2. Nonlinear Systems of Two Parabolic Equations (Reaction-Diffusion Equations)

2.1. Reaction-Diffusion and Mathematical Biology Systems of the Form $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + F(u,w)$, $\frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + G(u,w)$

Preliminary comments. Similar systems of equations are frequent in the theory of heat and mass transfer of reacting media, the theory of chemical reactors, combustion theory, mathematical biology, and biophysics.

Systems of this form are invariant under translations in independent variables (and under the change of x to $-x$) and admit traveling-wave solutions, $u = u(kx - \lambda t)$ and $w = w(kx - \lambda t)$. These solutions and also degenerate solutions where one of the sought functions is zero are not considered further on.

The functions $f(\varphi)$, $g(\varphi)$, and $g(\varphi)$ appearing below are arbitrary functions of an argument $\varphi = \varphi(u,w)$; the equations are arranged in order of complicating this argument.

1. $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + u\exp\left(k\frac{w}{u}\right)f(u)$, $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \exp\left(k\frac{w}{u}\right)[wf(u) + g(u)]$.

2. $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(bu-cw) + g(bu-cw)$, $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(bu-cw) + h(bu-cw)$.

3. $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + e^{\lambda u}f(\lambda u - \sigma w)$, $\frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + e^{\sigma w}g(\lambda u - \sigma w)$.

4. $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right)$, $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wg\left(\frac{u}{w}\right)$.

5. $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right)$, $\frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + wg\left(\frac{u}{w}\right)$.

6. $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right) + g\left(\frac{u}{w}\right)$, $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(\frac{u}{w}\right) + h\left(\frac{u}{w}\right)$.

7. $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right) + \frac{u}{w}h\left(\frac{u}{w}\right)$, $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wg\left(\frac{u}{w}\right) + h\left(\frac{u}{w}\right)$.

8. $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + u^3f\left(\frac{u}{w}\right)$, $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + u^3g\left(\frac{u}{w}\right)$.

9. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - u^3f\left(\frac{u}{w}\right)$, $\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + aw - u^3g\left(\frac{u}{w}\right)$.

10. $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + u^nf\left(\frac{u}{w}\right)$, $\frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + w^ng\left(\frac{u}{w}\right)$.

11. $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right)\ln u + ug\left(\frac{u}{w}\right)$, $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(\frac{u}{w}\right)\ln w + wh\left(\frac{u}{w}\right)$.

12. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{w}{u}\right) - wg\left(\frac{w}{u}\right) + \frac{u}{\sqrt{u^2+w^2}}h\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(\frac{w}{u}\right) + ug\left(\frac{w}{u}\right) + \frac{w}{\sqrt{u^2+w^2}}h\left(\frac{w}{u}\right).$$

13. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{w}{u}\right) + wg\left(\frac{w}{u}\right) + \frac{u}{\sqrt{u^2-w^2}}h\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(\frac{w}{u}\right) + ug\left(\frac{w}{u}\right) + \frac{w}{\sqrt{u^2-w^2}}h\left(\frac{w}{u}\right).$$

14. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^n w^m), \qquad \frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + wg(u^n w^m).$$

15. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + u^{1+kn}f(u^n w^m), \qquad \frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + w^{1-km}g(u^n w^m).$$

16. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + cu\ln u + uf(u^n w^m), \qquad \frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + cw\ln w + wg(u^n w^m).$$

17. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2+w^2) - wg(u^2+w^2), \qquad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(u^2+w^2) + ug(u^2+w^2).$$

18. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2-w^2) + wg(u^2-w^2), \qquad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(u^2-w^2) + ug(u^2-w^2).$$

19. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2+w^2) - wg(u^2+w^2) - w\arctan\left(\frac{w}{u}\right)h(u^2+w^2),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(u^2+w^2) + ug(u^2+w^2) + u\arctan\left(\frac{w}{u}\right)h(u^2+w^2).$$

20. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2-w^2) + wg(u^2-w^2) + w\,\mathrm{artanh}\left(\frac{w}{u}\right)h(u^2-w^2),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(u^2-w^2) + ug(u^2-w^2) + u\,\mathrm{artanh}\left(\frac{w}{u}\right)h(u^2-w^2).$$

21. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + u^{k+1}f(\varphi), \qquad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + u^{k+1}[f(\varphi)\ln u + g(\varphi)], \qquad \varphi = u\exp\left(-\frac{w}{u}\right).$$

22. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2+w^2) - wg\left(\frac{w}{u}\right), \qquad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + ug\left(\frac{w}{u}\right) + wf(u^2+w^2).$$

23. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2-w^2) + wg\left(\frac{w}{u}\right), \qquad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + ug\left(\frac{w}{u}\right) + wf(u^2-w^2).$$

24. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(u^2+w^2, \frac{w}{u}\right) - wg\left(\frac{w}{u}\right), \qquad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(u^2+w^2, \frac{w}{u}\right) + ug\left(\frac{w}{u}\right).$$

25. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(u^2-w^2, \frac{w}{u}\right) + wg\left(\frac{w}{u}\right), \qquad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(u^2-w^2, \frac{w}{u}\right) + ug\left(\frac{w}{u}\right).$$

26. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + F(u,w), \qquad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bF(u,w).$$

27. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(bu - cw) + g(bu - cw) + c\Phi(u, w),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(bu - cw) + h(bu - cw) + b\Phi(u, w).$$

2.2. Reaction-Diffusion and Mathematical Biology Systems of the Form $\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + F(u, w)$, $\frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + G(u, w)$

Preliminary comments. Similar systems of equations are frequent in the theory of heat and mass transfer of reacting media, the theory of chemical reactors, combustion theory, mathematical biology, and biophysics. The values $n = 1$ and $n = 2$ correspond to equations with axial and central symmetry, respectively.

The functions $f(\varphi)$, $g(\varphi)$, and $g(\varphi)$ appearing below are arbitrary functions of an argument $\varphi = \varphi(u, w)$; the equations are arranged in order of complicating this argument.

1. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(bu - cw) + g(bu - cw),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(bu - cw) + h(bu - cw).$$

2. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + e^{\lambda u}f(\lambda u - \sigma w), \quad \frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + e^{\sigma w}g(\lambda u - \sigma w).$$

3. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf\left(\frac{u}{w}\right), \quad \frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wg\left(\frac{u}{w}\right).$$

4. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf\left(\frac{u}{w}\right), \quad \frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wg\left(\frac{u}{w}\right).$$

5. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf\left(\frac{u}{w}\right) + \frac{u}{w}h\left(\frac{u}{w}\right),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial w}\right) + wg\left(\frac{u}{w}\right) + h\left(\frac{u}{w}\right).$$

6. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + u^k f\left(\frac{u}{w}\right), \quad \frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + w^k g\left(\frac{u}{w}\right).$$

7. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf\left(\frac{u}{w}\right)\ln u + ug\left(\frac{u}{w}\right),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf\left(\frac{u}{w}\right)\ln w + wh\left(\frac{u}{w}\right).$$

8. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(x, u^k w^m), \quad \frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wg(x, u^k w^m).$$

9. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + u^{1+kn}f(u^n w^m), \quad \frac{\partial w}{\partial t} =$$
$$\frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + w^{1-km}g(u^n w^m).$$

10. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + cu\ln u + uf(x, u^k w^m),$$
$$\frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + cw\ln w + wg(x, u^k w^m).$$

11. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(u^2 + w^2) - wg(u^2 + w^2),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(u^2 + w^2) + ug(u^2 + w^2).$$

12. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(u^2 - w^2) + wg(u^2 - w^2),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(u^2 - w^2) + ug(u^2 - w^2).$$

13. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(u^2 + w^2) - wg\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(u^2 + w^2) + ug\left(\frac{w}{u}\right).$$

14. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(u^2 - w^2) + wg\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(u^2 - w^2) + ug\left(\frac{w}{u}\right).$$