Frank Ramsey's Notes on Saving and Taxation

Edited by Pedro Garcia Duarte

Editor's Introduction

There are two sets of undated and until now unpublished notes by Ramsey that support the idea that both of his papers published in the late 1920s, which are the predecessors of modern discussions on economic growth, optimal taxation, and monetary economics, were part of a common research agenda, as discussed in the preceding article. These notes are deposited at the Archives of Scientific Philosophy of the University of Pittsburgh, Hillman Library (Papers of Frank Plumpton Ramsey, series "Undergraduate Materials," box 6, folder 7, "Ramsey Economics") and are published here for the first time by permission of that university (all rights reserved).

The first set of notes contains two sections (numbered IV and V) that most likely were the ones cut out by Keynes from the published version of Ramsey's 1928 paper that is divided into three sections.[1] In the next section I reprint these notes, leaving out most of the passages struck out by Ramsey in his manuscript.[2] I sometimes include additional notes explaining some passages, mostly as footnotes to Ramsey's text. Lastly, in a few instances Ramsey underlined words, which are here italicized.

1. The pages of these notes are numbered from 31 to 40. According to the classification system of the University of Pittsburgh, this set of annotations is numbered 006-07-01.

2. At a few places I do include, in double parentheses (( )), the text struck out by Ramsey.

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In these notes Ramsey modifies his equations to consider the case of a tax on savings. In section IV he explores the distinction between exhaustive and transfer expenditures, which is, as argued in the preceding essay in this issue, in line with Arthur Cecil Pigou's analysis in his 1928 book on public finance (chap. 3). He also considers, in the simple case present in the published article of a linear production function (i.e., the case of constant and independent returns to capital and labor), different tax rates on earned and unearned incomes. Another interesting point is that these notes constitute additional material to important discussions about Ramsey's enterprise in economics. For instance, with respect to the issue of whether or not Ramsey had a notion of a representative agent, as we now understand it, here we see Ramsey referring to the utility function "of an eternal community" rather than of an individual (see Duarte 2008).

In section V of the manuscript we see Ramsey again following the discussion in Pigou's 1928 book about the extent to which "a uniform income tax should be remitted on savings." It was in this context that Pigou was one of the first to refer to Ramsey's 1927 result that lower tax rates should be applied to the uses of income for which the demand is relatively elastic (Duarte 2008). In the manuscripts Ramsey disagrees with Pigou's use of his result and Pigou's conclusion that "as the demand for savings is more elastic than that for consumption . . . there should be differentiation in favour of savings." As Ramsey makes clear, he does not "dispute the soundness of this conclusion," but he believes that his result cannot be used to support it (see the full argument below). Ramsey then plays with his equations in order to find an answer to Pigou's query. The mathematics gets "extremely complicated" and Ramsey then considers the special case of a fixed labor supply, discussing in detail the several hypotheses made in order to derive the set of equations in this manuscript.

The second set of notes are derivations of equations, some of which Ramsey used in the first set. I tried to add to these equations the side notes that Ramsey wrote on some of his manuscript's pages. At some places, indicated in footnotes, his handwriting is not completely clear. I should also warn readers that it is possible that I have made errors in editing some of his equations, as some are not easy to read. The last pages of his manuscript are particularly unclear and not fully developed. Here I indicate these passages in footnotes.

For instance, in Ramsey's annotations he seems to make a distinction between the return to capital, a cursive *r*, and the interest rate used as discount factor, a printed *r*. However, in a few places he seems to use both interchangeably and therefore I make no such distinction in the edited notes. Lastly, in both sets of notes I kept Ramsey's numbering of equations.

In the second set of notes we can see Ramsey crafting his mathematical analysis on exempting savings from income taxes. He considers many special cases, some discussed in the first set of notes, others not. In some places Ramsey even considers particular functional forms for the utility function, which is something that he marginally used in the printed article for the case of no taxation (p. 555). These are notes that are hard reading by themselves. Instead they should be explored together with the published article and the first set of notes. When compared with this first set, they make clear Ramsey's compromise between mathematical derivations and sound economic reasoning obtained from them, as Ramsey wrote to Harrod on 27 March 1929: "I did a very elaborate treatment of taxation and savings which was cut out by Maynard; rightly as it was too involved in comparison with the conclusions which were feeble" (Besomi 2003, 104, letter 158).

Therefore, Ramsey's two published economics articles and the set of notes published here taken together highlight important aspects of the approach to economics of the young mathematician by *métier* who was strongly drawn to economic science (Keynes [1933] 1972, 324).

First Set of Notes

IV

Let us now consider how to modify our equations in order to show the effect of taxation ((on saving)), supposing that the public as a whole saves as if it were discounting future utilities at a rate ρ, but otherwise keeping its utility schedules unaltered. Suppose an income tax is levied at a constant rate λ but is remitted on savings to a constant extent μ i.e. the total tax paid is λ times consumption plus $\lambda - \mu$ times savings, then it is not difficult to see that equations (2) and (3) become[3]

3. Editor's note: From the published article, equation (2) is $v(a) = \frac{\partial f}{\partial a} u(x)$, where $a(t)$ and $x(t)$ denote the total rates of labor and consumption at time t, $v(a) = \frac{dV(a)}{da}$ is the marginal disutility of labor, and $f(a, c)$ is the production function, with $c(t)$ being the stock of capital at time t. Equation (3) is $\frac{d}{dt} u(x(t)) = -\frac{\partial f}{\partial c} \cdot u(x(t))$, where $u(x(t)) = \frac{dU(x(t))}{dx(t)}$ is the marginal utility of consumption. Using modern terminology, equation (2) gives us the intratemporal relationship between labor and consumption while (3) represents the optimality condition for intertemporal allocation of consumption. It is important to mention also that both equations refer to the case of no discounting. Later in the paper, Ramsey (1928, 553–54) shows that when time-discounting is introduced (again, with no taxes), equation (2) is unaltered, but (3) becomes $\frac{d}{dt} u(x(t)) = -u(x(t))\{\frac{\partial f}{\partial c} - \rho\}$.

$$v(a) = (1-\lambda)\,\frac{\partial f}{\partial a}\,u(x) \tag{11}$$

and

$$\frac{d}{dt}\,u(x) = \left\{\frac{(1-\lambda)}{(1-\mu)}\,\frac{\partial f}{\partial c} - \rho\right\} u(x) \tag{12}$$

The new form of equation (1) depends on what the government does with the revenue it raises:[4] it is convenient to consider two alternative hypotheses. The first is that the revenue is all *transferred* by the government back to the public in the form of pensions, dividends on war loan etc to be spent or saved by the public at its discretion. This income will then be supposed[5] liable to tax as is the case at present with war loan dividends so that whether it is saved or spent will be affected by the rates of taxation just as in the case of any other income. On this hypothesis equation (1) will be unaltered, as on balance no income is taken from the public.

The second hypothesis is that the government raises money to spend itself on purposes separate from those on which the public spends its money, and that this government expenditure does not alter the utility of private incomes. In this case we shall say that the revenue raised is *exhausted* by government, and equation (1) must be replaced by

$$(1-\mu)\frac{dc}{dt} + x = (1-\lambda)f(a,c) \tag{13}$$

Let us take first the hypothesis that the revenue is all exhausted; then if $\rho = 0$ we can easily obtain an analogue to (4): for we have[6]

$$\begin{aligned}
\frac{d}{dx}\left(u(x)\cdot f(a,c)\right) &= \frac{du}{dx}f(a,c) + u(x)\frac{\partial f}{\partial a}\frac{da}{dx} + u(x)\frac{\partial f}{\partial c}\frac{dc}{dt}\frac{dt}{dx} \\
&= \frac{du}{dx}f(a,c) + \frac{v(a)}{(1-\lambda)}\,\frac{da}{dx} \\
&\quad - \frac{(1-\mu)}{(1-\lambda)}\,\frac{du(x)}{dt}\left\{\frac{(1-\lambda)f(a,c) - x}{1-\mu}\right\}\frac{dt}{dx} \\
&= \frac{x}{(1-\lambda)}\frac{du}{dx} + \frac{v(a)}{(1-\lambda)}\frac{da}{dx}
\end{aligned}$$

4. Editor's note: Ramsey (1928, 544) explains that equation (1) is the economy-wide resource constraint. It states that savings, or investment, ($dc(t)\,/\,dt$) plus consumption ($x(t)$) equal income, which is a function of labor ($a(t)$) and capital ($c(t)$): $\frac{dc(t)}{dt} + x(t) = f(a(t)), (c(t))$.

5. Editor's note: Unclear word.

6. Editor's note: In the following equation, Ramsey went from the first to the second line by using the expressions for $u(x)\cdot\frac{\partial f}{\partial a}$ and $u(x)\cdot\frac{\partial f}{\partial c}$ coming from (11) and (12), when $\rho = 0$. In the final line, he collected terms and used the fact that $\frac{du(x)}{dt}\frac{dt}{dx} = \frac{du}{dx}$.

so that $u(x)f(a,c)(1-\lambda) = xu(x) - U(x) + V(a) + K$ or[7]

$$(1-\mu)\frac{dc}{dt} = f(a,c)(1-\lambda) - x = \frac{K - U(x) + V(a)}{u(x)} \tag{14}$$

so that the onerous part of saving, as opposed to the proportion μ provided by the exchequer, is related to marginal and total utility in the same way as before; in the case of an eternal community K will as before be bliss, but taxation may reduce its value as it reduces the maximum obtainable utility if this is not equal to the maximum conceivable.

Further if we again make the assumption[8]

$$f(a,c) = pa + rc, \quad \rho \neq 0$$

we can conveniently allow for different rates of taxation λ_1 and λ_2 on earned and unearned incomes respectively as well as a rate of remission μ on savings.[9] λ in equation (11) has then to be interpreted as λ_1 and in equation (12) as λ_2 and instead of (13) we have

$$(1-\mu)\frac{dc}{dt} + x = (1-\lambda_1)pa + (1-\lambda_2)rc \tag{15}$$

We now put[10]

$$y = x - (1-\lambda_1)pa$$

$$w(y) = u(x) = \frac{v(a)}{(1-\lambda_1)p}$$

and

$$W(y) = U(x) - V(a)$$

(15) then gives $(1-\mu)\frac{dc}{dt} = (1-\lambda_2)rc - y$ and (12) $\frac{dw}{dt} + \left\{\frac{(1-\lambda_2)}{(1-\mu)}r - \rho\right\}w = 0$

Eliminating t and integrating for c in terms of w we get

7. Editor's note: Ramsey (1928, 544) denotes "by $U(x)$ the total rate of utility of a rate of consumption x, and by $V(a)$ the total rate of disutility of a rate of labour a."

8. Editor's note: Ramsey (1928) assumes in section II that returns to capital, r, and the returns to labor, p, are constant and independent in order to be able to (1) represent his solution in a diagram, (2) extend it to the case of an individual who lives a finite time, and (3) extend it to the case of time-discounting.

9. In this case μ might be greater than one of λ_1, λ_2 but not both, and the tax paid by a particular person could then be negative; i.e. he would receive a bonus.

10. Editor's note: In the equations to follow, as in the published paper (p. 550), y denotes the "unearned" income (which is equal to consumption minus "earned" or labor income), and $W(y)$ and $w(y)$ correspond to the total and marginal utility of unearned income, respectively. In the second equation Ramsey used a version of (11) in which λ is replaced by λ_1 and the fact that $\partial f(a,c) / \partial a$ is now equal to p. The third equation comes from the definition of $W(y) = \int w(y)dy$ and the fact that, from the first equation, $dy = dx\,(1-\lambda_1)p \cdot da$.

$$(1-\mu)\frac{dc}{dt} = \frac{K - \int w^{\sigma}(y)dy}{w^{\sigma}(y)}$$

where $\sigma = \frac{r(1-\lambda_2)}{\{(1-\lambda_2)r - \rho(1-\mu)\}}$.

Similarly if the revenue is all transferred, putting

$$y = x - pa$$

$$w(y) = u(x)$$

$$W(y) = U(x) - \frac{V(a)}{1-\lambda_1}$$

we get

$$\frac{dc}{dt} = \frac{K - \int w^{\sigma}(y)dy}{w^{\sigma}(y)}$$

where σ is now $\frac{r(1-\mu)}{\{(1-\lambda_2)r - \rho(1-\mu)\}}$.

V

In conclusion let us turn to the problem of raising a given revenue with minimum sacrifice, distributional and administrative considerations being disregarded.[11]

In the first place, if λ_1, λ_2, and μ can all be varied independently the problem is easy unless the revenue required is very large. The solution is evidently given by $\lambda_1 = 0$, $\lambda_2 = \mu$; i.e. there should be no taxation on earned income, and the tax on unearned income should be offset exactly by a bonus on saving. The revenue would in effect be raised entirely off capital existing at the time when tax is first imposed, and the taxation would have no "announcement effects," to use Prof. Pigou's phrase.

((Secondly)) But it may not be possible to tax earned and unearned incomes at such different rates, so let us suppose instead that they are taxed equally and inquire to what extent a uniform income tax should be remitted on savings. Prof. Pigou in discussing this question[12] refers to a result obtained by me[13] according to which uses of income for which the demand is relatively elastic should be taxed at a lower rate than those for which it is relatively inelastic; he concludes that as the demand for savings is more elastic than that for consumption, as far as announcement aspects

11. Editor's note: In the text struck out by Ramsey one reads: ((The last question I shall discuss is the bearing of the equations developed in section IV on the problem of raising . . .)).

12. A Study in Public Finance p. 138.

13. See *Economic Journal* March 1927 p. 59.

go there should be differentiation in favour of savings. This means, or should mean, that not merely should savings be altogether exempted from income tax but that they should actually be rewarded with a bonus; for ((as he has pointed out)) on Prof. Pigou's view,[14] unless exempted from income tax they are really taxed twice, once when originally made and again when they earn interest. I ((cannot, however, agree with)) do not dispute the soundness of this conclusion, ((and)) but I ((am sure)) want to point out that ((my result)) the result of mine to which he refers really cannot be used to support it. The reason for this is that in a problem covering a considerable term of years saving cannot be considered simply as a use of income with its own utility. Its utility is indirect and arises from the consumption it makes possible later; it is therefore a part of the process of production rather than of that of consumption, and it is evident that the ((kind of)) reasoning by which I proved the result to which Prof. Pigou refers ((simpl[y])) cannot be applied at all. The only case in which it can be applied is when we are concerned with a very short period and a system of taxes which are only to last for that short period and create no expectation of similar taxation in future. In this case we should, as I showed in my paper,[15] take the elasticity of demand for saving as being infinite.

Prof. Pigou's solution must therefore be set aside and we must tackle the problem by means of our equations. It is, however, ((unbearably)) extremely complicated unless we assume that the supply of work is fixed. In this case if all the revenue is transferred back to the public there will be no sacrifice at all provided $\mu = \lambda$, since the two equations to determine x and c are[16]

$$\frac{dc}{dt} + x = f(a,c) \tag{1}$$

and

$$\frac{d}{dt}u(x) = -\left\{ \frac{(1-\lambda)}{(1-\mu)} \frac{\partial f}{\partial c} - \rho \right\} u(x) \tag{12}$$

which are unaltered by raising λ, the rate of income-tax, to any extent provided we always keep μ equal to it. In this case, therefore, income tax

14. Op. cit p. 136.

15. p. 59.

16. Editor's note: The pages of Ramsey's manuscript containing this paragraph are clearly misnumbered. The text, as presented here, follows from page 37 to page 39 and then back to page 38 and finally page 40.

should be completely remitted on savings, but these should receive no further bonus.[17]

If, on the other hand, all the revenue is exhausted this conclusion is still I think true provided that $\rho = 0$ and that, as we have often assumed before, $f(a, c) = pa + rc$, but it ceases to be in the least obvious. All indeed that I have been able to prove is that $\mu = \lambda$ gives a stationary value of the sacrifice not that it gives a true minimum. There seems, however, no reason to doubt that the stationary value is in fact a minimum.

Equations (13) and (14) give us, if we absorb $V(a)$ in B,[18]

$$(1-\mu)\frac{dc}{dt} = (1-\lambda)(pa+rc) - x = \frac{B-U(x)}{u(x)} \tag{16}$$

The rate of revenue obtained at any time is given by

$$\lambda(pa+rc) - \mu\frac{dc}{dt}$$

and the total revenue obtained throughout time discounted at the rate of interest r is given by[19]

$$\begin{aligned} R &= \int_0^\infty \left(\lambda pa + \lambda rc - \mu\frac{dc}{dt}\right)e^{-rt}dt \\ &= \lambda\frac{pa}{r} + \lambda c_0 + (\lambda-\mu)\int_0^\infty \frac{dc}{dt}\,e^{-rt}dt \end{aligned}$$

since ce^{-rt} vanishes at ∞.[20]

Now if we put $\frac{1-\lambda}{1-\mu} = \kappa$, we have

$$u(x) = u(x_0)e^{-\kappa rt},\ \frac{dt}{dx} = -\frac{u'(x)}{\kappa ru(x)}$$

and therefore using (16)

17. Editor's note: This paragraph in the manuscript finished as "bonus, and this."

18. B is ((unaffected by)) independent of λ and μ since on our present assumptions it is the maximum *conceivable* rate of utility, which is bound to be obtainable or approachable in the limit.

19. Editor's note: In writing the exponential of the following equation Ramsey seems to want to make a distinction between the return to capital (a cursive r) and the interest rate (a printed r) used as the discount rate, as already mentioned. However, he used a cursive r to denote the interest rate in the sentence preceding the equation. I make no such distinction here.

20. If it did not, some of our savings would be being left to earn compound interest for ever, and so might as well never have been made.

$$R = \frac{\lambda(pa + rc_0)}{r} + \frac{(\lambda - \mu)}{(1-\mu)u^{1/\kappa}(x_0)} \int_{0\ ((x_0))}^{\infty} \frac{B - U(x)}{u(x)} u^{1/\kappa}(x)dt$$
$$= \frac{\lambda(pa + rc_0)}{r} + \frac{(1-\kappa)}{\kappa r \cdot u^{1/\kappa}(x_0)} \int_{x_0}^{\infty} \left\{ \frac{B - U(x)}{u^2(x)} u^{1/\kappa}(x) - u'(x) \right\} dx \tag{17}$$

Now the total amount, L, by which utility falls short of bliss is given by

$$L = \int_0^{\infty} (B - U(x))dt$$
$$= \frac{1}{\kappa \cdot r} \int_{x_0}^{\infty} \left\{ \frac{B - U(x)}{u(x)} - u'(x) \right\} dx \tag{18}$$

We wish to show that the minimum value of L subject to (17) and to

$$(1-\lambda)(pa + rc_0) - x_0 = \frac{B - U(x_0)}{u(x_0)} \tag{19}$$

(from (16)) is given by $\lambda = \mu$ or $\kappa = 1$.

Differentiating (18) and putting $dL = 0$ we get

$$r \cdot L \cdot dk = \frac{B - U(x_0)}{u(x_0)} \cdot u'(x_0)dx_0 \tag{20}$$

Differentiating (19)

$$-(pa + rc_0)d\lambda - dx_0 = -\frac{u(x_0)}{u(x_0)}dx_0 - \frac{B - U(x_0)}{u^2(x_0)}u'(x_0)dx_0$$

from which using (20) we get

$$r \cdot L \cdot d\kappa = (pa + rc_0)u(x_0)d\lambda \tag{21}$$

If now the equation got by differentiating (17) and putting $\kappa = 1$ in the result is an algebraic consequence of (20) and (21), it will follow that $\kappa = 1$ gives at least a stationary value of L.

But differentiating (17) and putting $\kappa = 1$ in the result we get

$$(pa + rc_0)d\lambda - \frac{d\kappa}{u(x_0)} \int_{x_0}^{\infty} \left\{ \frac{B - U(x)}{u(x)} - u'(x) \right\} dx = 0$$

or $r \cdot L \cdot d\kappa = (pa + rc_0)u(x_0)d\lambda$ which is identical with equation (21). The required result is therefore established.

It is worth stating explicitly that we have assumed in this argument that the rates of tax once imposed are never to be varied, and that by a given revenue we mean a given supply of goods discounted to their present value at the rate of interest, which is the rate at which the government could borrow or lend so as to obtain the use of its revenue in such quantities at such times as it desired.[21]

Second Set of Notes: "Savings Problem Abstract"

(1) No tax:

$$\frac{dc}{dt} + x = f(a,c) \tag{1}$$

$$v(a) = \frac{\partial f}{\partial a}u(x) \tag{2}$$

$$-\frac{d}{dt}\log u(x) = \frac{\partial f}{\partial c} - \rho \tag{3}$$

(2) Tax revenue R rate λ with remission of μ on savings:

$$R = \lambda f(a,c) - \mu\frac{dc}{dt} \tag{0$'$}$$

$$v(a) = (1-\lambda)\frac{\partial f}{\partial a}u(x) \tag{2$'$}$$

$$-\frac{d}{dt}\log u(x) = \frac{(1-\lambda)}{(1-\mu)}\frac{\partial f}{\partial c} - \rho \tag{3$'$}$$

if S of revenue exhausted

$$\frac{dc}{dt} + x = f(a,c) - S \tag{1$'$}$$

[Side note:] esp. $S = 0$ or R

$$\frac{dc}{dt}(1-\mu) + x = (1-\lambda)f \quad \text{[end of side note]}$$

if nat debt[22] = B

21. Editor's note: On the back of this last page of the manuscript Ramsey wrote: "*Memo*: correct III α to say equil may be attained not with $r = \rho$ but with bliss."

22. Editor's note: Ramsey probably refers here to the national debt, with the following equation being a government budget constraint.

$$R_1 = \lambda\{f(a,c) + r_0 B\} - \mu\left\{\frac{dc}{dt} + \frac{dB}{dt}\right\} \tag{0''}$$

(3) conversion into money of constant marginal utility (money by dashes):[23]

if $\lambda = \mu = 0 \quad c' = u \cdot c \quad$ etc

if $b = f - a\frac{\partial f}{\partial a} - c\frac{\partial f}{\partial c} =$ real[24]

$$\frac{dc'}{dt} + x' = b' + a' + \rho c \tag{4}$$

(4) Derived equation if $\rho = 0$; $S = R$:

$$(1-\mu)\frac{dc}{dt} = \frac{B - U(x)}{u(x)} + V(a) \tag{5}$$

{workout $\frac{d}{dx}(uf)$}

[Side note:] λ on consumption r on saving = λ on both $-\mu$ on saving $\therefore\ \lambda - \mu = r$, $1 - \mu = 1 + r - \lambda$ [end of side note]

(5) Solution if $f(a,c) = g(a) + rc$ and either S or $R - S$ a known f[u]n[ction] of a, x.[25]

Take S first

$$v(a) = (1-\lambda)g'(a) \cdot u(x) \tag{2'}$$

gives x as f[u]n[ction] of a or conversely let

$$y = x + S - g(a)$$

$$w(y) = u(x)$$

$$\left(W(y) = \int w(y)dy = U(x) - \frac{V(a)}{1-\lambda} \ \text{ if } \ S = 0\right)$$

Then $\frac{dc}{dt} = rc - y$, $\frac{dw}{dt} + \left\{\frac{(1-\lambda)}{(1-\mu)}r - \rho\right\} w = 0$[26]

$$\therefore rc - y + \frac{dc}{dw}\left\{\frac{(1-\lambda)}{(1-\mu)}r - \rho\right\} w = 0$$

$$\frac{dc}{dw} + \frac{r}{\frac{(1-\lambda)}{(1-\mu)}r - \rho}\,\frac{c}{w} = \frac{y}{w\left\{\frac{(1-\lambda)}{(1-\mu)}r - \rho\right\}}$$

23. Editor's note: Unclear word.

24. Editor's note: Unclear word; it looks like "reat" or "sent."

25. Editor's note: On the left side of the manuscript page with case (5), Ramsey wrote vertically, in the middle of the page, "These equations can be solved geometrically."

26. Editor's note: Here Ramsey seems to make the already mentioned distinction between the return to capital and the interest rate. However, in the following equations he seems to use both interchangeably.

Or

$$c \cdot w^{r/\frac{(1-\lambda)}{(1-\mu)}r-\rho} = \int \frac{y}{\frac{(1-\lambda)}{(1-\mu)}r-\rho}\, w^{\frac{r}{\frac{(1-\lambda)}{(1-\mu)}r-\rho}-1}\, dy + B$$

$$= \frac{y}{r}\, w^{r/\frac{(1-\lambda)}{(1-\mu)}r-\rho} - \frac{1}{r}\int w^{r/\frac{(1-\lambda)}{(1-\mu)}r-\rho}\, dy + \frac{B}{r}$$

Or

$$\frac{dc}{dt} = rc - y = \frac{B - \int w^{r/\frac{(1-\lambda)}{(1-\mu)}r-\rho}\, dy}{w^{r/\frac{(1-\lambda)}{(1-\mu)}r-\rho}(y)} \tag{6}$$

$$\left(\text{also } w(y) = Ae^{-\left\{\frac{(1-\lambda)}{(1-\mu)}r-\rho\right\}t}\right)$$

Secondly if $R - S$ = known f[u]n[ction] take

$$y = x - (1-\lambda)g(a) - (R-S)$$

$$w(y) = u(x)$$

$$\left(W(y) = U(x) - V(a) \text{ if } R - S = 0\right)$$

$$(1-\mu)\frac{dc}{dt} = (1-\lambda)rc - y \quad \frac{dw}{dt} + \left\{\frac{(1-\lambda)}{(1-\mu)}r - \rho\right\} w = 0$$

$$\therefore\ (1-\lambda)rc - y \ + (1-\mu)\frac{dc}{dw}\left\{\frac{(1-\lambda)}{(1-\mu)}r - \rho\right\} w = 0$$

$$\frac{dc}{dw} + \frac{r(1-\lambda)}{r(1-\lambda) - \rho(1-\mu)}\frac{c}{w} = \frac{y}{w(1-\lambda r - 1 - \mu\rho)}$$

$$\text{or } \ (1-\mu)\frac{dc}{dt} = (1-\lambda)rc - y \ = \frac{B - \int w^{r(1-\lambda)/(1-\lambda)r-\rho(1-\mu)}dy}{w^{r(1-\lambda)/(1-\lambda)r-\rho(1-\mu)}(y)} \tag{7}$$

Special Case

$a = \text{const}, S = R,\ f(a,c) = b + rc,\ u(x) = 1/x^{\alpha} = (1/A^{\alpha})e^{-\sigma t}, \sigma = r\left\{\frac{(1-\lambda)}{(1-\mu)}\right\}$

$$\therefore\ x = Ae^{\sigma/\alpha t}$$

$$(1-\mu)\frac{dc}{dt} + x = (1-\lambda)(b + rc)$$

$$\frac{dc}{dt} - \sigma c = \frac{(1-\lambda)}{(1-\mu)}b - \frac{A}{(1-\mu)}e^{\sigma/\alpha t}$$

$$\therefore\ c = Be^{\sigma/\alpha t} - \frac{(1-\lambda)b}{(1-\mu)\sigma} + \frac{A}{\left(1-\frac{1}{\alpha}\right)\frac{\sigma}{(1-\mu)}}e^{\sigma/\alpha t}$$

if to ∞ B = 0.

$$A = \left(1-\frac{1}{\alpha}\right)\frac{\sigma}{(1-\mu)}\left\{c_0 + \frac{b}{r}\right\}$$

$$= \left(1-\frac{1}{\alpha}\right)(1-\lambda)\left\{rc_0 + b\right\}$$

[Side note:] $\alpha > 1$ or no spending [end of side note]

$$\text{Utility} = \text{Bliss} - \frac{1}{(\alpha-1)}\frac{1}{x^{\alpha-1}}$$

$$(\alpha-1)\,\text{Loss of } U = \frac{1}{A^{\alpha-1}}e^{-\frac{(\alpha-1)}{\alpha}\sigma t}$$

$$\int_0^\infty \text{Loss} = \frac{1}{A^{\alpha-1}\cdot\frac{(\alpha-1)}{\alpha}\sigma} = \frac{\alpha(1-\mu)}{(\alpha-1)r(1-\lambda)}$$

$$\cdot\frac{\alpha^{\alpha-1}}{(\alpha-1)^{\alpha-1}(1-\lambda)^{\alpha-1}\left\{rc_0+b\right\}^{\alpha-1}}$$

$$= K\cdot\frac{(1-\mu)}{(1-\lambda)^{\alpha}}$$

$$\text{Revenue} = \lambda b + \lambda rc - \mu\frac{dc}{dt}$$

$$= \lambda\cdot\frac{rA}{\left(1-\frac{1}{\alpha}\right)\sigma(1-\mu)}e^{\sigma/\alpha t} - \frac{\mu}{(1-\alpha)(1-\mu)}e^{\sigma/\alpha t}$$

$$= \frac{A}{\alpha-1}e^{\sigma/\alpha t}\left(\frac{\alpha\lambda}{1-\lambda} - \frac{\mu}{1-\mu}\right)$$

Discounted at rate r and integrated to ∞

$$= \frac{A}{\alpha-1}\left(\frac{\alpha\lambda}{1-\lambda} - \frac{\mu}{1-\mu}\right)\frac{1}{r-\frac{\sigma}{\alpha}}$$

$$= K\left(\frac{\alpha\lambda}{1-\lambda} - \frac{\mu}{1-\mu}\right)\frac{(1-\mu)(1-\lambda)}{\alpha(1-\mu)-(1-\lambda)}$$

$$= K\cdot\frac{\alpha\lambda(1-\mu)-\mu(1-\lambda)}{\alpha(1-\mu)-(1-\lambda)}$$

$1 - \mu = \mu_0$, $1 - \lambda = \lambda_0$, loss $= \frac{\mu_0}{\lambda_0 \alpha}$

$$K \cdot \text{rev} = c = \frac{\alpha(1-\lambda_0)\mu_0 - (1-\mu_0)\lambda_0}{\alpha\mu_0 - \lambda_0} < 1. \text{ Take small}$$

$$\therefore\ \mu_0 = \frac{A_0(1-c)}{\alpha(1-\lambda_0) + \lambda_0 - c\lambda} = \frac{\lambda_0(1-c)}{\alpha(1-c) - (\alpha-1)\lambda_0}$$

$$\frac{1}{\text{loss}}\ \alpha\lambda_0^{\alpha-1}\left\{\alpha(1-c) - (\alpha-1)\lambda_0\right\}$$

for min loss

$$\alpha(\alpha-1)(1-c) - \alpha(\alpha-1)\lambda_0 = 0$$

$$\therefore\ \lambda_0 = 1-c, \quad \lambda = c$$

$$u_0 = \frac{(1-c)^2}{1-c} = 1-c, \quad \mu = c$$

$$\therefore\ \lambda = \mu$$

$\therefore$ savings should be exempted entirely.

$$f(a,c) = pa + rc$$

$\lambda_0 = 1-\lambda, \quad \mu_0 = 1-\mu, \quad \sigma = \frac{\lambda_0}{\mu_0} r$
If

$$u(x) = \alpha - x = Ae^{-\sigma t}$$

$$v(a) = \alpha - \beta = \lambda_0 p(\alpha - x)$$

$$= \lambda_0 pAe^{-\sigma t}$$

$a = \beta + \lambda_0 pAe^{-\sigma t}$ if $R = S$.

$$\mu_0 \frac{dc}{dt} + x = \lambda_0\left\{rc + p\beta + \lambda_0 p^2 Ae^{\sigma t}\right\}$$

$$\therefore\ \frac{dc}{dt} - \sigma c = p\beta\frac{\lambda_0}{\mu_0} - \frac{\alpha}{\mu_0} + \frac{Ae^{-\sigma t}\left\{\lambda_0^2 p^2 + 1\right\}}{\mu_0}$$

$$\therefore\ c = \frac{a}{\lambda_0 r} - \frac{p\beta}{r} - \frac{A}{2\lambda_0 p}\left\{\lambda_0^2 p^2 + 1\right\}e^{-\sigma t} + 0e^{\sigma t}$$

$$A = 2 \cdot \frac{\lambda_0(\alpha - \lambda_0 p\beta) - 4\lambda_0 rc_0}{\lambda_0^2 p^2 + 1}$$

$$c = c_0 + \frac{1}{r} \cdot \left\{\frac{\alpha}{\lambda_0} - p\beta - rc_0\right\}\left\{1 - e^{-\sigma t}\right\}$$

Const $- U(x) = \frac{1}{2}(\alpha - x)^2 = \frac{1}{2}A^2 e^{-2\sigma t}$

Loss throughout time $= \frac{A^2}{4\sigma} = \frac{\mu_0}{\lambda_0} \cdot \frac{\left\{\alpha - \lambda_0\left(rc_0 + p\beta\right)\right\}^2}{\left\{1 + \lambda_0^2 p^2\right\}^2}$

$$\text{Revenue} = \left(1 - \lambda_0\right)\left(rc + pa\right) - \left(1 - \mu_0\right)\frac{dc}{dt}$$

$$= \left(1 - \lambda_0\right)\left\{rc_0 + \left(\frac{a}{\lambda_0} - p\beta - rc_0\right)\left(1 - e^{-\sigma t}\right) + p\beta + p^2\lambda_0 Ae^{-\sigma t}\right\} -$$

$$- \left(1 - \mu_0\right)\frac{\lambda_0}{\mu_0}\left(\frac{\alpha}{\lambda_0} - p\beta\right)$$

$$= \frac{\left(1 - \mu_0\right)}{\mu_0}\lambda_0 p\beta + \left\{\frac{\left(1 - \lambda_0\right)}{\lambda_0} - \frac{\left(1 - \mu_0\right)}{\mu_0}\right\}\alpha + \left(1 - \lambda_0\right)e^{-\sigma t} \cdot$$

$$\cdot \left\{\frac{2p^2\lambda_0}{p^2\lambda_0^2 + 1} \cdot \left(\alpha - \lambda_0 p\beta - \lambda_0 rc_0\right) - \frac{\left(\alpha - \lambda_0 p\beta - \lambda_0 rc_0\right)}{\lambda_0}\right\}$$

$$= \frac{\left(1 - \mu_0\right)}{\mu_0}\lambda_0 p\beta + \left\{\frac{1}{\lambda_0} - \frac{1}{\mu_0}\right\}\alpha +$$

$$+ \frac{\left(1 - \lambda_0\right)}{\lambda_0}\frac{2p^2\lambda_0^2 - 1}{p^2\lambda_0^2 + 1}\left(\alpha - \lambda_0 p\beta - \lambda_0 rc_0\right)$$

$$u(x) = \alpha - x = Ae^{-\sigma t}, \quad x = \alpha + Ae^{-\sigma t}$$

$$v(a) = a - \beta = \lambda_0 p\left(\alpha - x\right)$$

$$= \lambda_0 pAe^{-\sigma t}, \quad \sigma = \lambda_0\frac{r}{\mu_0}$$

$$a = \beta + \lambda_0 pAe^{-\sigma t}$$

debt D to be paid off ($S = 0$)

$$\frac{dc}{dt} + x = pa + rc$$

$$\frac{dc}{dt} - rc = p\beta + \left(\lambda_0 p^2 + 1\right)^2 Ae^{-\sigma t} - \alpha$$

$$c = \frac{\alpha - p\beta}{r} - \frac{A\mu_0}{r\left(\lambda_0 - \mu_0\right)}\left(\lambda_0 p^2 + 1\right)e^{-\sigma t}$$

$$\therefore\ c = c_0 + \frac{\alpha - p\beta}{r}\,c_0\left(1 - e^{-\sigma t}\right)$$

$$A = \frac{\left(\alpha - p\beta - rc_0\right)}{\left(1 + \lambda_0 p^2\right)}\frac{\left(\lambda_0 + \mu_0\right)}{\mu_0}$$

$$\text{Const} - U(x) = \frac{1}{2}\left(\alpha - x\right)^2 = \frac{1}{2}A^2 e^{-2\sigma t}$$

[Side note]: If work fixed, $p^2\lambda_0$ term omitted; then loss = f[u]n[ction] of $\left(\frac{\lambda_0}{\mu_0}\right)$; min if this = 1 [end of side note]

$$\therefore \text{ Loss} = \frac{A^2}{4\sigma} = \frac{\mu_0}{4\lambda_0} \cdot \frac{\left(\lambda_0 + \mu_0\right)^2}{\mu_0^2} \frac{\left(\alpha - p\beta - rc_0\right)^2}{\left(1 + p^2\lambda_0\right)^2}$$

$$= \frac{\left(\lambda_0 + \mu_0\right)^2}{\lambda_0\mu_0\left(1 + p^2\lambda_0\right)^2} \cdot \frac{\left(\alpha - p\beta - rc_0\right)^2}{4}$$

$$\frac{dD}{dt} + \left(1 - \lambda_0\right)\left(pa + rc + rD\right) - \left(1 - \mu_0\right)\left(\frac{dc}{dt} + \frac{dD}{dt}\right) = rD$$

$$\therefore \mu_0\frac{dD}{dt} - \lambda_0 rD = \left(1 - \mu_0\right)\frac{dc}{dt} - \left(1 - \lambda_0\right)\left(pa + rc\right)$$

$$= \left\{\left(1 - \mu_0\right)\frac{\lambda_0}{\mu_0}\left(\alpha - p\beta - rc_0\right) - \left(1 - \lambda_0\right)\lambda_0 p^2 A + \left(1 - \lambda_0\right)\left(\alpha - p\beta\right)\right\}e^{-\sigma t}$$

$$- \left(1 - \lambda_0\right)\left(p\beta + rc_0\right)$$

$$\text{Loss of utility} = \int_0^\infty \left(B - U(x) + V(a)\right)dt$$

$$= \int_{x_0}^{X} \left(B - U(x) + V(a)\right)\frac{-u'(x)}{u(x)}\frac{\left(1 - \mu\right)}{\left(1 - \lambda\right)}\frac{1}{r}dx$$

$$= \frac{1}{r}\frac{\left(1 - \mu\right)}{\left(1 - \lambda\right)}\left\{\int_{x_0}^{X} \frac{B - U(x)}{u(x)} \cdot (-u'(x))dx + \int_0^{a_0} V(a)\frac{v'(a)}{v(a)}da\right\}$$

Initial conditions

(1) $S = 0$.

$$rc_0 - x_0 + pa_0 = \frac{B - \int_{x_0 - pa_0}^{X} w^{\frac{1-\mu}{1-\lambda}}(y)dy}{w^{\frac{1-\mu}{1-\lambda}}\left(x_0 - pa_0\right)}$$

where $w(y) = u(x) = \frac{v(a)}{(1-\lambda)p}$, $\qquad dy = dx - pda$

$$\therefore rc_0 + pa_0 = x_0 + \frac{B - \int_{x_0}^{X} u^{\frac{1-\mu}{1-\lambda}}(x)dx + \frac{1}{(1-\lambda)^{\frac{\lambda-\mu}{1-\lambda}}p^{\frac{\lambda-\mu}{1-\lambda}}} \cdot \int_0^{a^0} v^{\frac{1-\mu}{1-\lambda}}(a)da}{u^{\frac{1-\mu}{1-\lambda}}\left(x_0\right)}$$

(2) $S = R$

$$\left(1 - \lambda\right)\left(rc_0 + pa_0\right) = x_0 + \frac{B - U\left(x_0\right) + V\left(a_0\right)}{u\left(x_0\right)}$$

Revenue

(1) $S = 0$ reckoned by marginal utility

$$\text{Rev} = \int_0^\infty \left\{\lambda\left(pa + rc\right) - \mu\frac{dc}{dt}\right\} u(x)dt$$

$$= \frac{1}{r}\int_{x_0}^{X} \left\{\lambda\left(pa + rc\right) - \mu\frac{dc}{dt}\right\}(-u'(x))\frac{(1-\mu)}{(1-\lambda)}dx$$

$$= \frac{1}{r}\frac{(1-\mu)}{(1-\lambda)}\int_{x_0}^{\infty}\left\{\lambda x + (\lambda - \mu)\frac{dc}{dt}\right\}\cdot(-u'(x))\,dx \quad \text{etc}$$

(2) $S = R$. Revenue reckoned at marginal utility

$$\text{Rev} = \int_0^\infty \left\{\lambda\left(pa + rc\right) - \mu\frac{dc}{dt}\right\} u(x)dt$$

$$= \int_0^\infty \frac{\lambda}{1-\lambda}x \cdot u(x)dt + \frac{1-\mu}{1-\lambda}\int_0^\infty \frac{dc}{dt}u(x)dt$$

$$= \frac{\lambda}{(1-\lambda)}\frac{(1-\mu)}{(1-\lambda)r}\left\{x_0 u(x_0) + B - U(x_0)\right\} + \frac{\lambda-\mu}{(1-\lambda)(1-\mu)}\text{ loss}$$

if a is fixed

$$\text{Rev} = \lambda\frac{(1-\mu)}{(1-\lambda)}\frac{1}{r}(rc_0 + pa_0)u(x_0) + \frac{\lambda-\mu}{(1-\lambda)(1-\mu)}\text{ loss}$$

$$\text{[Define:] } v = \frac{1-\mu}{1-\lambda}$$

$$r\text{Rev} = \frac{\lambda v}{1-\lambda}(rc_0 + pa_0)u(x_0) + \frac{1-v}{(1-\lambda)}\int_{x_0}^{X}\frac{B-U}{u}(-u')dx$$

for min loss and given rev

$$0 = dv\left\{\int_{x_0}^{X}\frac{B-U}{u}\cdot(-u')dx\right\} + v\frac{(B-U)u'}{u}dx_0$$

$$+ d\lambda\left(rc_0 + pa_0\right) = \frac{(B-U)u'}{u^2}dx_0$$

$$0 = \left(\frac{v}{(1-\lambda)^2}d\lambda + \frac{\lambda}{(1-\lambda)}dv\right)(rc_0 + pa_0)u(x_0) + \frac{\lambda}{1-\lambda}v(rc_0 + pa_0)u'dx_0 +$$

$$+ \frac{1-v}{1-\lambda}\frac{B-U}{u}u'dx_0 + \left\{-\frac{dv}{1-\lambda} + \frac{1-v}{(1-\lambda)^2}d\lambda\right\}\int_{x_0}^{X}\frac{B-U}{u}\cdot(-u')dx_0$$

[Define:] $Z = \int_{x_0}^{X} \frac{B-U}{u}(-u')dx$

$$\begin{array}{ccc} ((d\lambda)) & ((d\nu)) & ((dx_0)) \end{array}$$

$$\begin{vmatrix} 0 & Z & ((\nu))\,\frac{B-U}{u}\,((u')) \\ rc_0 + pa_0 & 0 & -\frac{(B-U)((u'))}{u^2} \\ \frac{((\nu))}{(1-\lambda)^{((2))}}(rc_0 + pa_0)u + \left(\left(\frac{1-\nu}{(1-\lambda)^2}\right)\right)Z & \frac{\lambda}{((1-\lambda))}(rc_0 + pa_0)u - \frac{Z}{((1-\lambda))} & \frac{\lambda((\nu))}{((1-\lambda))}(rc_0 + pa_0)((u')) + \frac{((1-\nu))}{((1-\lambda))}\left(\left(\frac{(B-U)u'}{u}\right)\right) \end{vmatrix} = 0$$

if $\lambda = \mu, \quad \nu = 1$

[Some additional equations which are hard to make sense of.]

Attempt to include variable rate of discount.

$$\frac{d}{dt}\{u(x)\varphi(t)\} = -\frac{\partial f}{\partial c}u(x)\varphi(t)$$

$$\frac{d}{dt}\log(u(x)\varphi(t)) = -\frac{\partial f}{\partial c}$$

$$\frac{d}{dt}\log u(x) + \frac{\varphi'(t)}{\varphi(t)} = -\frac{\partial f}{\partial c}$$

$$u(x)\cdot\varphi(t) = Ae^{-rt}$$

$$u(x) = \frac{Ae^{-rt}}{\varphi(t)}$$

$$w(y) = \frac{Ae^{-rt}}{\varphi(t)} = A\psi(t), \qquad y = \chi(t)$$

$$y = rc - \frac{dc}{dt}$$

$$\frac{d}{dt}(w(y)\cdot\varphi(t)) = -r\cdot w(y)\cdot\varphi(t)$$

$$\frac{d}{dt}(c\cdot w(y)\cdot\varphi(t)) = \left(\frac{dc}{dt} - rc\right)w(y)\varphi(t) = -y\cdot w(y)\varphi(t)$$

[Side note:][27] $\left[\frac{d}{dt}w(y)=\right]w'(y)\frac{dy}{dt}=A\left(-rw(y)-\frac{\varphi'(t)}{\varphi(t)}w(y)\right)$ [end of side note]

$$\frac{dc}{dt}-rc=\chi(t)$$

$$c=\left(\left(Ae^{-rt}\right)\right)+e^{rt}\int_{((0))t}^{((t))\infty}e^{-rt}\chi(t)dt$$

$$\frac{dc}{dt}=re^{rt}\int_{((0))t}^{((t))\infty}e^{-rt}\chi(t)dt=re^{rt}\int_{t}^{\infty}\chi(t)e^{-rt}dy$$

$$\frac{dc}{dt}=r\int_{0}^{\infty}\chi(t)e^{-rt}dt=r\int_{0}^{\infty}ye^{-rt}dt=r\int_{t}^{\infty}\frac{w'(y)}{Aw(y)\left(-r-\frac{\varphi'(t)}{\varphi(t)}\right)}dy$$

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27. Editor's note: On the right-hand side of the following equation, Ramsey seems to have mistakenly multiplied the term in square brackets by A. In this equation, I inserted the left-hand-side term in square brackets.