulerian formulation as the $n = 3$ special case. The generalization is mathematically established by introducing three new matrix decompositions: two for proper Orthogonal matrices and one for Skew-Symmetric matrices. Finally, this paper investigates the common edge between Orthogonal and Skew-Symmetric matrices, and further introduces the set of the *Ortho-Skew* and the *Ortho-Skew-Hermitian* matrices, $\Im_e$ and $\Im_o$, which represent the analogous extension to the matrix field of the imaginary unit $i = \sqrt{-1}$. It is shown, for example, that they satisfy the Euler and the Moivre formulae.

Several important studies have been carried out to analyze dynamics in spaces of dimension higher than three. These developments are motivated by practical needs of nuclear and sub-nuclear physics, as well as cosmology. However, all these dynamical theories, such as the restricted relativity (4-D space, which uses the Lorentz transformations to rotate [5]), the projective relativity (5-D space, [6]), and the conformal relativity (6-D space, [7]), all include a corresponding mathematical tool to describe the time-varying orientation in n-D.

From a mathematical point of view, a fundamental aspect of the orientation is described by the Cayley Trasforms, and several new insights that came out later, as embedded in the Group Theories [5, 8, 9, 10] especially the Orthogonal Special Group $SO(n)$ and the Lie Group. Other physics studies (as in [11, 12, 13]) have contributed closed form expressions for orientation useful in modern relativity theories. Finally, even in recent research, as for instance in [14, 15], attention is still focused on *orientation* and not on *rigid rotation*. Due to the Euler Theorem, which for the $n = 3$ case establishes the equivalence of orientation and rotation, a natural confusion has resulted, and many individuals do not appreciate the distinct nature of *Orientation* (*General Rotation* in the Group Theory) and *Rotation* concepts. For these reasons, the terminology adopted in this study does not follow that of the Group Theory. Moreover, the references to *Grassmann Algebra* is here minimized in order to be directly understood even by readers who do not have any knowledge of *Exterior Algebra* and *Differential Forms* [9, 10].

Since the rigid rotation concept is strictly related to the concept of the n-D vector cross-product, as presented and used in [17], the first-and-next section re-introduces its definition and emphasizes some of its properties that will be extensively used in this paper.

n-Dimensional Vector Cross-Product

Reference [16] has presented a new technique to estimate optimally the spacecraft attitude. This led the development of a new nonsingular and general method [17] to compute the eigenvectors of any matrix. This is accomplished by extending the 3×3

Skew-Symmetric matrix $\tilde{\mathbf{v}}$ [‡]

$$\tilde{\mathbf{v}} = \begin{bmatrix} 0 & -\mathbf{v}(3) & +\mathbf{v}(2) \\ +\mathbf{v}(3) & 0 & -\mathbf{v}(1) \\ -\mathbf{v}(2) & +\mathbf{v}(1) & 0 \end{bmatrix} \tag{1}$$

which performs the vector cross product in 3-D space, to any n-D space. The $n \times n$ Skew-Symmetric matrix $\tilde{V}$, which generalizes $\tilde{\mathbf{v}}$ in the n-D space, is built using $(n-2)$ vectors $\mathbf{v}_k$, $k = 1, \ldots, (n-2)$, that can be arranged to form the $n \times (n-2)$ matrix

$$V = [\mathbf{v}_1 \ \vdots \ \mathbf{v}_2 \ \vdots \ \cdots \ \vdots \ \mathbf{v}_{n-2}].$$

The elements of the Skew-Symmetric matrix $\tilde{V}$ are computed as follows

$$\begin{cases} \tilde{V}(i,j) = (-1)^{i+j} \det[V^{(ij)}] = -\tilde{V}(j,i) & \text{for} \quad 1 \le i < j \le n \\ \tilde{V}(i,i) = 0 & \text{for} \quad 1 \le i \le n \end{cases} \tag{2}$$

where the $(n-2) \times (n-2)$ square matrix $V^{(ij)}$ is the matrix obtained by deleting the ith and the jth rows from the matrix V. Let $\mathbf{v}_{n-1}$ be any n-long non-zero vector such that $V^{\mathrm{T}} \mathbf{v}_{n-1} \neq 0_{(n-2),n}$, then matrix $\tilde{V}$ is such that the product $\tilde{V} \mathbf{v}_{n-1} = \mathbf{v}_n$, which represents the extension to n-D space of the vector cross product, outputs a vector $\mathbf{v}_n$ which is a null vector of matrix V^{T} and Orthogonal to the vector $\mathbf{v}_{n-1}$, that is, $[V \ \vdots \ \mathbf{v}_{n-1}]^{\mathrm{T}} \mathbf{v}_n = 0_{(n-1),n}$, where $[V \ \vdots \ \mathbf{v}_{n-1}]$ denotes the $n \times (n-1)$ matrix formed with V and $\mathbf{v}_{n-1}$.

Note that the formulation provided by equation (2) can also be seen as an exterior product (see [9, 10] for its definition in the Grassman algebra) which uses the Levi-Civita tensor $\epsilon_{k_1, k_2, \cdots, k_n}$. In an Orthogonal n-D space the components of the Levi-Civita tensor are zero if at least two indices are equal and ± 1 if all the n indices are distinct. In particular $\epsilon_{k_1, k_2, \ldots, k_n} = +1$ if the index sequence $(k_1, k_2, \ldots, k_n)$ is an even permutation of $(1, 2, \ldots, n)$, and $\epsilon_{k_1, k_2, \ldots, k_n} = -1$, otherwise. For instance, in an 4-D space and in tensor notation, we write the following three exterior products[§]

$$\begin{cases} G_{i,j} & = \epsilon_{i,j,k,\ell} \ \mathbf{v}_k \ \mathbf{v}_\ell & \text{(matrix that coincides with our } \tilde{V}) \\ G_i & = \epsilon_{i,j,k,\ell} \ \mathbf{v}_j \ \mathbf{v}_k \ \mathbf{v}_\ell & \text{(vector called the vector product)} \\ G & = \epsilon_{i,j,k,\ell} \ \mathbf{v}_i \ \mathbf{v}_j \ \mathbf{v}_k \ \mathbf{v}_\ell & \text{(scalar called the mixed product or determinant)} \end{cases} \tag{3}$$

Let us examine some of the properties of the $n \times n$ Skew-Symmetric matrix given by equation (2), when the $(n-2)$ vectors $\mathbf{v}_i$ are unit vectors and mutually Orthogonal. Let $C = [\, \hat{\mathbf{c}}_1 \vdots \hat{\mathbf{c}}_2 \vdots \ \cdots \ \vdots \hat{\mathbf{c}}_n \,]$ be an $n \times n$ proper Orthogonal matrix [$C^{\mathrm{T}} C = I_n$, and $\det(C) = +1$] which we decompose as

$$C = [\, A \ \vdots \ P \,] \quad \text{where:} \begin{cases} A = [\, \hat{\mathbf{a}}_1 \vdots \hat{\mathbf{a}}_2 \vdots \ \cdots \ \vdots \hat{\mathbf{a}}_{n-2} \,] \equiv [\, \hat{\mathbf{c}}_1 \vdots \hat{\mathbf{c}}_2 \vdots \ \cdots \ \vdots \hat{\mathbf{c}}_{n-2} \,] \\ P = [\, \hat{\mathbf{p}}_1 \vdots \hat{\mathbf{p}}_2 \,] \equiv [\, \hat{\mathbf{c}}_{n-1} \vdots \hat{\mathbf{c}}_n \,] \end{cases} \tag{4}$$

‡ In some texts, as in [15], $\tilde{\mathbf{v}}$ is indicated by $[\mathbf{v}\times]$ or $-[[\mathbf{v}]]$.

§ Where sums are performed over repeated indices.

Hence $A^{\mathrm{T}}A = I_{n-2}$, $P^{\mathrm{T}}P = I_2$, and $A^{\mathrm{T}}P = 0_{(n-2),2}$. From this equations and $CC^{\mathrm{T}} = I_n$ it follows that

$$AA^{\mathrm{T}} + PP^{\mathrm{T}} = I_n \tag{5}$$

We construct the $n \times n$ Skew-Symmetric matrix $\tilde{A}$ from A according to equation (2). It follows that $\tilde{A}A = 0_{n,(n-2)}$ and the relationship

$$\tilde{A}\,P = \tilde{A}\,[\hat{\mathbf{p}}_1 \vdots \hat{\mathbf{p}}_2] = [\hat{\mathbf{p}}_2 \vdots -\hat{\mathbf{p}}_1] = P\,J_2, \tag{6}$$

where $J_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the 2×2 symplectic matrix ($J_2 J_2^{\mathrm{T}} = I_2$, $J_2 J_2 = -I_2$). Premultiplying equation (6) by $\tilde{A}$ and postmultiplying by P^{T} we obtain $\tilde{A}\tilde{A}PP^{\mathrm{T}} = \tilde{A}PJ_2P^{\mathrm{T}} = PJ_2J_2P^{\mathrm{T}} = -PP^{\mathrm{T}}$ that, together equation (5), implies $(\tilde{A}\tilde{A} + I_n)PP^{\mathrm{T}} = (\tilde{A}\tilde{A} + I_n)(I_n - AA^{\mathrm{T}}) = 0_n$, which, finally, leads to the relationship

$$\tilde{A}\tilde{A} + I_n = AA^{\mathrm{T}}, \tag{7}$$

that will be used later.

The $n \times n$ Skew-Symmetric matrix $\tilde{A}$ and the Symmetric matrix AA^{T} are both invariant with respect to the Orthogonal transformation

$$B = A\,H \tag{8}$$

where H is an $(n-2) \times (n-2)$ Orthogonal matrix ($HH^{\mathrm{T}} = I_{n-2}$). The matrix H transforms the $(n-2)$ mutually Orthogonal unit vectors $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, $\ldots$, $\hat{\mathbf{a}}_{n-2}$, into another set of $(n-2)$ Orthogonal unit vectors spanning the same subspace. Therefore, the Symmetric matrix

$$AA^{\mathrm{T}} = BB^{\mathrm{T}}$$

depends only on the subspace spanned by these $(n-2)$ vectors and, not on the choice of orthonormal basis spanning that space. We note also from equation (5) that AA^{T} can depend only on $\hat{\mathbf{a}}_{n-1}$ and $\hat{\mathbf{a}}_n$, which are the columns of P. This implies that matrix AA^{T} can be built using any set of $(n-2)$ unit-vectors Orthogonal to the plane identified by the two directions, $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$, of matrix P. If $AA^{\mathrm{T}} = BB^{\mathrm{T}}$, then equation (7) implies

$$\tilde{A}\tilde{A} = \tilde{B}\tilde{B} \tag{9}$$

Therefore, $\tilde{A}\tilde{A}$ is also independent of Orthogonal transformations performed in the null space of matrix P and defined by equation (8). Equation (7) allows us to write $\tilde{A}\tilde{A}\tilde{A} = -\tilde{A}$ while equation (9) allows us to write $\tilde{B}\tilde{A}\tilde{A} = \tilde{B}\tilde{B}\tilde{B} = -\tilde{B}$. Subtracting these two equations we obtain $(\tilde{A} - \tilde{B})(\tilde{A}\tilde{A} + I_n) = (\tilde{A} - \tilde{B})AA^{\mathrm{T}} = 0_n$ that must hold for any A and B matrices. This implies that

$$\tilde{A} = \tilde{B} \tag{10}$$

which completes the demonstration of the independence of $\tilde{A}$ from the Orthogonal transformations defined in equation (8).

Equations (5) and (7) imply $\tilde{A}\tilde{A} = -PP^{\mathrm{T}}$ which demonstrates that the matrix $\tilde{A}$ could be constructed using P instead of A and, therefore, using 2 vectors instead of $(n-2)$. Without demonstration, this is accomplished by the relationship

$$\begin{cases} \tilde{A}(i,j) = \det[P_{(ji)}] = -\tilde{A}(j,i) & \text{for} \quad 1 \le i < j \le n \\ \tilde{A}(i,i) = 0 & \text{for} \quad 1 \le i \le n \end{cases} \tag{11}$$

where the 2×2 matrix $P_{(ji)}$ is the matrix obtained by taking only the jth and the ith rows from the $n \times 2$ matrix P. We highlight here that the matrix $\tilde{A}$, constructed by equation (11) implies the evaluation of $n(n-1)/2$ determinants of 2×2 matrices, while, using equation (2), the computation of $n(n-1)/2$ determinants of $(n-2) \times (n-2)$ matrices, is required. Therefore, $\tilde{A}$ can be built more conveniently using equation (11) instead of equation (2), when $n > 4$, while for $n = 4$, equations (2) and (11) require the same computational loads. Finally, note that in equation (11) the sign terms $(-1)^{i+j}$, which appear in equation (2) and which represent the elements of the Levi-Civita tensor, disappear.

However, the use of the equation (2) or equation (11) can be completely avoided since equation (6) can be written in the important simple expression

$$\tilde{A} = P\,J_2\,P^{\mathrm{T}} \tag{12}$$

Equation (12) can be demonstrated by post-mupltiplying equation (6) by P^{T} and using the property $AA^{\mathrm{T}} + PP^{\mathrm{T}} = I_n$, obtaining $\tilde{A}PP^{\mathrm{T}} = \tilde{A}\,(I_n - AA^{\mathrm{T}}) = \tilde{A} = P\,J_2\,P^{\mathrm{T}}$. Note that equation (6) states that the complex unit vectors $\frac{\sqrt{2}}{2}(\hat{\mathbf{p}}_1 \pm i\hat{\mathbf{p}}_2)$ are the eigenvectors of $\tilde{A}$ associated with the eigenvalues $\lambda = \mp i$. Since $\tilde{A}A = 0_{n,(n-2)}$, then the other $(n-2)$ eigenvectors are the columns of A, all associated with $\lambda = 0$.

Equation (12) also represents the matrix expression of the "2-form" (see [9, 10]), or the exterior product of two vectors (1-form) which, with the notation of Grassmann Algebra, can be writen as

$$\tilde{A} = [\,\hat{\mathbf{p}}_1 \wedge \hat{\mathbf{p}}_2\,] = P\,J_2\,P^{\mathrm{T}} \tag{13}$$

Rotation Matrix in the n-D space

Let us accept the "*linear hypothesis*", that is, the fact that in the n-D space, the rigid rotation is performed about an $(n-2)$-D subspace. This implies that the rotation matrix R has to satisfy the $(n-2)$ relationships

$$R\,\hat{\mathbf{a}}_k = \hat{\mathbf{a}}_k \tag{14}$$

$(k = 1, ..., n-2)$ associated with $(n-2)$ Orthogonal eigenvectors $\hat{\mathbf{a}}_k$,[¶] all associated with the eigenvalue $\lambda = +1$. In analogy with the well known 3-D case, the remaining two eigenvalues are complex and they identify the plane of rotation, while the associated eigenvalues identify the angle of rotation. Therefore, we can write

$$R\,\frac{\sqrt{2}}{2}\,(\hat{\mathbf{p}}_1 \pm i\hat{\mathbf{p}}_2) = (\cos\Phi \mp i\sin\Phi)\,\frac{\sqrt{2}}{2}\,(\hat{\mathbf{p}}_1 \pm i\hat{\mathbf{p}}_2) \tag{15}$$

[¶]Which identify the $(n-2)$-D subspace of rotation.

Matrix R can be expressed in terms of eigenvalues and eigenvectors as $R = \sum_{k=1}^{n} \lambda_k \mathbf{w}_k \mathbf{w}_k^\dagger$, therefore

$$\begin{aligned} R = \sum_{k=1}^{n-2} \hat{\mathbf{a}}_k \hat{\mathbf{a}}_k^{\mathrm{T}} + & \frac{1}{2}(\cos\Phi - i\sin\Phi)(\hat{\mathbf{p}}_1 + i\hat{\mathbf{p}}_2)(\hat{\mathbf{p}}_1^{\mathrm{T}} - i\hat{\mathbf{p}}_2^{\mathrm{T}}) + \\ + & \frac{1}{2}(\cos\Phi + i\sin\Phi)(\hat{\mathbf{p}}_1 - i\hat{\mathbf{p}}_2)(\hat{\mathbf{p}}_1^{\mathrm{T}} + i\hat{\mathbf{p}}_2^{\mathrm{T}}) \end{aligned} \tag{16}$$

which can be written as ($A = [\,\hat{\mathbf{a}}_1 \vdots \hat{\mathbf{a}}_2 \vdots \cdots \vdots \hat{\mathbf{a}}_{n-2}\,]$)

$$\begin{aligned} R = & \ AA^{\mathrm{T}} + (\hat{\mathbf{p}}_1\hat{\mathbf{p}}_1^{\mathrm{T}} + \hat{\mathbf{p}}_2\hat{\mathbf{p}}_2^{\mathrm{T}})\cos\Phi + (\hat{\mathbf{p}}_2\hat{\mathbf{p}}_1^{\mathrm{T}} - \hat{\mathbf{p}}_1\hat{\mathbf{p}}_2^{\mathrm{T}})\sin\Phi = \\ = & \ AA^{\mathrm{T}} + PP^{\mathrm{T}}\cos\Phi + PJ_2P^{\mathrm{T}}\sin\Phi \end{aligned} \tag{17}$$

and, using equation (5), finally we obtain the searched expression

$$R(P,\Phi) = I_n + (\cos\Phi - 1)PP^{\mathrm{T}} + PJ_2P^{\mathrm{T}}\sin\Phi \tag{18}$$

which represents, as it will demonstrated in the next section, the closed form expression of the $n \times n$ Orthogonal matrix performing the rigid rotation in the n-D space. Equation (12) allows us to write

$$R(P,\Phi) = I_n + PP^{\mathrm{T}}(\cos\Phi - 1) + \tilde{A}(P)\sin\Phi \tag{19}$$

and, using equation (7) and equation (9), $R(P,\Phi)$ can also be given as $R(A,\Phi)$, by the following two expressions

$$\begin{cases} R(A,\Phi) = I_n + \tilde{A}\tilde{A}(1 - \cos\Phi) + \tilde{A}\sin\Phi \\ R(A,\Phi) = I_n\cos\Phi + AA^{\mathrm{T}}(1 - \cos\Phi) + \tilde{A}\sin\Phi \end{cases} \tag{20}$$

where the first equation explicitly shows the close connection between the n-D vector cross-product and the rigid rotation matrices, while the second form is the most familiar with the known formula for the 3-D space.

The proof that $R(P,\Phi)$ performs a rigid rotation in n-D is given in the next section.

Proof That $R(A,\Phi)$ Performs a Rigid Rotation

Considering the properties $A^{\mathrm{T}}A = I_{n-2}$, $\tilde{A}A = 0_{n,(n-2)}$, and equation (7), the demonstration that the R matrix is Orthogonal, that is, $RR^{\mathrm{T}} = I_n$, is immediate.

The demonstration that R performs a rigid rotation is accomplished by the two following steps: first, it is demonstrated the *rotation*, that is, that any vector $\mathbf{v}$ describes a cone about the subspace identified by the $(n-2)$ principal axes $\hat{\mathbf{a}}_k$, and second, it is demonstrated the *rigidity* of the motion, that is, that the distance between any two vectors $\mathbf{v}$ and $\mathbf{w}$, does not change during the rotation.

The *rotation* is mathematically described by the equation $\hat{\mathbf{a}}_k^{\mathrm{T}}\mathbf{v} = \hat{\mathbf{a}}_k^{\mathrm{T}}R\mathbf{v}$, which can be written as $\hat{\mathbf{a}}_k^{\mathrm{T}}(I_n - R)\mathbf{v} = 0$, that must hold for any of the $(n-2)$ directions $\hat{\mathbf{a}}_k$ and for any value of Φ. Substituting the first expression for R given in equation (20), and using the property $\hat{\mathbf{a}}_k^{\mathrm{T}}\tilde{A} = 0_{1,n}$, the *rotation* (about any of the $\hat{\mathbf{a}}_k$ and for any value of Φ) is demonstrated. This allows us to write

$$\left(\sum_{k=1}^{n-2} c_k \hat{\mathbf{a}}_k^{\mathrm{T}}\right) \mathbf{v} = \left(\sum_{k=1}^{n-2} c_k \hat{\mathbf{a}}_k^{\mathrm{T}}\right) R\mathbf{v} = \left(\sum_{k=1}^{n-2} c_k \hat{\mathbf{a}}_k^{\mathrm{T}}\right) \mathbf{v}(\Phi) \tag{21}$$

where the c_k coefficients can take any values, which completes the demonstration. Equation (21) implies a vector $\mathbf{v}$ coning about an $(n-2)$-D subspace. For example, this result tells us that in the 4-D space the matrix R is such that any vector $\mathbf{v}$ describes a cone about the plane, *the infinite plane*, identified by the two principal axes $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$. In the n-D space, the vector $\mathbf{v}$ describes $(n-2)$ arcs of cones about $(n-2)$ Orthogonal directions and, based on equation (21), about any other direction belonging to the subspace defined by these directions.

The *rigidity* is easily demonstrated since any Orthogonal transformation preserves the lengths, $\mathbf{v}^{\mathrm{T}}\mathbf{w} = \mathbf{v}^{\mathrm{T}}R^{\mathrm{T}}R\mathbf{w} = \mathbf{v}^{\mathrm{T}}\mathbf{w}$. Thus the distance between $\mathbf{v}$ and $\mathbf{w}$ does not change during rotation. This completes the demonstration that R performs a rigid rotation.

Finally, the invariance of matrix $\tilde{A}$, from the Orthogonal transformations $B = AH$ [see equation (8)], also demonstrates that $R(A,\Phi) = R(B,\Phi)$. Therefore, the rotation matrix R does not change if the principal axes are re-oriented in the subspace defined by the principal axes themselves. This results comes out because the null space of A coincides with the null space of B and, this null space is nothing else that the plane of rotation, identified by the $n \times 2$ matrix P. Therefore, this plane of rotation is what characterize the rigid rotation. In fact, we *rotate on a plane* in any dimensional space. Hence, the plane of rotation is the invariant with respect to the dimensional space, which demonstrates that the rigid rotation is planar in nature.

Inverse Problem

The *inverse problem*, that is, how to compute the $(n-2)$ *principal axes* $\hat{\mathbf{a}}_k$ defined by A (or the *principal plane* defined by P) and the *principal angle* Φ associated with a given rotation matrix $R(A,\Phi) = R(P,\Phi)$ can completely be solved by the eigenanalysis of the matrix R by using equation (14) and equation (15). However, when only the principal angle Φ is required, the complete eigenanalysis of R is unnecessary. In fact, since the mathematical operator *trace* is an invariant with respect to similarity transformations, it is possible to write that $\mathrm{tr}[R] = \sum_{i=1}^{n} \lambda_i = (n-2) + 2\cos\Phi$ and, therefore,

$$\cos\Phi = \frac{\mathrm{tr}[R] + 2 - n}{2} \tag{22}$$

which allows the computation of the principal angle Φ directly from the matrix R.

Properties of Orthogonal Rotations

Let $R_k(P_k, \Phi_k)$ be a set of m matrices ($k = 1, \ldots, m$), built with m *Orthogonal* principal planes of rotation P_k. In the 3-D space two Orthogonal planes passing through the origin will always share a line, while in higher dimensional spaces it is possible that they share only the point of the origin of the coordinates. For instance, in a 4-D space defined by the coordinates x, y, z, and w, the planes $(x = 0,\ y = 0)$ and $(z = 0,\ w = 0)$ have only the origin in common. We use here the word *Orthogonal* with this meaning. Associated with any two Orthogonal principal planes of rotation, P_i and P_j

$$P_i^{\mathrm{T}} P_j = 0_2,$$

and using the expression for the rotation matrix provided by equation (18), it is easy to see that $(R_i - I_n)(R_j - I_n) = 0_n,\ (i, j = 1, \ldots, m,$ and $i \neq j)$, which implies

$$R_i R_j = R_i + R_j - I_n. \tag{23}$$

which demonstrates that the set of the matrices built with Orthogonal planes of rotation constitutes an Abelian Group, since $R_i R_j = R_j R_i$.

The Orthogonal property of equation (23) can easily be generalized to the product of m matrices $R_k(P_k, \Phi_k)$ built with Orthogonal planes of rotation P_k to obtain

$$\prod_{k=1}^{m} R_k(P_k, \Phi_k) = \sum_{k=1}^{m} R_k(P_k, \Phi_k) - (m - 1)\ I_n \tag{24}$$

This equation contains the very unusual property that the product among a set of m Orthogonal rotation matrices can be expressed by their sum! As a consequence of this, then the order of the matrix product is not important! It is unusual, in fact, to see that subsequent rotations, which is described by the product of rotation matrices - a typically non linear phenomenon - becomes here linear!

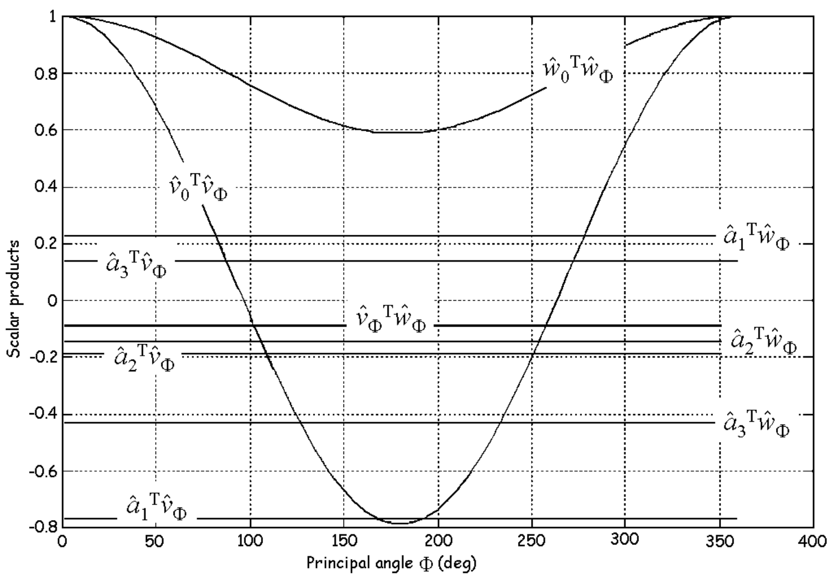

FIG. 1. Rigid rotation in 5-D space

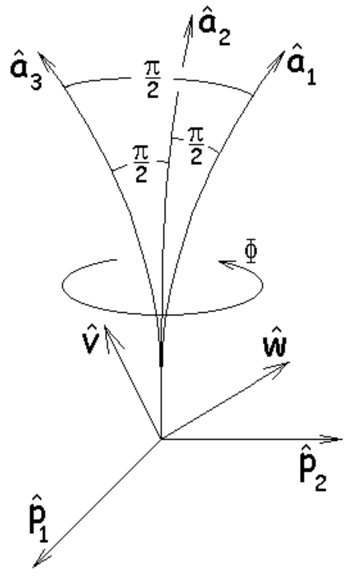

FIG. 2. Coning in 5-D

Figure 1 shows, for a 5-D space, the rotation of two random directions $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ by a matrix performing the rigid rotation about three random principal axes ($\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and

$\hat{\mathbf{a}}_3$) and by a principal angle varying from 0 to 360°. This figure shows that the distance $\hat{\mathbf{v}}^{\mathrm{T}}\hat{\mathbf{w}}$, as well as the distances $\hat{\mathbf{v}}^{\mathrm{T}}\hat{\mathbf{a}}_k$ and $\hat{\mathbf{w}}^{\mathrm{T}}\hat{\mathbf{a}}_k$ $(k = 1, 2, 3)$, are all constant during the rotation. This means, as stated before, that the directions $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ describe a cone about the $(n-2)$-D subspace defined by $A = [\hat{\mathbf{a}}_1 \vdots \hat{\mathbf{a}}_2 \vdots \hat{\mathbf{a}}_3]$ or, equivalently, about the null space of $P = [\hat{\mathbf{p}}_1 \vdots \hat{\mathbf{p}}_2]$. The angle between the initial positions, $\hat{\mathbf{v}}_0$ and $\hat{\mathbf{w}}_0$, with the positions associated with the angle Φ, that is, $\hat{\mathbf{v}}_\Phi = R\hat{\mathbf{v}}_0$ and $\hat{\mathbf{w}}_\Phi = R\hat{\mathbf{w}}_0$ has a cone-type typical behavior.

By bending the axes, Figure 2 artistically provides a "way to see" the geometry of coning about the subspace identified by the three Orthogonal axes $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$.

Orientation Matrices

An orientation matrix C is an $n \times n$ proper [det(C)=+1] Orthogonal ($CC^{\mathrm{T}} = I_n$) matrix. The rows of C describe the directions (the axes) of the oriented frame with respect to another frame of reference whose orientation is described by the unity matrix I_n. In general, an orientation matrix has n_p complex conjugate eigenvalue pairs $\lambda_k^{(\mp)} = \cos\Phi_k \mp i\sin\Phi_k$, $(k = 1, \ldots, n_p)$, and $n_a = n - 2n_p$ eigenvalues $\lambda_k = 1$, $(k = 1, \ldots, n_a)$. Associated with the n_p complex eigenvalues there are n_p eigenvectors $\left[\frac{\sqrt{2}}{2}(\hat{\mathbf{p}}_1^{(k)} \pm i\hat{\mathbf{p}}_2^{(k)})\right]$ which identify n_p proper planes $P_k = [\hat{\mathbf{p}}_1^{(k)} \vdots \hat{\mathbf{p}}_2^{(k)}]$, while associated with the n_a real eigenvalues there are n_a eigenvectors $\hat{\mathbf{a}}_k$ describing n_a proper axes. Therefore, the eigenanalysis of an orientation matrix can be expressed as

$$\begin{cases} C\,\dfrac{\sqrt{2}}{2}(\hat{\mathbf{p}}_1^{(k)} \pm i\hat{\mathbf{p}}_2^{(k)}) = (\cos\Phi_k \pm i\sin\Phi_k)\,\dfrac{\sqrt{2}}{2}(\hat{\mathbf{p}}_1^{(k)} \pm i\hat{\mathbf{p}}_2^{(k)}) & (k = 1, \ldots, n_p) \\ C\,\hat{\mathbf{a}}_k = \hat{\mathbf{a}}_k & (k = 1, \ldots, n_a) \end{cases} \tag{25}$$

As it is well known, the *orientation* can be expressed by a Skew-Symmetric matrix Q, associated to C by the Cayley Transforms‖ (Cayley Conformal Mapping), which consist of the relationships

$$C = \begin{cases} = (I_n - Q)(I_n + Q)^{-1} \\ = (I_n + Q)^{-1}(I_n - Q) \end{cases} \quad \text{and} \quad Q = \begin{cases} = (I_n - C)(I_n + C)^{-1} \\ = (I_n + C)^{-1}(I_n - C) \end{cases} \tag{26}$$

called *forward* and *inverse* transformations, respectively. The matrices C and Q, satisfying equation (26), have the same eigenvector matrix. In fact, let W be the eigenvector matrix of C, and Λ_C and Λ_Q the eigenvalue matrices of C and Q, respectively. Since C is Orthogonal then W is Orthogonal too, then $WW^\dagger = I_n$ and $C = W\Lambda W^\dagger$. Now, applying the inverse transformation we have

$$\begin{aligned} Q &= (I_n - C)(I_n + C)^{-1} = (WI_nW^\dagger - W\Lambda_C W^\dagger)(WI_nW^\dagger + W\Lambda_C W^\dagger)^{-1} = \\ &= [W(I_n - \Lambda_C)W^\dagger][W(I_n + \Lambda_C)W^\dagger]^{-1} = W(I_n - \Lambda_C)(I_n + \Lambda_C)^{-1}W^\dagger = W\Lambda_Q W^\dagger \end{aligned}$$

‖The sign adopted here is such that in 3-D the orientation matrix C coincide with a matrix performing the rigid rotation about the principal axis by the principal angle, while usually, the coincidence with the transpose is adopted.

which demonstrates that: 1) C and Q have the same eigenvector matrix, and 2) that their eigenvalues are related by the bilinear transformation

$$\lambda^{(C)} = \frac{1-\lambda^{(Q)}}{1+\lambda^{(Q)}} \qquad \Longleftrightarrow \qquad \lambda^{(Q)} = \frac{1-\lambda^{(C)}}{1+\lambda^{(C)}}$$

These equations imply the eigenvalue associations

$$\lambda^{(C)} = \cos\Phi \pm i\sin\Phi \qquad \Longleftrightarrow \qquad \lambda^{(Q)} = \mp i\tan\left(\frac{\Phi}{2}\right)$$

Actually, Cayley Transforms are nothing else than a bilinear transformation between real matrices.

The eigenanalysis of the Skew-Symmetric *orientation* matrix Q can, therefore, be written as

$$\begin{cases} Q\,\dfrac{\sqrt{2}}{2}(\hat{\mathbf{p}}_1^{(k)} \pm i\hat{\mathbf{p}}_2^{(k)}) = \mp i\tan\left(\dfrac{\Phi_k}{2}\right)\,\dfrac{\sqrt{2}}{2}(\hat{\mathbf{p}}_1^{(k)} \pm i\hat{\mathbf{p}}_2^{(k)}) & (k=1,\ldots,n_p) \\ Q\,\hat{\mathbf{a}}_k = \mathbf{0} & (k=1,\ldots,n_a) \end{cases} \tag{27}$$

The important difference between C and Q consists in the fact that Q may become singular, which occurs when one (or more) of its eigenvalues $\lambda_k^{(Q)} = \mp i\tan\left(\dfrac{\Phi_k}{2}\right)$ becomes infinite.

The eigenanalysis of equations (25) and (27) allows us to provide an expression of C and Q in terms of their eigenvalues and eigenvectors

$$\begin{cases} C = \displaystyle\sum_{k=1}^{n} \lambda_k^{(C)}\hat{\mathbf{w}}_k\hat{\mathbf{w}}_k^\dagger = \sum_{k=1}^{n_a}\hat{\mathbf{a}}_k\hat{\mathbf{a}}_k^{\mathrm{T}} + \sum_{k=1}^{n_p} P_k(I_2\cos\Phi_k + J_2\sin\Phi_k)\;P_k^{\mathrm{T}} \\ Q = \displaystyle\sum_{k=1}^{n} \lambda_k^{(Q)}\hat{\mathbf{w}}_k\hat{\mathbf{w}}_k^\dagger = \sum_{k=1}^{n_p} P_k J_2 P_k^{\mathrm{T}}\tan\left(\frac{\Phi_k}{2}\right) = \sum_{k=1}^{n_p}\tilde{A}_k(P_k)\tan\left(\frac{\Phi_k}{2}\right) \end{cases} \tag{28}$$

Equation (18) allows us to write $\displaystyle\sum_{k=1}^{n_p} P_k(I_2\cos\Phi_k + J_2\sin\Phi_k)\;P_k^{\mathrm{T}} = \sum_{k=1}^{n_p} R_k(P_k,\Phi_k) - n_p\,I_n + \sum_{k=1}^{n_p} P_kP_k^{\mathrm{T}}$ and, since $\displaystyle\sum_{k=1}^{n_p} P_kP_k^{\mathrm{T}} + \sum_{k=1}^{n_a}\hat{\mathbf{a}}_k\hat{\mathbf{a}}_k^{\mathrm{T}} = I_n$, we obtain

$$\begin{cases} C = \displaystyle\sum_{k=1}^{n_p} R_k(P_k,\Phi_k) - (n_p-1)\,I_n \\ Q = \displaystyle\sum_{k=1}^{n_p} S_k(P_k,\Phi_k) \end{cases} \tag{29}$$

where the matrices

$$\begin{cases} R_k(P_k,\Phi_k) = I_n + (\cos\Phi_k - 1)P_kP_k^{\mathrm{T}} + P_kJ_2P_k^{\mathrm{T}}\sin\Phi_k \\ S_k(P_k,\Phi_k) = P_kJ_2P_k^{\mathrm{T}}\tan\left(\dfrac{\Phi_k}{2}\right) \end{cases} \tag{30}$$

represents the $n \times n$ Skew-Symmetric *rotation* matrix associated with the Orthogonal rotation matrix $R_k(P_k, \Phi_k)$. Matrices $S_k(P_k, \Phi_k)$ have one pure imaginary eigenvalue pair $\lambda_k^{(Q)} = \mp i \tan\left(\dfrac{\Phi_k}{2}\right)$ and a set of $(n-2)$ eigenvalues $\lambda = 0$. Equation (29) demonstrates that the general rotation, as the planar rotation, depends on the parameters associated with its complex eigenvalues/eigenvectors only. Finally, equation (29) allows us to extend the Euler's Theorem to any n-D space.

The relationships between $R_k(P_k, \Phi_k)$ and $S_k(P_k, \Phi_k)$ are the classic Cayley Transforms

$$R_k = \begin{cases} = (I_n - S_k)(I_n + S_k)^{-1} \\ = (I_n + S_k)^{-1}(I_n - S_k) \end{cases} \qquad S_k = \begin{cases} = (I_n - R_k)(I_n + R_k)^{-1} \\ = (I_n + R_k)^{-1}(I_n - R_k) \end{cases} \tag{31}$$

which remember that C stands to Q (for general rotation) as R_k stands to S_k (for planar rotation), and the exponential relationships

$$R_k = e^{[\Phi_k / \tan(\Phi_k/2)]\, S_k} \qquad \Longleftrightarrow \qquad S_k = \frac{\tan(\Phi_k/2)}{\Phi_k}\, \ln(R_k) \tag{32}$$

Extension to the n-D Spaces of the Euler's Theorem

Equation (29) tells us that the Euler's Theorem (any orientation can be achieved by only one rigid rotation**) is a property that holds in the 2-D and 3-D spaces, only, because $n_p = 1$. However, this coincidence between geometrical displacement (orientation) and rigid rotation operator (matrix) has also caused the use of *orientation* and *rigid rotation* expressions, without any distinction. These two concepts start differing from one another in dimensional spaces greater than three.

Several publications exist which claim to extend the Euler's Theorem to n-D spaces (see, for instance, [14]), however, most of them actually generalize the dynamics in the n-D spaces by providing the expression of the angular velocity. Unfortunately, the dynamics problem deal with the *orientation* of a proper reference frame as a function of time. On the contrary, Euler's Theorem is a geometrical property and, therefore, it can be considered as a static problem. Even in Group Theory, the so called *rotation group* actually identify the *orientation group* of which the set of the rigid rotation matrices is only a subset. For all these reasons, the confusion between the concepts of *orientation* and *rigid rotation* still holds.

As already stated, the n-D rigid rotation is performed about an $(n-2)$-D figure identified by the space spanned by the A matrix. This figure has two perpendicular Orthogonal directions which identify the plane of rotation (matrix P). Therefore, any direction belonging to the space spanned by A is not affected by the rotation itself. This implies that only two Orthogonal directions (axes of the reference frame) can be taken

**In Ref. [4], the original latin presentation states that "*Quomodocunque sphaera circa centrum suum conuertatur, semper assignari potest diameter, cuius directio in situ translato conueniat cum situ initiali.*"

to the final orientation at each subsequent rotation. Therefore, in the most common case that there is no axis already at its final position, a number of

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} = (n-1)/2 & \text{if} \quad n \quad \text{is odd} \\ = n/2 & \text{if} \quad n \quad \text{is even} \end{cases} \tag{33}$$

rigid rotations are needed in order to reach a given orientation, where the function $\lfloor x \rfloor$ rounds x to the nearest integer towards zero. This result can also be derived by analyzing the eigenvalues of the orientation and the rigid rotation matrices. In fact, in the n-D space the orientation is identified by a matrix which has, in the most general case, $n/2$ complex conjugate eigenvalue pairs if n is even, or $(n-1)/2$ complex conjugate eigenvalue pairs and the eigenvalue $\lambda = 1$ if n is odd. An n-D rotation matrix has one complex conjugate eigenvalue pair ($\lambda^{(R)} = \cos\Phi \mp i \sin\Phi$) associated with the plane of rotation (matrix P), and $(n-2)$ eigenvalues $\lambda = 1$, associated with the principal axes (matrix A). Now, a multiplication between two n-D rotation matrices (subsequent rotations) outputs a matrix which has, in general, two complex conjugate eigenvalue pairs and $(n-4)$ eigenvalues $\lambda = 1$. This implies that, in order to fill the eigenvalue matrix with complex conjugate eigenvalue pairs, $\lfloor n/2 \rfloor$ subsequent rotations are needed.

Equation (29) and the property given in equation (24) demonstrate that the $n \times n$ orientation matrix C can be decomposed by a set of n_p Orthogonal rigid rotation matrices $R_k(P_k, \Phi_k)$, whose expression is given in equation (18), that is, by the relationship

$$C = \prod_{k=1}^{n_p} R_k(P_k, \Phi_k) = \sum_{k=1}^{n_p} R_k(P_k, \Phi_k) - (n_p - 1)\, I_n \tag{34}$$

This equation implies that the orientation C can be described by $n_p!$ different sequences of Orthogonal rotations, where the Orthogonality is expressed by $(i, j = 1, \ldots, n_p,\ i \neq j)$

$$(R_i - I_n)\,(R_j - I_n) = 0_n \qquad \Longrightarrow \qquad R_i\, R_j = R_i + R_j - I_n \tag{35}$$

The second of equation (29) shows how to decompose the orientation expressed by Q by the sum of a set of n_p Skew-Symmetric matrices $S_k(P_k, \Phi_k)$. In this case the Orthogonality property is

$$S_i(P_i, \Phi_i)\, S_j(P_j, \Phi_j) = 0_n \tag{36}$$

Equation (34) is, therefore, nothing else that the mathematical expression of the generalized Euler's Theorem to the n-D spaces. This Theorem can be expressed as follows:

THE GENERALIZED EULER'S THEOREM: Regardless of the way a coordinate system is re-oriented from its original orientation, in the n-dimensional space, it is always possible to find a minimum sequence of $n_p \leq \lfloor n/2 \rfloor$ rigid rotations, where n_p is the number of complex eigenvalue pairs of the re-orientation matrix, performed on a set of n_p Orthogonal planes, which ends the initial orientation to the final orientation.

The expression for C given in equation (34) and the expression for Q given in the second of equation (29), represent also two new matrix decompositions of the proper Orthogonal and of the Skew-Symmetric matrices, respectively. In particular, equation (34) highlights the important fact that subsequent rigid rotations, typically a non linear phenomenon, becomes linear when the rotations matrices constitute an Orthogonal set. In fact, in this case subsequent rigid Orthogonal rotations can be expressed by the sum instead of the product!

For sake of clarity, an example in 4-D space of the decomposition introduced by equation (34), is given in the following.

Numerical Example of Orientation Decomposition

Consider the proper Orthogonal orientation matrix

$$C = \begin{bmatrix} 0.1003 & 0.2496 & -0.8894 & -0.3697 \\ 0.9593 & -0.0238 & -0.0153 & 0.2810 \\ -0.1172 & -0.8638 & -0.3828 & 0.3059 \\ -0.2366 & 0.4370 & -0.2495 & 0.8311 \end{bmatrix} \tag{37}$$

the eigenanalysis of C yields the eigenvalues $(\cos\Phi_k \mp i\sin\Phi_k)$ and the eigenvectors $\frac{\sqrt{2}}{2}(\hat{\mathbf{p}}_1^{(k)} \pm i\hat{\mathbf{p}}_2^{(k)})$ where

$$\begin{cases} \begin{cases} \cos\Phi_1 = -0.7123 \\ \sin\Phi_1 = +0.7019 \end{cases} & \frac{\sqrt{2}}{2}(\hat{\mathbf{p}}_1^{(1)} \pm i\hat{\mathbf{p}}_2^{(1)}) = \begin{Bmatrix} 0.2923 \\ -0.5417 \\ -0.2923 \\ 0.1888 \end{Bmatrix} \pm i \begin{Bmatrix} -0.4168 \\ 0.0499 \\ -0.5629 \\ -0.0832 \end{Bmatrix} \\ \\ \begin{cases} \cos\Phi_2 = +0.9747 \\ \sin\Phi_2 = +0.2235 \end{cases} & \frac{\sqrt{2}}{2}(\hat{\mathbf{p}}_1^{(2)} \pm i\hat{\mathbf{p}}_2^{(2)}) = \begin{Bmatrix} -0.4368 \\ -0.4462 \\ 0.3036 \\ -0.1339 \end{Bmatrix} \pm i \begin{Bmatrix} -0.2237 \\ 0.0704 \\ 0.0739 \\ 0.6630 \end{Bmatrix} \end{cases} \tag{38}$$

Associated with the eigenvalues, the principal angles

$$\begin{cases} \Phi_1 = \mathrm{ATAN2}(\sin\Phi_1, \cos\Phi_1) = 2.3635 \\ \Phi_2 = \mathrm{ATAN2}(\sin\Phi_2, \cos\Phi_2) = 0.2254 \end{cases} \tag{39}$$

are introduced, and associated with the eigenvectors, the principal (rotation) planes $P_1 = [\hat{\mathbf{p}}_1^{(1)} \vdots \hat{\mathbf{p}}_2^{(1)}]$ and $P_2 = [\hat{\mathbf{p}}_1^{(2)} \vdots \hat{\mathbf{p}}_2^{(2)}]$, are given

$$P_1 = \begin{bmatrix} 0.4134 & -0.5894 \\ -0.7661 & 0.0706 \\ -0.4134 & -0.7961 \\ 0.2669 & -0.1177 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} -0.6177 & -0.3164 \\ -0.6310 & 0.0995 \\ 0.4294 & 0.1045 \\ -0.1893 & 0.9376 \end{bmatrix} \tag{40}$$

Now, using equation (18), the rigid rotation matrices

$$R_1(P_1, \Phi_1) = \begin{bmatrix} 0.1125 & 0.3170 & -0.9128 & -0.2314 \\ 0.9100 & -0.0135 & 0.0024 & 0.4144 \\ -0.1088 & -0.8947 & -0.3779 & 0.2119 \\ -0.3840 & 0.3143 & -0.1547 & 0.8543 \end{bmatrix} \tag{41}$$

and

$$R_2(P_2, \Phi_2) = \begin{bmatrix} 0.9878 & -0.0674 & 0.0235 & -0.1383 \\ 0.0493 & 0.9897 & -0.0177 & -0.1334 \\ -0.0084 & 0.0309 & 0.9951 & 0.0940 \\ 0.1474 & 0.1226 & -0.0948 & 0.9769 \end{bmatrix} \tag{42}$$

are obtained. It is easy to see that $C = R_1 R_2 = R_2 R_1 = R_1 + R_2 - I_4$.

The General Exponential Relationship with Orientation

The Generalized Euler Theorem also allows us to extend the exponential relationship, given in equation (32) and which holds for rigid rotations only, to orientations in n-D spaces. In fact, equation (32) states that a constant scalar α_k, such that $R_k = e^{(\alpha_k S_k)}$, exists for rigid rotation, where

$$\alpha_k = \frac{\Phi_k}{\tan(\Phi_k/2)} \tag{43}$$

it is easy to see that it not possible to find a constant scalar α such that $C = e^{\alpha Q}$. However, the Generalized Euler Theorem provides us a tool to find the closed-form expression of a general exponential relationship associated with an orientation matrix C. In fact, for general rotation identified by the matrix C, which has eigenvector matrix W, we can write that

$$C = W \Lambda_C W^\dagger = \prod_{k=1}^{n_p} R_k = \prod_{k=1}^{n_p} e^{(\alpha_k S_k)} = e^{\left(\sum_k \alpha_k S_k\right)} = e^{W \Lambda_E W^\dagger} = e^E \tag{44}$$

where

$$E = W \Lambda_E W^\dagger \tag{45}$$

is the Skew-Symmetric exponential matrix which has the same eigenvector matrix of C and Q, and which has an eigenvalue matrix Λ_E with elements $\pm i \Phi_k$. For the sake of clarity, in the following equation the involved eigenvalues are summarized

$$\begin{cases} \Lambda_C \quad \rightarrow \quad \lambda_k^{(C)} = \begin{cases} = \cos\Phi_k \pm i \sin\Phi_k & (n_p \text{ for } C, \quad 1 \text{ for } R_k) \\ = 1 & (n_a \text{ for } C, \quad (n-2) \text{ for } R_k) \end{cases} \\ \\ \Lambda_Q \quad \rightarrow \quad \lambda_{Q_k} = \begin{cases} = \pm i \tan(\Phi_k/2) & (n_p \text{ for } Q, \quad 1 \text{ for } S_k) \\ = 0 & (n_a \text{ for } Q, \quad (n-2) \text{ for } S_k) \end{cases} \\ \\ \Lambda_E \quad \rightarrow \quad \lambda_{E_k} = \begin{cases} = \pm i \Phi_k & (n_p \text{ for } E) \\ = 0 & (n_a \text{ for } E) \end{cases} \end{cases} \tag{46}$$

Multiple Rigid Rotations Matrices

The rotation matrix has one complex eigenvalue pair and the remaining $(n-2)$ are all ones. The product of $g \leq \lfloor n/2 \rfloor$ Orthogonal rotation matrices output a matrix which has, in general, g complex eigenvalue pairs ($n \geq g > 1$) while the remaining $(n-2g)$ eigenvalues are all ones. These matrices perform *multiple rigid rotations* about an $(n-2g)$-D subspace.

Let us to see an interesting property of these matrices, by analyzing the projection on the kth plane (P_k) of a rotated point $\mathbf{v} = \left[\prod_{i=1}^{g} R_i\right] \mathbf{v}_0$, where $\mathbf{v}_0$ identifies any position

$$P_k^{\mathrm{T}}\mathbf{v} = P_k^{\mathrm{T}} \left[\sum_{i=1}^{g} R_i - (g-1) I_n\right] \mathbf{v}_0 = P_k^{\mathrm{T}} R_k \mathbf{v}_0 = (I_2 \cos\Phi_k + J_2 \sin\Phi_k) P_k^{\mathrm{T}} \mathbf{v}_0 \qquad (47)$$

which demonstrates that the projected point belong to a circle. Now, varying the angles Φ_i, $(i = 1, \ldots, g)$, the vertex of the vector $\mathbf{v}_0$ describes an g-D surface. The projections of this surface on the g Orthogonal planes of rotations are still circles.

Multiple rigid rotation matrices (at least, the orientation matrices) can be seen as complex rigid rotation matrices. In particular, if n is odd, the orientation matrix can be seen as a multiple rotation matrix performing the rigid rotation about an axis (a 1-D figure) while, if n is even, the orientation performs a complex rigid rotation about a point (a 0-D figure). Figure 3 summarizes these properties for all of the multiple rigid rotation matrices, that is for all the matrices connecting rotation and orientation.

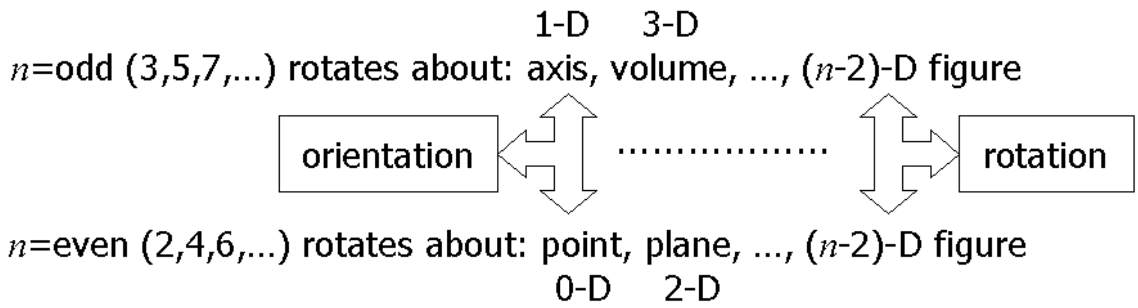

FIG. 3. From Rigid Rotation to Orientation

Rotation in n-D Complex Spaces

In n-D complex spaces, equation (18) can still be used, but the matrix transpose (that is, P^{T}) must be replaced by the transpose conjugate, getting $P^{\dagger}$. With this simple modification, equation (18) assumes the important general form

$$R(P, \Phi) = I_n + (\cos\Phi - 1)\, PP^{\dagger} + PJ_2P^{\dagger}\, \sin\Phi \qquad (48)$$

which represents the closed-form expression of the matrix performing the rigid rotation (on the complex plane defined by P, and by the angle Φ), in any n-D complex space.

Just for example, let us consider a particular 4-D complex space which has three real Orthogonal axes and one pure imaginary, as that introduced by the restricted relativity ($x_1 = x$, $x_2 = y$, $x_3, = z$, and $x_4 = i\,c\,t$, where c represents the speed light). Choosing the "space-time" coordinate plane as the rotation plane

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & i \end{bmatrix} \qquad (49)$$

then, the rigid rotation matrix can be simply obtained using equation (48)

$$R = \begin{bmatrix} \cos\Phi & 0 & 0 & -i\,\sin\Phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\sin\Phi & 0 & 0 & \cos\Phi \end{bmatrix} \tag{50}$$

In the complex space, the $n \times n$ Skew-Symmetric *rotation* matrix $S(P,\Phi)$, associated with the rotation matrix $R(P,\Phi)$, becomes Skew-Hermitian

$$S(P,\Phi) = P\,J_2\,P^{\dagger}\tan\left(\frac{\Phi}{2}\right) \tag{51}$$

Analogously, the decompositions provided for n-D real spaces, by means of equation (34) for C, and by the second of equation (29) for Q, still hold for n-D complex spaces, but $R(P,\Phi)$ and $S(P,\Phi)$ must be evaluated by using equations (48) and (51), respectively.

The Ortho-Skew and the Ortho-Skew-Hermitian Matrices

It is known that a proper Orthogonal matrix C has the eigenvalues, $\lambda_k^{(\mp)} = \cos\Phi_k \mp i\,\sin\Phi_k$, which all belong to the unit-radius circle. The eigenvalues of the associated Skew-Symmetric matrix Q consists of pure imaginary pairs $\lambda_k^{(\mp)} = \mp i\,\tan(\Phi_k/2)$. In particular, for Symmetricity, when the dimensional space is odd, then C has one eigenvalue $\lambda = 1$, while the Q matrix has one eigenvalue $\lambda = 0$.

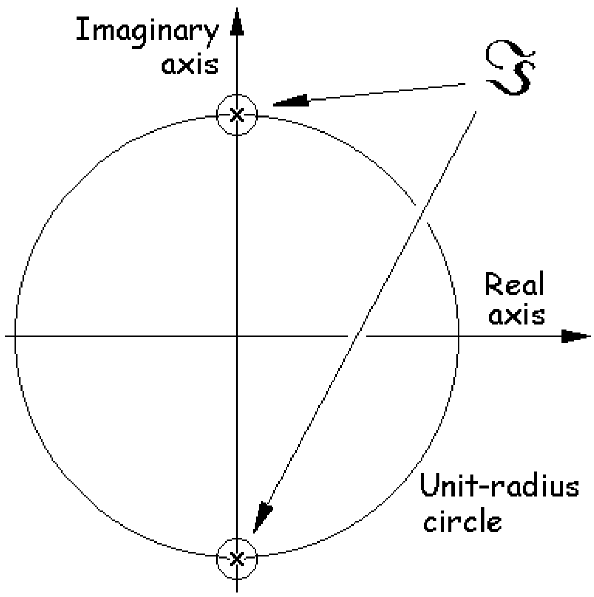

FIG. 4. Eigenvalue existence field of the Ortho-Skew-Hermitian matrices

The intersection between the unit-radius circle with the imaginary axis (see Fig. 4) represents, therefore, the field of existence of a set of matrices which are both Orthogonal and Skew-Symmetric, the *Ortho-Skew* $\Im_e$ matrices.[††] These matrices have to satisfy the constraints

$$\Im_e^{\mathrm{T}}\Im_e = \Im_e\Im_e^{\mathrm{T}} = I_n \qquad \text{and} \qquad \Im_e + \Im_e^{\mathrm{T}} = 0_n \tag{52}$$

which imply $\Im_e \in \mathcal{SO}(n)$, $\Im_e \in \mathcal{SU}(n)$, and the fundamental condition

$$\Im_e\Im_e = -I_n \tag{53}$$

the Cayley Transforms of equation (23), and another similar conditions

$$\Im_e = \begin{cases} = (\Im_e - I_n)(\Im_e + I_n)^{-1} \\ = (\Im_e + I_n)^{-1}(\Im_e - I_n) \end{cases} \qquad \text{and} \qquad \Im_e = \begin{cases} = (I_n + \Im_e)(I_n - \Im_e)^{-1} \\ = (I_n - \Im_e)^{-1}(I_n + \Im_e) \end{cases} \tag{54}$$

The Ortho-Skew matrices $\Im_e$ have, therefore, only pure imaginary eigenvalue pairs $\lambda_k = \mp i$ with algebraic multiplicity equal to $n/2$. This implies that the $\Im_e$ matrices exist in the *even* space only (this is why they are indicated with the subscript e). In this way we can see the eigenvalues of $\Im_e$ as belonging to the unit-radius circle ($\Im_e$ is Orthogonal, then $\det[\Im_e] = +1$), and to the imaginary axis ($\Im_e$ is Skew-Symmetric, then $\mathrm{tr}[\Im_e] = 0$). The eigenanalysis of the $\Im_e$ matrices can be performed by using either equation (25) and equation (27), with eigenvalues $\mp i$, only, because they are both Orthogonal and Skew-Symmetric. This implies that they can be decomposed by both the equation (34) and the equations (29,30), obtaining [$R_k = R_k(P_k, \pi/2)$, and $S_k = S_k(P_k, \pi/2)$]

$$\Im_e = \prod_{k=1}^{n/2} R_k = \sum_{k=1}^{n/2} R_k - (n/2 - 1)\, I_n = \sum_{k=1}^{n/2} S_k = \sum_{k=1}^{n/2} P_k J_2 P_k^{\mathrm{T}} \tag{55}$$

as it is easy to verify. Equation (55) provides also a general method to construct Ortho-Skew matrices. Note that the sympletic matrices $J_{2n} = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$ constitute only a particular subset of the Ortho-Skew matrices.

It is possible, however, to extend the Ortho-Skew matrix set to odd spaces, getting the *Ortho-Skew-Hermitian* matrix set

$$\Im_o = \sum_{k=1}^{(n-1)/2} P_k J_2 P_k^{\dagger} \pm i\, \mathbf{p}\mathbf{p}^{\dagger} \tag{56}$$

where $P_k = [\,\hat{\mathbf{p}}_1^{(k)} \,\vdots\, \hat{\mathbf{p}}_2^{(k)}\,]$ contains the two Orthogonal directions identifying the k-th plane (that can be real or complex), and where the unit-vector $\mathbf{p}$ is Orthogonal to all the P_k planes, $\mathbf{p}^{\dagger} P_k = 0_{1,2}$, and $k = 1, \ldots, (n-1)/2$.

If n is even, then an $(n+1)$-D Ortho-Skew-Hermitian matrix $\Im_o$ can also be built, for instance, by an n-D Ortho-Skew matrix $\Im_e$. In fact, an $(n+1) \times (n+1)$ Ortho-Skew-Hermitian matrix can be obtained by inserting a zero column and a zero row in the Ortho-Skew matrix $\Im_e$ at position m (where $m = 1, \ldots, n+1$), and by inserting

[††]Name suggested to me by Dr. John L. Junkins.

the imaginary unit $\pm i$ in the position $\Im_o(m,m)$. For instance, for $m=n$, $\Im_o$ can be obtained as

$$\Im_o = \begin{bmatrix} \Im_e & 0_{n,1} \\ 0_{1,n} & \pm i \end{bmatrix} \tag{57}$$

where $0_{n,1}$ and $0_{1,n}$ are a column and a row zero vectors with n elements, respectively. Matrix $\Im_o$ is no more either real and Skew-Symmetric, but it is Orthogonal and Skew-Hermitian, since it satisfies the conditions

$$\Im_o^\dagger \Im_o = \Im_o \Im_o^\dagger = I_n \qquad \text{and} \qquad \Im_o + \Im_o^\dagger = 0_n \tag{58}$$

and the fundamental condition of (52) still holds as

$$\Im_o \Im_o = -I_n \tag{59}$$

.

Extension of the Imaginary Unit to the Matrix Field

The Ortho-Skew matrices $\Im_e$ are real. This matrix set, together with the Ortho-Skew-Hermitian matrix set $\Im_o$, can be considered the extension to the matrix field of the imaginary number $i=\sqrt{-1}$. In fact, these matrices, since now identified by $\Im$ (that is, either $\Im_e$ and $\Im_o$), satisfy most of the known complex identities. They are:

(1) First of all, subsequent powers of i and $\Im$ follow an identical structure

$$i^k = \begin{cases} +i & (\text{for } k = 1+4m) \\ -1 & (\text{for } k = 2+4m) \\ -i & (\text{for } k = 3+4m) \\ +1 & (\text{for } k = 4m) \end{cases} \quad \Longrightarrow \quad \Im^k = \begin{cases} +\Im & (\text{for } k = 1+4m) \\ -I_n & (\text{for } k = 2+4m) \\ -\Im & (\text{for } k = 3+4m) \\ +I_n & (\text{for } k = 4m) \end{cases} \tag{60}$$

where m can be any integer number.

(2) The $\Im$ matrices satisfy the Euler's formula (where $e^{I_n\vartheta} = I_n e^{\vartheta}$)

$$e^{\vartheta + i\varphi} = e^{\vartheta}\left(\cos\varphi + i\,\sin\varphi\right) \quad \Longrightarrow \quad e^{I_n\vartheta + \Im\varphi} = e^{I_n\vartheta}\left(I_n\,\cos\varphi + \Im\,\sin\varphi\right) \tag{61}$$

(3) In particular, when $\vartheta=0$, equation (61) implies a similarity associated with the trigonometric functions

$$\begin{cases} 2\,\cos\varphi = e^{i\varphi} + e^{-i\varphi} \\ 2\,i\,\sin\varphi = e^{i\varphi} - e^{-i\varphi} \end{cases} \quad \Longrightarrow \quad \begin{cases} 2\,I_n\,\cos\varphi = e^{\Im\varphi} + e^{-\Im\varphi} \\ 2\,\Im\,\sin\varphi = e^{\Im\varphi} - e^{-\Im\varphi} \end{cases} \tag{62}$$

(4) The *polar expression* of the complex number $z = a + i\,b$ is

$$z = \xi\left(\cos\varphi + i\,\sin\varphi\right) \quad \text{where:} \quad \begin{cases} \xi = \sqrt{a^2+b^2} \\ a = \xi\,\cos\varphi \\ b = \xi\,\sin\varphi \end{cases} \tag{63}$$

and, analogously, the *polar expression* of the real matrix $Z = I_n\,a + \Im\,b$ is

$$Z = \Xi\left(I_n\,\cos\varphi + \Im\,\sin\varphi\right) \quad \text{where:} \quad \begin{cases} \Xi = I_n\,\vartheta + \Im\,\varphi \\ \vartheta = a\,\cos b + b\,\sin b \\ \varphi = -a\,\sin b + b\,\cos b \end{cases} \tag{64}$$

(5) The similarity for *polar expression* implies a similarity for the Moivre Formula

$$z^{k/j} = \xi^{k/j} \left[\cos\left(\frac{k\varphi}{j}\right) + i \, \sin\left(\frac{k\varphi}{j}\right) \right] \tag{65}$$

where k and j can be any integer number, to a real matrix form, getting

$$Z^{k/j} = \Xi^{k/j} \left[I_n \, \cos\left(\frac{k\varphi}{j}\right) + \Im \, \sin\left(\frac{k\varphi}{j}\right) \right] \tag{66}$$

where $Z = I_n \, a + \Im \, b$. Note that, for $k = 1$, and $j > k$, the Moivre formula computes the roots of z while, for $j = 1$, and $k > j$, the Moivre formula computes the powers of z.

Based on the above, it is possible to complete this study by outlining that

$$i = \sqrt{-1} \;\text{ is the }\; 1 \times 1 \;\text{ Ortho-Skew-Hermitian matrix.}$$

Conclusions

This paper presents the general mathematical formulation of the rigid rotation matrix for any n-dimensional real or complex space, and which demonstrates that the rigid rotation is planar in nature. The rigid rotation is shown to depend on an angle (principal angle), which identify the amplitude of the rotation, and on a set of 2 principal axes, identifying the plane of rotation, that is, its spatial orientation. This fact suggests us to replace the common sentence of *rotation about an axis* (which holds true in 3-D space only) with the sentence *rotation on a plane*, because the plane of rotation is, in fact, an invariant with respect to the dimensional space. The inverse problem, that is, how to compute these *principal rotation parameters* from the rotation matrix, is also treated.

Then, Euler's Theorem is extended to the n-D spaces, by expressing the orientation as a product of a minimum set of rigid rotations. This relationship, which shows that the concepts of rigid rotation and orientation become distinct starting from a 4-D space, consists of a new decomposition for proper Orthogonal matrices that can be canonically expressed by *either a product or a sum of the same rotation matrix set.* A numerical example of this decomposition for 4-D space is given. Skew-Symmetric matrices, as representing orientation, can also be similarly decomposed by a sum of a set of Skew-Symmetric matrices describing rigid rotations on Orthogonal planes. Then the multiple rigid rotations matrices, which represent the connection path between rigid rotation and orientation, are introduced and the Skew-Symmetric matrix representing the general exponential relationship with orientation, is also presented.

Finally, the *Ortho-Skew* real matrices, which are both Orthogonal and Skew-Symmetric and which exist in the even dimensional spaces, and the *Ortho-Skew-Hermitian* matrices, which are both Orthogonal and Skew-Hermitian and which exist in the odd dimensional spaces, are introduced. These matrices, which satisfy most of the known complex identities, represent a striking analogy to the imaginary unit for the matrix field.

Acknowledgments

This work is dedicated to the memory of Carlo Arduini, deeply missed scientist, teacher, and friend. Carlo, thank you.

The author offers heartfelt thanks to my wife Andreea for her indiscriminate love, to the poet Silvano Agosti for his freedom of tough, and to John L. Junkins, to whom no thanks can ever be adeguate. I feel indebted also to many important researchers such as F. Landis Markley, Malcolm D. Shuster, and Beny Neta, for many discussions containing directly or indirectly contributing to this work.

The author is grateful to the Italian Space Agency for its continued support.

References

[1] RODRIGUEZ, M.O. "Des Lois Géométriques Qui Régissent les Déplacement d'un Système Solide dans L'espace, et de la Variation des Coordonnées Provenant de ces Déplacements Considérés Independamment des Causes qui Peuvent les Produire," *Journal de Mathématique Pures et Appliquées* (Liouville), 5 (1840), pp. 380-440.

[2] CAYLEY, A. "On the Motion of Rotation of a Solid Body," *Cambridge Mathematical Journal*, III (1843), pp. 224-232. Also in *The Collected Mathematical Papers of Arthur Cayley*, I . Cambridge, MA: The Cambridge University Press, 1889. Reprinted by Johnson Reprint Corp., New York, 1963, pp. 28-35.

[3] CAYLEY, A. "On Certain Results Relating to Quaternions," *Cambridge Mathematical Journal*, III (1843), pp. 131-145. Also in *The Collected Mathematical Papers of Arthur Cayley*, I . Cambridge, MA: The Cambridge University Press, 1889. Reprinted by Johnson Reprint Corp., New York, 1963, pp. 123-126.

[4] EULER, L. "Formulae Generales pro Trandlatione Quacunque Corporum Rigidorum," *Novi Acad. Sci. Petrop.*, 20 (1775), pp. 189-207.

[5] ARCIDIACONO, G. "Relatività e Cosmologia," Vol. I: *Le Teorie Relativistiche di Einstein*, Libreria Eredi Virgilio Veschi, Roma, 1979, pp. 24-43, and pp. 54-59.

[6] FANTAPPIÉ, L. "Su una Nuova Teoria di Relatività Finale," Rend. Acc. Lincei, ser 8, Vol. 17, fasc. 5 (1954).

[7] CARTAN, E. "Leçons sur la Théorie des Spineurs," Vol. I, *Actualité Scientifiques*, n. 643, Paris, Hermann, 1938.

[8] ARCIDIACONO, G. "Relatività e Cosmologia," Vol. II: *Teoria dei Gruppi e Modelli di Universo*, Libreria Eredi Virgilio Veschi, Roma, 1979, pp. 11-21.

[9] ABRAHAM, R., MARSDEN, J.E., and RATIU, T. "Manifolds, Tensor Analysis, and Applications," Springer Verlag, Applied Math Sciences No. 75., 1983, Addison Wesley.

[10] LOVELOCK, D., and RUND, H. "Tensors, Differential Forms, and Variational Principles," Dover Publications, pp. 131-136, 1989.

[11] ARCIDIACONO, G. "Sui Gruppi Ortogonali negli Spazi a Tre, Quattro, Cinque Dimensioni," Portugaliae Mathematica, Vol. 14, fasc. 2, 1955.

[12] ARCIDIACONO, G. "Sulle Trasformazioni Finite dei Gruppi delle Rotazioni," Coll. Math. XV, 1963, pp. 259-271.

[13] FANTAPPIÉ, L. "Sulle Funzioni di una Matrice," *Anais da Academia Brasileira de Ciências*, n. 1, 26 (1954).

[14] BAR-ITZHACK, I.Y. "Extension of Euler's Theorem to n-Dimensional Spaces," IEEE *Transactions on Aerospace and Electronic Systems*, T-AES/25/6//30692, Vol. 25, No. 6, November 1989, pp. 439-517.

[15] SHUSTER, M.D. "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, October-December 1993, pp. 439-517.

[16] MORTARI, D. "ESOQ: A Closed-Form Solution to the Wahba Problem," *Journal of the Astronautical Sciences*, Vol. 45, No. 2, April-June 1997, pp. 195-204.

[17] MORTARI, D. "n-Dimensional Cross Product and Its Application to Matrix Eigenanalysis," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 3, May-June 1997, pp. 509-515.