Chapter 2

Why are laws mathematical?[1]

> Our experience hitherto justifies us in believing that nature is the realization of the simplest conceivable mathematical ideas”
>
> Einstein[2]

The “Unreasonable Effectiveness of Mathematics” in Describing, Explaining and Predicting the Physical World[3]

With a more or less implicit recall of Pythagorism, Leibniz once wrote that mathematics can be differentiated from music only because it a a form of *conscious* calculation, whereas music represents *unconscious calculation*. We can add that mathematics has a relationship to other sciences that is similar to music’s relationship to other arts, as it is the most abstract but perhaps most effective instrument for understanding the world. From physics to biology, psychology to economics, there is no empirical science today which has not, in some way, been mathematized, and Immanuel Kant had already noted that

> Since in every theory of nature there can be only as much science, properly speaking, as there is *a priori* knowledge, it follows that the theory of nature can contain as much science, properly speaking, as mathematics that can be applied to it.[2]

A few simple examples from the history of science will serve to illustrate the odd, and up until now, mysterious phenomenon on the basis of which entire parts of mathematics, which were initially invented and constructed without any applicative purpose, later proved to be highly useful in predicting, describing, and explaining new and unexpected natural phenomena, and therefore in bringing to light “areas of knowledge” which had been previously completely obscure.

What renders the problem particularly difficult is that it does not seem easily resolvable by invoking one current philosophical position over another on the ontology of mathematics, given that *prima facie*, the applicability issue creates puzzles for all such positions. Within a *constructivist* philosophy of mathematics, for example, one must explain why mathematics – regarded as a *creation of ours* –

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[1] I would like to thank the audience at Copenhagen for the interesting questions raised during the discussion of a previous version of this chapter.

[2] I. Kant**,** *Metaphysische Anfangsgründe der Naturwissenschaft*, Akademie Textausgabe, Bd. IV, Berlin 1968, p. 470.

enables us to discover properties and entities belonging to a world which, like the physical world, we certainly did *not* create. In the context of a *Platonic* conception – in which mathematics is considered more the fruit of the *discovery* of facts that are abstract and independent from ourselves, rather than an *invention* of ours – it is difficult to explain how we can come into cognitive contact with this reign of abstract entities, which are by definition *causally inert*. Even if one avoided the presupposition that "knowing" necessarily implied more or less direct causal interaction with the known entity, mathematical Platonism would seem to *duplicate* the problems to be resolved. It should in fact be explained why the physical world (which, in being extended in space-time, is certainly *different* from the abstract world of mathematics) should "reflect" the essential structures of the latter.[3] Finally, even if we invoked an *empiricist* or *naturalistic* explanation of the applicability of certain mathematical structures, which could be the evolutionary fruit of the long-term adaptation of the human brain to objects with dimensions comparable to those of our bodies, we should in any case clarify why these structures have also proved themselves to be indispensable to the investigation of entities much smaller and larger than ourselves (from subatomic particles to the entire observable universe.)

The most famous example of the applicability of mathematical notions to the physical world is perhaps given by *conical sections* or conics (circles, ellipses, hyperbolas and parabolas) which are so named because they are obtained by intersecting a circular cone with a plane at different angles (see Figure 1 in the next page). In calling AA' one of the two generators of the cone, we can immediately see that by cutting one of the two layers of the cone with a plane that is perpendicular to axis HH', we obtain a circle (which in the limit degenerates into point O), while by tilting the plane a bit with respect to the horizontal line we obtain an *ellipse*, which in the figure is indicated by FDE. If the position of the intersecting plane is parallel to the direction of one of the two generators, we obtain a *parabola*: in Figure 1, the parabola GIK is the intersection between the cone and a plane parallel to AA'. Finally, the two branches of the hyperbola R'S'T' and RST can be obtained by a plane which cuts *both* layers of the cone.

[3] On Platonism in the philosophy of mathematics, see M. Piazza's essay, *Intorno ai numeri*, Bruno Mondadori, Milano 2000.

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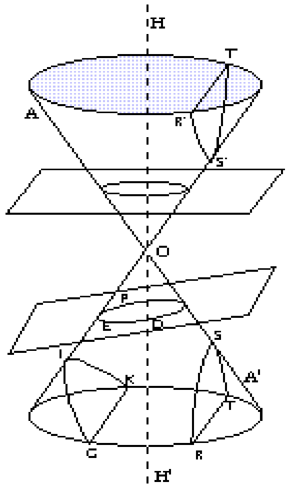

Fig. 1

The study of the properties of conics, which dates to the Third Century B.C., was performed by the Greek mathematician Apollonius, apparently without any applicative purpose. Almost twenty centuries after Apollonius, one of these curves, the *ellipse*, was used by Kepler to describe the orbit of all the planets around the sun, which occupies one of its two focal points. Incidentally, this event was particularly revolutionary in the history of ideas, given that the perfection and immutability that ancient astronomers attributed to non-corruptible astral bodies led pre-modern astronomers to hypothesize that this orbit was necessarily *circular*. The same "anticipatory" fate met the geometries which did not respect the Euclidean postulate of parallels, or the so-called *non-Euclidean geometries*, which allow for the existence of more than one parallel, or no parallels, to a given line.[4] These geometries were invented (or discovered) by mathematicians in the Nineteenth Century without any particular thought towards application, and were later utilized by Einstein at the beginning of the Twentieth Century to describe the universe on a large scale, whose spatio-temporal curvature appears variable – and different from zero, as predicted by these geometries – in the presence of matter. The extraordinary degree of experimental confirmation achieved during the Twentieth Century by the physical theory that uses these geometries, *the theory of general relativity*, seems to fully corroborate the ancient Pythagorean conception

[4] As is known, given a straight line r and point P external to it, according to *Euclidean geometry* there is only one straight line parallel to r that passes through P.

of the universe, later expanded upon by Kepler and Galilei, on the basis of which the huge world surrounding us is written in geometrical characters.[5]

To cite a different, but more familiar example, when Bertrand Russell published his works on logic, the mathematicians of his time considered them completely useless, from the applicative point of view, as well as from the point of view concerning the progress of pure mathematics. Paradoxically, it was *also* thanks to the logic codified in Russell and Whitehead's *Principia Mathematica* (1910-1913) that the theory of recursive functions and computability matured, as perfected by the logicians Alan Turing and Alonzo Church during the 1930's. These scientists created the mathematical theory that is at the basis of the functioning of the personal computers that we all use today. The difference between this example and the preceding ones is that computers, obviously, are not objects existing in nature. However, they can be utilized for comprehending how both the external world and our minds function through *simulations*: how, in particular, the cognitive capacities of the mind can be simulated by "neural networks," that try to represent the causal structure exemplified by the real connections between the neurons of our brains.

Another good example is offered by imaginary or complex numbers, which were introduced to calculate the square roots of negative numbers and to solve some algebraic equations, and have highly important applications in all fields of engineering and physics, especially quantum mechanics, which studies matter at the microscopic levels of atoms and subatomic particles. It was precisely this extraordinary power of imaginary numbers that "troubled" the young Törless, so that in a novel published in 1906 by the German writer Robert Musil, this character said: [6]

> But what is strange is that with these imaginary or in any case impossible values, calculations can be made anyway that are perfectly real and that eventually allow us to have something concrete in our hands.

I could give other examples regarding the discovery of new families of elementary particles, which also came about from considerations of symmetry based on the theory of groups,[7] but what I have said so far should be enough to convince the reader of the importance of the question that will be the focus of our chapter: *why do mathematical inventions or discoveries, which are often achieved without any applicative purpose, just as often prove themselves to be fundamental instruments for explaining and describing the physical world?* An answer to this question would be of crucial interest to us, because it could also be interpreted as an explanation as to why physical laws are mathematically formulated.

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[5] See the citation from Galilei's *Saggiatore* in the previous chapter, p. 19.

[6] Musil R., (1906/2001), *The Confusions of Young Torless*, transl. by S. Whiteside, Penguin Book, New York, p.56.

[7] For other interesting examples taken from particle physics see the excellent book by M. Steiner, *The Applicability of Mathematics as a Philosophical Problem*, Harvard University Press, Cambridge, Massachusetts, 1998, especially Chapter 4.

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Apart from their intrinsic interest, I also believe that these issues – which from Kant onwards have been strangely neglected, both by philosophers of mathematics, as well as by the vast majority of philosophers of science who have written about the laws of nature – serve as an important key to responding to a question which has accompanied philosophic-scientific literature on the laws of nature from its birth:[8] Are laws discovered or do we postulate them in a conventional way for the purpose of structuring and predicting the phenomena of the external world?

The Three Ingredients for "Producing" the Laws of Physics

If one had to explain what a law of nature was in a way that best connected conceptual rigor with conciseness, he or she should use a technical term, "*differential equation*," which, unfortunately, involves the disadvantage of forcing us to introduce the fundamentals of calculus. A road this long can fortunately be avoided, given that for our purposes it would be enough to define a physical law as an *instrument of calculation* (an algorithm) that permits us to pass from certain experimental observations of a certain group of phenomena (initial or boundary conditions) to other observations that can be performed at different times. Following a useful indication by the philosopher William Whewell, which was partly expanded upon by the astronomer John Barrow,[9] we can reduce the ingredients necessary to defining the notion of the law of nature in the mathematical sense to *three*:

1. the *algorithmic structure*, given by the formula (or differential equation) which represents the law properly speaking;
2. the *initial or boundary conditions*, or the initial numeric data to which we apply the law;
3. the constant quantities left unchanged by the application of the algorithm, or the *constants of nature*.

In this respect, Whewell affirmed that the investigation of a physical phenomenon consisted of the identification of a relationship between phenomena ("a colligation of facts") occurring in three stages, which he called the determination of the independent variable (2 above), the identification of the formula or function connecting the independent variable to a dependent variable (number 1 above) and the determination of coefficients (3).[10]

I submit that a philosophical discussion on the laws of nature that does not take these three ingredients into account would not be complete, or would run the risk of distancing itself from what physicists and scientists generally mean by "natural

8 See Chapter 1, "The Notion of the Law of Nature in the Seventeenth Century: The Role of the "Other" Bacon," concerning the Seventeenth-Century division between the "realists" and the "conventionalists."

9 See J. Barrow, *The World within the World*, cit., p. 279.

10 See R. Butts, *William Whewell's Theory of Scientific Method*, Pittsburgh University Press, Pittsburgh, 1968, pp. 210-211.

law." One can certainly admit that not all laws of nature have a mathematical formulation: "all metal objects rust" or "all crows are black" certainly express natural regularities which we call laws, and which cannot be formulated in quantitative language. Nevertheless, the "original sin" that has tainted many philosophical analyses of the nature of laws has consisted of considering "qualitative" propositions of this type to be paradigmatic examples of natural laws. Note that these propositions do not refer to any of the three components mentioned above, and it is perhaps precisely for this reason that philosophical debates on laws sometimes cause the reader to feel that they have degenerated into purely "academic" disputes, in the sense that they center on topics which seem far from both the practice and real problems that interest scientists.

Here, I will examine mainly the first two components of the laws of nature, given that the third one does not have much to do with the fundamental question of this chapter. Obviously, as a partial integration of what was affirmed in the preceding paragraph, it would be correct to observe that to the degree in which one also find laws *outside* of physics (a problem which we will deal with later on), the three components above might not be present. Nevertheless, a restriction of our attention to the way in which the laws of nature are codified in physics is justified not only by the fact that the latter is the science in which the notion of law has a more important role, but also by the undeniable observation that the mathematization of physics is at a much more advanced stage than in the other empirical sciences. The problem of why laws are mathematical can therefore be posed especially within physics, and it is plausible to believe that what is *philosophically significant* about physical laws can apply without much difficulty to the more "qualitative" laws of the other sciences.

The Algorithmic View of Laws as a First Attempt to Explain Why the Laws of Physics Are Mathematical

In returning to the three "ingredients" of a law of physics, I will now focus on the first one, which refers to a law as an algorithm: a mathematically formulated law in effect functions like a "bridge" connecting two "river banks," in which we find experimental data, expressed quantitatively as a result of measurements. On one side of the "river" we find the initial data (*the input*) and on the other the predictions (*the output*), which are the results of the calculation. Since the predictions, or the output, are typically obtained in a purely deductive way, as a result of the application of the law to the initial data, the metaphor of laws as a computer program or algorithm appears to be initially justified. If the initial data satisfy a few mathematical conditions which we do not need to consider here,[11] a law effectively permits us, in a finite number of steps, to transform the initial input into final predictions (the output) in a purely "mechanical" fashion.

[11] It involves ascertaining that the functions representing the data are differentiable at least twice.

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What interests us, obviously, is whether this metaphor drawn from computer science can help us to understand why the world can be described by mathematical laws. To this end, I will begin by supposing that any physical system subject to laws is perfectly comparable to a machine, made up of certain parts describable in the language of physics, in the same sense in which the *hardware* of a real computer is a group of electrical microcircuits built upon a silicon chip, and connected appropriately. Affirming that the laws governing the temporal evolution of a physical system serve as the *software* of the system is tantamount to presupposing the following analogy: Just as the physicist, who in order to pass from initial measurements to predictions, performs calculations with formulas expressing natural laws, *a well-identified physical system*, in passing from one initial state to a successive one on the basis of how these states are *causally* connected, *"performs a calculation" in a certain sense.* If the observable universe in its entirety can be treated as this sort of system, one could say that the universe passes from one state to a successive one by "calculating" it on the basis of its laws, which we can therefore call *the software of the universe.*

"Performing the calculation" or simply "calculating," should be understood in the most liberally possible sense; in fact, no physical system, not even a calculator, literally "calculates," if by this term we mean an *intentional*, goal-directed activity. A computer that manipulates symbols actually carries out a few physical transformations that *we interpret* as calculations, on the basis of a task that *we* have its operating system perform in relation to its central processing unit. Natural phenomena as well, like the beats of our hearts or the movement of the earth around the sun, can serve to measure time, and therefore to calculate in a certain sense, but this functioning obviously presupposes our intentional attribution of function.

Therefore, affirming that a natural system "calculates" a certain future *state* is no more metaphorical than affirming that "a computer calculates," given that both affirmations presuppose the intentional attribution of a *function* to an inanimate object. If by "physical state" we mean a description of a system at a certain time, all we can say about a physical system (including a computer) is that it passes from one state to another, and nothing more. In short, therefore, it is not possible to refute the identification of the laws of a physical system with algorithms performed by a computer on the basis of the claim that a computer is able to perform calculations, whereas this operation would be precluded in any other physical system. Either *both* a computer *and* any other physical system compute or they both do not; to the extent that we are willing to claim that a computer "calculates," we could say the same about any physical system, *provided that the operations of the latter can be perfectly simulated by another computer taking in the same input and producing the same output*.

Having put aside this first objection to the view equating the laws of nature to algorithms, we nonetheless encounter another view that is much more important: not all natural laws establish a link between states of the world that are temporally ordered (*laws of succession*), as the notion of algorithm would instead seem to presuppose. Rather, a few natural laws, called *laws of coexistence*, limit the compatibility of physical states *coexisting at any single instant*, so that any connection possibly transmitted by a causal or light signal is excluded. That is, unlike sequential computer programs, or even those which govern the functioning of the computer in a "parallel"[12] mode through "synchronizations" achieved by *causal interactions*, the laws of coexistence exclude any casual link between the magnitudes or properties of the natural phenomena that they relate.

Let us consider, for example, the law of universal gravity, which relates the attractive force F between two bodies to their masses M_1 and M_2, to their distance r and to the constant of gravity G:

$$F = G\,(M_1 \times M_2)/r^2 \qquad [1]$$

In continuing with the above metaphor, this equation serves as a bridge between the information relative to the masses of the two bodies and the distance separating their centers of mass to the attractive force exerted between them.[13] In the context of Newtonian mechanics, however, we generally hypothesize that force F explicates itself instantly or "at a distance": the "attraction" that the sun exerts upon the earth could be conceived of as a signal traveling at infinite speed. It follows that [1] does not connect two states of the Earth-Sun system as can be identified in different instances, but rather properties which both bodies manifest simultaneously and instantaneously, without any possibility of establishing a line of temporal evolution between them. Even though one can, of course, use force and distance to calculate the masses or viceversa (this aspect of the algorithmic nature of scientific laws is performed by a physicist or by a machine that is used to make the appropriate calculations), a physical, two-body system cannot implement [1] if we regard it as an algorithm, since the parts of the system would have to communicate instantaneously. If the approach to laws as algorithms is meant to apply not just to scientific laws, or the laws of science, for which it is adequate, but is intended (i) as an explanation of the applicability of mathematics to nature as well as (ii) a characterization of laws of nature as opposed to laws of science, then it must overcome this difficulty: laws of nature that are of coexistence cannot be regarded as algorithms, because they cannot be implemented by any physical system.

Another example of a law of coexistence is offered by Boyle's law, which links pressure P and volume V of a perfect gas[14] to its temperature T and to a constant k:

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[12] To explain the difference between parallel and sequential, imagine a group of cars entering a toll area simultaneously through several entrances (in parallel), or the same cars entering in succession through a single entrance (sequentially).

[13] Naturally, the bridge in question does not necessarily link the magnitudes in the way illustrated in the text. For example, if given the force and the two masses of the bodies, we can calculate their distance, etc. The expert reader will also note that this equation and the ones that follow are not written in differential form.

[14] A perfect or ideal gas is a *model* of a real gas, which we can resort to in the hypothesis that the forces of cohesion, or other forces acting on their molecules, are negligible.

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$$P\,V \;=\; k\,T \qquad [2]$$

This law tells us that by doubling the volume of gas contained, for example, in a cylinder, its pressure halves or vice versa, in a way that respects the constancy of the product (as long as during this transformation temperature T remains the same), but does not say anything with regard to the temporal evolution of the system, given that it links properties exemplified by the gas in question at the same instant.

To further illustrate the difference between these laws of coexistence and a law of succession, we will now consider the Galilean law of free fall. For the sake of simplicity, I will suppose that our test particle has no initial velocity, and that its initial position coincides with the origin of the coordinates s. The final space s traveled by the particle can then be obtained by the formula

$$s_f \;=\; \tfrac{1}{2}\,g\,t^2 \qquad [3]$$

expressing the proportionality between the distances traveled and the squares of the times taken to travel them.[15]

In sum, while in type [3] laws of succession the formula refers to temporally extended processes, whose single stages *manifest themselves in successive instances,* laws like Newton's and Boyle's relate states of physical systems occurring *simultaneously and instantaneously*. It follows that the *identification between laws and algorithms*, as impressive as it may be, *is not sufficiently general, given that the algorithms manage to account for only the laws of succession, but not for those of coexistence.*

As a consequence, it would seem legitimate to conclude that this identification should be seen at best as a heuristic instrument, and that it cannot be utilized to explain why the laws of nature are mathematical, as the physicists and philosophers Paul Davies and John Barrow propose, by expanding a few technical results owed to Andrej Kolmogorov, Gregory Chaitin and Ray Solomonoff.[16]

To further elaborate on this criticism and to discuss possible ways to circumvent it, it would be opportune to present the view that equates natural laws to algorithms in more detail. The idea essentially consists of assuming that the temporal evolution of every physical system passing through a finite number of states can be described by a string of *real numbers*, which correspond to measurements of values of certain physical magnitudes that the system possesses (temperature, pressure, velocity, etc.) It is essential to note that these strings can be either *ordered* or completely "random."

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15 The factor of proportionality is given by the acceleration of gravity *g* multiplied by ½.

16 Besides the previously-cited essay by Barrow, consider J. Barrow, *Perché il mondo è matematico?*, Laterza, Roma-Bari 1992. The idea of considering the laws of nature as algorithms was also developed by D. Deutsch, *The Fabric of Reality*, Allen Lane The Penguin Book, London 1997. At the base of these theories is the work by Kolmogorov and Chaitin in the 1960s, and the essay by R. Solomonoff, *A Formal Theory of Inference, Part I*," in "Information and Control," VII, 1964, p. 1.

- In the first case, the string of numbers can be constructed on the basis of a precise criterion, as occurs in tests of mathematical ability that ask us to find the next number in the sequence.
- If the string is random, it *appears* to lack a generating criterion, where "appear" means that while we can demonstrate that a finite string *is not* causal simply by giving its generating law, we can never demonstrate that a string is random.[17]

For example, a list of numbers like

$$\{1, 4, 9, 16, 25, 36, \ldots\} \qquad [4]$$

is not formed by numbers in a random succession, given that it can obviously be obtained by squaring the positive integers:

$$\{1, 2, 3, 4, 5, 6, \ldots\} \qquad [5]$$

At this point, the link between the notions of algorithm and law should be clear. If the numbers in [5] are associated with certain physical magnitudes, like intervals of time measured in seconds (measurements of the independent variable, or input data) and if we find through experiment – as did Galilei when he rolled some spheres down an inclined plane – that the corresponding distances travelled in meters (measurements of the dependent variable, or output data) are proportional to the squares of the times, as in [4], then the *program*, which applied to the succession of times as inputs generating the succession "of distances" is precisely the *law* expressed by [3], modulo the constant factor ½ g.

Let us therefore imagine that the result of our measurements, or the output that constitutes the initial data, is represented by a sequence of numbers like the one in [4]. After converting this sequence into a group of binary numbers (binary digits, or bits), made up of only 0 and 1,[18] a search for laws becomes equivalent to asking (following Kolmogorov's important results) the length of the shortest program that can generate the sequence. This length, called *complexity*, will be equal to that of the list if the latter is made up of completely random numbers, and will be equal to the number of bits of the shortest program capable of generating the list, if the latter can indeed be generated in this way.[19]

Since a succession like [4] can be generated by the instruction "print the square of the first natural numbers n," and this instruction possesses, *also for a large number* n,[20] the same information as the complete list of the squares of the first

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[17] This fact comes from the insolvability of the halting problem of Turing machines. See, for example, H.. Rogers, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York 1967.

[18] In binary digits, list [4] is given by {1, 100, 1001, 10000…}. The symbols 0 and 1 correspond, respectively to the absence and presence of electricity.

[19] The *bit* is the elementary unit of information, which can have only the values 0 and 1.

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numbers *n*, [4] is called *algorithmically compressible*:[21] the information of the original list can in fact be condensed into a much smaller number of bits. Instead, in a list of numbers that is purely random, there is no rule or law for generating the sequence, and the sequence is called *algorithmically incompressible*. The complexity of these lists is equal to their lengths, because they are only generable by a complete enumeration of their elements.

In being equipped with these definitions, we can now consider Barrow's explanation of the reason for which the laws of nature are mathematical:[22]

> Science exists because the natural world seems algorithmically compressible. The mathematical formulas that we call laws of Nature are economic reductions of enormous sequences of data on the changes of the state of the world: this is what we mean by the intelligibility of the world. [. . .] Given that the physical world is algorithmically compressible, mathematics is useful for describing it: it is in fact the language of the abbreviation of the sequences. The human mind permits us to enter into contact with that world because the brain has the ability to compress complex sequences of sensorial data into a briefer form. These abbreviations allow for the existence of thought and memory. The natural limits of sensitivity that nature imposes upon our sensory organs impede us from overloading ourselves with information about the world. These limits function as safety valves for the mind.

In returning now to the difficulties mentioned above, we can observe that the aspect of compressibility, and therefore of descriptive economy, as Barrow highlights, is undoubtedly present *both* in the laws of coexistence *and* in those of succession. From this point of view, if the algorithmic thesis were limited to the claim, in Mach's tradition, that the laws of nature condense a very high and potentially infinite number of possible observations and measurements into their formulas, one should not hesitate to subscribe to it; if one needed to be further convinced, he or she need only read the memoirs of any experimental scientist.[23]
Could we then conclude that the laws of nature are mathematical because mathematics is an excellent instrument for compressing information? While this response surely has some truth to it, it is much more difficult to try to explain *why* this happens; in what sense does the claim that nature "is algorithmically compressible" explain why "the world is mathematical?" Barrow's attitude towards this question appears to oscillate between two positions:

[20] If *n* is one million or one billion, the instruction "print the square of the first million natural numbers" has approximately the same number of bits as "print the square of the first billion natural numbers," but the length of the two complete lists will be completely different.

[21] See J. Barrow, *Perché il mondo è matematico?*, cit., pp. 93-96.

[22] *Ibid.*

[23] See for example the data relative to Boyle's law, as appears in R. Harré, *Laws of Nature*, Duckworth, London 1993, p. 14.

(a) On the one hand, he attributes the compressibility to "the physical world in itself," whatever this may mean;
(b) On the other, he seems to suggest that it is the *mind* that filters sensorial information, by first compressing it in a "natural" way through perception and memory, and then by further processing it and compressing it through the construction of mathematical concepts, which are more or less freely invented.

Beginning with the second of the two interpretative hypotheses, we will note that it seems to depend on a constructivist view of mathematics, on the basis of which mathematics is a creation explicitly realized for an applicative purpose. In this way, the mystery of the applicability of geometry, algebra and calculus would shift from the philosophy of mathematics to the cognitive sciences and more generally, to the study of the human brain. From this point of view, mathematical activity, or at least the part of it that consists of trying to condense numerical data obtained through observation into formulas lacking redundancy, is the product of the human mind, in the same sense in which any human perception reduces the complexity of the object perceived through the mechanisms that filter information.

The problem with this second interpretation, as Barrow formulated it, lies not so much in the fact that he does not explicitly argue for mathematical constructivism or in favor of the view that mathematics is the art of compressing information, but in the fact that the notion of the brain's ability to compress information, for how interesting and plausible it may be, is not sufficiently developed. The claim that "mathematics is an effective instrument for compressing information" does not help us to understand why this happens with respect to the natural world, nor does it bring us closer to our aim of maintaining that the brain compresses information from the external world. What is, in fact, the relationship between mathematics' ability to compress information, and the human brain's capacity to filter the natural world? The suggestion of looking at the cognitive sciences and neurosciences is probably a step in the right direction, but in this form it is still too vague to be of help to us.

If we were to adopt the first hypothesis, which "reifies" the compressibility achieved by mathematics by referring it to some unknown properties of the physical world "in itself," we would still have to explain *why it is possible to represent relationships between observable properties of physical systems with numerical, algebraic or geometric structures*. This problem – which is preliminary both to the computational approach to the laws of nature and to any approach intended to explain the applicability of mathematics – has until now been almost completely neglected, and we will deal with it in the rest of this chapter.

For the purpose of granting hypothesis (a) every possibility of success, this problem can be temporarily put aside, remembering that a hypothesis on the "compressibility in itself" of the natural world is sometimes defended via a recent thesis by David Deutsch. According to this view, the applicability of mathematics is based on the fact, contingent and fortunate at the same time, that the simplest mathematical operations, like addition, can be simulated by natural processes, such

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as those that take place in the electrical circuits of a computer.[24] On the basis of a physical version of the so-called Turing-Church Principle, according to which every function that is "intuitively computable" can be computed by a Turing machine, Deutsch has maintained that any physical process consisting of a finite number of steps can in principle be simulated by a *quantum* Turing machine.[25]

This proposal, for how interesting it may be, seems very poorly founded. In fact, to affirm that the laws of nature render addition and multiplication possible due to the electrical impulses of a circuit can only mean affirming one of the following theses:

1. any computer, inasmuch as it is a physical system, is subject to limitations imposed by those physical laws that the computer itself must obey;
2. we can interpret certain operations performed by a physical system in finite time as a calculation; for example, we can use the shadow of a gnomon to determine the time of day or the fall of sand in an hourglass to measure the duration of a phenomenon.

In hypothesis (1), the thesis has undoubtedly interesting theoretical implications for the relationship between computer science and physics. However, from our point of view this thesis does not explain anything, given that it is only a condition necessary to the applicability of mathematics. Obviously, if the physical processes responsible for the simulation of simple mathematical operations were not "computable" (or, in particular, executable in finite time by microprocessors), we could not apply mathematics to the natural world, in the sense that we would not be able to use computers either to compute or to simulate other situations of the real world. *To claim that nature has mechanisms that can be used for performing addition*, and therefore that isomorphisms[26] exist between these physical processes and the arithmetical operation of addition, *is simply another way of formulating the problem that gives the title to this chapter, and not a solution.* Why in fact do these isomorphisms exist?

Hypothesis (2) takes us back to the subjective attribution of intentional properties to inanimate objects that has been already discussed, and which does not seem to shed any particular light on our problem. We must admit that from the point of view of an abstract and purely mathematical formulation of Turing's theory, the anthropocentric concept of "computability" lacks *epistemic*

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[24] D. Deutsch, *Quantum Theory, the Church-Turing Principle, and the Universal Quantum Computer*, in "Proceedings of The Royal Society", A 400, 1985, pp. 97-117, and J. Barrow, *Perché il mondo è matematico?* cit., p. 78.

[25] A Turing machine is a tape made up of an infinite number of squares, read one at a time by a head that moves back and forth along the tape from left to right. On the basis of internal instructions, the head of the machine can read, erase, and write a symbol in each of the squares.

[26] An isomorphism is a bijective function f between two sets that preserves the relations between their elements. A bijective function $f: A \to B$ associates distinct elements of A to distincts element of B, and is such that all elements of B originate from elements of A.

characteristics (that is, characteristics referring to cognitive human powers).[27] Nevertheless, in its applications to physical systems, the affirmation that a physical system effectively "computes" values in a way that is *structurally indistinguishable* from the operations of a Turing machine seems to have some validity, depending on our particular interpretation of the system in question, and renders the physical version of the Turing-Church principle explicatively weak for our purposes.

Additionally, the purely "behavioristic," black-box type consideration of a physical system (on the basis of which only the input and output of the system are considered, and the nature of the intermediate states are ignored) in theoretical physics appears needlessly restrictive. If in the case of the intersubjective study of the mind, the interpretation of the latter as a sort of "black box" is justified by the need for an intersubjective foundation of the hypotheses that must explain our behavior, in the case of physical systems this need does not arise, since attributing causal powers to the physical states between input and output does not involve, as in the case of introspection, affirmations which are purely "subjective" and not objectively controllable.

The Reducibility of the Laws of Coexistence to the Laws of Succession

Deutsch's interesting claim regarding the possibility of perfectly simulating any physical system undergoing a finite number of changes with a quantum Turing machine cannot be further discussed here, given that it would take us too far from our purposes.[28] I have said that the algorithmic view of the laws of nature, which uses the metaphor of the world as a giant computer that in every instant computes a temporally successive state on the basis of its cosmic "software," does not succeed in accounting for all those nomic relationships between phenomena that I have referred to as being of coexistence, and which are not identifiable with real and proper *sequential* programs. Might we not try to overcome this difficulty by maintaining that every law of coexistence is expressible through an equivalent combination of laws of succession? Otherwise, might it not be possible that the laws of succession are in any case *more fundamental* than the laws of coexistence, in the sense that the nomic correlations between events coexisting in space depend on those that characterize the laws of succession in some way?

To defend this hypothesis, one could begin by observing that post-Einstein physics has tried to ban action at a distance. The speed of light, in fact, is often interpreted as being the maximum limit for the speed of a signal, so that the concept of *absolute simultaneity*, as would be required by action at a distance, is

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[27] This observation is emphasized in J. Earman, *A Primer on Determinism*, Reidel, Dordrecht 1986, p. 124.

[28] For an initial elaboration on this, see M. Dorato, *Chi ha bisogno di una fisica non-computabile della mente?*, in *Prospettive della logica e della filosofia della scienza*, edited by V.M. Abrusci, C. Cellucci, R. Cordeschi and V. Fano, ETS, Pisa 1998, pp. 245-254.

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not valid. As a consequence, in the fundamental theories of contemporary physics, in which the concept of *field* is truly fundamental, *all interactions propagate at finite speed*. In desiring to salvage the hypothesis of the reducibility of *all* laws of nature to sequential algorithms executed by physical systems, one could then maintain that to the extent to which the laws of coexistence presuppose that precisely *instantaneous* causal action excluded by contemporary field theories, we would have good reason to do without it!

Despite the unexceptionable correctness of this last observation, the plan to eliminate all laws of coexistence from physics does not appear to be easily achievable, given that at least some of them, at closer analysis, do not at all presuppose action that is instantaneous, or "at a distance."[29] As a result, a law that sets forth relationships between properties as exemplified by a physical system in one instant does not necessarily involve the existence of a causal interaction between the two. For example, let us consider Gauss' law concerning the flow Φ of an electric field $\mathbf{E}_0$, generated by a charge q at the center of a closed surface, through the surface itself:

$$\Phi(\mathbf{E}_0) = q / \varepsilon_0 \qquad [6]$$

In this formula, subscript "0" refers to the fact that the charge is in the vacuum, so ε_0 is the dielectric constant of the vacuum. This law connects two physical magnitudes, $\Phi(\mathbf{E}_0)$ and q, which, *to the extent that they belong to the same plane of simultaneity*, are *coexistent*.[30] Of course, the charge generates or "causes" the flux *at a delayed time*, but in the form in which the relationship between the two magnitudes is stated in [6], the two magnitudes are coexistent, because no specification is made of the possibly different times at which one considers them. *Qua* coexisting events, the flux crossing the surface and the simultaneous state of the charged body cannot be causally connected, given that, if they were, the spatio-temporal regions in which they were located would have to exchange *information* at a velocity faster than that of light, in contrast with the spirit of the theory of special relativity.

We can therefore conclude our discussion on the relationships between laws of coexistence and causal laws of succession in the following way: until physics

[29] For an updated discussion on the fact that the laws of coexistence do not imply non-locality, see J. Earman, *Bangs, Crunches, Whimpers and Shrieks*, Oxford University Press, Oxford 1995, Chapter 5. See also M. Dorato, *Earman on "Bangs, Crunches, Whimpers, and Shrieks,"* in "British Journal for the Philosophy of Science", IL, 2, 1998, pp. 338-347.

[30] The equivalent differential formulation,

$$div\,\boldsymbol{E}_0 = \rho_0$$

(Maxwell's equation), relates the divergence of an electric field to the charge density in the vacuum. To be sure, the flow through the surfaces of an infinitesimal volume that contains an infinitesimal electric charge *dq* is generated by the charge. However, at any instant of time the charge *coexists* with with the flow through the surface on a plane of simultaneity having an infinitesimal area.

forces us to reintroduce signals propagating faster than light, the nomic relationship of coexistence between physical phenomena (or types of physical events) must be considered "just as fundamental" as the causal one. "Just as fundamental" in this case means that:

i) the fact that a single physical event *a* is the *cause* of a single event *b* does not necessarily presuppose a nomic relationship between *A* and *B* (the capital letter indicates that we are referring to *types* of events);
ii) as a result of the existence of "a limiting signal" like that of light, a *law* of coexistence between *a* and *b* does not necessarily presuppose a causal link.

Contrary to the thesis that I have just expressed in i), we should take into account that, on the basis of some interesting reductionist and physicalist theories of causality, a causal process is characterized by the existence of laws of conservation of some physical magnitudes (including momentum, angular momentum, electrical charge, etc.), as exemplified by pairs of events or interacting processes.[31] For example, if we imagine that the causal interaction between a rock thrown at a window and the window itself is characterized by *the conservation of momentum*, the product of the velocity of the rock times its mass (i.e., its momentum) must correspond, if such a magnitude must be conserved through the interaction, to the sum of the products of the velocities of the single fragments of glass times their respective masses.

Within such a physicalist conception of causality or causal processes, the causal interaction of two physical events would therefore seem to *require the existence of a law of conservation as its necessary condition.* Nevertheless, for reasons which will become clear only later on, I will instead maintain that, especially outside of physics, but sometimes also within it, many causal relationships between single events do not at all presuppose a law. It follows that, as affirmed in 1), a causal relationship existing between two single events cannot be reduced to a nomic relationship (causal singularism).

Similarly, ii) above claims that laws of coexistence do not presuppose laws of succession or causal processes, precisely because the former refer to pairs of simultaneous events, or events belonging, as we say in jargon, to spatial hypersurfaces.[32]

Nonetheless, even in this case there is a possible objection that should be considered. We could hypothesize that the existence of any nomic link between events belonging to spatial hypersurfaces is evidence of the fact that the link in

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[31] For this theory of causal processes, see P. Dowe, *Wesley Salmon's Process Theory of Causality and the Conserved Quantity Theory*, in "Philosophy of Science," LIX, 1992, pp. 195-216; W. Salmon, *Causality Without Counterfactuals*, in "Philosophy of Science," LXI, 1994, 297-312; P. Dowe, *Causality and Conserved Quantities: A Reply to Salmon*, in "Philosophy of Science," LXII, 1995, pp. 321-333.

[32] These hypersurfaces correspond to instantaneous sections of space-time, instants of "cosmic time," which in certain models of general relativity allow us to subdivide all events into past and future (with respect to those instants).

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question originated from a single common "ancestor," or *common cause*, which explains the very existence of the nomic correlation across space. This hypothesis seems to be suggested by the example of Gauss' law [6], given that my classification of it as a law of coexistence might be criticized for neglecting the fact [33] that the flow is *generated* through the closed surface by the charge placed inside it, *through a process that is causal and not instantaneous*.

According to Reichenbach's Principle of the Common Cause, in the presence of a statistical correlation between spatially separate events (for example, a certain number of simultaneous cases of indigestion in different people who have dined together), and in the hypothesis that such spatially separate events are not directly causally related, one can assume the existence of an event in their common past, or the so-called "common cause" (spoiled milk utilized for the preparation of a dessert that everyone has eaten). The common cause would explain away the coincidences given by the collective illness, since without the existence of such an event in the common past, the probability of a simultaneous illness for a certain number of people would be extremely low. Similarly, on a more theoretical level, the presence of cosmic background radiation, for example, that is homogeneous and isotropic, has suggested models of the early universe that predict a highly rapid expansion (the so-called "inflationary" models). In this way, we could explain why regions of space, that would otherwise be causally disconnected, instead present such important "correlations" between the physical magnitudes associated with the radiation in question.

We could therefore argue that every law of coexistence has been generated, and therefore presupposes and depends on laws of succession, or on "almost parallel" causal processes that run between them and that originated in the past. For this, see Figure 2.

[33] J. Earman, *Bangs, Crunches, Whimpers and Shrieks*, cit., Chapter 5.

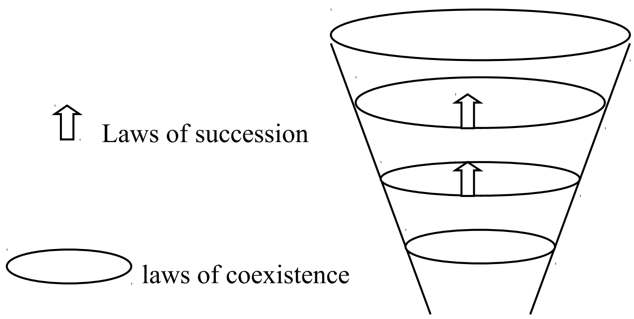

Fig. 2

Even though it would not be implausible to argue that nomic links between simultaneous events can be explained by common causal processes that have generated and "preserve" the relationships between the events in question, a remaining obstacle to implementing this plan would be the so-called *quantum correlations*. Such correlations nomologically link the measurement results of two identical particles fired in opposite directions from a common source, and revealed by apparatuses placed in causally disconnected regions. These correlations cannot be explained by common causes that render them probabilistically independent of each other, nor can they be explained, without running into difficult and controversial conceptual questions, by postulating symmetrical causal dependencies between the measurement results, given that their temporal order is not an invariant for different inertial observers.[34]

In not being able to go into the details of these difficult problems, I will only emphasize that the plan to "substitute" every law of coexistence with causal laws that invoke a common cause for the nomic correlations between simultaneous events does not have enough empirical support at this time. More importantly, even if we could maintain that for every nomic correlation between simultaneous events, laws of succession existed that implicated a past common causal process, this would not be the same as maintaining that these correlations *do not exist now.* Explaining a law of coexistence by affirming that it derives (or that it is

[34] For a primary introduction to the problem of quantum non-locality, see J. Cushing and E. McMullin (eds.), *Philosophical Consequences of Quantum Theory*, The University of Notre Dame, South Bend, Indiana 1989.

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determined) by laws of succession is not the same as denying that the laws of coexistence exist. *In brief, the laws of coexistence seem difficult to reduce (and therefore eliminate) in favor of laws of temporal succession.*

If, given this irreducibility, we wanted to try to salvage the algorithmic conception of the laws of nature by restricting the applicability of the latter to the laws of succession, the philosophical theory we were discussing would become very weak, because the laws of coexistence should be excluded. These observations would seem to definitively confute the interpretation of natural laws as algorithms.

The reason that the algorithmic conception of laws implicitly tends to underestimate the role of the laws of coexistence with respect to those of succession should also be clear: predictions *per se* are only permitted by the former, and not by the latter, and predictive accuracy is considered the most important cognitive fruit of science.

Note that laws like [1] or [2] above do not explicitly depend on time: from the cognitive or applicative point of view, they have properties that are very different from laws like [3]. In [2], by keeping the temperature of the gas constant and doubling its volume, we can “predict” that the pressure exercised by the gas on the walls of the vessel will halve only in the sense that this fact can be *deduced* from the form of the law. However, the “prediction” in question does not imply, as with genuine predictions, that the modification of pressure is *temporally successive* to the modification of the volume. The changes in pressure and volume are in fact *simultaneous*, because equation [2] describes an *equilibrium state* of the system.[35] For instance, a decrease in the density of the gas – owing, for example, to an increase in its volume – *simultaneously* provokes a lower average molecular velocity, and therefore an overall lower pressure: the impacts of the molecules against themselves and against the walls of the recipient are less violent and less frequent. Alternatively, we can say that if we modified the system by diminishing the pressure, if the temperature remained the same we would *simultaneously* cause an increase in volume. For example, if we took 50 grams of air to the summit of a mountain 1000 meters high, if the temperature remained the same, the air would occupy a greater volume than at sea level, because at 1000 meters the pressure is lower.

In brief, the algorithmic structure involved in the laws of coexistence enables us to calculate the value of certain magnitudes (F or M_i in [1]; P or T in [2]) on the basis of the other magnitude, which we suppose to be known or given. However, while the laws of succession allow us to interpret the determination of the final states by the initial ones in a causal sense, the laws of coexistence imply a determination that is only “logical,” or deductive (the “cause,” if we could speak of such a concept, would be simultaneous to its effect). What I have claimed so far could be then summarized as such: *every scientific prediction is the result of a calculation performed on the basis of an algorithm yielded by a law of nature, but*

[35] This remains true despite the fact that it is easier to intervene on the system from the outside, modifying the volume rather than the pressure.

not every calculation useful for determining the value of a magnitude through a law is a prediction.

Before continuing our discussion, it would be useful to try to understand why the predictability of the temporal evolution of a physical system is rendered possible by the mathematical aspect that we attribute to the laws of nature, an aspect which validates the numerous efforts scientists have made to find a quantitative formulation of laws. The application of a law of succession allows us to calculate what the state of a physical system will be (relative to the units of measurement that interest us) in any instant of time, assuming the initial conditions, if the solution to the differential equation that expresses it

(1) exists
(2) is unique
and if
(3) slight imperfections in the measurement of the initial conditions do not increase with the passing of time in the successive states.

This last condition, known as *stability*, is especially responsible for the predictability, also in the long run, that constitutes the strong point of the scientific description of the world. The other two conditions, *existence* and *uniqueness* of the solutions, instead render possible what is known as the *deterministic description* of the world, i.e., that the state of the system at any given time, plus the algorithm given by the law, univocally fixes the state of the system at any other time. If the ability to predict constitutes the essential characteristic of empirical science, then a discipline lacking in laws would not be a science.[36]

[36] Predictability, despite frequent assertions to the contrary, should not be seen as a necessary condition for the presence of determinism. One the one hand, abandoning the condition of stability severely limits the predictive possibilities of a system of equations, but does not necessarily cancel them, as meteorology shows. On the other hand, phenomena that depend sensitively on initial conditions show how an unforeseeable evolution can be compatible with the existence of a single deterministic output. This input (initial data), being specifiable only within certain intervals of experimental error, can give origin to widely divergent future evolutions, which over time amplify small initial differences. Indeterminism, finally, is given by a "one-to-many" correspondence between initial and final data: many future possible events correspond to a single present. Determinism requires the "same cause-same effect" principle; non-linear phenomena, depending as they do on small variations in the initial conditions, respect this principle, but violate the one according to which "similar effects follow similar causes." Finally, indeterminism violates the first principle, given that different effects can follow perfectly identical causes.

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Measurement as a Necessary Premise for the Mathematization of Laws

We can now proceed to the problem of the *quantitative* modelling of experimental data, which will turn out to be essential for making progress not only towards understanding why mathematics can describe nature, but also to discovering what the laws of nature are and how science operates. A law of physics is nothing but an algorithm for passing from a certain set of observations or properties of a system to different observations and properties of that system. It would then follow that if this data were not expressed quantitatively to begin with, the expressions that relate them (the laws) could not be applied to predict the future, and no arithmetical operation, from the most elementary to the most complex, could ever be executed. It is therefore rather strange that, up until now, this aspect has not been considered important to the clarification of why laws have a mathematical nature, especially if we consider that *measurement is definitely one of the most important, if not the main, scientific activity.*

As mentioned at the beginning of this chapter, the path that has been preferred until now for explaining the effectiveness of mathematics has led to the attempt to clarify the nature of mathematical knowledge. This entails trying to ascertain whether mathematical truths literally reveal the properties of abstract and causally inert entities (including functions, groups, classes, etc.), as Platonists would have it, or whether mathematics is essentially a creation of ours (constructivism), or instead, as Galilei believed, it is the language in which nature is objectively written.

I will at least partially work around these difficult questions, venturing down a much less-beaten path, which will lead us to attempt to establish what type of relationship exists within the qualitative fabric of our phenomena, as they appear to our perceptions, and their quantitative treatment, which constitutes the presupposition of their measurement. The key notion I will appeal to is that of *isomorphism* between real objects and mathematical models,[37] a notion which for our purposes can be considered equivalent to postulating the objective existence of structural and formal similarities between certain aspects of the phenomenal world and the mathematical structures used to describe them. I will provide evidence for the existence of these isomorphisms with a two-pronged argument, i.e., with the thesis that supports the empirical origin of all mathematical concepts on the one hand, and, on the other, with the hypothesis by the mathematician Saunders Mac Lane, which is examined and elaborated upon here, on the basis of which *mathematics is essentially the knowledge of the forms of objects*, a thesis which seems supported by a few results obtained in the field of the cognitive sciences, and which I will now illustrate.

As we saw in the previous chapter, the most characteristic difference between ancient and post-Newtonian physics lies in the widespread use that the latter makes of *mathematics*. While Aristotle's physics was purely qualitative – we have seen

[37] See note 28.

that all Greek mathematics, even Archimedes' work, made no reference to *physis*-- Newton's *Mathematical Principles of Natural Philosophy* is full of formulas and theorems. It must be agreed that ancient astronomy – *a discipline that is both empirical and mathematized* – constitutes a notable exception to the transition from the "world of more or less" to the "universe of precision," to which the historian of modern science Alexandre Kovré referred in characterizing the scientific revolution.[38] However, it is doubtless that the progress of our knowledge of nature has evolved from *purely qualitative classifications* of natural phenomena (hot and cold, light and dark, large and small, etc.) to their quantitative descriptions. Such descriptions have been achieved thanks to the application of mathematics to measurement results that have been obtained through instruments capable of yielding more and more precise experimental data.

To this regard, one of the most important logicians and philosophers of science of the Twentieth Century, Rudolf Carnap, has maintained that the evolution of our scientific understanding of nature passes through three essential stages:

1. the *classification* of phenomena on the basis of the presence or absence of certain qualities or properties;
2. a *comparative* analysis in terms of "*more*" or "*less*";
3. the construction of *quantitative concepts*, which represents the final destination.[39]

We can exemplify this "Comtean law of the three stages" using the case of "possible-probable," whose quantitative aspect is the subject of Carnap's *Logical Foundations of Probability* (see previous note). However, the following considerations are also valid for fundamental physical magnitudes, including the length of an object, its weight, or the interval of time between events, even if, for reasons of space, I will limit myself to extending the remarks I will be making on the notion of probability only to the metric concept of *length*.

1. CLASSIFICATION–Let us suppose that a judge must make some decisions regarding the depositions of some witnesses whose credibility he or she has reason to doubt. In wishing to separate what is believable from what seems implausible, the judge will start by *classifying* all the events described by the witnesses by relying on the qualitative concept of *possible*, the direct opposite of *impossible*. This dichotomy is normally used to *separate* phenomena that are either compatible or incompatible with some generalizations we normally take for granted: logical, natural, or social. For example, if the version told by one witness contains contradictions, the facts stated are automatically classified as "impossible."

[38] See A. Koyré, "Du monde de l'«à-peu.près» a l'univers de la precision", in *Etudes d'histoire de la pensée philosophique*, Paris, Gallimard, 1973.

[39] This division into three, present in R. Carnap, *Logical Foundations of Probability*, The University of Chicago Press, Chicago 1950, was later also examined by C. Hempel, *Fundamentals of Concept Formation in Empirical Science,* The University of Chicago Press, Chicago 1952.

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In proceeding now to the physical concept of length, if we are dealing with spatial dimensions, we will begin by classifying objects that we measure as *long* or *short*, with respect to standard objects conventionally regarded as measures of length. These measures tend to involve parts of our body, including feet and inches (once known as "thumbs"), the latter still present in the Anglo-Saxon system of measurement. This is the first state of the formation of concepts, which Carnap refers to as *classificatory*.

2. COMPARISON–In listening to more than one version of the same unknown events by different witnesses, our judge will plausibly proceed to *compare* them, ordering all the available reconstructions according to their being of *greater, smaller* or *equal* probability, *therefore implicitly assigning different "degrees"* to the concept of *possible* (plausible), which was previously used in a purely classificatory way. Similarly, the possibility of comparing the length of different objects will permit us to order them, so that one will be longer or shorter than another.

This constitutes the second stage of concept formation, which Carnap calls *comparative*, and it is particularly important because it already has a very rich logical structure.

Once we have recognized that a scientific concept admits different *degrees*, it is possible to order all cases in which it is exemplified through relations of *comparison* (majority or minority) and of *equivalence.* The comparison of minority for example, is characterized by two properties:

a. Transitivity. In our two examples, the relation "being less plausible" (denoted by "$<_p$", while the relation "being less long" will be referred to by "$<_l$") is *transitive,* in the sense that if the version of the facts a is less plausible than version b, and if the latter is less plausible than c, then it necessarily follows that a is less plausible than c. Similarly, if body a is shorter than b, and the latter is shorter than c, from the meaning of "shorter," it necessarily follows that $a <_l c$.

b. Asymmetry. Relations of comparison are also *asymmetrical*, in the sense that the order of the two relata is not "interchangeable": if $a <_p b$ or if $a <_l b$, by the very meaning of the relations in question, it follows that $b <_p a$ or $b <_l a$ are not also possible. The same property is obviously valid for the comparisons of majority ("greater than").

Since we cannot assume that for any version V, V is necessarily more plausible or less plausible than any other version, we must introduce another relation, *the relation of equivalence*, to denote the notion of "being equally plausible" ("$=_p$"). Similarly, for the concept of length we can introduce the equivalence relation "being of the same length" ("$=_l$").

Any relation of equivalence is characterized by the satisfaction of three properties:

a. Reflexivity. Every version of the facts is trivially as plausible as itself: $a =_p a$;

b. Transitivity: if $a =_p b$ and $b =_p c$, then $a =_p c$;

c. *Symmetry*: if $a =_p b$, then also $b =_p a$.

The equivalence relation '$=_p$' allows us to divide all our versions into classes of equivalence, the elements of which all have the property of "being equally plausible." These properties also hold for length: "being of the same length" or "being equally long" is also reflexive, transitive and symmetric, and therefore a relation of equivalence. All objects belonging to the same class are equally long and no two objects belonging to different classes of equivalence can be congruent.

3. QUANTIFICATION

Let us suppose that the judge is now willing to *bet* on the veracity of some of the testimony. To allow the *force* of his or her conviction to emerge and to be able to decide rationally on one or other of the available versions,[40] he or she must pass from the qualitative to the quantitative plane. Essentially this means that he or she needs to reach a *numerical measurement* of what had previously been evaluated only in terms of "being plausible" or "implausible" (classification) or having *smaller* or *equal* probability (comparison). We need to introduce a *real-valued function P, assigning a real number between zero and one, including extremes, to the likelihood of the testimony in question.* Obviously, the choice of the extremes of the numerical interval is partially arbitrary, given that he or she could just as well choose a range from 0 to 100, like the one used for percentages. An event that is contradictory or *impossible*, on which the judge would bet no money at all, would therefore become an event of *zero* probability, while a testimonial account that was certainly true would receive the maximum allowed probability, namely *one*.

Between these two extremes, there is what is more or less probable, expressed as fractions ranging from numbers very close to zero (denoting a version of the facts that is almost impossible) to numbers very close to one (designating a version that is almost certain), and where the vague "more or less" can be estimated with much more precision, the more reliable the method is for calculating the probability in question.[41]

Of course, it would not be possible to assign a precise real number to the plausibility of a version in all cases, but if, for example, the judge were to decide between only two versions available, the first (V_1) being twice as probable with respect to the second (V_2), then

$$P(V_1) = 2P(V_2).$$

[40] For a study of how a wager can lead to a numeric value between 0 and 1, see R. Festa, *Cambiare opinione: temi e problemi di epistemologia bayesiana*, CLUEB, Bologna 1996.

[41] In this sense, probability, regarded as the measurement and quantification of the possible, represents, as the mathematician and philosopher Hermann Weyl affirmed, the *new modality of science*. "Modality" or "modal propositions" here mean the different ways in which a proposition *P* can be true (necessarily true, actually true, or possibly true). If *P* is "I have 1 euro in my pocket," and this is true, the truth of *P* is actual but its negation is not contradictory. If *P* is instead "2 + 2 = 4," then *P* is necessarily true, because its negation is contradictory.

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If it were possible to establish that V_1 or V_2 were mutually exclusive (either one or the other was possible, but not both) and exhaustive versions of the facts (no other version were possible), then the sum of their probabilities, as we will also see in what follows, must equal certainty (it rains or it does not), and the sum of the two probabilities must equal 1:

$$P(V_1) + P(V_2) = 1$$

In this case it is already possible to assign to $P(V_2)$ the value 1/3, since $3P(V_2)=1$; $P(V_1)$ will be twice as much, and therefore 2/3.

Similarly, to quantify the concept of length, we introduce a real-valued function L, which associates tangible objects to real numbers giving them their relative *length* on a given scale. Constructing a scale presupposes the conventional attribution of a unit of measurement to some chosen object which functions as standard (an inch, a foot, the platinum-iridium bar in Paris measuring 1 meter, etc.). The *congruence* of r times the length of the standard object with the object to be measured O means that O has length r (r being a real number), relative to the chosen unit.

The representability of the order of plausibility or the order of length in numerical terms that we have constructed through the two relationships "$<_p$" and "$=_p$" can then be obtained by requiring that function P (that is, L) satisfy certain constraints. Namely:

$$\text{if } a =_p b\text{, then } P(a) = P(b) \qquad [7.1]$$

$$\text{if } a =_l b \text{ then } L(a) = L(b) \qquad [7.2]$$

In [7.1] and in [7.2] the sign "=" after the two words "then" has the same meaning, given that it indicates *equality between real numbers*, but it has a different meaning both from "$=_p$", which stands for "equally probable" and from "$=_l$", which stands for "having the same length."

We then have

$$\text{if } a <_p b\text{, then } P(a) < P(b) \qquad [8.1]$$

$$\text{if } a <_l b \text{ then } L(a) < L(b) \qquad [8.2]$$

where "<" indicates the usual relationship of "being less than" which holds between numbers, and therefore has a different meaning from both "$<_p$" and "$<_l$".

Despite this difference in meaning, the expressions [7.1 and 7.2] and [8.1 and 8.2] allow us to *represent* all the significant *relations* between the different versions on the one hand (their being more or less probable), and the different lengths of objects on the other through "analogous" relations (analogous in a sense to specify) among the corresponding real numbers. The values of the probability function P are in fact expressed as real numbers between zero and one, and we can apply the normal arithmetical operations, including sum and product to these numbers.

Suppose that for any possible testimony there is a real number in the closed interval [0, 1] that measures its degree of plausibility. Suppose furthermore that any two different testimonies are assigned different numbers of the interval and that any real number in the interval corresponds to some testimony. In this case, the function *P*

$$P\colon E \rightarrow \mathrm{R}$$

mapping the set of testimonies *E* onto the set of real numbers between zero and one (R) would be *bijective*.[42] We could say that a bijection *P* created an isomorphic correspondence between the two sets, provided that *P* *conserved* all the relationships and operations between the two groups. The same reasoning would apply to the bijective function *L* and to the comparative relations "having the same length," or "being less long." The meaning of the term "conserve" in this case could be explained by the following example. If as before we wanted to evaluate the plausibility of two versions which could not both be true, in the sense that they were mutually exclusive, we would have to introduce beforehand an operation of "sum" $+_p$ defined on elements of *E*, and then apply *P* to the result of the sum of "$a +_p b$", however it is defined. The conservation by the bijection *P* of the operation in question ("summing" versions or conjoining them as in "$a +_p b$") would imply that we had

$$P(a +_p b) = P(a) + P(b) \qquad [9.1]$$

where, once again, the two operations "$+_p$" e "+" do not have the same meaning, given that the second is the usual sum between numbers and the first is the union between the two sets representing the two different testimonies (the classes of equivalence thereof). Obviously, if the two testimonies were mutually exclusive and exhaustive, in the sense that it were possible to exclude any other, the probability of their union would be equal to the maximum, that is 1, because it is certain that either *a* or *b* occurred, and the two versions could not be simultaneously true.

In the case of the concepts "long/short," the "sum" of the length of the two objects (say, two rulers) would be obtained by simply putting them one next to one another, as if we had a ruler twice as long as the original one:

$$L\,(a +_l b) = L(a) + L(b) \qquad [9.2]$$

[42] See note 29.

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In the case of another additive magnitude, weight, the operation of "summing"$_w$ the weights of two objects would be given by putting both on a scale, as if they were a single object. Also in this case, with obvious terminology

$$W(a +_w b) = W(a) + W(b) \qquad [9.3]$$

Despite the fact that in the case of physical magnitudes that are not additive, like temperature, the procedure would be a bit more complicated (we cannot mix two gases of a different temperature and expect the temperature of the mixture to equal the sum of the temperature of the single, original gases), the procedures introduced are substantially the same,[43] so that the method illustrated here could be applied to any measurable magnitude.

In simplifying and summarizing our discourse, our comprehension of nature through a quantitative method passes through four stages. We first introduce a comparative order via asymmetrical and transitive relations ("being longer," "being hotter," "being heavier," "being earlier than," etc.), properties which are generally ascertainable through direct observation. We then determine classes of equivalence, formed by entities having reflexive, symmetrical, and transitive relations among one another ("being of equal length," "equal temperature," "equal weight," or "being simultaneous," etc). To measure this purely qualitative order quantitatively, we therefore introduce appropriate units of measurement, and construct a bijective function which assigns numbers to the group of bodies or events whose properties we would like to measure. As illustrated above, for probability and length, this is typically done on the basis of procedures of measurement of an *operative* type. Finally, by taking the properties exemplified by [7.1 and 7.2], [8.1 e 8.2], and [9.1 e 9.2] into account, we order the classes of objects on a numerical basis.

Naturally, it is not always possible to order phenomena in a quantitative way, but where we can do so, the advantages are undeniable, and it is not difficult to explain why. During the Renaissance, for example, the Italian natural philosopher Bernardino Telesio believed that the fundamental explicative principles of nature were, besides matter, *hotness* and *coldness*. However, this purely qualitative account, besides lacking in predictive power, presented, unlike the assignment of quantitative concepts or magnitudes to physical systems, the disadvantage of *subjectivity*. As we know, if we submerge our hand in a basin of water at room temperature after having left a sauna, the water will seem *cold*, while if we come out of a refrigerated room, the *same* water will seem just as *hot*. How can we reach

[43] Non-additive magnitudes are those whose sum does not correspond to "the joining" of the corresponding bodies: for temperature, mixing two gases at different temperatures in general does not yield the sum of the numbers corresponding to the temperatures of the single gases. It is therefore necessary to associate the size in question to the volume occupied by a substance which is dilated in a cylinder, or to the adiabatic transformation of the volume of a gas. The same happens for the hardness of bodies, see R. Carnap, *Philosophical Foundations of Physics: an Introduction to the Philosophy of Science*, M. Gardner (ed.), Basic Books, New York 1966.

an agreement on the basis of qualitative and subjective concepts like hot and cold? Only *after* having created a conventional scale from 0 to 100 degrees Celsius, or from 32 to 212 degrees Fahrenheit, and with the use of a graduated instrument (a thermometer) can we find an objective expression corresponding to our sensations of hot and cold. As we have seen, this occurs through the assignment of a *number* measuring the temperature of a body, which allows us to compare our impressions of "hotter" or "less hot" in a precise way.

In defending the advantages of a quantitative description of nature from the numerous attacks that still come from certain schools of thought, I do not intend to discount the importance of classificatory or purely quantitative concepts. After all, our learning a natural language and therefore the first concepts like dog, cat, window, bird, etc., coincides with subdividing everything that exists into groups or different classes without individuals in common amongst themselves. In natural sciences like zoology and botany, in which the classifications use strain to identify a group of *natural genera*, the classificatory concepts help us to find the best way to systemize our knowledge of plants and animals. In the study of inorganic matter, Mendeleev's Periodic Table of Elements illustrates how important it is to try to base classificatory concepts (the elements) on quantitative differences. On the other hand, even the physics of elementary particles passed through various phases in which a general criterion of classification was sought for the numerous particles discovered.

It would not be therefore legitimate to affirm that all sciences *necessarily* first pass through a qualitative classification of nature (the hot-cold, night-day, male-female, even-odd opposites characterize many primitive cosmogonies based on the exclusive "or") to then proceed to a comparison by degrees (more or less) and in their maturity, finally arrive at quantitative concepts or magnitudes. However, it is undeniable that, as Hempel emphasizes, the use of quantitative concepts does not only help us to *differentiate* cases that in a given classification would result indiscriminate – for us "it is hot" both at 31 and 35 degrees Celsius, but at the latter temperature it could be, for example, much more risky to exposure ourselves to the sun. The principal advantage of the formulation of natural laws in quantitative terms consists of the fact that it becomes *possible to apply arithmetic and more advanced mathematical theories.*[44] In sum, an important difference between natural and scientific language lies in the systematic use that the latter makes of quantitative concepts, not only classificatory and comparative ones.[45]

The further question of whether, according to Carnap, the difference between quality and quantity can *only* be posed in linguistic-conceptual terms (and there is nothing in nature which objectively corresponds to it)[46] will be examined in later

[44] See C. Hempel, cited work, pp. 72-73.

[45] Natural languages contain numerals for counting and are therefore partly quantitative; in the same way, scientific languages, from a certain point of view, can be considered a part of the natural ones.

[46] «The difference between qualitative and quantitative is not a difference in nature, but a difference in our conceptual system, that is, our language, as we say, if for language we intend a system of concepts.» (R. Carnap, cited works, pp. 81-82).

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chapters. What I can say at this point inclines us towards Carnap's view, since in physics, the scales and units of measurement are at least in part conventional, even when they are introduced on the basis of known laws of nature.[47] As a result, to attribute, for example, *certain numbers over others* (32 degrees Fahrenheit rather than 0 Celsius or 273.15 Kelvin) to the freezing temperature of water seems to be the fruit of a choice which is also dictated by convenience.

In any case, it is of fundamental importance to emphasize the fact that *relations* such as "being longer," "being of higher temperature" and "being heavier" that we have attributed to physical systems *are objective*, or *valid independently of our mind and of the language that we adopt*. From here it would follow that comparative relations between homogeneous measurements of physical magnitudes are independent of the choice of scales. Therefore, it makes sense to institute *comparative relations between the qualitative properties of physical systems and their relative magnitudes, which constitute the framework of the mathematization of the world, and therefore the possibility of formulating natural laws in a qualitative way, where the term "law" indicates "relation between magnitudes defined on the basis of certain fundamental units."*

These observations help us, at least in part, to understand why the method of processing and mathematically transforming empirical data, and the application of the deductive consequences of the abstract, mathematical model to nature is successful. However, the following question remains unanswered: to what degree does isomorphic correspondence between magnitudes and numerical relations allow us to explain why mathematics is successful in predicting new aspects of reality? The hypothesis which I will put forward here, essentially for the purpose of stimulating further research, *is that the ISOMORPHIC MAPPING* between *objectively existent physical magnitudes* (mass, length, temperature, charge, etc.) and *the numerical structures by which we represent them is also valid*, even if only partially, between *the physical laws and the relations existing among the magnitudes of physical entities which such laws refer to.* It remains to be explained if and why such a partial structural resemblance also exists at the level of the *fundamental* laws of nature: would it not be a ridiculous anthropomorphism to suppose that a creation of ours (mathematics) described, explained and predicted the external world in a highly accurate way?

An Explicative Hypothesis on the Applicability of Mathematics

To respond to the question posed above, I will first of all note that *natural language* also satisfactorily describes the outside world, and that mathematics can be seen as a "specialization" of this language, i.e., as a particular, more abstract and more rigorously defined type of language. From this point of view, the

[47] The thermometric scale introduced by Lord Kelvin utilizes Boyle's law and departs from the concept of absolute zero, that is, from the lowest temperature that can be hypothesized as based on the laws of thermodynamics. This temperature is equal to -273, 15 degrees Celsius.

"representational" efficacy of mathematics should not be more mysterious than that of natural languages.

One could object by saying that the *epistemologies* of natural language and those of mathematics are very different, insofar as the latter makes use of *a priori* knowledge in a much more systematic and extended way, while within the former the referential and causal contact with the *empirical* world apparently plays a much more important role (*a posteriori* knowledge). However, accepting the importance of *a priori* knowledge in mathematical proofs does not at all exclude a radical form of mathematical constructivism, somewhat suggesting that fundamental mathematical concepts originate in our experience. Mathematics can be therefore conceived of as an abstract elaboration *originating* from our fundamental experience as living beings that are biologically equipped with a body that moves in surrounding space. This hypothesis implies seriously taking the constraints that accompanied our evolutionary history into account. As Henri Poincaré wrote:[48]

> An immobile being could have never acquired the notion of space: in not being able to correct the effects of changes in external objects through his or her movements, he or she would have had no reason to distinguish them from their changes in state.

Once again, the fact that the *justification* of mathematical theorems is typically *a priori*, and therefore independent of experience, does not imply that fundamental mathematical concepts are not empirically derived and do not have an intuitive meaning. If this were not the case, the applicability of mathematics to the world of phenomena would constitute an unsolvable mystery.

Of course, we must admit that a Platonic conception of mathematics, intended as an activity aimed at the discovery of an abstract universe existing in a non-spatio-temporal way, and in a certain sense therefore "pre-existent" to the beginning of *homo sapiens*, is not incompatible with the Darwinian theory of the origin of man, strictly speaking. However, Platonism renders the problem of the applicability of mathematics even more difficult to resolve.

Independently of the notorious difficulty raised by Paul Benacerraf concerning the difficulties in explaining our ability to discover something about this abstract (and therefore causally inert) Platonic realm without some sort of *causal* interaction with it raises the question as to why the physical world should resemble the structures of the mathematical, abstract universe? We need only reconsider the criticism that Aristotle had already levied upon the Platonic dualism between the world of ideas and the phenomenal world, which we mentioned at the beginning of this chapter. In our case, the duplication of the properties of the physical world in a mathematical universe existing prior to the appearance of minds on earth would only duplicate the problems to be resolved. In fact, an explanation of the question as to why the physical world reproduces the structural characteristics of an abstract world disconnected and independent from it seems as unavoidable as irresolvable.

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[48] H. Poincaré, *La Science et l'Hypothèse*, Flammarion, Paris 1902, p. 78.

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It is therefore highly plausible to hypothesize that mathematics can be applied to experience only because they originate and *derive* from the latter and in particular, as Kant had suggested, from our intuition of *space* (geometry) and *time* (arithmetic). The same distinction that has been present throughout this chapter, between the laws of coexistence and the laws of succession, recalls the different way of intuiting phenomena in space (following the order of what exists simultaneously) and in time (following the order of succession). Therefore, even in its most abstract developments, mathematics does not lose the intuitive element constituted by the perception of visible space, and by the activity of counting physically distinguishable objects within our visual field in succession.
Just to give an example, we will note that despite its mathematical complications, a structure like Hilbert's space, which is particularly important in the formulation of one of the fundamental theories of physics of our time (quantum mechanics) eventually makes reference to a concept which is of undeniably intuitive significance, that of *vector space*, in which entities (vectors) are in the simplest case (denoted by a three-dimensional space) identified da triple from real numbers (x, y, z, see fig. 3):

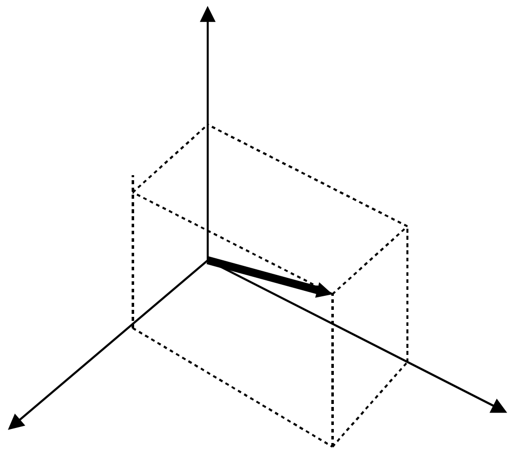

Fig. 3

Considering the intuitive origin of mathematics, whose sources are arithmetic and geometry, we can therefore understand Mac Lane's definition of *mathematics as the science of shape.*

> The external world exhibits patterns that repeat themselves and so can be captured as forms. The same form can appear in different concrete guises, and so can effectively be described abstractly, without reference to any one concrete realization. Such an abstract consideration of the form as

> such makes it possible to deduce its various properties, independently of the different guises. The mathematics lies in these deductions, even when the investigations go far beyond the original appearance of the forms. Then combination of different forms leads to new mathematical objects, which is some cases can then be used to understand facts of the physical and social world. This is why mathematics is effective: the world exhibits regularities which can be described, independently of the world, by forms which can be studied and then reapplied.[49]

The fundamental point is that these forms exist either in the objects that surround us or in our perceptive re-elaboration of them, and do not exist *independently* of them, as the Platonist believes. In a way in which I cannot discuss in here detail, we perceive the forms and "abstract" them *from objects*, and only later conceptually elaborated in a deductive and *a priori* way. In this sense, they can only be studied independently from the physical world. The fact that these deductive developments also render themselves "useful" for predicting the existence of properties or entities later effectively found in the phenomenal world cannot but be explained by the intuitive origins of the fundamental concepts of mathematics.

The hypothesis that I would like to put forward is that the concept of form, and the consequential abstract ability of the human mind to which Mac Lane makes implicit reference, is related to the processes of visual and tactile perception of solid objects and their shapes, and to our subsequent imagining of them, through the reactivation of neural circuits indispensable to their perception. From this point of view, the forms/shapes to which Mac Lane refers, if they can be interpreted, as we believe, realistically, definitely recalls the Aristotelian theory of perception and knowledge, as was re-examined by the late Nineteenth-Century philosopher and psychologist Franz Brentano.

In Aristotle's *De Anima*, on which Brentano wrote his dissertation, sensation is compared to the impression of a wax seal through the passage from potentiality to actuality. In perceiving an object, we "absorb" only its shape, in the same sense in which the matter that the seal is made of, gold or iron, is completely irrelevant with respect to the imprint that it leaves upon the wax. The deductive elaboration of the properties of shapes is independent from the physical world, Mac Lane affirms, analogous to the fact that from the point of view of the shape of a geometrical object, *which* concrete object it exemplifies is completely irrelevant. The shape of the wax seal is potentially even in our minds (in dispositional representations already activated by past experiences)[50] and it is actualized through an *abstraction* that is also a representation of the *shape* of the perceived object that preserves its spatial properties.

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49 S. Mac Lane, "The Reasonable Effectiveness of Mathematical Reasoning," in *Mathematics and Science*, R. Mickens (ed.), World Scientific, Singapore 1990, pp. 115-122.

50 For the concept of *dispositional representation*, we defer to A. Damasio, *Descartes' Error*, Papermac, London 1994

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This objectivistic conception of the form or shape of objects need not presuppose a naive theory of perception, according to which a perception is a mere photographic copy of the outside world that does not presuppose any active re-elaboration on the part of the human brain. Nor does it require a distinction between objective or primary qualities (shape, number) and qualities or properties that are subjective or secondary (taste, odor). In this context, it is enough to emphasize that, just as to Mac Lane the abstract property of mathematics depended upon recurring form/shapes as exemplified by the natural world, the perceptive process also involves forms existing in real objects, not as pure dispositions but as actual, occurrent properties of the objects. If this were not the case, mathematics would apply only to ghosts produced by our brains, which would give meaning to and interpret a physical stimulus which, in itself, would be completely de-structured. Therefore, the applicability of mathematics to the physical world would be even more mysterious.

To clarify our thoughts on this controversial point of view, I will take the liberty of including a citation by Barry Smith on Brentano, which surprisingly recalls the words of Mac Lane's citation above.[51]

> The mathematical concept of curve is already in my sensorial representation of an object in the shape of an upturned nose, and in this sense it is already the thing itself. [. . .] Therefore in the moment that the intellect grasps mathematical concepts, it does not recognize something that is separate from tangible matter, but [as Brentano says] "only knows something that is not separate from itself in a separate way."

If it is true that mathematics has its origins in the primitive experience of space and time, as Poincaré[52] had well understood, it is especially the concept of experiential space, and the motion of solid bodies within it, that is fundamental to all geometrical applications of advanced mathematical theories, which not by chance often presuppose the visualization of certain mathematical structures characterizing them.

In arriving now at more recent data on the way in which we visually perceive the world, space and the spatial shapes of objects are predominantly represented to us in the right hemisphere of the brain, as a result of mental images capable of preserving the fundamental spatial relations of external objects (the form/shape that Aristotle, Brentano, and Mac Lane variously refer to). In considering that elementary geometry is no more than the study of the shapes of objects, apart from weight, colors, tastes, etc., it is not at all implausible *that the quasi-perceptive format of visual and tactile mental images is the base material upon which Euclidean elementary geometry was developed.* Not by chance, this format is again invoked so that geometry can be explained to children of every generation. There

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[51] B. Smith, *Austrian Philosophy. The Legacy of Franz Brentano*. Open Court, La Salle, Ill., 1994, p. 40.

[52] See for example H. Poincaré, *La valeur de la Science*, Flammarion, Paris, 1905, p. 67.

is no reason to believe that more advanced geometry does not also retain at least some intuitive content based on visual mental images.

Today, there is sufficient scientific evidence of the fact that the act of perceiving a round object and imagining it partly involves the same cerebral mechanisms,[53] and that the mental representation of a circular object in an image preserves a few of its topological and spatial properties. For example, some studies of the brains of macaques[54] have unequivocally demonstrated that the same spatial organization of the brain during perception (the way in which the neurons are laid out) render it plausible to hypothesize that the representative *medium* (the topographical layout of neurons of the occipital cortex) influences the nature of mental representation, and therefore the mental images that we usually associate with circles, triangles, or squares.

In a way similar to geometry, which is regarded as the study of the shapes of objects, the activity of counting, and therefore the epistemic conditions of arithmetic, also presuppose the ability to identify certain configurations that are spatially identifiable by sharply delineated forms/shapes. That is, we must presuppose the ability to individuate shapes from space also in arithmetic. According to Mario Piazza,[55]

> In order to count, objects must be assigned an *outline*, or a border, just as in a few paintings by Cézanne the objects are outlined by turquoise borders: for this reason it is so difficult to count waves, amoebas, clouds, and even our thoughts.

We can therefore affirm that arithmetic does not presuppose only time as an internal intuition as Kant intended it, or, as Hermann von Helmholz believed:[56]

> The ability to preserve in our memories the succession in which conscious acts follow one another in time.

The perception of space, and in it the *separate and identifiable shapes* of objects, is what allows us to create a correspondence between these shapes and *numerals*, which are the signs extended in space-time that we use to designate *numbers*, regarded as abstract entities. From our experience of listing concrete objects by grouping them, we know that it is not necessary to put 5 and 7 sheep together to prove that 5 + 7 = 12, given that this operation is valid for any object that is discrete and separable from others; the definition of what counts as an "object" depends on the context and on our cognitive purposes.

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[53] See F. Ferretti, *Pensare vedendo*, Nis, Rome 1998.

[54] Ibid, p. 86-87.

[55] M. Piazza, cited works.

[56] H. von Helmholtz, "Zahlen und Messen", in *Philosophische Aufsaetze*, Fues's Verlag, Leipzig, 1887, pp. 17-52, transl. by C.L. Bryan, "Counting and Measuring", Van Nostrand, 1930.

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In sum, the problem of explaining the applicability of mathematics to the natural world can be dealt with most plausibly in a constructivist philosophy of mathematics, which sees the latter as a non-arbitrary human invention, rooted in our spatio-temporal experience of the external world. In particular, all the complex applications of geometry to physics – from the study of the spaces of variable curvatures to fiber bundles, from the vector spaces of linear algebra to group theory – presupposes a background of intuitive data *that originated with our experience of concrete objects*, and is therefore furnished by our perception of shapes, as codified in the perceptions of external objects by our visual and tactile apparatuses.

The mental images that form the fabric of our thoughts and that prove to be so important to the resolution of problems in mathematics and physics[57] preserve their intuitive content because in a sense, they are isomorphic to the objects from which they derive. I have claimed that the isomorphic correspondence between the magnitudes of phenomena, and the numeric structures with which we associate them is at the root of the process of quantification of the world, and the secret of success of modern science. My hypothesis is that the isomorphic correspondence in question depends on the *empirical and intuitive origins* of the two fundamental pillars of mathematics, *geometry* and *arithmetic.*

This reference to the intuitive, and not purely formal nature of mathematics[58] can also account for an important characterization of the contemporary laws of physics in terms of symmetry or invariance,[59] notions which I will later discuss in reference to a theory of natural laws of skeptic orientation. In effect, the same notion of the symmetry of an object can be described intuitively as the congruence or invariance of a *shape* following its motions in space (translations, rotations, reflections, etc.) For example, a hexagon rotating at an particular angle around its center (60 degrees and multiples thereof) remains unchanged or *invariant*, in the sense that after a rotation at that angle it is congruent to itself.

Let us now note the following properties:

i. if the combination of the two rotations is another rotation;
ii. if to each rotation it is possible to associate its inverse;
iii. if a rotation "identity" exists which leaves the object unchanged;
iv. if the operations of rotation are associative;
v. then the set of rotations form an abstract structure called a *group*.

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[57] See J. Hadamard, *An Essay on the Psychology of Invention in the Mathematical Field*, Princeton University Press, Princeton, New Jersey, 1949.

[58] On these themes, I have been influenced by two stimulating essays by G. Longo, "The Constructed Objectivity of Mathematics and the Cognitive Subject," and "The Reasonable Effectiveness of Mathematics and its Cognitive Roots," available on the internet at http://www.dmi.ens.fr/users/longo.

[59] For this interpretation, see van Fraassen B., cited works, and E. Castellani, *Simmetria e natura*, Laterza, Rome-Bari 2000.

Considering the numerous applications of the algebraic theory of groups, not only in the study of the symmetry of crystals,[60] but also in physical geometry and in particle physics, the applicative versatility of this concept can only be understood by taking into account its intuitive origins, which are related to the group of motions that a solid body (its shape) can make in space while keeping its identity unchanged.

The main question that I will tackle in the next chapter regards the ontological status of scientific laws. We will see how a proper emphasis on the fact that the fundamental relationship between a mathematical law and the phenomena it represents has to do with their relational structures can be helpful in evaluating the various philosophical positions debated in contemporary literature.

[60] J. Burkhardt, *Die Symmetrie der Kristalle*, Birkhäuser Verlag, Basel 1988.