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An Algorithm for Solving Second Order Linear Homogeneous Differential Equations

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In this paper we present an algorithm for finding a "closed-form" solution of the differential equation $y''+ay'+by$, where a and b are rational functions of a complex variable x, provided a "closed-form" solution exists. The algorithm is so arranged that if no solution is found, then no solution can exist.

1. Introduction

In this paper we present an algorithm for finding a "closed-form" solution of the differential equation $y''+ay'+by$, where a and b are rational functions of a complex variable x, provided a "closed-form" solution exists. The algorithm is so arranged that if no solution is found, then no solution can exist.

The first section makes precise what is meant by "closed-form" and shows that there are four possible cases. The first three cases are discussed in sections 3, 4 and 5 respectively. The last case is the case in which the given equation has no "closed-form" solution. It holds precisely when the first three cases fail.

In the second section we present conditions that are necessary for each of the three cases. Although this material could have been omitted, it seems desirable to know in advance which cases are possible.

The algorithm in cases 1 and 2 is quite simple and can usually be carried out by hand, provided the given equation is relatively simple. However, the algorithm in case 3 involves quite extensive computations. It can be programmed on a computer for a specific differential equation with no difficulty. In fact, the author has worked through several examples using only a programmable calculator. Only in one example was a computer necessary, and this was because intermediate numbers grew to 20 decimal digits, more than the calculator could handle. Fortunately, the necessary conditions for case 3 are quite strong so this case can often be eliminated from consideration.

The algorithm does require that the partial fraction expansion of the coefficients of the differential equation be known, thus one needs to factor a polynomial in one variable over the complex numbers into linear factors. Once the partial fraction expansions are known, only linear algebra is required.

Using the MACSYMA computer algebra system, see, for example, Pavelle & Wang (1985), Bob Caviness and David Saunders of Rensselear Polytechnic Institute programmed the entire algorithm (see Saunders (1981)). Meanwhile, the algorithm has

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been implemented also in the MAPLE computer algebra system, see, for example, Char *et al.* (1985), by Carolyn Smith (1984).

This paper is arranged so that the algorithm may be studied independently of the proofs. In section 1, parts 1 and 2 are necessary to understand the algorithm, parts 3 and 4 are devoted to proofs. In the other sections, part 1 describes the algorithm, part 2 contains examples, and the remaining parts contain proofs.

Since the first appearance of this paper as a technical report, a number of papers have appeared on the same problem: Baldassarri (1980), Baldassarri & Dwork (1979), Singer (1979, 1981, 1985).

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1. The Four Cases

In the first part of this section we define precisely what we mean by "closed-form" solution. In the second part we state the four possible cases that can occur. These cases are treated individually in the latter sections. The third part is devoted to a brief description of the Galois theory of differential equations. This theory is used in the proofs of the theorems of the present chapter and those of sections 4 and 5. Part 4 contains a proof of the theorem stated in part 2.

1.1. LIOUVILLIAN EXTENSIONS

The goal of this paper is to find "closed-form" solutions of differential equations. By a "closed-form" solution we mean, roughly, one that can be written down by a first-year calculus student. Such a solution may involve esponentials, indefinite integrals and solutions of polynomial equations. (As we are considering functions of a complex variable, we need not explicitly mention trigonometric functions, they can be written in terms of exponentials. Note that logarithms are indefinite integrals and hence are allowed.) A more precise definition involves the notion of Liouvillian field.

DEFINITION. Let $\mathbf{F}$ be a differential field of functions of a complex variable x that contains $\mathbb{C}(x)$. (*Thus* $\mathbf{F}$ is a field and the derivation operator $'$ $(= d/dx)$ carries $\mathbf{F}$ into itself). $\mathbf{F}$ is said to be *Liouvillian* if there is a tower of differential fields

$$\mathbb{C}(x) = \mathbf{F}_0 \subseteq \mathbf{F}_1 \subseteq \cdots \subseteq \mathbf{F}_n = \mathbf{F}$$

such that, for each $i = 1, \ldots, n$,

either $\mathbf{F}_i = \mathbf{F}_{i-1}(\alpha)$ where $\alpha'/\alpha \in \mathbf{F}_{i-1}$
($\mathbf{F}_i$ is generated by an exponential of an integral over $\mathbf{F}_{i-1}$)

or $\mathbf{F}_i = \mathbf{F}_{i-1}(\alpha)$ where $\alpha' \in \mathbf{F}_{i-1}$
($\mathbf{F}_i$ is generated by an integral over $\mathbf{F}_{i-1}$)

or $\mathbf{F}_i$ is finite algebraic over $\mathbf{F}_{i-1}$.

A function is said to be *Liouvillian* if it is contained in some Liouvillian differential field. Suppose that η is a (non-zero) Liouvillian solution of the differential equation

$y''+ay'+by$, where $a, b \in \mathbb{C}(x)$. It follows that every solution of this differential equation is Liouvillian. Indeed, the method of reduction of order produces a second solution, namely $\eta \int (e^{-\int a}/\eta^2)$. This second solution is evidently Liouvillian and the two solutions are linearly independent. Thus any solution, being a linear combination of these two, is Liouvillian.

We may use a well-known change of variable to eliminate the term involving y' from the differential equation. Set $z = e^{\frac{1}{2}\int a} y$. Then $z'' + (b - \frac{1}{4}a^2 - \frac{1}{2}a')z = 0$. This new equation still has coefficients in $\mathbb{C}(x)$ and evidently y is Liouvillian if and only if z is Liouvillian. Thus no generality is lost by assuming that the term involving y' is missing from the differential equation.

1.2 THE FOUR CASES

In the remainder of this paper we shall consider the equation

$$y'' = ry, \quad r \in \mathbb{C}(x).$$

We shall refer to this equation as "the DE". To avoid triviality, we assume that $r \notin \mathbb{C}$. By a solution of the DE is always meant a non-zero solution.

THEOREM. *There are precisely four cases that can occur.*

Case 1. *The DE has a solution of the form $e^{\int \omega}$ where $\omega \in \mathbb{C}(x)$.*

Case 2. *The DE has a solution of the form $e^{\int \omega}$ where ω is algebraic over $\mathbb{C}(x)$ of degree 2, and case 1 does not hold.*

Case 3. *All solutions of the DE are algebraic over $\mathbb{C}(x)$ and cases 1 and 2 do not hold.*

Case 4. *The DE has no Liouvillian solution.*

It is evident that these cases are mutually exclusive, the theorem states that they are exhaustive. The proof of this theorem will be presented in part 1.4.

1.3. THE DIFFERENTIAL GALOIS GROUP

Here we present a brief summary of the Picard–Vessiot theory of differential equations (see Kaplansky (1957), or Chapter 6 of Kolchin (1973)), which is tailored specifically to the DE $y'' = ry$.

Suppose that η, ζ is a fundamental system of solutions of the DE (where η, ζ are functions of a complex variable x). Form the differential extension field $\mathbf{G}$ of $\mathbb{C}(z)$ generated by η, ζ, thus

$$\mathbf{G} = \mathbb{C}(x)\langle \eta, \zeta \rangle = \mathbb{C}(x)(\eta, \eta', \zeta, \zeta').$$

Then the Galois group of $\mathbf{G}$ over $\mathbb{C}(x)$, denoted by $G(\mathbf{G}/\mathbb{C}(x))$, is the group of all differential automorphisms of $\mathbf{G}$ that leave $\mathbb{C}(x)$ invariant. (An automorphism σ is differential if $\sigma(a') = (\sigma a)'$ for every $a \in \mathbf{G}$.) We refer the reader to the references cited above for a proof that the Fundamental Theorem of Galois Theory holds in this context.

There is an isomorphism of $G(\mathbf{G}/\mathbb{C}(x))$ with a subgroup of $GL(2)$, the group of invertible 2×2 matrices with coefficients in $\mathbb{C}$. Let $\sigma \in G(\mathbf{G}/\mathbb{C}(x))$. Then

$$(\sigma\eta)'' = \sigma(\eta'') = \sigma(r\eta) = r\sigma\eta.$$

Hence, $\sigma\eta$ is also a solution of the DE and so is a linear combination $\sigma\eta = a_\sigma\eta + c_\sigma\zeta$, a_σ, $c_\sigma \in \mathbb{C}$, of η, ζ. Similarly, $\sigma\zeta = b_\sigma\eta + d_\sigma\zeta$ for some b_σ, $d_\sigma \in \mathbb{C}$.

$$c : \sigma \to \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$$

is immediately seen to be an injective group homomorphism.

This representation $c: G(\mathbf{G}/\mathbb{C}(x)) \to GL(2)$ certainly does depend on the choice of fundamental system η, ζ. If η_1, ζ_1 is another fundamental system, then there is a matrix $X \in GL(2)$ such that $(\eta_1, \zeta_1) = (\eta, \zeta)X$. Therefore,

$$\mathbf{G} = \mathbb{C}(x)\langle\eta, \zeta\rangle = \mathbb{C}(x)\langle\eta_1, \zeta_1\rangle \quad \text{and} \quad c_1(\sigma) = X^{-1}c(\sigma)X.$$

The representation $G(\mathbf{G}/\mathbb{C}(x)) \to GL(2)$ is determined by the DE only up to conjugation. By abuse of language, we allow ourselves to speak of any one of these conjugate groups as the Galois group of the DE. If a fundamental system η, ζ is fixed, then we refer to $c(G(\mathbf{G}/\mathbb{C}(x))) \subseteq GL(2)$ as the Galois group of the DE relative to η, ζ.

Fix a fundamental system η, ζ of solutions of the DE and let $G \subseteq GL(2)$ be the Galois group relative to η, ζ. Let $W = \eta\zeta' - \eta'\zeta$ be the Wronskian of η, ζ. A simple computation, using the DE, shows that $W' = 0$, so W is a (non-zero) constant and is left fixed by any element of $G(\mathbf{G}/\mathbb{C}(x))$. Let $\sigma \in G(\mathbf{G}/\mathbb{C}(x))$, then, using the notation above,

$$\begin{aligned} W = \sigma W &= (a_\sigma\eta + c_\sigma\zeta)(b_\sigma\eta' + d_\sigma\zeta') - (a_\sigma\eta' + c_\sigma\zeta')(b_\sigma\eta + d_\sigma\zeta) \\ &= (a_\sigma d_\sigma - b_\sigma c_\sigma)W = \det c(\sigma) \cdot W. \end{aligned}$$

Thus $G \subseteq SL(2)$, the group of 2×2 matrices with determinant 1.

Recall that a subgroup G of $GL(2)$ is an algebraic group if there exist a finite number of polynomials

$$P_1, \ldots, P_r \in \mathbb{C}[X_1, X_2, X_3, X_4] \quad \text{such that} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

if and only if

$$P_1(a, b, c, d) = \cdots = P_r(a, b, c, d) = 0.$$

One of the principal facts in the Picard–Vessiot theory is that the Galois group of a differential equation is an algebraic group. For a proof in all generality, see the references cited above. Here we sketch a proof in the special case that we are considering.

Let Y, Z, Y_1, Z_1 be indeterminates over $\mathbb{C}(x)$ and consider the substitution homomorphism

$$\mathbb{C}[x, Y, Z, Y_1, Z_1] \to \mathbb{C}[x, \eta, \zeta, \eta', \zeta'].$$

The kernel of this mapping is a prime ideal $\mathbf{p}$. Any element

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $SL(2)$ induces an automorphism of $\mathbb{C}[x, Y, Z, Y_1, Z_1]$ over $\mathbb{C}[x]$ by the formula

$$(Y, Z, Y_1, Z_1) \to (aY + cZ, bY + dZ, aY_1 + cZ_1, bY_1 + dZ_1).$$

Moreover, $A \in G$ if and only if $\mathbf{p}$ is carried into itself. The ideal $\mathbf{p}$ is finitely generated, say $\mathbf{p} = (q_1, \ldots, q_s)$, where $q_1, \ldots, q_s$ are linearly independent over $\mathbb{C}$. Let n be the maximum of the degrees of $q_1, \ldots, q_s$ in x, Y, Z, Y_1, Z_1 and let V be the vector space over $\mathbb{C}$ of all polynomials in $\mathbb{C}[x, Y, Z, Y_1, Z_1]$ of degree n or less. Evidently the action of $SL(2)$ on

$\mathbb{C}[x, Y, Z, Y_1, Z_1]$ restricts to V. If $q_1, \ldots, q_s, q_{s+1}, \ldots, q_t$ is a basis of V, then there exist polynomials $P_{ij} \in \mathbb{C}[X_1, X_2, X_3, X_4]$ such that the result of the action of A on q_i is

$$\sum_{j=1}^{t} P_{ij}(a, b, c, d)q_j.$$

It follows that $A \in G$ if and only if $P_{ij}(a, b, c, d) = 0$ for $i = 1, \ldots, s$, $j = s+1, \ldots, t$. Therefore G is an algebraic group.

1.4. PROOF

In this section we shall prove the theorem that was stated in 1.2. We shall use several facts about algebraic groups. Suitable references are Borel (1956), Kaplansky (1957), and Chapter 5 of Kolchin (1973). The following result is contained in Kaplansky (1957, p. 31).

LEMMA. *Let G be an algebraic subgroup of $SL(2)$. Then one of four cases can occur.*

Case 1. *G is triangulisable.*

Case 2. *G is conjugate to a subgroup of*

$$D^{\dagger} = \left\{\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in \mathbb{C}, c \neq 0\right\} \cup \left\{\begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| c \in \mathbb{C}, c \neq 0\right\},$$

and case 1 *does not hold.*

Case 3. *G is finite and cases* 1 *and* 2 *do not hold.*

Case 4. $G = SL(2)$.

Proof. Denote the component of the identity of G by G°. First we note that any two-dimensional Lie algebra is solvable, hence either dim $G = 3$ (in which case $G = SL(2)$) or else G° is solvable. In the latter case, G° is triangulisable by the Lie–Kolchin Theorem. Assume that G° is triangular.

If G° is not diagonalisable, then G° contains a matrix of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \neq 0$ (since an algebraic group contains the unipotent and semi-simple parts of all of its elements). Since G° is normal in G, any matrix in G conjugates $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ into a triangular matrix. A direct computation shows that only triangular matrices have this property. Thus G itself is triangular. This is case 1.

Assume next that G° is diagonal and infinite, so G° contains a non-scalar diagonal matrix A. Because G° is normal in G, any element of G conjugates A into a diagonal matrix. A direct computation shows that any matrix with this property must be contained in $D^{\dagger}$. Therefore either G is diagonal, this being case 1, or else G is contained in $D^{\dagger}$, this being case 2.

Finally we observe that if G° is finite (and therefore $G^{\circ} = \{1\}$), then G must also be finite. This is case 3. This proves the lemma.

We shall now prove the theorem of section 2.

Let η, ζ be a fundamental system of solutions of the DE and let G be the Galois group relative to η, ζ. Set $\mathbf{G} = \mathbb{C}(x)\langle \eta, \zeta \rangle$.

Case 1. G is triangulisable. We may assume that G is triangular. Then, for every

$\sigma \in G(\mathbf{G}/\mathbb{C}(x))$, $\sigma\eta = c_\sigma \eta$, where $c_\sigma \in \mathbb{C}$, $c_\sigma \neq 0$. Therefore $\sigma\omega = \omega$, where $\omega = \eta'/\eta$, which implies that $\omega \in \mathbb{C}(x)$.

Case 2. G is conjugate to be a subgroup of $D^\dagger$. We may assume that G is a subgroup of $D^\dagger$. If $\omega = \eta'/\eta$ and $\phi = \zeta'/\zeta$, then, for every $\sigma \in G(\mathbf{G}/\mathbb{C}(x))$, either $\sigma\omega = \omega$, $\sigma\phi = \phi$ or $\sigma\omega = \phi$, $\sigma\phi = \omega$. Thus ω is quadratic over $\mathbb{C}(x)$.

Case 3. G is finite. In this case $\mathbf{G}$ has only a finite number of differential automorphisms $\sigma_1, \ldots, \sigma_n$. Since the elementary symmetric function of $\sigma_1\eta, \ldots, \sigma_n\eta$ are invariant under $G(\mathbf{G}/\mathbb{C}(x))$, η is algebraic over $\mathbb{C}(x)$. Similarly, ζ is algebraic over $\mathbb{C}(x)$. Because every solution of the DE is contained in $\mathbf{G}$, every solution of the DE is algebraic.

Case 4. $G = SL(2)$. Suppose that the DE had a Liouvillian solution. Then, as pointed out in 1.1, every solution of the DE is Liouvillian. Thus $\mathbf{G}$ is contained in a Liouvillian field. It follows that G° is solvable (Kolchin, 1973, p. 415). Since $G^\circ = SL(2)$ is not solvable, the DE can have no Liouvillian solution.

This proves the theorem.

2. Necessary Conditions

In this section we discuss some easy conditions that are necessary for cases 1, 2, or 3 to hold. These conditions give a sufficient condition for case 4 to hold (namely when the necessary conditions for cases 1, 2, and 3 fail). Throughout, we shall consider the DE $y'' = ry$, $r \in \mathbb{C}(x)$.

2.1. THE NECESSARY CONDITIONS

Since r is a rational function, we may speak of the poles of r, by which we shall always mean the poles in the finite complex plane $\mathbb{C}$. If $r = s/t$, with $s, t \in \mathbb{C}[x]$, relatively prime, then the poles of r are the zeros of t and the order of the pole is the multiplicity of the zero of t. By the order of r at ∞ we shall mean the order of ∞ as a zero of r, thus the order of r at ∞ is $\deg t - \deg s$.

THEOREM. *The following conditions are necessary for the respective cases to hold.*

Case 1. *Every pole of r must have even order or else have order* 1. *The order of r at ∞ must be even or else be greater than* 2.

Case 2. *r must have at least one pole that either has odd order greater than* 2 *or else has order* 2.

Case 3. *The order of a pole of r cannot exceed* 2 *and the order of r at ∞ must be at least* 2. *If the partial fraction expansion of r is*

$$r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j},$$

then $\sqrt{1+4\alpha_i} \in \mathbb{Q}$, for each i, $\sum_j \beta_j = 0$, and if

$$\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j,$$

then $\sqrt{1+4\gamma} \in \mathbb{Q}$.

2.2. EXAMPLES

Airey's Equation $y''=xy$ has no Liouvillian solution (i.e. case 4 holds). This is clear because the necessary conditions for cases 1, 2, and 3 all fail. More generally, $y''=Py$, where $P\in\mathbb{C}[x]$ has odd degree, has no Liouvillian solution.

For Bessel's Equation

$$y''=\frac{4(n^2-x^2)-1}{4x^2}\,y,\quad n\in\mathbb{C}$$

(in self-adjoint form), only cases 1, 2, and 4 are possible.

For Weber's Equation

$$y''=(\tfrac{1}{4}x^2-\tfrac{1}{2}-n)y,\quad n\in\mathbb{C},$$

only cases 1 and 4 are possible.

2.3. PROOF

In this section we prove the theorem of Section 1.

Case 1. In this case the DE has a solution of the form $\eta=e^{\int\omega}$ where $\omega\in\mathbb{C}(x)$. Since $\eta''=r\eta$, it follows that $\omega'+\omega^2=r$ (the Riccatti Equation). Both ω and r have Laurent series expansions about any point c of the complex plane, for ease of notation we take $c=0$. Say

$$\omega=bx^\mu+\cdots,\quad \mu\in\mathbb{Z},\quad b\neq 0$$
$$r=\alpha x^\nu+\cdots,\quad \nu\in\mathbb{Z},\quad \alpha\neq 0.$$

(The dots represent terms involving x raised to powers higher than that shown.) Using the Riccatti Equation, we find that

$$\mu bx^{\mu-1}+\cdots+b^2x^{2\mu}+\cdots=\alpha x^\nu+\cdots.$$

As we need to show that every pole of r either has order 1 or else has even order, we may assume that $\nu\leqslant -3$. Since $\alpha\neq 0$, $-3\geqslant\nu\geqslant\min(\mu-1,2\mu)$. It follows that $\mu<-1$ and $2\mu<\mu-1$. Since $b^2\neq 0$, $2\mu=\nu$, which implies that ν is even. For use in section 3.3, we remark that if r has a pole of order $2\mu\geqslant 4$ at c, then ω must have a pole of order μ at c.

Now consider the Laurent series expansions of r and ω at ∞.

$$\omega=bx^\mu+\cdots,\quad \mu\in\mathbb{Z},\quad b\neq 0$$
$$r=\alpha x^\nu+\cdots,\quad \nu\in\mathbb{Z},\quad a\neq 0.$$

(The dots represent terms involving x raised to a power lower than that shown. The order of r at ∞ is $-\nu$.) As we need to show that either the order of r at ∞ is $\geqslant 3$ or else is even, we may assume that $\nu\geqslant -1$. Using the Riccatti Equation, we have

$$\mu bx^{\mu-1}+\cdots+b^2x^{2\mu}+\cdots=\alpha x^\nu+\cdots.\ .$$

Just as above, $-1\leqslant\nu\leqslant\max(\mu-1,2\mu)$, $\mu>-1$, $2\mu>\mu-1$. Since $b^2\neq 0$, $2\mu=\nu$, so ν is even. For use in section 3,3, we remark that if r has a pole of order $2\mu\geqslant 0$ at ∞, then ω has a pole of order μ at ∞.

This verifies the necessary conditions for case 1.

Case 2. We analyse this case by considering the differential Galois group that must obtain. By section 1.4 the group must be conjugate to a subgroup G of $D^\dagger$, which is not triangulisable (otherwise case 1 would hold). Let η,ζ be a fundamental system of solutions of the DE relative to the group G. For every $\sigma \in G(\mathbf{G}/\mathbb{C}(x))$, either $\sigma\eta = c_\sigma \eta$, $\sigma\zeta = c_\sigma^{-1}\zeta$ or $\sigma\eta = -c_\sigma^{-1}\zeta$, $\sigma\zeta = c_\sigma \eta$. Evidently $\eta^2\zeta^2$ is an invariant of $G(\mathbf{G}/\mathbb{C}(x))$ and therefore $\eta^2\zeta^2 \in \mathbb{C}(x)$. Moreover, $\eta\zeta \notin \mathbb{C}(x)$, for otherwise G would be a subgroup of the diagonal group, which is case 1.

Writing

$$\eta^2\zeta^2 = \prod (x-c_i)^{e_i} \quad (e_i \in \mathbb{Z}),$$

we have that at least one exponent e_i is odd. Without loss of generality we may assume that

$$\eta^2\zeta^2 = x^e \prod (x-c_i)^{e_i}$$

and that e is odd. Let

$$\theta = (\eta\zeta)'/(\eta\zeta) = \tfrac{1}{2}(\eta^2\zeta^2)'/(\eta^2\zeta^2) = \tfrac{1}{2}ex^{-1} + \cdots,$$

where the dots represent terms involving x to non-negative powers. Since $\eta'' = r\eta$ and $\zeta'' = r\zeta$,

$$\theta'' + 3\theta'\theta + \theta^3 = 4r\theta + 2r'.$$

Let $r = \alpha x^\nu + \cdots$ be the Laurent series expansion of r at 0, where $\alpha \neq 0$ and $\nu \in \mathbf{Z}$. From the equation above we obtain

$$(e - \tfrac{3}{4}e^2 + \tfrac{1}{8}e^3)x^{-3} + \cdots = 2\alpha(e+\nu)x^{-1} + \cdots.$$

If $\nu > -2$, then $0 = 8e - 6e^2 + e^3 = e(e-2)(e-4)$. This contradicts the fact that e is odd. Therefore $\nu \leqslant -2$. If $\nu < -2$, then $e + \nu = 0$, so ν is odd.

This verifies the necessary conditions for case 2.

Case 3. In this case the DE has a solution η that is algebraic over $\mathbb{C}(x)$. η has a Puiseaux series expansion about any point c in the complex plane, for ease of notation we take $c = 0$. Then $\eta = ax^\mu + \cdots$, where $a \in \mathbb{C}$, $a \neq 0$, $\mu \in \mathbb{Q}$. Since $r \in \mathbb{C}(x)$, $r = \alpha x^\nu + \cdots$, where $\alpha \neq 0$ and $\nu \in \mathbb{Z}$. The DE implies that

$$\mu(\mu-1)ax^{\mu-2} + \cdots = \alpha a x^{\nu+\mu} + \cdots.$$

It follows that $\nu \geqslant -2$, i.e. r has no pole of order greater than 2. If $\nu = -2$, then $\mu(\mu-1) = \alpha$. Because $\mu \in \mathbb{Q}$, we must have $\sqrt{1+4\alpha} \in Q$.

So far we have shown that the partial fraction expansion of r has the form

$$r = \sum_i \frac{\alpha_i}{(x-x_i)^2} + \sum_j \frac{\beta_j}{x-d_j} + P,$$

where $P \in \mathbb{C}[x]$ and $\sqrt{1+4\alpha_i} \in \mathbb{Q}$ for each i.

Next, we consider the series expansions about ∞,

$$\eta = ax^\mu + \cdots, \quad r = \gamma x^\nu + \cdots,$$

where the dots represent lower powers of x than those shown. From the DE we obtain

$$\mu(\mu-1)ax^{\mu-2} + \cdots = \nu\gamma a x^{\nu+\mu} + \cdots.$$

Just as above, we obtain $\nu \leqslant -2$ and therefore $P = 0$. But

$$r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j}$$
$$= (\sum_j \beta_j)x^{-1} + \gamma x^{-2} + \cdots,$$

where $\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j$. Therefore $\sum_j \beta_j = 0$ and $\mu(\mu-1) = \gamma$. Since $\mu \in \mathbb{Q}$, $\sqrt{1+4\gamma} \in \mathbb{Q}$.

This completes the proof of the theorem stated in section 2.1.

3. The Algorithm for Case 1

The first part of this section is devoted to a description of the algorithm. It is somewhat complicated to describe in full generality, yet, as the examples in part 2 show, it is often quite easy to apply. The third part is devoted to a proof that the algorithm is correct.

3.1. DESCRIPTION OF THE ALGORITHM

The goal of this algorithm is to find a solution of the DE of the form $\eta = Pe^{\int \omega}$, where $P \in \mathbb{C}[x]$ and $\omega \in \mathbb{C}(x)$. Since η may be written as $\eta = e^{\int (P'/P + \omega)}$, this is of the form described in section 1.2. The first step on the algorithm consists of determining "parts" of the partial fraction expansion of ω. In the second step we put these "parts" together to form a candidate for ω. The maximum number of candidates possible is $2^{\rho+1}$ where ρ is the number of poles of r. If there are no candidates, then case 1 cannot hold. The third and last step is applied to each candidate for ω and consists of searching for a suitable polynomial P. If one is found, then we have the desired solution of the DE. If, for each candidate for ω, we fail to find a suitable P, then case 1 cannot hold.

We assume that the necessary condition of section 2.1 for case 1 holds, and we denote by Γ the set of poles of r.

Step 1. For each $c \in \Gamma \cup \{\infty\}$ we define a rational function $[\sqrt{r}]_c$ and two complex numbers α_c^+, α_c^- as described below.

(c_1) If $c \in \Gamma$ and c is a pole of order 1, then

$$[\sqrt{r}]_c = 0, \quad \alpha_c^+ = \alpha_c^- = 1.$$

(c_2) If $c \in \Gamma$ and c is a pole of order 2, then

$$[\sqrt{r}]_c = 0.$$

Let b be the coefficient of $1/(x-c)^2$ in the partial fraction expansion for r. Then

$$\alpha_c^{\pm} = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4b}.$$

(c_3) If $c \in \Gamma$ and c is a pole of order $2\nu \geqslant 4$ (necessarily even by the conditions of section 2.1), then $[\sqrt{r}]_c$ is the sum of terms involving $1/(x-c)^i$ for $2 \leqslant i \leqslant \nu$ in the Laurent series expansion of $\sqrt{r}$ at c. There are two possibilities for $[\sqrt{r}]_c$, one being the negative of the other, either one may be chosen. Thus

$$[\sqrt{r}]_c = \frac{a}{(x-c)^\nu} + \cdots + \frac{d}{(x-c)^2}.$$

In practice, one would not form the Laurent series for $\sqrt{r}$, but rather would determine $[\sqrt{r}]_c$ by using undetermined coefficients. Let b be the coefficient of $1/(x-c)^{\nu+1}$ in r minus the coefficient of $1/(x-c)^{\nu+1}$ in $([\sqrt{r}]_c)$. Then

$$\alpha_c^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a}+\nu\right).$$

(∞_1) If the order of r at ∞ is >2, then

$$[\sqrt{r}]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1.$$

(∞_2) If the order of r at ∞ is 2, then

$$[\sqrt{r}]_\infty = 0.$$

Let b be the coefficient of $1/x^2$ in the Laurent series expansion of r at ∞. (If $r = s/t$, where $s, t \in \mathbb{C}[x]$ are relatively prime, then b is the leading coefficient of s divided by the leading coefficient of t.) Then

$$\alpha_\infty^{\pm} = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4b}.$$

(∞_3) If the order of r at ∞ is $-2\nu \leqslant 0$ (necessarily even by the conditions of section 2.1), then $[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leqslant i \leqslant \nu$ in the Laurent series for $\sqrt{r}$ at ∞. (Either one of the two possibilities may be chosen.) Thus

$$[\sqrt{r}]_\infty = ax^\nu + \cdots + d.$$

Let b be the coefficient of $x^{\nu-1}$ in r minus the coefficient of $x^{\nu-1}$ in $([\sqrt{r}]_\infty)^2$. Then

$$\alpha_\infty^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a}-\nu\right).$$

Step 2. For each family $s = (s(c))_{c \in \Gamma \cup \{\infty\}}$, where $s(c)$ is $+$ or $-$, let

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}.$$

If d is a non-negative integer, then

$$\omega = \sum_{c \in \Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

is a candidate for ω. If d is not a non-negative integer, then the family s may be removed from consideration.

Step 3. This step should be applied to each of the families retained from Step 2, until success is achieved or the supply of families has been exhausted. In the latter event, case 1 cannot hold.

For each family, search for a monic polynomial P of degree d (as defined in Step 2) that satisfies the differential equation

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0.$$

This is conveniently done by using undetermined coefficients and is a simple problem in linear algebra, which may or may not have a solution. If such a polynomial exists, then

$\eta = Pe^{\int\omega}$ is a solution of the DE. If no such polynomial is found for any family retained from Step 2, then Case 1 cannot hold.

3.2. EXAMPLES

Example 1. Consider the DE $y'' = ry$ where

$$r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x}$$
$$= x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}.$$

Since r has a single pole (at 0) and the order there is 4, the necessary conditions of section 2.1 for case 2 do not hold. Evidently the necessary conditions for case 3 also do not hold. We apply the algorithm for case 1 to this DE.

The order of r at the pole 0 is $2\nu = 4$. Therefore $[\sqrt{r}]_0 = a/x^2$, and $a^2 = 1$. We choose $a = 1$, so $[\sqrt{r}]_0 = 1/x^2$. $b = -5 - 0 = -5$, and therefore $\alpha_0^{\pm} = \frac{1}{2}(\pm(-5/1) + 2)$, which gives $\alpha_0^+ = -3/2$ and $\alpha_0^- = 7/2$.

At ∞, $\nu = 1$, and $[\sqrt{r}]_\infty = ax + d$. Comparing r and $[\sqrt{r}]_\infty^2 = a^2x^2 + 2adx + d^2$ we see that $a^2 = 1$ and $2ad = -2$. We choose $a = 1$, $d = -1$. Thus $[\sqrt{r}]_\infty = x - 1$. $b = 3 - 1 = 2$, and $\alpha^{+\infty} = 1/2$, $\alpha_\infty^- = -3/2$.

There are four families to consider.

$s(0) = +,$	$s(\infty) = +,$	$d = 1/2 - (-3/2) \quad = 2$
$s(0) = +,$	$s(\infty) = -,$	$d = -3/2 - (-3/2) = 0$
$s(0) = -,$	$s(\infty) = +,$	$d = 1/2 - 7/2 \quad = -3$
$s(0) = -,$	$s(\infty) = -,$	$d = -3/2 - 7/2 \quad = -5.$

Only the first two remain for consideration.

We shall treat the second family first, since $d = 0$ in that case. The candidate for ω is

$$\omega = [\sqrt{r}]_0 + \frac{\alpha_0^+}{x} - [\sqrt{r}]_\infty = \frac{1}{x^2} - \frac{3}{2x} - x + 1.$$

We now search for a monic polynomial P of degree 0 such that

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0.$$

Since $P = 1$, the existence of P is a question of whether or not $\omega' + \omega^2 - r = 0$. But the coefficient of $1/x$ in $\omega' + \omega^2 - r$ is -6. Thus no such polynomial P can exist.

The only remaining family is the first family. The candidate for ω is

$$\omega = [\sqrt{r}]_0 + \frac{\alpha_0^+}{x} + [\sqrt{r}]_\infty = \frac{1}{x^2} - \frac{3}{2x} + x - 1.$$

We now search for a monic polynomial P of degree 2 that satisfies the linear differential equation given above. Writing $P = x^2 + ax + b$, we easily determine that $a = 0$, $b = -1$ provides a solution.

Therefore a solution of the DE is given by

$$\eta = Pe^{\int\omega} = (x^2 - 1)e^{\int(1/x^2 - 3/(2x) + x - 1)}$$
$$= x^{-3/2}(x^2 - 1)e^{-1/x + x^2/2 - x}.$$

Example 2. In this example we begin the discussion of Bessel's Equation

$$y'' = \left(\frac{4n^2-1}{4x^2} - 1\right)y, \quad n \in \mathbb{C}.$$

The necessary conditions of section 2.1 imply that case 3 cannot hold. Here we consider case 1, case 2 is worked out in section 3.2.

The only pole of r is at $c=0$ and the order there is 2. Thus

$$[\sqrt{r}]_0 = 0, \quad b = (4n^2-1)/4, \quad \alpha_0^{\pm} = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4b} = \tfrac{1}{2} \pm n.$$

At ∞, r has order 0 and $[\sqrt{r}]_\infty = i$. Evidently $b=0$ so $\alpha_\infty^{\pm} = 0$.

There are four families to consider.

$$\begin{array}{lll} s(0) = +, & s(\infty) = +, & d = -1/2 - n \\ s(0) = +, & s(\infty) = -, & d = -1/2 - n \\ s(0) = -, & s(\infty) = +, & d = -1/2 + n \\ s(0) = -, & s(\infty) = -, & d = -1/2 + n. \end{array}$$

A necessary condition that case 1 holds is that $-1/2 \pm n$ be a non-negative integer, i.e. that n be half an odd integer. We claim that this condition is also sufficient.

If n is negative, and half an odd integer, then $m = -1/2 - n \in N$. This corresponds to the first family, in which case $\omega = -m/x + i$. We need to find a polynomial P of degree m such that

$$\begin{aligned} 0 &= P'' + 2\omega P' + (\omega' + \omega^2 - r)P \\ &= P'' + 2\left(-\frac{m}{x} + i\right)P' - \frac{2im}{x}P. \end{aligned}$$

It is straightforward to verify that

$$P = \sum_{j=0}^{m} \frac{1}{(-2i)^{m-j}} \frac{(2m-j)!}{j!\,(m-j)!} x^j$$

is the desired polynomial. A solution to Bessel's Equation is given by $\eta = x^{-m}Pe^{ix}$.

If n is positive, then $m = -1/2 + n$ is a non-negative integer. This corresponds to the third family. In this case $\omega = -m/x + i$, and we are back to the case considered above.

Example 3. In this example we treat the general situation where r is a polynomial of degree 2. We may write $r = (ax+d)^2 + b$ for some $a, b, d \in \mathbb{C}$ (a and d are determined by r only up to sign, we choose either of the two possibilities). We claim that the DE has a Liouvillian solution if and only if b/a is an odd integer.

The necessary condition of section 2.1 implies that only cases 1 and 4 are possible. We consider case 1.

Evidently $[\sqrt{r}]_\infty = ax + d$ and $\alpha_\infty^{\pm} = \tfrac{1}{2}(\pm(b/a) - 1)$. There are no poles. Thus d equals α_∞^+ or α_∞^-, so one of these two numbers must be a non-negative integer for case 1 to hold. It follows that b/a must be an odd integer, which is the necessity part of our claim.

For sufficiency, we may assume that $b/a = 2n+1$ is positive, since a may be replaced by $-a$. Case 1 will hold provided that there is a monic polynomial P of degree n such that

$$\begin{aligned} 0 &= P'' + 2\omega P' + (\omega' + \omega^2 - r)P \\ &= P'' + 2(ax+d)P' - 2naP. \end{aligned}$$

If we write

$$P = \sum_{i=0}^{n} P_i x^i$$

and substitute, we obtain a system of linear equations in $P_0, \ldots, P_{n-1}$ ($P_n = 1$) that has a solution, namely

$$P_i = \frac{(2n+1)(i+1)}{n-i} P_{i+1} + \frac{(i+2)(i+1)}{2a(n-i)} P_{i+2} \quad (i = n-1, \ldots, 0)$$

where $P_{n+1} = 0$ and $P_n = 1$.

A special case of this example is Weber's Equation

$$y'' = (\tfrac{1}{4}x^2 - \tfrac{1}{2} - n)y, \quad n \in \mathbb{C}.$$

Here $a = -1/2$, $b = -1/2 - n$, $d = 0$. Thus $b/a = 2n+1$ is an odd integer if and only if n is an integer.

3.3. PROOF

In case 1, the DE has a solution of the form $\eta = e^{\int \theta}$, with $\theta \in \mathbb{C}(x)$. Since $\eta'' = r\eta$, we have

$$\theta' + \theta^2 = r \quad \text{(Riccatti Equation)}.$$

We shall determine the partial fraction expansion of θ using the Laurent series expansion of r and this Riccatti Equation.

For $c \in \mathbb{C}$, we denote the "component at c" of the partial fraction expansion of θ by

$$[\theta]_c + \frac{\alpha}{x-c} = \sum_{i=2}^{\nu} \frac{a_i}{(x-c)^i} + \frac{\alpha}{x-c}.$$

In order to simplify the notation, we assume that $c = 0$ and drop the subscript "0". We shall also need to consider the Laurent series expansion of θ about 0

$$\theta = [\theta] + \frac{\alpha}{x} + \bar{\theta}.$$

Here $\bar{\theta} = * + *x + \cdots$, where the * denotes a complex number whose particular value is irrelevant to our discussion.

We assume that the necessary conditions for case 1 (see section 2.1) are satisfied, in particular we assume that the poles of r are either of even order or else of order 1. We split our proof into parts, depending on the nature of r at 0. This parallels the division of Step 1 of the algorithm.

(c_1) Suppose that 0 is a pole of r of order 1, so $r = */x + \cdots$. The Riccati equation becomes

$$-\frac{\nu a_\nu}{x^{\nu+1}} + \cdots + \frac{a_\nu^2}{x^\nu} + \cdots = \frac{*}{x} + \cdots.$$

Since $a_\nu^2 \neq 0$, $\nu \leqslant 1$ and $[\theta] = 0$.

Substituting $\theta = \alpha/x + \bar{\theta}$ into the Riccati Equation, we have

$$-\frac{\alpha}{x^2} + \bar{\theta}' + \frac{\alpha^2}{x^2} + \frac{2\alpha}{x}\bar{\theta} + \bar{\theta}^2 = \frac{*}{x} + \cdots.$$

Therefore $-\alpha+\alpha^2=0$, so $\alpha=0$ or $\alpha=1$. Were $\alpha=0$, the left-hand side of this equation would have 0 as an ordinary point; however, the right-hand side has a pole at 0. We conclude that $\alpha=1$ and the component of the partial fraction expansion of θ at 0 is (in the notation of the algorithm)

$$\frac{\alpha^{\pm}}{x}, \quad \text{where } \alpha^{\pm}=1.$$

(c_2) Suppose that r has a pole at 0 of order 2, say

$$r=\frac{b}{x^2}+\frac{*}{x}+\cdots.$$

As in (c_1), $[\theta]=0$ and $-\alpha+\alpha^2=b$. Thus the component of the partial fraction expansion of θ at 0 is

$$\frac{\alpha^{\pm}}{x}, \quad \text{where } \alpha^{\pm}=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{1+4b}.$$

(c_3) Suppose that r has a pole at 0 of order $2\mu\geqslant 4$. In section 2.3, we showed that $\nu=\mu$. Recall from section 3.1 that

$$[\sqrt{r}]=\frac{a}{x^{\nu}}+\cdots+\frac{*}{x^2},$$

where we have dropped the subscript "0".

Let $\bar{r}=\sqrt{r}-[\sqrt{r}]$. Then $r=[\sqrt{r}]^2+2\bar{r}[\sqrt{r}]+\bar{r}^2$. From the Riccatti Equation we obtain the following formula

$$(\&)\qquad \begin{aligned}([\theta]-[\sqrt{r}])\cdot([\theta]+[\sqrt{r}])& \\ =-[\theta]'+\frac{\alpha}{x^2}-\bar{\theta}'-\frac{2\alpha}{x}[\theta]-2\bar{\theta}[\theta]& \\ -\frac{\alpha^2}{x^2}-\frac{2\alpha}{x}\bar{\theta}-\bar{\theta}^2+2\bar{r}[\sqrt{r}]+\bar{r}^2.&\end{aligned}$$

An examination of the right-hand side of this equation determines that it is free of terms involving $1/x^i$ for $i=\nu+2,\ldots,2$ (since $\nu\geqslant 1$). This implies that the left-hand side is 0. Indeed, since

$$([\theta]-[\sqrt{r}])+([\theta]+[\sqrt{r}])=2[\theta],$$

at least one of the factors involves $1/x^{\nu}$. Were the other factor non-zero, it would involve $1/x^i$ for some $i\geqslant 2$. The product would then involve $1/x^{\nu+i}$ for some $i\geqslant 2$, which is absurd. Hence $[\theta]=\pm[\sqrt{r}]$.

The coefficient of $1/x^{\nu+1}$ in the right-hand side of (&) is $\pm\nu a\mp 2\alpha a+b$, where b is the coefficient of $1/x^{\nu+1}$ in $2\bar{r}[\sqrt{r}]+\bar{r}^2=r-[\sqrt{r}]^2$. Therefore $\alpha^{\pm}=\frac{1}{2}(\pm b/a+\nu)$. We have shown that if 0 is a pole of r of order $2\nu\geqslant 4$, then the component of the partial fraction expansion of θ at 0 is

$$\pm[\sqrt{r}]+\frac{\alpha^{\pm}}{x}, \quad \text{where} \quad \alpha^{\pm}=\frac{1}{2}\left(\pm\frac{b}{a}+\nu\right).$$

(c_4) Finally, we must consider what happens when 0 is an ordinary point of r. As in (c_1), $[\theta]=0$ and $-\alpha+\alpha^2=0$. Contrary to the situation in (c_1), however, we cannot conclude that $\alpha\neq 0$. Hence the component of the partial fraction expansion of r at 0 is either 0 or $1/x$.

We collect together what we have proven so far. Let Γ be the set of poles of r. Then

$$\theta = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + \sum_{i=1}^{d}\frac{1}{x-d_i} + R,$$

where $R \in \mathbb{C}[x]$, $s(c) = +$ or $-$, and $[\sqrt{r}]_c$, $\alpha_c^{s(c)}$ are as in the statement of the algorithm.
Next we consider the Laurent series about ∞. Suppose that

$$\theta = R + \frac{\alpha_\infty}{x} + \cdots.$$

(∞_1) If r has order $\nu > 2$ at ∞, then

$$r = \frac{*}{x^\nu} + \frac{*}{x^{\nu+1}} + \cdots.$$

The Riccatti Equation implies that $R = 0$ and $-\alpha_\infty + \alpha_\infty^2 = 0$, so $\alpha_\infty = 0$ or 1.
(∞_2) If r has order 2 at ∞, then

$$r = \frac{b}{x^2} + \frac{*}{x^3} + \cdots.$$

The Riccatti Equation implies that $R = 0$ *and* $-\alpha_\infty + \alpha_\infty^2 = b$, hence

$$\alpha_\infty = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4b}.$$

(∞_3) In the other cases, the order of r at ∞ must be even, by the necessary conditions of section 2. Following an argument similar to that used in (c_3) we find that

$$R = \pm[\sqrt{r}]_\infty, \qquad \alpha_\infty = \frac{1}{2}\left(\pm\frac{b}{a} - \nu\right),$$

where -2ν is the order of r at ∞, a is the leading coefficient of $[\sqrt{r}]_\infty$ and b is the coefficient of $1/x^{\nu-1}$ in $r - [\sqrt{r}]_\infty^2$.
We now know that the partial fraction expansion of θ has the form

$$\theta = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty + \sum_{i=1}^{d}\frac{1}{x-d_i}.$$

Moreover, the coefficient of $1/x$ in the Laurent series expansion of θ at ∞ is $\alpha_\infty^{s(\infty)}$. Thus

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)} \in \mathbb{N}.$$

Let

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty,$$

and

$$P = \prod_{i=1}^{d}(x-d_i).$$

Then $\theta = \omega + P'/P$. Again, using the Riccatti Equation, we obtain

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0.$$

The converse, namely that if P is a solution of this equation, then θ satisfies the Riccatti Equation, is a simple verification. It follows that if P is a solution of this equation, then $\eta = Pe^{\int\omega}$ is a solution of the DE $y'' = ry$.
This proves that the algorithm for case 1 is correct.

4. The Algorithm for Case 2

Following the pattern of section 3, we shall describe the algorithm in section 4.1, give examples in section 4.2 and the proof in section 4.3. The algorithm and its proof assume that case 1 is known to fail.

4.1. DESCRIPTION OF THE ALGORITHM

Just as for case 1, we first collect data for each pole c of r and also for ∞. The form of the data is a set E_c (or E_∞) consisting of from one to three integers. Next we consider families of elements of these sets, perhaps discarding some and retaining others. If no families are retained, case 2 cannot hold. For each family retained we search for a monic polynomial that satisfies a certain linear differential equation. If no such polynomial exists for any family, then case 2 cannot hold. If such a polynomial does exist, then a solution to the DE has been found.

Let Γ be the set of poles of r.

Step 1. For each $c \in \Gamma$ we define E_c as follows.

(c_1) If c is a pole of r of order 1, then $E_c = \{4\}$.

(c_2) If c is a pole of r of order 2 and if b is the coefficient of $1/(x-c)^2$ in the partial fraction expansion of r, then

$$E_c = \{2 + k\sqrt{1+4b} \mid k = 0, \pm 2\} \cap \mathbb{Z}.$$

(c_3) If c is a pole of r of order $\nu > 2$, then $E_c = \{\nu\}$.

(∞_1) If r has order >2 at ∞, then $E_\infty = \{0, 2, 4\}$.

(∞_2) If r has order 2 at ∞ and b is the coefficient of x^{-2} in the Laurent series expansion of r at ∞, then

$$E_\infty = \{2 + k\sqrt{1+4b} \mid k = 0, \pm 2\} \cap \mathbb{Z}.$$

(∞_3) If the order of r at ∞ is $\nu < 2$, then $E_\infty = \{\nu\}$.

Step 2. We consider all families $(e_c)_{c \in \Gamma \cup \{\infty\}}$ with $e_c \in E_c$. Those families all of whose coordinates are even may be discarded. Let

$$d = \tfrac{1}{2}\Big(e_\infty - \sum_{c \in \Gamma} e_c\Big).$$

If d is a non-negative integer, the family should be retained, otherwise the family is discarded. If no families remain under consideration, case 2 cannot hold.

Step 3. For each family retained from Step 2, we form the rational function

$$\theta = \tfrac{1}{2} \sum_{c \in \Gamma} \frac{e_c}{x-c}.$$

Next we search for a monic polynomial P of degree d (as defined in Step 2) such that

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0.$$

If no such polynomial is found for any family retained from Step 2, then case 2 cannot hold.

Suppose that such a polynomial is found. Let $\phi = \theta + P'/P$ and let ω be a solution of the equation

$$\omega^2 + \phi\omega + (\tfrac{1}{2}\phi' + \tfrac{1}{2}\phi^2 - r) = 0.$$

Then $\eta = e^{\int \omega}$ is a solution of the DE $y'' = ry$.

4.2. EXAMPLES

Example 1. Consider the DE $y'' = ry$ where

$$r = \frac{1}{x} - \frac{3}{16x^2}.$$

The necessary conditions of section 2 show that cases 1 and 3 cannot hold. (The order of r at ∞ is 1.) The only pole of r is at 0 and the order there is 2. The coefficient of $1/x^2$ in the partial fraction expansion of r is $b = -3/16$. Since $2\sqrt{1+4b} = 1$ is an integer, $E_0 = \{1, 2, 3\}$. The order of r at ∞ is 1 and $E_\infty = \{1\}$.

We have three families to consider.

$$\begin{aligned} e_0 &= 2, & e &= 1, & d &= -1/2 \\ e_0 &= 3, & e &= 1, & d &= -1 \\ e_0 &= 1, & e &= 1, & d &= 0. \end{aligned}$$

Only the third family need remain in consideration. For this family, $\theta = 1/2x$ and we need to find a monic polynomial P, of degree 0, such that

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0.$$

Evidently P must be 1, so the existence of P is a question of whether or not $\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'$ is zero. That expression does happen to be 0, so $P = 1$ is the desired polynomial.

Next we form

$$\phi = \theta + P'/P = \frac{1}{2x}.$$

The equation for ω is

$$0 = \omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi - r\right) = \omega^2 - \frac{1}{2x}\omega + \frac{1}{16x^2} - \frac{1}{x}.$$

The roots are

$$\omega = \frac{1}{4x} \pm \frac{1}{\sqrt{x}}.$$

It follows that

$$\eta = e^{\int\omega} = e^{\int(1/(4x) + 1/\sqrt{x})} = x^{1/4}e^{2\sqrt{x}}$$

is a solution of the DE. (And $x^{1/4}e^{-2\sqrt{x}}$ is also a solution.)

Example 2. In this example we finish consideration of Bessel's Equation

$$y'' = \left(\frac{4n^2-1}{4x^2} - 1\right)y, \quad n \in \mathbb{C},$$

that was started in section 3.2. In that section we observed that case 3 cannot hold and that case 1 holds if and only if n is half an odd integer. Here we treat case 2 and make the assumption that n is not half an odd integer.

The only pole of r is at 0 and the order there is 2. Since

$$2\sqrt{1+4b} = 2\sqrt{1+4(4n^2-1)/4} = 4n,$$

either $E_0 = \{2\}$ or $E_0 = \{2, 2+4n, 2-4n\}$, depending on whether $4n$ is an integer or not. If $4n$ is not an integer, then there is only one case to consider.

$$e_0 = 2, \quad e_\infty = 0, \quad d = -1.$$

Thus if $4n$ is not an integer, case 2 cannot hold. If $4n$ is an integer, there are three cases to consider.

$$\begin{aligned} e_0 &= 2, & e_\infty &= 0, & d &= -1 \\ e_0 &= 2+4n, & e_\infty &= 0, & d &= -1-2n \\ e_0 &= 2-4n, & e_\infty &= 0, & d &= -1+2n. \end{aligned}$$

In order that d be a non-negative integer, it is necessary that n be half an integer. Since n is not half an odd integer, n must be half an even integer, that is n is an integer. But, for such n, both e_0 and e_∞ are even. Hence all families are discarded and case 2 cannot hold.

In this example, and in Example 2 of section 3.2, we have shown that Bessel's Equation has a Liouvillian solution if and only if n is half an odd integer.

4.3. PROOF

For the proof of the algorithm for case 2 we shall rely heavily on the differential Galois group of the DE. In case 2, this group is (conjugate to) a subgroup of

$$D^\dagger = \left\{\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in \mathbb{C},\ c \neq 0\right\} \cup \left\{\begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| c \in \mathbb{C},\ c \neq 0\right\}.$$

Moreover, we may assume that case 1 does not hold, so the differential Galois group is not triangulisable. Let η, ζ be a fundamental system of solutions of the DE corresponding to the subgroup of $D^\dagger$. For any differential automorphism σ of $\mathbb{C}(x)\langle\eta, \zeta\rangle$ over $\mathbb{C}(x)$, either $\sigma\eta = c\eta$, $\sigma\zeta = c^{-1}\zeta$ or $\sigma\eta = -c^{-1}\zeta$, $\sigma\zeta = c\eta$, for some $c \in \mathbb{C}$, $c \neq 0$. Evidently $\sigma(\eta^2\zeta^2) = \eta^2\zeta^2$, therefore $\eta^2\zeta^2 \in \mathbb{C}(x)$. Moreover, $\eta\zeta \notin \mathbb{C}(x)$ since case 1 does not hold.

We write

$$\eta^2\zeta^2 = g \prod_{c\in\Gamma} (x-c)^{e_c} \prod_{i=1}^{m} (x-d_i)^{f_i},$$

where Γ is the set of poles of r and the exponents e_c, f_i are integers. Our goal is to determine these exponents.

Let

$$\phi = (\eta\zeta)'/(\eta\zeta) = \tfrac{1}{2}(\eta^2\zeta^2)'/(\eta^2\zeta^2) = \tfrac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} + \tfrac{1}{2}\sum_{i=1}^{m} \frac{f_i}{x-d_i}.$$

Because $\phi = \eta'/\eta + \zeta'/\zeta$, it follows that

$$(*) \qquad \phi'' + 3\phi\phi' + \phi^3 = 4r\phi + 2r'.$$

We first determine e_c for $c \in \Gamma$. In order to simplify the notation, we assume that $c = 0$.

(c_1) Suppose that 0 is a pole of r of order 1. The Laurent series expansions of r and ϕ at 0 are of the form

$$\begin{aligned} r &= \alpha x^{-1} + \cdots \quad (\alpha \neq 0) \\ \phi &= \tfrac{1}{2}ex^{-1} + f + \cdots \quad (e \in \mathbb{Z}, f \in \mathbb{C}). \end{aligned}$$

Substituting these series into the equation (*) and retaining all those terms that involve x^{-3} and x^{-2}, we obtain the following.

$$ex^{-3} + \cdots - \tfrac{3}{4}e^2x^{-3} - \tfrac{3}{2}efx^{-2} + \cdots + \tfrac{1}{8}e^3x^{-3} + \tfrac{3}{4}e^2fx^{-2} + \cdots$$
$$= 2\alpha ex^{-2} + \cdots - \alpha x^{-2} + \cdots.$$

Therefore $e-\frac{3}{4}e^2+\frac{1}{8}e^3=0$, so $e=0, 2, 4$. Also $-\frac{3}{2}ef+\frac{3}{4}e^2f=2\alpha e-\alpha$. Because $\alpha\neq 0$, $e\neq 0, 2$. Hence, e must be 4.

(c_2) Suppose that 0 is a pole of r of order 2 and that b is the coefficient of $1/x^2$ in the Laurent series for r. That is

$$r=bx^{-2}+\cdots,\qquad \phi=\tfrac{1}{2}ex^{-1}+\cdots.$$

Equating the coefficients of x^{-3} on the two sides of equation (*), we obtain

$$e-\tfrac{3}{4}e^2+\tfrac{1}{8}e^3=2eb-4b.$$

The roots of this equation are $e=2$, $e=2\pm 2\sqrt{1+4b}$. Of course, the latter two roots may be discarded in the case that they are non-integral.

(c_3) Finally we consider the possibility that 0 is a pole of r of order $\nu>2$. Then $r=x^{-\nu}+\cdots$ and $\phi=\frac{1}{2}ex^{-1}+\cdots$. Equating the coefficients of $x^{-\nu-1}$ in (*) we obtain $0=2\alpha e-2\alpha\nu$, hence $e=\nu$.

In determining the exponents f_i we may use the calculation of (c_1) above if we replace α by 0 (since d_i must be an ordinary point of r). We find that $f_i=0$, 2, or 4. We cannot exclude the possibility that $f_i=2$, but we can, of course, exclude the possibility $f_i=0$.

We have shown so far that

$$\eta^2\zeta^2=\prod_{c\in\Gamma}(x-c)^{e_c}P^2,$$

where $e_c\in E_c$ (as defined in section 4.1) and $P\in\mathbb{C}[x]$.

$$\text{Set}\quad \theta=\tfrac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c},\qquad \text{so}\quad \phi=\theta+P'/P.$$

The next step in our proof is to determine the degree d of P, which we do by examining the Laurent series expansion of ϕ at ∞ and using equation (*).

$$\phi=\tfrac{1}{2}e_\infty x^{-1}+\cdots,\qquad e_\infty=\sum_{c\in\Gamma}e_c+2d.$$

(∞_1) Suppose that the order of r at ∞ is 2. As in (c_1) we find that $e_\infty=0$, 2 or 4.

(∞_2) Suppose that the order of r at ∞ is 2 and that b is the coefficient of x^{-2} in the Laurent series expansion of r at ∞. Then, as in (c_2), $e_\infty=2$, $2\pm 2\sqrt{1+4b}$ and e_∞ is integral.

(∞_3) Suppose that the order of r at ∞ is $\nu<2$. As in (c_3), it follows that $e_\infty=\nu$.

Note that at least one of the e_c ($c\in\Gamma$) is odd, since $\eta\zeta\notin\mathbb{C}(x)$.

Using equation (*) and the equation $\phi=\theta+P'/P$, we obtain

$$P'''+3\theta P''+(3\theta^2+3\theta'-4r)P'+(\theta''+3\theta\theta'+\theta^3-4r\theta-2r')P=0.$$

This is a linear homogeneous differential equation for P, so there is a polynomial solution if and only if there is a monic polynomial which is a solution.

Now let ω be a solution of the equation

$$(**)\qquad \omega^2-\phi\omega+\tfrac{1}{2}\phi'+\tfrac{1}{2}\phi^2-r=0.$$

To complete the proof we need to show that $\eta=e^{\int\omega}$ is a solution of the DE $y''=ry$.

From (**) we obtain (by differentiation)

$$(2\omega-\phi)\omega'=\phi'\omega-\tfrac{1}{2}\phi''-\phi\phi'+r'.$$

The factor $(2\omega-\phi)$ cannot be zero. Indeed, if $\phi=2\omega$, then $\omega'+\omega^2-r=0$ (from (**)) so $\eta=e^{\int\omega}$ is a solution of the DE. But $\omega=\frac{1}{2}\phi\in\mathbb{C}(x)$. This is case 1, which was assumed to fail. Using (**) and (*) we have

$$2(2\omega-\phi)(\omega'+\omega^2-r)=-\phi''-3\phi\phi'-\phi^3+4r\phi+2r'=0.$$

Thus $\omega'+\omega^2=r$ so $\eta=e^{\int\omega}$ is a solution of the DE.

This completes the proof that the algorithm for case 2 is correct.

5. The Algorithm for Case 3

Following the pattern established in the previous two sections, we describe the algorithm in section 5.1 and give examples in section 5.2. The proof of the algorithm requires a knowledge of the finite subgroups of $SL(2)$ and their invariants, which is provided in section 5.3. The proof of the algorithm is presented in section 5.4.

In case 3, the DE has only algebraic solutions and we assume that cases 1 and 2 are known to fail. (It is possible for the DE to have only algebraic solutions and for cases 1 or 2 to apply. For example, case 1 gives the solution $\eta=x^{1/4}$ to the DE $y''=-(3/16x^2)y$, then reduction of order gives $\zeta=x^{3/4}$ as a second solution.)

5.1. DESCRIPTION OF THE ALGORITHM

Let η be a solution of the DE $y''=ry$ and set $\omega=\eta'/\eta$. Then, as we shall show in section 5.4, ω is algebraic over $\mathbb{C}(x)$ of degree 4, 6 or 12. It is the minimal polynomial for ω that we shall determine. We are unable to determine the minimal equation for η (which would be of degree 24, 48 or 120).

There are two possible methods for finding the minimal equation for ω. We could find a polynomial of degree 12 and then factor it. We shall prove that if ω is any solution of the 12th degree polynomial found by our method, then $\eta=e^{\int\omega}$ is a solution of the DE, hence any one of the irreducible factors may be used. This is the most direct method; however, the factorisation can be a formidable problem, even with the assistance of a computer. We illustrate this by example, in section 5.2. The alternative is to first attempt to find a 4th degree equation for ω, then a 6th degree equation and finally a 12th degree equation. The advantage is that if an equation is found, then it is guaranteed to be irreducible.

In our description of the algorithm, we shall combine the various possibilities, denoting by n the degree of the equation being sought. As before, we denote by Γ the set of poles of r. Recall that, by the necessary conditions of section 2, r cannot have a pole of order >2.

Step 1. For each $c\in\Gamma\cup\{\infty\}$ we define a set E_c of integers as follows.

(c_1) If c is a pole of r of order 1, then $E_c=\{12\}$.

(c_2) If c is a pole of r of order 2 and if α is the coefficient of $1/(x-c)^2$ in the partial fraction expansion of r, then

$$E_c=\left\{6+\frac{12k}{n}\sqrt{1+4\alpha}\,\middle|\,k=0,\pm1,\pm2,\ldots,\pm\frac{n}{2}\right\}\cap Z.$$

(∞) If the Laurent series for r at ∞ is

$$r=\gamma x^{-2}+\cdots\quad(\gamma\in\mathbb{C},\text{ possibly }0),$$

then

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4\gamma} \mid k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap Z.$$

Step 2. We consider all families $(e_c)_{c \in \Gamma \cup \{\infty\}}$ such that $e_c \in E_c$. For each such family, define

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right).$$

If d is a non-negative integer, the family is retained, otherwise the family is discarded. If no families are retained, then ω cannot satisfy a polynomial equation of degree n with coefficients in $\mathbb{C}(x)$.

Step 3. For each family retained from step 2, form the rational function

$$\theta = \frac{n}{12} \sum_{c \in \Gamma} \frac{e_c}{x-c}.$$

Also define

$$S = \prod_{c \in \Gamma} (x-c).$$

Next search for a monic polynomial $P \in \mathbb{C}[x]$ of degree d (as defined in step 2) such that when we define polynomials $P_n, P_{n-1}, \ldots, P_{-1}$ recursively by the formulas below, then $P_{-1} = 0$ (identically).

$$P_n = -P$$
$$P_{i-1} = -SP_i' + ((n-i)S' - S\theta)P_i - (n-i)(i+1)S^2 r P_{i+1} \qquad (i = n, n-1, \ldots, 0).$$

This may be conveniently done by using undetermined coefficients for P. If no such polynomial P is found for any family retained from step 2, then ω cannot satisfy a polynomial equation of degree n with coefficients in $\mathbb{C}(x)$.

Assume that a family and its associated polynomial P has been found. Let ω be a solution of the equation

$$\sum_{i=0}^{n} \frac{S^i P_i}{(n-i)!} \omega^i = 0.$$

Then $\eta = e^{\int \omega}$ is a solution of the DE.

5.2. EXAMPLES

Example 1. Our first example illustrates the alternative technique mentioned at the beginning of the last section, namely to bypass the search for equations of degrees 4 and 6 for ω and proceed directly to the search for an equation of degree 12.

We consider the hypergeometric equation $y'' = ry$ where

$$r = -\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}.$$

The necessary conditions of section 2 show that all four cases are possible.

Applying the algorithm for case 1, we find that

$$\alpha_0^+ = 3/4, \qquad \alpha_0^- = 1/4$$
$$\alpha_1^+ = 2/3, \qquad \alpha_1^- = 1/3$$
$$\alpha_\infty^+ = 2/3, \qquad \alpha_\infty^- = 1/3,$$

and $d = \alpha_\infty^\pm - \alpha_0^\pm - \alpha_1^\pm$ can never be a non-negative integer. Case 1 fails.

Applying the algorithm for case 2, we find that

$$E_0 = \{2, 3, 1\}$$
$$E_1 = \{2\}$$
$$E_\infty = \{2\},$$

and $d = e_\infty - e_0 - e_1$ can never be a non-negative integer. Case 2 fails.

We apply the algorithm for case 3, searching for an equation of degree 12 for ω, thus $n = 12$ in the algorithm.

At $c = 0$, $\alpha = -3/16$ and $\sqrt{1+4\alpha} = 1/2$ (or $-1/2$). Hence $E_0 = \{3, 4, 5, 6, 7, 8, 9\}$. At $c = 1$, $\alpha = -2/9$ and $\sqrt{1+4\alpha} = 1/3$. So $E_1 = \{4, 5, 6, 7, 8\}$. At ∞, $\gamma = -2/9$ and $E_\infty = \{4, 5, 6, 7, 8\}$.

Following the instructions of step 2, we now form the expression $d = e_\infty - e_0 - e_1$ for every choice of $e_\infty \in E_\infty$, $e_0 \in E_0$, $e_1 \in E_1$. We discard those families for which d is a negative integer. Only four possibilities remain.

$$\begin{array}{llll} e_\infty = 7, & e_0 = 3, & e_1 = 4, & d = 0 \\ e_\infty = 8, & e_0 = 3, & e_1 = 4, & d = 1 \\ e_\infty = 8, & e_0 = 3, & e_1 = 5, & d = 0 \\ e_\infty = 8, & e_0 = 4, & e_1 = 4, & d = 0. \end{array}$$

We now consider the first possibility, following step 3. We set $\theta = 3/x + 4/(x-1)$, $S = x^2 - x$, and search for a monic polynomial P of degree 1 that satisfies the conditions given in step 3. Of course, $P = 1$.

The computations are far too complicated to be accurately done by hand; however, they are easily programmed into a computer. Since P_i is always a polynomial ($i = 12, \ldots, -1$) whose degree is easily predicted (in this example $\deg P_i = 12 - i$) arrays of coefficients may be manipulated to carry through the computations. In order to avoid roundoff error, we computed $12^{12-i}P_i$ using 33 digit integer arithmetic. The results follow.

$$P_{12} = -1$$
$$P_{11} = 7x - 3$$
$$P_{10} = (1/12)(-536x^2 + 459x - 99)$$
$$P_9 = (3!/(3\cdot 12^2))(18544x^3 - 23799x^2 + 10260x - 1485)$$
$$P_8 = (4!/(16\cdot 12^2))(-127488x^4 + 217972x^3 - 140879x^2 + 40770x - 4455)$$
$$P_7 = (5!/(2\cdot 12^3))(174080x^5 - 371748x^4 + 320305x^3 - 138975x^2 + 30375x - 2673)$$
$$P_6 = (6!/12^5)(-8257536x^6 + 21145136x^5 - 22757500x^4 + 13168377x^3 - 4318083x^2 + 760347x - 56133)$$
$$P_5 = (7!/(2\cdot 12^5))(7929856x^7 - 23673984x^6 + 30564708x^5 - 22107287x^4 + 9668646x^3 - 2555280x^2 + 377622x - 24057)$$
$$P_4 = (8!/(16\cdot 12^6))(-26421152x^8 + 900984832x^7 - 1356734768x^6 + 1177673400x^5 - 644082327x^4 + 227124972x^3 - 50398362x^2 + 6429780x - 360855)$$
$$P_3 = (9!/(3\cdot 12^8))(174483046x^9 - 6688997376x^8 + 11509039440x^7 - 11656902184x^6 + 7654170465x^5 - 3376695033x^4 + 1000183626x^3 - 191681802x^2 + 21552885x - 1082565)$$

$$P_2 = (10!/(2\cdot 12^9))(-2281701376x^{10}+9713634848x^9-18799438080x^8 +21766009616x^7-16683774768x^6+8840413683x^5 -3277319535x^4+838780110x^3-141739470x^2+14270175x -649539)$$

$$P_1 = (11!/12^{10})(1342177280x^{11}-6282018816x^{10}+13507531776x^9-17598922384x^8 +15426848952x^7-9546427017x^6+4252638672x^5-1362816657x^4 +307684656x^3-46576539x^2+4251528x-177147)$$

$$P_0 = (12!/12^{12})(-8589934592x^{12}+43838865408x^{11}-103681720320x^{10} +150145637824x^9-148170380976x^8+104901110964x^7 -54596424249x^6-21032969490x^5-5948563455x^4 +1203654816x^3-165278151x^2+13817466x-531441)$$

$$P_{-1} = 0$$

Therefore $\eta = e^{\int \omega}$ is a solution of the DE, where ω is a solution of the equation

$$\sum_{i=0}^{12} \frac{(x^2-x)^i P_i}{(12-i)!} \omega^i = 0.$$

Professors Caviness and Saunders of Rensselaer Polytechnic Institute kindly offered to attempt a factorisation of this polynomial for ω. They used the exceedingly powerful system for algebraic manipulation called MACSYMA at MIT. The program took less than 5 minutes to write but took 3 minutes of CPU time to execute. The result is that the polynomial above is the cube of the following polynomial.

$$(x^2-x)^4\omega^4-(1/3)(x^2-x)^3(7x-3)\omega 3+(1/24)(x^2-x)^2(48x^2-41x+9)\omega^2 -(1/432)(x^2-x)(320x^3-409x^2+180x-27)\omega +(1/20736)(2048x^4-3484x^3+2313x^2-702x+81)$$

Example 2. In this example we consider the DE $y'' = ry$, where

$$r = -\frac{5x+27}{36(x-1)^2}.$$

The necessary conditions of section 2 show that all four cases are possible.

Note that the partial fraction expansion of r has the form

$$r = -\frac{2}{9(x+1)^2} + \cdots - \frac{2}{9(x-1)^2} + \cdots$$

and the Laurant series for r about ∞ is

$$r = -\frac{5}{36x^2} + \cdots.$$

Applying the algorithm for case 1 we find that

$$\begin{aligned} \alpha_{-1}^{+} &= 2/3, & \alpha_{-1}^{-} &= 1/3 \\ \alpha_1^{+} &= 2/3, & \alpha_1^{-} &= 1/3 \\ \alpha_\infty^{+} &= 5/6, & \alpha_\infty^{-} &= 1/6. \end{aligned}$$

For no choice of signs is $d = \alpha_\infty^{\pm} - \alpha_{-1}^{\pm} - \alpha_1^{\pm}$ a non-negative integer, thus case 1 cannot hold.

Applying the algorithm for case 2 we find that $E_{-1} = E_1 = E_\infty = \{2\}$, and case 2 does not hold.

We now apply the algorithm for case 3, attempting to find an equation of degree 4 over $\mathbb{C}(x)$ that is satisfied by ω.

From step 1 we have that

$$E_{-1} = \{4, 4, 6, 7, 8\}, \quad E_1 = \{4, 5, 6, 7, 8\} \quad \text{and} \quad E_\infty = \{2, 4, 6, 8, 10\}.$$

There are four families with the property that $d = \frac{1}{3}(e_\infty - e_{-1} - e_1)$ is a non-negative integer, namely

$$\begin{array}{llll} e_\infty = 8, & e_{-1} = 4, & e_1 = 4, & d = 0, \\ e_\infty = 10, & e_{-1} = 4, & e_1 = 6, & d = 0, \\ e_\infty = 10, & e_{-1} = 5, & e_1 = 5, & d = 0, \\ e_\infty = 10, & e_{-1} = 6, & e_1 = 4, & d = 0. \end{array}$$

The first possibility gives

$$\theta = \frac{1}{3}\left(\frac{4}{x+1} + \frac{4}{x-1}\right) = \frac{8x}{3(x^2-1)}.$$

Setting $S = x^2 - 1$, we have $S\theta = \frac{8}{3}x$, $S^2 r = -\frac{1}{36}(5x^2+27)$. We then have

$$\begin{aligned} P_4 &= -1 \\ P_3 &= (8/3)x \\ P_2 &= -(1/3)(15x^2+1) \\ P_1 &= (1/9)(50x^3+14x) \\ P_0 &= -(1/54)(125x^4+134x^2-3) \\ P_{-1} &= 0. \end{aligned}$$

Let ω be a solution of the equation

$$S\omega^4 = \tfrac{8}{3}xS\omega^3 - \tfrac{1}{6}(15x^2+1)S\omega^2 + \tfrac{1}{27}(25x^3+7x)S\omega - \tfrac{1}{1296}(125x^4+134x^2-3).$$

If we make the substitution $6S\omega = z + 4x$, the equation simplifies to

$$z^4 = 6(x^2-1)z^2 - 8x(x^2-1)z + 3(x^2-1)^2.$$

Then

$$\eta = e^{\int \omega} = (x^2-1)^{1/3} \exp\left(\int (z/(x^2-1))\, dx\right)$$

is a solution of the DE.

5.3. FINITE SUBGROUPS OF SL(2)

In this section we determine the finite subgroups of $SL(2)$, up to conjugation, and their invariants. This work is classical, being found in the work of Klein, Jordan and others. For the sake of completeness we sketch the results here in the form needed in the subsequent section.

THEOREM 1. *Let G be a finite subgroup of $SL(2)$. Then either*

(*i*) *G is conjugate to a subgroup of the group*

$$D^{\dagger} = D \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot D,$$

where D is the diagonal group, or

(*ii*) *the order of G is 24 (the "tetrahedral" case), or*

(*iii*) *the order of G is 48 (the "octahedral" case), or*

(*iv*) *the order of G is 120 (the "icosahedral" case).*

In the last three cases G contains the scalar matrix -1.

The geometric names were used by Klein; however, our proof will be entirely algebraic.

Proof. We assume that G is not conjugate to a subgroup of $D^{\dagger}$. Let H be the set of scalar matrices in G, thus $H=\{1\}$ or $\{1,-1\}$, so the order of H is 1 or 2. For any $x \in G-H$ (i.e. $x \in G$ and $x \notin H$) we denote by $Z(x)$ the centraliser of x in G and by $N(x)$ the normaliser of $Z(x)$ in G.

Let $x \in G-H$. Since x is of finite order, x is diagonalisable. (The Jordan form of a non-diagonalisable matrix in $SL(2)$ must be $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.) Since the centraliser in $SL(2)$ of a diagonal non-scalar matrix is D (by direct computation) $Z(x)$ must be the intersection of G and a conjugate of D. Hence $Z(x)=Z(y)$ if and only if $y \in Z(x)$. Using this fact and the fact that $Z(gxg^{-1})=gZ(x)g^{-1}$ we may conclude that (for arbitrary $x, y, g, g' \in G$) either

$$gZ(x)g^{-1} \cap g'Z(y)g'^{-1} = H \quad \text{or} \quad gZ(x)g^{-1} = g'Z(y)g'^{-1}$$

and in the latter case $y \in g'^{-1}gZ(x)g^{-1}g'$. In addition $gZ(x)g^{-1}=g'Z(x)g'^{-1}$ if and only if $g'^{-1}g \in N(x)$. Therefore we may write G as a disjoint union

$$G = \bigcup_{i=1}^{s} \bigcup (gZ(x_i)g^{-1}-H) \,\dot\cup\, H \quad \text{(disjoint)},$$

where the inner union is taken over all cosets $gN(x_i)$ in $G/N(x_i)$, s is some natural number and $x_1, \ldots, x_s \in G-H$.

The group $N(x_i)$ is easy to describe since x_i is diagonalisable. First note that the only matrices in $SL(2)$ that conjugate a diagonal non-scalar matrix into a diagonal matrix are the matrices in $D^{\dagger}$ (by direct computation). It follows that $N(x_i)$ is the intersection of G and a conjugate of $D^{\dagger}$, in particular the index of $Z(x_i)$ in $N(x_i)$, $[N(x_i):Z(x_i)]$, is either 1 or 2.

Let $M=\operatorname{ord}(G/H)$ and $e_i=\operatorname{ord}(Z(x_i)/H)$. The representation of G as a disjoint union gives the following formulas.

$$M \cdot \operatorname{ord} H = \sum_{i=1}^{s} [G:N(x_i)](e_i \cdot \operatorname{ord} H - \operatorname{ord} H) + \operatorname{ord} H,$$

or

$$M = \sum_{i=1}^{s} \frac{M}{[N(x_i):Z(x_i)] \cdot e_i}(e_i-1)+1,$$

or

(#)
$$\frac{1}{M} = \sum_{i=1}^{s} \frac{1}{[N(x_i):Z(x_i)]}\left(\frac{1}{e_i}-1\right)+1.$$

Certainly $s \neq 0$ since $G \neq H$. If $s = 1$, then

$$1/M \geqslant 1/([N(x_1):Z(x_1)]e_1) = 1/\text{ord}\,(N(x_1)/H), \text{ so } G = N(x_1).$$

This contradicts the fact that G is not conjugate to a subgroup of $D^\dagger$.

Since $e_i \geqslant 2$ ($i = 1, \ldots, s$) we have

$$0 < \frac{1}{M} \leqslant 1 - \tfrac{1}{2} \sum_{i=1}^{s} \frac{1}{[N(x_i):Z(x_i)]}$$

so

$$\sum_{i=1}^{s} \frac{1}{[N(x_i):Z(x_i)]} < 2.$$

Because

$$[N(x_i):Z(x_i)] = 1 \text{ or } 2,$$

there are only three solutions of this inequality.

$$\begin{aligned} &s = 2, \quad [N(x_1):Z(x_1)] = 1, \quad [N(x_2):Z(x_2)] = 2 \quad , \\ &s = 2, \quad [N(x_1):Z(x_1)] = [N(x_2):Z(x_2)] = 2, \\ &s = 3, \quad [N(x_1):Z(x_1)] = [N(x_2):Z(x_2)] = [N(x_3):Z(x_3)] = 2. \end{aligned}$$

For all solutions $[N(x_2):Z(x_2)] = 2$. Thus G contains a conjugate of a matrix in $D^\dagger - D$, i.e. the conjugate of a matrix of the form $\begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix}$. The square of such a matrix is -1. Hence ord $H = 2$.

The first solution gives $1/M = 1/e_1 + 1/(2e_2) - 1/2$, so $e_1 = 3$, $e_2 = 2$ and $M = 12$, so ord $G = 24$. (The point being that $M > 2e_2$, since G is not conjugate to a subgroup of D, and therefore $e_1 \geqslant 3$.)

The second solution gives $1/M = 1/(2e_1) + 1/(2e_2)$, which is impossible since $M > 2e_2$.

The third solution gives

$$\frac{2}{M} = \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} - 1.$$

Assuming that $e_1 \leqslant e_2 \leqslant e_3$ we find that $e_1 < 3$ so $e_1 = 2$ and

$$\frac{2}{M} = \frac{1}{e_2} + \frac{1}{e_3} - \frac{1}{2}.$$

Also $e_2 = 3$ since $M > 2e_3$. The solutions are

$$\begin{aligned} e_1 = 2, \quad e_2 = 3, \quad &e_3 = 3, \quad M = 12, \quad \text{ord}\, G = 24, \\ &e_3 = 4, \quad M = 24, \quad \text{ord}\, G = 48, \\ &e_3 = 5, \quad M = 60, \quad \text{ord}\, G = 120. \end{aligned}$$

This proves the theorem.

In the following sequence of theorems we shall explicitly determine the three "geometric" groups. To that end we need the following lemma.

LEMMA. *Let G be a finite subgroup of $SL(2, C)$ that is not conjugate to a subgroup of $D^\dagger$. Let $H = \{1, -1\}$. Then G/H has no normal cyclic subgroup.*

PROOF. If xH is a generator of a normal cyclic subgroup of G/H then the group generated by x and $-x$ is diagonalisable. Since this group would be normal in G, G would be conjugate to a subgroup of $D^\dagger$.

THEOREM 2. *Let G be a subgroup of $SL(2, C)$ of order 24 that is not conjugate to a subgroup of $D^{\dagger}$. Let $H = \{1, -1\}$. Then G/H is isomorphic to A_4, the alternating group on 4 letters. Moreover, G is conjugate to the group generated by the matrices*

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \phi\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix},$$

where ξ is a primitive 6th root of 1 and $3\phi = 2\xi - 1$.

PROOF. Since ord G/H is 12, and because of the previous lemma, G/H has 4 Sylow 3-groups, and G/H acts by conjugation on the set of these Sylow 3-groups. This action induces a homomorphism $G/H \to S_4$ (the symmetric group on 4 letters). The subgroup of the image consisting of those permutations that leave a particular Sylow 3-group fixed must have index 4 since G/H acts transitively. Therefore the order of the image is divisible by 4. It follows that the order of the kernel is 1, 2 or 3. By the previous lemma, the order of the kernel must be 1, so G/H is isomorphic to a subgroup of S_4. Now consider the composite homomorphism $G/H \to S_4 \to \{1, -1\}$, with the last arrow being given by $\sigma \to$ signum (σ). By the previous lemma, G/H cannot have a normal subgroup of order 6 (since a subgroup of order 6 contains a unique subgroup of order 3 which would be normal in G/H). Therefore the composite homomorphism has trivial image and G/H is isomorphic to A_4.

Let $\tau: G \to A_4$ be a homomorphism with kernel H. Let $A \in \tau^{-1}(123)$. We may conjugate G so that A is a diagonal matrix. Thus

$$A = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}.$$

Since $\tau A^3 = (1)$, $A^3 \in H$. However, $\tau A \neq (1)$ and $\tau A^2 \neq (1)$, thus $A \notin H$ and $A^2 \notin H$. Replacing A by $-A$, if necessary, we may assume that ξ is a primitive 6th root of 1.

Let $B \in \tau^{-1}(12)(34)$. Since $\tau(AB) \neq \tau(BA)$, B cannot be a diagonal matrix, i.e. not both B_{12} and B_{21} are zero. In fact neither is zero because if one were zero and the other non-zero, then B would have infinite order.

We may conjugate G by $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ without affecting A. If we choose $c = B_{21}$ and $d = 2B_{12}$, then B has the form

$$B = \begin{pmatrix} \phi & \psi \\ 2\psi & -\chi \end{pmatrix}.$$

Now $\tau B^2 = (1)$ so $B^2 \in H$. A direct computation shows that $\chi = \phi$.

Next we observe that $\tau(BA^2) = \tau(AB)^2$ so $BA^2 = \pm(AB)^2$. We perform the computation and discover that $\phi(\xi^2 - 1) = \pm\xi^4$ (using the fact that $\psi \neq 0$). Replacing B by $-B$, if necessary, we may assume that $\phi(\xi^2 - 1) = \xi^4$, hence $3\phi = 2\xi - 1$ (using the relation $\xi^2 = \xi - 1$).

Next we use the fact that $\det B = 1$ to obtain the formula $\phi^2 + 2\psi^2 = -1$, or $3\psi = \pm(2\xi - 1)$. If necessary, we conjugate G by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so that $3\psi = 2\xi - 1 = 3\phi$. This proves the theorem.

The group of this theorem is called the tetrahedral group.

THEOREM 3. *Let G be a subgroup of $SL(2)$ of order 48 that is not conjugate to a subgroup of $D^{\dagger}$. Let $H=\{1,-1\}$. Then G/H is isomorphic to $\mathbf{S}_4$, the symmetric group on 4 letters. Moreover, G is conjugate to the group generated by the matrices*

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \qquad \phi\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

where ξ is a primitive 8th root of 1 and $2\phi=\xi(\xi^2+1)$.

PROOF. Since ord $G/H=24$, and because of the previous lemma, G/H has 4 Sylow 3-groups. The action of G/H on the set of Sylow 3-groups (via conjugation) induces a homomorphism $G/H \to \mathbf{S}_4$. The image contains a subgroup of index 4, namely the subgroup of permutations leaving a particular Sylow 3-group fixed, since G/H acts transitively on the set of Sylow 3-groups. Hence the order of the image is divisible by 4, so the order of the kernel is 1, 2, 3 or 6. Were the order of the kernel 6, then the kernel would contain a unique subgroup of order 3 which would be normal in G. This contradicts the lemma. Indeed, the lemma implies that ord ker $=1$, so G/H is isomorphic to $\mathbf{S}_4$.

Let $\tau: G \to \mathbf{S}_4$ be a homomorphism with kernel H and let $A \in \tau^{-1}(1234)$. We may conjugate G so that A is a diagonal matrix

$$A=\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}.$$

Since $\tau A^4=(1)$, $\xi^4=\pm 1$. However, were $\xi^4=1$, then $\xi^2=\pm 1$ and $A^2 \in H$. But $\tau A^2 \neq (1)$. Hence ξ is a primitive 8th root of 1.

Let $B \in \tau^{-1}(12)$. Since $\tau(AB) \neq \tau(BA)$, B cannot be a diagonal matrix, thus not both B_{12} and B_{21} are zero. In fact, neither is zero since B has finite order. We may conjugate G, without disturbing A, by $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, where $c^2=B_{21}$ and $d^2=B_{12}$. Then B has the form

$$B=\begin{pmatrix} \phi & \psi \\ \psi & -\chi \end{pmatrix}.$$

Using the fact $\tau B^2=(1)$, i.e. $B^2 \in H$, we obtain easily that $\chi=\phi$.

Because $\tau(BA^3)=\tau(AB)^2$, $BA^3=\pm(AB)^2$. Making this computation, and using the fact that $\psi \neq 0$, we find that $\phi(\xi^2-1)=\pm\xi$, or $2\phi=\pm\xi(\xi^2+1)$. Replacing B by $-B$, if necessary, we may assume that $2\phi=\xi(\xi^2+1)$. Then $2\phi^2=-1$. Now we use the fact that $1=\det B=-\phi^2-\psi^2$ to conclude that $2\psi^2=-1$. Conjugate G, if necessary, by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so that $\psi=\phi$.

Because $\tau A, \tau B$ generate $\mathbf{S}_4$ and the group generated by A, B contains H, we can conclude that A, B generate G. This proves the theorem.

The group of this theorem is called the octahedral group.

THEOREM 4. *Let G be a subgroup of $SL(2)$ of order 120 that is not conjugate to a subgroup of $D^{\dagger}$. Let $H=\{1,-1\}$. Then G/H is isomorphic to $\mathbf{A}_5$, the alternating group on 5 letters. Moreover, G is conjugate to the group generated by the matrices*

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \qquad \begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix},$$

where ξ is a primitive 10th root of 1, $5\phi=3\xi^3-\xi^2+4\xi-2$, and $5\psi=\xi^3+3\xi2-2\xi+1$.

PROOF. The proof that G/H is isomorphic to $\mathbf{A}_5$ may be found in Burnside (1955, 127, p. 161–2).

Let $\tau: G \to \mathbf{A}_5$ be a homomorphism with kernel H and let $A \in \tau^{-1}(12345)$. We may conjugate G so that A is a diagonal matrix

$A = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$. Since $\tau A^5 = (1)$, $\xi^5 = \pm 1$. Replacing A with $-A$, if necessary, we may assume that $\xi^5 = -1$. Evidently ξ is a primitive 10th root of 1.

Let $B \in \tau^{-1}(12)(34)$. As in the proof of Theorem 3, we may assume that B has the form

$$B = \begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix}.$$

Because $\tau(A^4B) = \tau(BA)^2$, $A^4B = \pm(BA)^2$. Making this computation we find that $\phi(1+\xi^3) = \pm\xi^4$, or $5\phi = \pm(3\xi^3 - \xi^2 + 4\xi - 2)$. Replacing B by $-B$, if necessary, we may assume that the plus sign obtains. Now we use the fact that $1 = \det B$ to conclude that $5\psi = \pm(\xi^3 + 3\xi^2 - 2\xi + 1)$. Conjugate G by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, if necessary, so that the plus sign obtains.

Note that $\tau A, \tau B$ generate $\mathbf{A}_5$. (This group generated by τA and τB contains an element of order 5, an element of order 2 and an element of order 3. Thus the order of this group is divisible by 30. Since $\mathbf{A}_5$ is simple, this group must be $\mathbf{A}_5$.) Also the group generated by A, B contains H. Therefore A, B generate G. This proves the theorem.

The group described in this theorem is called the icosahedral group.

For use in the next section, we also need to know the invariants of the three "geometric" groups.

THEOREM 5. *Let G be the Galois group of the DE $y'' = ry$ and let η, ζ be a fundamental system of solutions relative to the group G.*

(i) *If G is the tetrahedral group, then $(\eta^4 + 8\eta\zeta^3)^3 \in \mathbb{C}(x)$.*
(ii) *If G is the octahedral group, then $(\eta^5\zeta - \eta\zeta^5)^2 \in \mathbb{C}(x)$.*
(iii) *If G is the icosahedral group, then $\eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11} \in \mathbb{C}(x)$.*

PROOF. (i) Consider the tetrahedral group, using the notation of Theorem 2. Recall that $\xi^3 = -1$, $\xi^2 = \xi - 1$ and $3\phi = 2\xi - 1$.

$\eta^4 + 8\eta\zeta^3$ is carried into $\xi^4(\eta^4 + 8\eta\zeta^3)$ by the matrix $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$. The matrix $\phi\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ carries

$$\eta^4 + 8\eta\zeta^3 = \eta \cdot (\eta + 2\zeta) \cdot (\eta + 2\xi^2\zeta) \cdot (\eta - 2\xi\zeta)$$

into

$$\begin{aligned} &\phi(\eta+2\zeta) \cdot 3\phi\eta \cdot \phi(2\xi-1)(\eta - 2\xi\zeta) \cdot \phi(1-2\xi)(\eta + 2\xi^2\zeta) \\ &\quad = -3 \cdot \phi^4 \cdot (2\xi - 1)^2 \cdot (\eta 4 + 8\eta\zeta^3) \\ &\quad = -3 \cdot (-1/3)^2 \cdot (-3) \cdot (\eta^4 + 8\eta\zeta^3) = \eta^4 + 8\eta\zeta^3. \end{aligned}$$

Thus $(\eta^4 + 8\eta\zeta^3)^3$ is an invariant of G and therefore is in $\mathbb{C}(x)$.

(ii) Consider the octahedral group, using the notation of Theorem 3. Recall that $\xi^4 = -1$ and $2\phi = \xi(\xi^2 + 1)$.

$\eta^5\zeta-\eta\zeta^5$ is carried into $\xi^4(\eta^5\zeta-\eta\zeta^5)$ by the matrix $\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}$. The matrix $\phi\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ carries

$$\eta^5\zeta-\eta\zeta^5 = \eta\cdot\zeta\cdot(\eta+\zeta)\cdot(\eta-\zeta)\cdot(\eta+\xi^2\zeta)\cdot(\eta-\xi^2\zeta)$$

into

$$\begin{aligned}\phi(\eta+\zeta)\cdot(\eta-\zeta)\cdot 2\phi\eta\cdot 2\phi\zeta\cdot\phi(1+\xi^2)(\eta-\xi^2\zeta)\cdot\phi(1-\xi^2)(\eta+\xi^2\zeta)\\ = 4\cdot\phi^6\cdot(1-\xi^4)\cdot(\eta^5\zeta-\eta\zeta^5)\\ = 8\cdot(-1/2)^3\cdot(\eta^5\zeta-\eta\zeta^5) = -(\eta^5\zeta-\eta\zeta^5).\end{aligned}$$

Thus $(\eta^5\zeta-\eta\zeta^5)^2$ is an invariant of G and therefore is in $\mathbb{C}(x)$.

(iii) Consider the icosahedral group and use the notation of Theorem 4. First we collect some easily derivable formulas.

$$\xi^5=-1, \qquad \xi^4=\xi^3-\xi^2+\xi-1,$$
$$5\phi^2=\xi^3-\xi^2-3, \qquad 5\psi^2=-\xi^3+\xi^2-2,$$
$$5\phi\psi=2\xi^3-2\xi^2-1=5(\phi^2-\psi^2).$$

The matrix $\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}$ carries $\eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11}$ into itself. The matrix $\begin{pmatrix}\phi & \psi\\ \psi & -\phi\end{pmatrix}$ carries

$$\begin{aligned}\eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11} = \eta\zeta\cdot(\eta^2-\eta\zeta-\zeta^2)\cdot(\eta^2+\xi^3\eta\zeta+\xi\zeta^2)\cdot\\ (\eta^2-\xi^2\eta\zeta-\xi^4\zeta^2)\cdot(\eta^2+\xi\eta\zeta-\xi^2\zeta^2)\cdot(\eta^2-\xi^4\eta\zeta+\xi^3\zeta^2)\end{aligned}$$

into

$$\begin{aligned}\phi\psi(\eta^2-\eta\zeta-\zeta^2)\cdot 5\phi\psi\eta\zeta\cdot(-\xi)(\eta^2-\xi^2\eta\zeta-\xi^4\zeta^2)\cdot(\xi^4)(\eta^2+\xi^3\eta\zeta+\xi\zeta^2)\cdot\\ (-1)(\eta^2+\xi\eta\zeta-\xi^2\zeta^2)\cdot(-1)(\eta^2-\xi^4\eta\zeta+\xi^3\zeta^2)\\ = 5\cdot(\phi\psi)^2\cdot(\eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11}) = \eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11}.\end{aligned}$$

Thus $\eta^{11}\zeta-11\eta^6\zeta^6-\eta\zeta^{11}$ is an invariant of G and therefore is in $\mathbb{C}(x)$.

This proves the theorem.

5.4. PROOF OF THE ALGORITHM

We must prove the validity of four separate algorithms. We must show that the algorithms for finding a 4th, 6th and 12th degree equation for ω are correct for the tetrahedral, octahedral and icosahedral groups and that the equation obtained is irreducible, and finally that the algorithm for finding a 12th degree equation is all-inclusive (although the equation obtained need not be irreducible). In so far as is possible, we carry out the proofs simultaneously.

We begin by showing that the equations obtained for ω in the tetrahedral, octahedral and icosahedral cases are minimal. Throughout we assume that the Galois group G of the DE $y''=ry$ is the tetrahedral, octahedral or icosahedral group. We also fix a fundamental system of solutions η, ζ of the DE relative to the group G and set $\omega=\eta'/\eta$.

THEOREM 1. *Let η_1 be any solution of the DE and let $\omega_1=\eta_1'/\eta_1$.*

(*i*) *If G is the tetrahedral group, then*

$$\deg_{\mathbb{C}(x)}\omega_1 \geqslant 4 \quad \textit{and} \quad \deg_{\mathbb{C}(x)}\omega = 4.$$

(ii) *If G is the octahedral group, then*

$$\deg_{\mathbb{C}(x)}\omega_1 \geqslant 6 \quad \textit{and} \quad \deg_{\mathbb{C}(x)}\omega = 6.$$

(iii) *If G is the icosahedral group, then*

$$\deg_{\mathbb{C}(x)}\omega_1 \geqslant 12 \quad \textit{and} \quad \deg_{\mathbb{C}(x)}\omega = 12.$$

PROOF. Since ω is left fixed by the group G_1 generated by $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$, where ξ is a primitive 6th, 8th or 10th root of 1 in the tetrahedral, octahedral or icosahedral cases, the degree of ω over $\mathbb{C}(x)$ is $\leqslant [G:G_1] = 4$, 6 or 12. The reverse inequality is proven more generally, as indicated in the statement of the theorem.

Let G_1 be the subgroup of G that fixes η_1. Complete η_1 to a fundamental system of solutions η_1, ζ_1 of the DE and conjugate G to XGX^{-1} so that XGX^{-1} is the Galois group of the DE relative to η_1, ζ_1. Then XG_1X^{-1} consists of matrices of the form $\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}$. Since XG_1X^{-1} is finite, $d = 0$ and $c^m = 1$, where m is the order of G_1. Evidently XG_1X^{-1} is a subgroup of the cyclic group

$$\left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c^m = 1 \right\}$$

and therefore is cyclic. Hence G_1/H (where H is the centre of G) is isomorphic to a cyclic subgroup of $\mathbf{A}_4$ in the tetrahedral case, of $\mathbf{S}_4$ in the octahedral case, and of $\mathbf{A}_5$ in the icosahedral case. So ord $G_1/H \leqslant 3, 4, 5$ or ord $G_1 \leqslant 6, 8, 10$. *Thus*

$$\deg_{\mathbb{C}(x)}\omega_1 = [G:G_1] \geqslant 4, 6, 12.$$

This proves the theorem.

Throughout the remainder of this section we shall be considering a certain differential equation written recursively, namely

$$(\#)_n \qquad \begin{aligned} a_n &= -1 \\ a_{i-1} &= -a_i' - za_i - (n-i)(i+1)ra_{i+1} \quad (i = n, \ldots, 0) \end{aligned}$$

By a solution of $(\#)_n$ is meant a function z such that when $a_n, \ldots, a_{-1}$ are defined as above, then a_{-1} is (identically) zero.

THEOREM 2. *Let z be a solution of* $(\#)_n$, *and let* ω *be any solution of*

$$\omega^n = \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!}\,\omega^i.$$

Then $\eta = e^{\int\omega}$ *is a solution of the DE* $y'' = ry$.

Proof. Let

$$A = -w^n + \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!} w^i = \sum_{i=0}^{n} \frac{a_i}{(n-i)!} w^i \quad (a_n = -1),$$

where w is an indeterminate. We claim that

$$\frac{\partial^{k+1}A}{\partial w^{k+1}}(w^2 - r) = \frac{\partial^{k+1}A}{\partial w^k\,\partial x} + [(n-2k)w + z]\frac{\partial^k A}{\partial w^k} + k(n-k+1)\frac{\partial^{k-1}A}{\partial w^{k-1}} \quad (k = 0, 1, \ldots).$$

For $k=0$, we have

$$\begin{aligned}
\frac{\partial A}{\partial w}(w^2-r) &= \left(\sum_{i=1}^{n} \frac{ia_i}{(n-i)!} w^{i-1}\right)(w^2-r) \\
&= na_n w^{n+1} + \sum_{i=0}^{n-1} \frac{ia_i}{(n-i)!} w^{i+1} - \sum_{i=0}^{n-1} \frac{(i+1)ra_{i+1}}{(n-1-i)!} w^i \\
&= nwA - \sum_{i=0}^{n-1} \frac{(n-i)a_i}{(n-i)!} w^{i+1} - \sum_{i=0}^{n-1} \frac{(n-i)(i+1)ra_{i+1}}{(n-i)!} w^i \\
&= nwA + a_{n-1}A - \sum_{i=0}^{n-1} \frac{a_{n-1}a_i}{(n-i)!} w^i - \sum_{i=0}^{n} \frac{a_{i-1}}{(n-i)!} w^i \\
&\quad - \sum_{i=0}^{n} \frac{(n-i)(i+1)ra_{i+1}}{(n-i)!} w^i \\
&= (nw+z)A - \sum_{i=0}^{n} \frac{1}{(n-i)!}[za_i + {}^{a}{}_{i-1} + (n-i)(i+1)ra_{i+1}]w^i \\
&= (nw+z)A + \sum_{i=0}^{n} \frac{a_i'}{(n-i)!} w^i = (nw+z)A + \frac{\partial A}{\partial x}.
\end{aligned}$$

Our claim now follows by induction.

To show that $\eta = e^{\int \omega}$ is a solution of the DE is equivalent to showing that $\omega' + \omega^2 = r$. We assume that $\omega' + \omega^2 - r \neq 0$ and force a contradiction.

Since $A(\omega) = 0$, we have

$$\frac{\partial A}{\partial w}(\omega)\omega' + \frac{\partial A}{\partial x}(\omega) = 0.$$

Therefore

$$\frac{\partial A}{\partial w}(\omega)(\omega' + \omega^2 - r) = -\frac{\partial A}{\partial x}(\omega) + (n\omega + z)A(\omega) + \frac{\partial A}{\partial x}(\omega) = 0.$$

Hence

$$\frac{\partial A}{\partial w}(\omega) = 0.$$

Assuming that

$$\frac{\partial^{k-1} A}{\partial w^{k-1}}(\omega) = \frac{\partial^k A}{\partial w^k}(\omega) = 0,$$

we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial^k A}{\partial w^k}(\omega)\right) = \frac{\partial^{k+1} A}{\partial w^{k+1}}(\omega)\omega' + \frac{\partial^{k+1} A}{\partial w^k x}(\omega).$$

Thus

$$\begin{aligned}
&\frac{\partial^{k+1} A}{\partial w^{k+1}}(\omega)(\omega' + \omega^2 - r) \\
&\quad = -\frac{\partial^{k+1} A}{\partial w^k \, \partial x}(\omega) + \frac{\partial^{k+1} A}{\partial w^k \, \partial x}(\omega) + [(n-2k)\omega + z]\frac{\partial^k A}{\partial w^k}(\omega) + k(n-k+1)\frac{\partial^{k-1} A}{\partial w^{k-1}}(\omega) \\
&\quad = 0,
\end{aligned}$$

so

$$\frac{\partial^{k+1} A}{\partial w^{k+1}}(\omega) = 0.$$

The desired contradiction follows from the fact that

$$\frac{\partial^n A}{\partial w^n}(\omega) = -n! \neq 0.$$

This proves the theorem.

THEOREM 3.

(*i*) *Suppose that* $(\#)_4$ *has a solution* $z \in \mathbb{C}(x)$. *Then the polynomial*

$$w^4 - \sum_{i=0}^{3} \frac{a_i}{(4-i)!} w^i \in \mathbb{C}(x)[w]$$

is irreducible over $\mathbb{C}(x)$.

(*ii*) *Suppose that* $(\#)_6$ *has a solution* $z \in \mathbb{C}(x)$. *Then the polynomial*

$$w^6 - \sum_{i=0}^{5} \frac{a_i}{(6-i)!} w^i \in \mathbb{C}(x)[w]$$

is irreducible over $\mathbb{C}(x)$.

(*iii*) *Suppose that* $(\#)_{12}$ *has a solution* $z \in \mathbb{C}(x)$ *and that* $(\#)_4$ *and* $(\#)_6$ *do not have solutions in* $\mathbb{C}(x)$. *Then the polynomial*

$$w^{12} - \sum_{i=0}^{11} \frac{a_i}{(12-i)!} w^i \in \mathbb{C}(x)[w]$$

is irreducible over $\mathbb{C}(x)$.

PROOF. By Theorems 1 and 2, any root of the polynomial

$$w^n - \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!} w^i \quad (a_i \in \mathbb{C}(x))$$

must have degree 4, 6 or 12 over $\mathbb{C}(x)$. Statement (i) of the present theorem is clear. Statement (ii) follows from the fact that if a sextic is reducible, then one of the factors has degree $\leqslant 3$. To prove (iii) it suffices to show that if $\deg_{\mathbb{C}(x)} \omega = n$, then $(\#)_n$ has a solution $z \in \mathbb{C}(x)$.

Let $A \in \mathbb{C}(x)[w]$ be the minimal polynomial for ω over $\mathbb{C}(x)$. Let $\deg_w A = n$ and write

$$A = -w^n + \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!} w^i = \sum_{i=0}^{n} \frac{a_i}{(n-i)!} w^i \quad (a_n = -1).$$

Consider the polynomial

$$B = \frac{\partial A}{\partial w}(r - w^2) + \frac{\partial A}{\partial x} + (nw + z)A,$$

where

$$z = a_{n-1} \in \mathbb{C}(x).$$

The coefficient of w^{n+1} in B is

$$-na_n + na_n = 0,$$

and the coefficient of w^n in B is

$$-(n-1)a_{n-1} + a_n' + na_{n-1} + za_n = a_{n-1} - z = 0,$$

since $a_n = -1$ and $a_{n-1} = z$. Therefore $\deg_w B < n$. But

$$B(\omega) = \frac{\partial A}{\partial w}(\omega)(r-\omega^2) + \frac{\partial A}{\partial x}(\omega) + (n\omega + z)A(\omega)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}(A(\omega)) + (n\omega + z)A(\omega)$$

$$= 0.$$

Therefore $B = 0$. The coefficient of w^i in B is

$$0 = (i+1)\frac{a_{i+1}}{(n-1-i)!}r - (i-1)\frac{a_{i-1}}{(n+1-i)!} + \frac{a_i'}{(n-i)!} + n\frac{a_{i-1}}{(n+1-i)!} + z\frac{a_i}{(n-i)!}$$

$$= \frac{1}{(n-i)!}[(n-i)(i+1)ra_{i+1} + a_{i-1} + a_i' + za_i],$$

where $a_{-1} = 0$. These are precisely the equations of $(\#)_n$. This proves the theorem.

For any function b we denote by $l\delta b = b'/b$ the "logarithmic derivative" of b.

THEOREM 4. *Let F be any form (homogeneous polynomial) of degree n in solutions of the DE. Then $z = l\delta F$ is a solution of $(\#)_n$.*

PROOF. First we prove that if F_1 and F_2 are functions such that $l\delta F_1$ and $l\delta F_2$ are solutions of $(\#)_n$, then $l\delta(c_1F_1 + c_2F_2)$ is a solution of $(\#)_n$ for any $c_1, c_2 \in \mathbb{C}$. Let a_i^1, a_i^2, a_i^3 denote the sequences determined by $(\#)_n$ for $z = l\delta F_1, l\delta F_2, l\delta(c_1F_1 + c_1F_2)$ respectively.

We claim that

$$(c_1F_1 + c_2F_2)a_i^3 = c_1F_1a_i^1 + c_2F_2a_i^2.$$

This is clear for $i = n$. Also

$$\begin{aligned}(c_1F_1 + c_2F_2)a_{i-1}^3 &= (c_1F_1 + c_2F_2)[-a_i^{3\prime} - l\delta(c_1F_1 + c_2F_2)a_i^3 - (n-i)(i+1)ra_{i+1}^3] \\ &= -[(c_1F_1 + c_2F_2)a_i^3]' - (n-i)(i+1)r(c_1F_1 + c_2F_2)a_{i+1}^3 \\ &= -[c_1F_1a_i^1 + c_2F_2a_i^2]' - (n-i)(i+1)[c_1F_1a_{i+1}^1 + c_2F_2a_{i+1}^2] \\ &= c_1F_1a_{i-1}^1 + c_2F_2a_{i-1}^2 \quad (i = n, \ldots, 0).\end{aligned}$$

Therefore

$$(c_1F_1 + c_2F_2)a_{-1}^3 = c_1F_1a_{-1}^1 + c_2F_2a_{-1}^2 = 0,$$

which verifies our assertion.

To prove the theorem, we may assume that

$$F = \prod_{i=1}^{n} \eta_i,$$

where $\eta_1, \ldots, \eta_n$ are solutions of the DE.

Let $\omega_i = \eta_i'/\eta_i$ and denote by σ_{mk} the kth symmetric function of $\omega_1, \ldots, \omega_m$. Thus $\sigma_{mk} = 0$ for $k < 0$ or $k > m$, $\sigma_{m0} = 1$ and

$$\sigma_{mk} = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m} \omega_{i_1} \cdots \omega_{i_k}$$

for $1 \leqslant k \leqslant m$. First we claim that

$$\sigma_{mk}' = (m+1-k)r\sigma_{m,k-1} - \sigma_{m1}\sigma_{mk} + (k+1)\sigma_{m,k+1}.$$

This formula is easily checked for $m = 1$ and, for $m > 1$,

$$\begin{aligned}
\sigma'_{mk} &= (\sigma_{m-1,k} + \sigma_{m-1,k-1}\omega_m)' \\
&= (m-k)r\sigma_{m-1,k-1} - \sigma_{m-1,1}\sigma_{m-1,k} + (k+1)\sigma_{m-1,k+1} \\
&\quad + [(m+1-k)r\sigma_{m-1,k-2} - \sigma_{m-1,1}\sigma_{m-1,k-1} + k\sigma_{m-1,k}]\omega_m \\
&\quad + \sigma_{m-1,k-1}(r - \omega_m^2) \\
&= (m+1-k)r(\sigma_{m-1,k-1} + \sigma_{m-1,k-2}\omega_m) - (\sigma_{m-1,1} + \omega_m)(\sigma_{m-1,k} + \sigma_{m-1,k-1}\omega_m) \\
&\quad + (k+1)(\sigma_{m-1,k+1} + \sigma_{m-1,k}\omega_m) \\
&= (m+1-k)r\sigma_{m,k-1} - \sigma_{m1}\sigma_{mk} + (k+1)\sigma_{m,k+1},
\end{aligned}$$

which completes the induction.

Next we use induction on i to prove that

$$a_i = (-1)^{n-i+1}(n-i)!\,\sigma_{n,n-i}.$$

Evidently

$$a_{n-1} = z = l\delta F = \sum_{i=1}^{n} \omega_i = \sigma_{n1}.$$

Using $(\#)_n$, we have

$$\begin{aligned}
a_{i-1} &= -a'_i - za_i - (n-i)(i+1)ra_{i+1} \\
&= (-1)^{n-i}(n-i)!\,\sigma'_{n,n-i} + \sigma_{n1}(-1)^{n-i}(n-i)!\,\sigma_{n,n-i} \\
&\quad - (n-i)(i+1)r(-1)^{n-i}(n-1-i)!\,\sigma_{n,n-1-i} \\
&= (-1)^{n-i}(n-i)!\,[\sigma'_{n,n-i} + \sigma_{n1}\sigma_{n,n-i} - (i+1)r\sigma_{n,n-1-i}] \\
&= (-1)^{n-i}(n-i)!\,(n-i+1)\sigma_{n,n-i+1} \\
&= (-1)^{n-i}(n-i+1)!\,\sigma_{n,n-i+1}.
\end{aligned}$$

Hence

$$a_{-1} = (-1)^n(n+1)!\,\sigma_{n,n+1} = 0.$$

This completes the proof of the theorem.

THEOREM 5.

(i) *If G is the tetrahedral group, then $(\#)_4$ has a solution $z = l\delta u$, where $u^3 \in \mathbb{C}(x)$.*
(ii) *If G is the octahedral group, then $(\#)_6$ has a solution $z = l\delta u$, where $u^2 \in \mathbb{C}(x)$.*
(iii) *If G is either the tetrahedral group, the octahedral group or the icosahedral group, then $(\#)_{12}$ has a solution $z = l\delta u$, where $u \in \mathbb{C}(x)$.*

PROOF. This theorem is a corollary of Theorem 3 of the present section and Theorem 3 of the previous section. For part (i) we may take $u = \eta^4 + 8\eta\zeta^3$, for part (ii) we may take $u = \eta^5\zeta - \eta\zeta^5$ and for part (iii) we may take $u = (\eta^4 + 8\eta\zeta^3)^3$, $(\eta^5\zeta - \eta\zeta^5)^2$ or $\eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11}$.

We shall write

$$u^{12/n} = \prod_{c \in \mathbb{C}} (x-c)^{e_c} \in \mathbb{C}(x),$$

where $n = 4$, 6 or 12 and $e_c \in Z$. Our next step in the proof is to determine the various possibilities for e_c, as stated in step 1 of the algorithm. For ease of notation, we shall assume that $c = 0$. To this end we shall use the Laurent series for

$$z = l\delta u = \frac{n}{12}\, l\delta(u^{12/n}),$$

namely

$$z = \frac{n}{12} e x^{-1} + \cdots \quad (e = e_0 \in \mathbb{Z}, \text{ possibly } 0)$$

and for r, namely

$$r = \alpha x^{-2} + \beta x^{-1} + \cdots \quad (\alpha, \beta \in \mathbb{C}, \text{ possibly } 0).$$

(Note that, by the necessary conditions of section 2, r can have no pole of order exceeding 2.)

First we consider the possibility that $\alpha = 0$ and $\beta \neq 0$, corresponding to (c_1) of Step 1 of the algorithm.

THEOREM 6. *If* $\alpha = 0$ *and* $\beta \neq 0$, *then* $e = 12$.

PROOF. We write

$$z = \frac{n}{12} e x^{-1} + f + \cdots,$$

and treat e and f as indeterminates. Then

$$a_i = A_i x^{i-n} + B_i x^{i-n+1} + C_i f x^{i-n+1} + \cdots,$$

where A_i, B_i, C_i are polynomials in e with coefficients in $\mathbb{C}$. Using $(\#)_n$ we find that

$$A_n = -1, \qquad B_n = C_n = 0,$$

$$A_{i-1} = \left(n - i - \frac{n}{12} e\right) A_i,$$

$$B_{i-1} = \left(n - i - 1 - \frac{n}{12} e\right) B_i - (n-i)(i+1)\beta A_{i+1},$$

$$C_{i-1} = \left(n - i - 1 - \frac{n}{12} e\right) C_i - A_i,$$

for $i = n, \ldots, 0$.

We leave to the reader the verification that the solution to these equations is given by

$$A_i = -\prod_{j=0}^{n-i-1} \left(j - \frac{n}{12} e\right)$$

$$B_i = \beta \sum_{j=0}^{n-i-2} (j+1)(n-j) \prod_{\substack{k=0 \\ k \neq j}}^{n-i-2} \left(k - \frac{n}{12} e\right),$$

$$C_i = (n-i) \prod_{j=0}^{n-i-2} \left(j - \frac{n}{12} e\right) \quad (i = n, \ldots, 0)$$

because

$$0 = a_{-1} = A_{-1} x^{-n-1} + B_{-1} x^{-n} + C_{-1} f x^{-n} + \cdots,$$

$$0 = A_{-1} = -\prod_{j=0}^{n} \left(j - \frac{n}{12} e\right)$$

and

$$0 = B_{-1} + C_{-1} f$$

$$= \beta \sum_{j=0}^{n-1} (j+1)(n-j) \prod_{\substack{k=0 \\ k \neq j}}^{n-1} \left(k - \frac{n}{12} e\right) + f(n+1) \prod_{k=0}^{n-1} \left(k - \frac{n}{12} e\right).$$

The first equation implies that

$$e = \frac{12}{n} l$$

for some $l = 0, \ldots, n$. Suppose that $l \neq n$. Then the second equation gives

$$C = \beta(l+1)(n-l) \prod_{\substack{k=0 \\ k \neq l}}^{n-1} (k-l),$$

which implies that $\beta = 0$. This contradiction shows that $l = n$ and therefore $e = 12$. This proves the theorem.

Next we consider the possibility that $\alpha \neq 0$. This corresponds to (c_2) of Step 1 of the algorithm. As above we write $a_i = A_i x^{i-n} + \cdots$.

Lemma. *A_i is a polynomial in e with coefficients in $\mathbb{Q}[\alpha]$ whose degree is $n-i$ and whose leading coefficient is $-(-(n/12))^{n-i}$.*

Proof. Using $(\#)_n$ we have

$$A_n = -1,$$
$$A_{i-1} = \left(n - i - \frac{n}{12} e\right) A_i - (n-i)(i+1)\alpha A_{i+1}.$$

The lemma is immediate from these formulas.

The author did not succeed in finding a closed-form solution of these equations, thus we shall use an indirect argument.

Assume that $\alpha \neq -1/4$. Then the DE $y'' = ry$ has Puisseaux series solutions of the form

$$\eta_1 = x^{\mu_1} + \cdots, \qquad \mu_1 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\alpha},$$
$$\eta_2 = x^{\mu_1} + \cdots, \qquad \mu_2 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4a}.$$

By Theorem 4, $l\delta(\eta_1^i \eta_2^{n-i})$ is a solution of $(\#)_n$ for every $i = 0, \ldots, n$. Since

$$l\delta(\eta_1^i \eta_2^{n-i}) = (i\mu_1 + (n-i)\mu_2)x^{-1} + \cdots$$
$$= \left(\frac{n}{2} - \left(\frac{n}{2} - i\right)\sqrt{1+4\alpha}\right) x^{-1} + \cdots,$$

the polynomial A_{-1} must vanish for

$$\frac{12}{n} e = \frac{n}{2} - \left(\frac{n}{2} - i\right)\sqrt{1+4\alpha} \quad (i = 0, \ldots, n).$$

Theorem 7.

(*i*) *Assume that G is the tetrahedral group. Then e is an integer chosen from among $6 + k\sqrt{1+4\alpha}$, $k = 0, \pm 3, \pm 6$.*

(*ii*) *Assume that G is the octahedral group. Then e is an integer chosen from among $6 + k\sqrt{1+4\alpha}$, $k = 0, \pm 2, \pm 4, \pm 6$.*

(*iii*) *Assume that G is either the tetrahedral group, the octahedral group or the icosahedral group. Then e is an integer chosen from among $6 + k\sqrt{1+4\alpha}$, $k = 0, \pm 1, \ldots, \pm 6$.*

PROOF. (i) In this case $n=4$. If $\alpha \neq 1/4$, then we may use the Lemma and the remarks following it to obtain

$$0 = A_{-1} = \prod_{i=0}^{4}\left(\frac{e}{3} - 2 + (2-i)\sqrt{1+4\alpha}\right).$$

Thus

$$e = 6 + k\sqrt{1+4\alpha}, \qquad k = 0,\ \pm 3,\ \pm 6.$$

If $\alpha = -1/4$, we compute directly, using the recurrence relations given above.

$$\begin{aligned}
A_4 &= -1\\
A_3 &= \tfrac{1}{3}e\\
A_2 &= -\tfrac{1}{9}(e^2 - 3e + 9)\\
A_1 &= \tfrac{1}{27}(e^3 - 9e^2 + \tfrac{81}{2}e - 54)\\
A_0 &= -\tfrac{1}{81}(e^4 - 18e^3 + 135e^2 - 459e + \tfrac{1215}{2})\\
A_{-1} &= \tfrac{1}{243}(e^5 - 30e^4 + 360e^3 - 2160e^2 + 6480e - 7776)\\
&= \tfrac{1}{243}(e-6)^5.
\end{aligned}$$

(ii) In this case $n=6$. *If* $\alpha \neq -1/4$, then we may use the Lemma and the remarks following it to obtain

$$0 = A_{-1} = \prod_{i=0}^{6}\left(\frac{e}{2} - 3 + (3-i)\sqrt{1+4\alpha}\right).$$

Thus
$e = 6 + k\sqrt{1+4\alpha}, \qquad k = 0,\ \pm 2,\ \pm 4,\ \pm 6.$
If $\alpha = -1/4$, we compute directly.

$$\begin{aligned}
A_6 &= -1\\
A_5 &= \tfrac{1}{2}e\\
A_4 &= -\tfrac{1}{4}(e^2 - 2e + 6)\\
A_3 &= \tfrac{1}{8}(e^3 - 6e^2 + 24e - 24)\\
A_2 &= -\tfrac{1}{16}(e^4 - 12e^3 + 72e^2 - 192e + 216)\\
A_1 &= \tfrac{1}{32}(e^5 - 20e^4 + 180e^3 - 840e^2 + 2040e - 2016)\\
A_0 &= -\tfrac{1}{64}(e^6 - 30e^5 + 390e^4 - 2760e^3 + 11160e^2 - 24336e + 22320)\\
A_{-1} &= \tfrac{1}{128}(e^7 - 42e^6 + 756e^5 - 7560e^4 + 45360e^3 - 163296e^2 + 326592e - 279936)\\
&= \tfrac{1}{128}(e-6)^7.
\end{aligned}$$

(iii) In this case $n=12$. If $\alpha \neq -1/4$, then we may use the Lemma and the remarks following it to obtain

$$0 = A_{-1} = \prod_{i=0}^{12}\left(e - 6 + (6-i)\sqrt{1+4\alpha}\right).$$

Thus

$$e = 6 + k\sqrt{1+4\alpha}, \quad k = 0,\ \pm 1, \ldots, \pm 6.$$

If $\alpha = -1/4$, we compute directly. Using a programmable calculator we obtained the following.

$$
\begin{aligned}
A_{12} &= -1\\
A_{11} &= e\\
A_{10} &= -e^2+e-3\\
A_9 &= e^3-3e^2+\tfrac{21}{2}e-6\\
A_8 &= -e^4+6e^3-27e^2+45e-\tfrac{81}{2}\\
A_7 &= e^5-10e^4+60e^3-180e^2+315e-216\\
A_6 &= -e^6+15e^5-120e^4+540e^3-1485e^2+2241e-1485\\
A_5 &= e^7-21e^6+\tfrac{441}{2}e^5-1365e^4+5355e^3-13041e^2+\tfrac{36477}{2}e-11178\\
A_4 &= -e^8+28e^7-378e^6+3066e^5-16170e^4+56196e^3-125118e^2\\
&\quad +162378e-\tfrac{187677}{2}\\
A_3 &= e^9-36e^8+612e^7-6300e^6+42903e^5-199206e^4+628236e^3\\
&\quad -1293732e^2+\tfrac{3150495}{2}e-862488\\
A_2 &= -e^{10}+45e^9-945e^8+12060e^7-103005e^6+612927e^5-2566620e^4\\
&\quad -7453620e^3-\tfrac{28689795}{2}e2+\tfrac{33002235}{2}e-\tfrac{17213877}{2}\\
A_1 &= e^{11}-55e^{10}+\tfrac{2805}{2}e^9-21780e^8+228195e^7-1690227e^6\\
&\quad +\tfrac{18035325}{2}e^5-34613865e^4+\tfrac{187185735}{2}e^3-\tfrac{339306165}{2}e^2\\
&\quad +\tfrac{741729879}{4}e-92538045\\
A_0 &= -e^{12}+66e^{11}-2013e^{10}+37455e^9-\tfrac{945945}{2}e^8\\
&\quad -28176687e^6+137179251e^5-\tfrac{976923585}{2}e^4+1240169535e^3\\
&\quad -\tfrac{4261026627}{2}e^2+\tfrac{4446102717}{2}e-\tfrac{4261026627}{4}\\
A_{-1} &= e^{13}-78e^{12}+2808e^{11}-61776e^{10}+926640e^9-10007712e^8\\
&\quad +80061696e^7-480370176e^6+2161665792e^5-7205552640e^4\\
&\quad +17293326336e^3-28298170368e^2+28298170368e-13060694016\\
&= (e-6)^{13}.
\end{aligned}
$$

This proves the theorem.

Finally we consider what happens if $\alpha=\beta=0$, i.e. at an ordinary point of r. Using the previous theorem, we have that $(n/12)e$ is an integer.

Let Γ denote the set of poles of r. We have proven the following.

(i) In the tetrahedral case, $(\#)_4$ has a solution $z = l\delta u$, where

$$u^3 = P^3 \prod_{c\in\Gamma} (x-c)^{e_c},$$

$P \in \mathbb{C}[x]$ and e_c is an integer chosen from among $6+k\sqrt{1+4\alpha}$, $k=0, \pm 3, \pm 6$.

(ii) In the octahedral case, $(\#)_6$ has a solution $z = l\delta u$, where

$$u^2 = P^2 \prod_{c\in\Gamma} (x-c)^{e_c},$$

$P \in \mathbb{C}[x]$ and e_c in an integer chosen from among $6+k\sqrt{1+4\alpha}$, $k=0, \pm 2, \pm 4, \pm 6$.

(iii) In either the tetrahedral case, the octahedral case or the icosahedral case, $(\#)_{12}$ has a solution $z = l\delta u$, where

$$u = P \prod_{c \in \Gamma} (x-c)^{e_c},$$

$P \in \mathbb{C}[x]$ and e_c is an integer chosen from among $6+k\sqrt{1+4\alpha}$, $k=0, \pm 1, \ldots, \pm 6$.

Let $d = \deg P$. Then the Laurent series for z at ∞ has the form

$$z = \frac{n}{12}\left(\frac{12}{n} d + \sum_{c \in \Gamma} e_c\right) x^{-1} + \cdots$$

and the Laurent series for r at ∞ has the form

$$r = \gamma x^{-2} + \cdots.$$

(By the necessary conditions of section 2, the order of r at ∞ is at least 2.)

If we let

$$e_\infty = \frac{12}{n} d + \sum_{c \in \Gamma} e_c,$$

then, by a theorem analogous to Theorem 7, e_∞ satisfies the same conditions as does each e_c. Also

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

must be a non-negative integer. This is a restatement of step 2 of the algorithm.

We shall complete the proof of the algorithm by showing that the recursive relations of step 3 are identical with $(\#)_n$.

Let

$$\theta = \frac{n}{12} \sum_{c \in \Gamma} \frac{e_c}{x-c} \quad \text{and} \quad S = \prod_{c \in \Gamma} (x-c).$$

Then $z = l\delta u = P'/P + \theta$. Also set $P_i = S^{n-i} P a_i$. Using $(\#)_n$, we have

$$\begin{aligned}
P_n &= -P \\
P_{i-1} &= S^{n-i+1} P a_{i-1} \\
&= S^{n-i+1} P(-a_i' - z a_i - (n-i)(i+1) r a_{i+1}) \\
&= -S(S^{n-i} P a_i)' + (n-i) S^{n-i} S' P a_i + S^{n-i+1} P' a_i \\
&\quad - S(P' + P\theta)(S^{n-i} a_i) - (n-i)(i+1) S^2 r (S^{n-i-1} P a_{i+1}) \\
&= -S P_i' + ((n-i) - S\theta) P_i - (n-i)(i+1) S^2 r P_{i+1}.
\end{aligned}$$

This is precisely the equation of step 2 of the algorithm.

Finally, the equation

$$\omega^n = \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!} \omega^i$$

may be rewritten as

$$0 = -S^n P \omega^n + \sum_{i=0}^{n-1} \frac{S^n P a_i}{(n-i)!} \omega^i = \sum_{i=0}^{n} \frac{S^i P_i}{(n-i)!} \omega^i.$$

This completes the proof of the algorithm.

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