PRENILPOTENT PAIRS IN THE E_{10} ROOT LATTICE

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Abstract. Tits has defined Kac–Moody groups for all root systems, over all commutative rings with unit. A central concept is the idea of a prenilpotent pair of (real) roots. In particular, writing down his group presentation explicitly would require knowing all the Weyl-group orbits of such pairs. We show that for the hyperbolic root system E_{10} there are so many orbits that any attempt at direct enumeration is impractical. Namely, the number of orbits of prenilpotent pairs having inner product k grows at least as fast as (constant) $\cdot k^7$ as $k \to \infty$. Our purpose is to motivate alternate approaches to Tits' groups, such as the one in [2].

Kac–Moody groups generalize reductive algebraic groups to include the infinite dimensional case. Various authors have defined them in many ways, the most comprehensive approach being due to Tits [15]. Given a generalized Cartan matrix A, he defined a functor $\tilde{\mathfrak{G}}_A$ assigning a group to each commutative ring R with unit. The main result of [15] is that any functor from commutative rings to groups, having some properties that are reasonable to expect of anything called a Kac–Moody group, must agree with $\tilde{\mathfrak{G}}_A$ over every field. (See [15, Theorems 1 and 1′].)

(Actually Tits defined a group functor $\tilde{\mathfrak{G}}_D$ for a root datum D. For $\tilde{\mathfrak{G}}_A$ we use the root datum with generalized Cartan matrix A, which is "simply connected in the strong sense" [15, p. 551]. The difference between a root datum and its generalized Cartan matrix plays no role in this paper.)

Tits defined $\tilde{\mathfrak{G}}_A(R)$ by a complicated implicitly described presentation. The key relations are his generalizations of the Chevalley relations. He begins with the free product $*_\alpha(\mathfrak{U}_\alpha)$, where α varies over all real roots and each $\mathfrak{U}_\alpha$ is isomorphic to the additive group of R. This step requires knowing all the real roots, which is nontrivial but reasonably accessible (lemma 2). He imposes relations of the form

$$[X_\alpha(t), X_\beta(u)] = \prod_\gamma X_\gamma(v_\gamma)$$

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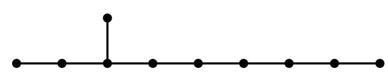
whenever $\alpha, \beta \in \Phi$ form a *prenilpotent pair* (see below). Here $\mathfrak{U}_\alpha = \{X_\alpha(t) : t \in R\}$ and similarly for the other roots, the γ's parameterizing the product are the real roots in $\mathbb{N}\alpha + \mathbb{N}\beta$ other than α and β, and the parameters v_γ depend on various choices like the ordering of the factors (and anyway are unimportant in this paper).

The definition of prenilpotency is that some element of the Weyl group W sends both α, β to positive roots, and some other element sends both to negative roots. When this holds, Prop. 1 of [15] and its proof show how to work out the Chevalley relation of α, β, at least in principle. It is similar to, but more complicated than, the working out of the structure constants of the Kac–Moody algebra. (In fact Hée [8] and Morita [12] have worked out all the possible types of the relations in closed form.) So the essence of writing down Tits' presentation is to list all the prenilpotent pairs. It would even be enough to find one representative of each W-orbit of prenilpotent pairs. Our main result, theorem 1 below, is that this is impossible in practice for the E_{10} root system, whose Dynkin diagram is

(1)



The argument suggests that the same holds for all hyperbolic root systems of rank > 3; see section 2.

This negative result is balanced by the fact that in many interesting cases, including E_{10}, most of the prenilpotent pairs can be ignored because their Chevalley relations follow from those of other prenilpotent pairs. There are two approaches to this. The first is due to Abramenko and Mühlherr [1][5][13], and applies to Kac–Moody groups associated to 2-spherical Dynkin diagrams, over fields, with some exceptions over $\mathbb{F}_2$ and $\mathbb{F}_3$. The second approach is due to the author [2][3]; see also [4]. It works over general rings, but requires some conditions on the diagram. Both approaches apply to all irreducible affine diagrams (of rank ≥ 3) and all simply laced diagrams without A_1 components, such as E_{10}. In both cases the result is that $\tilde{\mathfrak{G}}_A(R)$ is the direct limit of the family of groups $\tilde{\mathfrak{G}}_B(R)$, where B varies over the 1- and 2-node subdiagrams of A. So one may discard almost all of Tits' Chevalley-style relations, without changing the resulting group. In the author's approach, one even obtains an explicit presentation (often finite) given in terms of the Dynkin diagram, for example $\tilde{\mathfrak{G}}_{E_{10}}(R)$ and $\tilde{\mathfrak{G}}_{E_{10}}(\mathbb{Z})$ in theorem 1 and corollary 2 of [4].

Now we begin the E_{10}-specific material. We write Λ for the root lattice, i.e., the integer span of the simple roots. The generalized Cartan matrix A for E_{10} is got from (1) in the usual way: $A_{ii} = 2$ for each node i of the diagram, and $A_{ij} = -1$ or 0 according to whether distinct nodes i and j are joined or not. This matrix is symmetric, so it may be regarded as an inner product matrix on Λ. For $x \in \Lambda$, the norm of x means $x \cdot x$, usually written x^2. The Weyl group W acts on Λ by isometries. We will never refer to imaginary roots, so we follow Tits [15] in using "root" to mean "real root", i.e., "W-image of a simple root". Now we can state our main result:

Theorem 1. *Let $N(k)$ be the number of W-orbits of prenilpotent pairs of roots in the E_{10} root system with inner product k. Then for some positive constant C, we have $N(k) \geq Ck^7$ for all integers k.*

The constant is made effective in the proof, although we make no attempt to optimize it. The theorem says nothing if $k \leq 0$, but this case is uninteresting because there are no prenilpotent pairs with $k < -1$, by lemma 3 below. The proof shows that the problem of enumerating the prenilpotent pairs contains an infinite sequence of successively more difficult and less interesting classification problems in the theory of positive-definite quadratic forms. For example, the simplest such problem is the classification of positive-definite lattices of dimension 8 and determinant $k^2 - 4$, which becomes difficult and boring for quite small k. Hence our description of the direct enumeration of prenilpotent pairs as "impractical".

1. Proof

We will prove theorem 1 by converting it into a lattice-theoretic problem. First we need to describe the roots and prenilpotent pairs entirely in terms of the root lattice Λ.

Lemma 2. *The roots of the E_{10} root system are exactly the norm 2 vectors of Λ.*

Proof. The simple roots have norm 2 because the generalized Cartan matrix has 2's along its diagonal. The other roots are their W-images and therefore have norm 2 also. Now suppose a lattice vector r has norm 2; we must show it is a root. The reflection in r, namely $R : x \mapsto x - (x \cdot r)r$, preserves Λ because $x \cdot r \in \mathbb{Z}$ for all lattice vectors x. Also, Λ has signature $(9, 1)$, so the negative-norm vectors in $\Lambda \otimes \mathbb{R}$ fall into two components. Since $r^2 > 0$, R preserves each component. Vinberg [16] showed that W is the full group of lattice isometries that preserve each component, so $R \in W$. Since every reflection in W is conjugate to a simple reflection, r is W-equivalent to a simple root. So r is a root. $\square$

Lemma 3 ([4, Lemma 6])**.** *Two roots in the E_{10} root system form a prenilpotent pair if and only if their inner product is ≥ -1.* □

At this point the proof of theorem 1 becomes entirely lattice-theoretic, relying on the theory of integer quadratic forms to study certain sublattices of Λ. We fix $k \geq -1$ and consider prenilpotent pairs with inner product k. We write L for the integer span of such a prenilpotent pair; its inner product matrix is $\left(\begin{smallmatrix} 2 & k \\ k & 2 \end{smallmatrix}\right)$. We will write $\mathrm{O}(L)$ and $\mathrm{O}(\Lambda)$ for the orthogonal groups of L and Λ, and similarly for other lattices. The next lemma follows immediately from the previous two.

Lemma 4. *$N(k)$ equals the number of orbits of isometric embeddings $L \to \Lambda$, under the group $(\mathbb{Z}/2) \times W$, where $\mathbb{Z}/2$ acts on L by swapping its basis vectors and W acts on Λ in the obvious way. In particular, $N(k)$ is at least as large as the number of orbits of sublattices of Λ that are isometric to L, under the orthogonal group $\mathrm{O}(\Lambda)$.* □

We begin with an overview of a general method called gluing, used for studying the embeddings of one lattice into another. When considering any particular embedding $L \to \Lambda$, we will usually identify L with its image. In the current situation, one first studies the possibilities for the saturation $L^{\mathrm{sat}} := (L \otimes \mathbb{Q}) \cap \Lambda$. In the proofs below we will simplify this step away, by restricting to the case that L is already saturated. Then, assuming $\det L \neq 0$, one studies the possibilities for $L^{\perp}$. In this step we take advantage of the fact that Λ is unimodular: among other things, it implies that L^{sat} and $L^{\perp}$ have the same determinant. This limits $L^{\perp}$ to finitely many possibilities. For each candidate K for $L^{\perp}$, one then considers the possible ways to glue K to L^{sat} in a manner that yields Λ. Gluing means finding a copy of Λ between $K \oplus L^{\mathrm{sat}}$ and $K^* \oplus (L^{\mathrm{sat}})^*$, in which K and L^{sat} are saturated. (Asterisks indicates dual lattices.) This step boils down to analyzing the actions of $\mathrm{O}(K)$ and $\mathrm{O}(L)$ on the discriminant groups $\Delta(K)$ and $\Delta(L)$ of K and L, which are finite abelian groups defined below.

Here are the necessary definitions and background. A lattice K means a free abelian group equipped with a $\mathbb{Q}$-valued symmetric bilinear pairing. K is called integral if this pairing is $\mathbb{Z}$-valued, and K is called even if furthermore all vectors have even norm. For example, Λ is even. The determinant of the inner product matrix of K, with respect to a basis, is independent of basis, and is called the determinant $\det K$ of K. We will encounter only nondegenerate lattices, meaning those of nonzero determinant. So we assume nondegeneracy henceforth. The dual K^* of K means the set of vectors in $K \otimes \mathbb{Q}$ having integer inner product with all elements of K. When K is integral, we have $K \subseteq K^*$, and in this case we define the discriminant group

$\Delta(K)$ as K^*/K, an abelian group of order $\det K$. The $\mathbb{Z}$-valued inner product on K extends to a $\mathbb{Q}$-valued inner product on K^*, which descends to a $\mathbb{Q}/\mathbb{Z}$-valued inner product on $\Delta(K)$. Similarly, if K is even then we can regard the norm of an element of $\Delta(K)$ as a well-defined element of $\mathbb{Q}/2\mathbb{Z}$. (If K is not even then the norm is only well-defined mod $\mathbb{Z}$, but we will only encounter even lattices.) By the naturality of the constructions, $\mathrm{O}(K)$ acts on $\Delta(K)$, preserving these structures. We write $\mathrm{O}(\Delta(K))$ for the group of isometries of $\Delta(K)$, i.e., abelian group automorphisms that respect the $\mathbb{Q}/\mathbb{Z}$-valued inner product and $\mathbb{Q}/2\mathbb{Z}$-valued norm. So there is a natural map $\mathrm{O}(K) \to \mathrm{O}(\Delta(K))$.

The formulation of the theory of integer quadratic forms best suited for explicit computation is due to Conway and Sloane [9, ch. 15][11]. So we assume familiarity with their methods and we follow their conventions, including the unusual one of writing -1 for the infinite place of $\mathbb{Q}$ and defining $\mathbb{Z}_{-1}$ and $\mathbb{Q}_{-1}$ to be $\mathbb{R}$. For any place p of $\mathbb{Q}$ we write K_p for the p-adic lattice $K \otimes \mathbb{Z}_p$. The Sylow p-subgroup of $\Delta(K)$, with its norm form, is the same as the discriminant group of K_p. Two lattices K, K' are said to lie in the same genus if $K_p \cong K'_p$ for all places p. In the positive definite case, the *mass* of a genus means $\sum_K 1/|\mathrm{O}(K)|$, where K varies over the isometry classes in that genus. This definition makes sense because a genus contains only finitely many isometry classes. The mass is important for us because it is a lower bound for the number of isometry classes. We will compute it by using the Smith–Minkowski–Siegel mass formula, which avoids having to first enumerate the isometry classes in the genus.

Conway and Sloane gave an elaborate notational system for isometry classes of p-adic lattices, for example the symbols appearing in lemma 5(ii) below. For p a prime, a symbol $(p^e)^{\pm n}$ indicates an n-dimensional p-adic lattice that is got from some unimodular p-adic lattice U by multiplying the inner product by p^e. When $p = 2$, the symbol also has a subscript, discussed below. A chain of symbols $(p^e)^{\pm n}$ represents a direct sum decomposition of a p-adic lattice into Jordan constituents.

For any constituent, the sign $\pm$, together with the subscript when $p = 2$, describes the isometry class of U. The sign is defined as the Jacobi symbol $(\frac{\det U}{p}) = \pm$, which we recall is the Legendre symbol when p is odd. When $p = 2$ it is $+$ or $-$ according to whether $\det U \equiv \pm 1$ or ± 3 mod 8. In both cases, the sign is often suppressed when it is $+$. When $p = 2$ and U is even, the subscript is the formal symbol II and the Jordan constituent is said to have type II. If $p = 2$ and U is not even, then the subscript is an element of $\mathbb{Z}/8$, namely the trace (mod 8) of its inner product matrix, after diagonalization over $\mathbb{Z}_2$. (One shows that a diagonalization does exist, and that the trace is independent of the choice of diagonalization.) In this case the Jordan constituent is said to have type I.

Now we turn to the specific problem of enumerating the embeddings $L \to \Lambda$, where we recall that L has inner product matrix $\left(\begin{smallmatrix} 2 & k \\ k & 2 \end{smallmatrix}\right)$. In fact we will bound from below the number of saturated embeddings, meaning those whose images are saturated sublattices of Λ. We take $k \geq 3$ to make L indefinite, avoiding the special cases $k = -1, 0, 1, 2$. We factor $d := -\det L = k^2 - 4 > 0$ as $2^{e_2} 3^{e_3} 5^{e_5} \cdots$ and write f_p for the non-p-part d/p^{e_p} of d.

Lemma 5. *With L, d, e_p and f_p as above,*

(*i*) *e_2 is 0, 2 or ≥ 5.*

(*ii*) *There exists a genus of 8-dimensional positive-definite lattices K of determinant d, such that*

$$K_2 \cong \begin{cases} 1_{\mathrm{II}}^{-8} & \text{if } e_2 = 0 \\ 1_{\mathrm{II}}^{6}\, 2_{f_2-1}^{(\frac{f_2}{2})2} & \text{if } e_2 = 2 \\ 1_{\mathrm{II}}^{6}\, 2_{-1}^{1} \big(2^{e_2-1}\big)_{f_2}^{(\frac{f_2}{2})1} & \text{if } e_2 \geq 5 \end{cases}$$

$$K_p \cong \begin{cases} 1^{(\frac{f_p}{p})8} & \text{for } p > 2 \text{ when } e_p = 0 \\ 1^{(\frac{2}{p})7} \left(p^{e_p}\right)^{(\frac{2f_p}{p})1} & \text{for } p > 2 \text{ when } e_p > 0 \end{cases}$$

(*iii*) *Suppose K lies in this genus. Then there are at least*

$$\frac{2^{\text{number of odd primes dividing } d}}{4\,|\mathrm{O}(K)|}$$

$\mathrm{O}(\Lambda)$-orbits on the set of saturated sublattices of Λ that are isometric to L and have orthogonal complement isometric to K.

Proof. (*i*) If k is odd then so is $d = k^2 - 4$, so $e_2 = 0$. If k is divisible by 4 then d is divisible by 4 but not 8, so $e_2 = 2$. If k is twice an odd number then $d = 4(\mathrm{odd}^2 - 1)$ and the second factor is divisible by 8.

As preparation for (*ii*) and (*iii*), we give the p-adic invariants of L. Its determinant and signature are $-d$ and 0, and

$$L_2 \cong \begin{cases} 1_{\mathrm{II}}^{-2} & \text{if } e_2 = 0 \\ 2_{1-f_2}^{(\frac{-f_2}{2})2} & \text{if } e_2 = 2 \\ 2_1^1 \big(2^{e_2-1}\big)_{-f_2}^{(\frac{-f_2}{2})1} & \text{if } e_2 \geq 5 \end{cases}$$

$$L_p \cong \begin{cases} 1^{(\frac{-f_p}{p})2} & \text{if } p>2 \text{ and } e_p=0 \\ 1^{(\frac{2}{p})1}\left(p^{e_p}\right)^{(\frac{-2f_p}{p})1} & \text{if } p>2 \text{ and } e_p>0 \end{cases}$$

These can be worked out explicitly using the methods of §4.4 and §7 of [9, ch. 15]. (It helps to observe that if k is even or p is odd then $L_p \cong \langle 2\rangle \oplus \langle -d/2\rangle$.) Defining L^{neg} as L with all inner products negated, its local forms $L^{\mathrm{neg}}_{p\neq -1}$ are as follows. If $e_p=0$ then L^{neg}_p is isometric to L_p. If $e_p>0$ then the Conway–Sloane symbol of L^{neg}_p is got from that of L_p by multiplying each superscript by $(\frac{-1}{p})$ (if $p>2$), or negating subscripts (if $p=2$). When $p=2$ and a constituent has type II, the negation of its subscript II is taken to mean II again.

By Theorem 11 in §7.7 of [9, ch. 15], there exists a $\mathbb{Z}$-lattice K of determinant d, having specified local forms $K_{p=-1,2,3,\ldots}$, if and only if both the following hold. First, $\det K_p \in d\cdot(\mathbb{Q}_p^\times)^2$ for all places p. Second, the oddity formula holds:

$$\mathrm{signature}(K_{-1}) + \sum_{p\geq 3} p\text{-excess}(K_p) \equiv \mathrm{oddity}(K_2) \pmod 8.$$

Here the oddity (resp. p-excess) is a $\mathbb{Z}/8$-valued invariant of quadratic forms over $\mathbb{Q}_2$ (resp. $\mathbb{Q}_{p\,\mathrm{odd}}$), defined in §5.1 of [9, Ch. 15]. It is the sum of the oddities (resp. p-excesses) of the Jordan constituents, which can be read off from the Conway-Sloane notation as follows. The oddity of a 2-adic Jordan constituent of type II is always 0. For a type I constituent $(2^e)^{\pm n}_t$, the oddity is the subscript t, plus 4 if the sign is $-$ and e is odd. For odd p, the p-excess of $(p^e)^{\pm n}$ is $n(p^e-1)$, plus 4 if the sign is $-$ and e is odd.

Both the determinant condition and the oddity formula for the family of K_p's in (ii) can be verified as follows. First, K_{-1} and L_{-1} have signature 8 and determinants of opposite signs. Second, although we didn't say so, we constructed K_2 as $1^6_{\mathrm{II}}\oplus L^{\mathrm{neg}}_2$ and K_p as $1^{(\frac{-1}{p})6}\oplus L^{\mathrm{neg}}_p$ for $p>2$. Now, the $\mathbb{Z}_2$-lattice 1^6_{II} is isometric to the sum of three copies of $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, so it has determinant -1. And for odd p, the $\mathbb{Z}_p$-lattice $1^{(\frac{-1}{p})6}$ is isometric to $\langle 1,1,1,1,1,-1\rangle$, so it also has determinant -1. It follows that $\det K_p = -\det L^{\mathrm{neg}}_p = d$ for all p, verifying the determinant condition. For the oddity formula, note that the 2-adic lattice 1^6_{II} has oddity 0, and for $p>2$ the p-adic lattice $1^{(\frac{-1}{p})6}$ has p-excess equal to 0. Since L^{neg} exists, its local forms L^{neg}_p satisfy the oddity formula. Since the corresponding formula for the K_p has exactly the same terms, it also holds. So there exists a lattice K having those local forms.

(*iii*) In the language of §3 of [9, ch. 4], this is the question of how one may glue L to K to obtain Λ. Here are the details. Suppose that in addition to K, we are given a totally isotropic subgroup G of $\Delta(L \oplus K) = \Delta(L) \oplus \Delta(K)$ that projects isomorphically onto $\Delta(L)$ and onto $\Delta(K)$. Totally isotropic means that the natural $\mathbb{Q}/\mathbb{Z}$-valued inner product and $\mathbb{Q}/2\mathbb{Z}$-valued norm on $\Delta(L \oplus K)$ vanish identically on G. From K and G we will construct a saturated embedding $L \to \Lambda$ with $L^\perp \cong K$.

Before constructing the embedding, we explain why such a subgroup G exists. Recall from the proof of (*ii*) that each K_p was constructed as the sum of L_p^{neg} and a unimodular $\mathbb{Z}_p$-lattice. It follows that $\Delta(K_p)$ and $\Delta(L_p^{\text{neg}})$ are isomorphic as finite abelian groups equipped with $\mathbb{Q}_p/\mathbb{Z}_p$-valued inner products and $\mathbb{Q}_p/2\mathbb{Z}_p$-valued norms. These discriminant groups are the Sylow p-subgroups of $\Delta(K)$ and $\Delta(L^{\text{neg}})$, so it follows that $\Delta(K)$ and $\Delta(L^{\text{neg}})$ are isomorphic in the corresponding sense: there exists a group isomorphism $\Delta(L) \to \Delta(K)$ that negates norms and inner products. We may take G to be its graph.

Here is the construction of the embedding $L \to \Lambda$. Write $L \oplus_G K$ for the preimage of $G \subseteq \Delta(L) \oplus \Delta(K)$ in $L^* \oplus K^*$. This construction is called "gluing L to K by G". The resulting lattice is integral and even (since G is totally isotropic) and unimodular (since its index d sublattice $L \oplus K$ has determinant $-d^2$). By Theorem 5 of [14, §V.2], up to isometry Λ is the only even unimodular lattice of signature $(9, 1)$. So $L \oplus_G K \cong \Lambda$ and we have constructed a copy of Λ containing $L \oplus K$. Since $G \subseteq \Delta(L) \oplus \Delta(K)$ meets $\Delta(L)$ and $\Delta(K)$ trivially, both L and K are saturated in this copy of Λ. This completes the construction of a saturated embedding $L \to \Lambda$ whose orthogonal complement is isometric to K.

In fact we get such an embedding for every totally isotropic subgroup G of $\Delta(L) \oplus \Delta(K)$ that projects isomorphically to each summand. We can usually obtain many such subgroups from the one constructed above, by taking its images under transformations $f \oplus \mathrm{id}_{\Delta(K)}$ where f is an isometry of $\Delta(L)$. Distinct f's give distinct subgroups, because $G \to \Delta(K)$ is an isomorphism. The number of self-isometries of $\Delta(L)$ is at least 2^o, where o is the number of odd primes p dividing d. To see this, note that for each such p, the self-map of $\Delta(L)$ which negates the Sylow p-subgroup, and acts by the identity on all other Sylow subgroups, is an isometry. So $L^* \oplus K^*$ contains at least 2^o copies of Λ in which L and K are saturated.

It is possible for two of these subgroups G, G' to yield equivalent embeddings $L \to \Lambda$. "Equivalent" has the obvious meaning: that there is an isometry from $L \oplus_G K \cong \Lambda$ to $L \oplus_{G'} K \cong \Lambda$ that sends L to L. It is easy to see that this happens if and only if there are isometries of L and K, such that the induced isometry of $\Delta(L) \oplus \Delta(K)$ sends G to G'. To prove (*iii*) it therefore suffices to show that the 2^o many subgroups of $\Delta(L) \oplus \Delta(K)$ constructed in the previous paragraph represent at least $2^o/4|\mathrm{O}(K)|$ many orbits under the action of $\mathrm{O}(L) \times \mathrm{O}(K)$ on $\Delta(L) \oplus \Delta(K)$. And to prove this it suffices to prove that the action of $\mathrm{O}(L)$ on $\Delta(L)$ factors through a group of order ≤ 4.

For this we write down generators for $\mathrm{O}(L)$. Consider its action on the set of norm -1 vectors in $L \otimes \mathbb{R}$. There are two components, exchanged by negation, each a copy of 1-dimensional hyperbolic space (a copy of the real line). The subgroup of $\mathrm{O}(L)$ generated by the reflections in norm 2 roots is normal, and acts on each component as an infinite dihedral group D_∞. Since $\mathrm{O}(L)$ normalizes D_∞, it is generated by D_∞, the $\mathrm{O}(L)$-stabilizer of a Weyl chamber, and negation. Since the Weyl chamber is an interval, its $\mathrm{O}(L)$-stabilizer has order ≤ 2, so $\mathrm{O}(L)$ contains D_∞ of index ≤ 4. To finish the proof we check that D_∞ acts trivially on $\Delta(L)$. Suppose $x \in L^*$ and that $r \in L$ has norm 2. Then r's reflection sends x to $x - (x \cdot r)r$, which differs from x by an element of L, hence represents the same element of $\Delta(L)$ as x. □

Lemma 6. *The genus in lemma 5 has mass equal to*

$$\frac{d^{7/2}\,\zeta_d(4)}{30240\pi^4 \cdot 2^{\text{number of odd primes dividing } d}} \cdot \begin{cases} \frac{1}{272} & \text{if } e_2 = 0 \\ \frac{1}{512} & \text{if } e_2 = 2 \\ \frac{1}{1024} & \text{if } e_2 \geq 5 \end{cases}$$

where ζ_d *is defined as in* [11, §7], *by*

$$\zeta_d(s) = \prod_{\text{primes } p \nmid 2d} \left\{1 - \left(\frac{d}{p}\right)\frac{1}{p^s}\right\}^{-1} = \sum_{\substack{m \geq 1 \\ (m,2d)=1}} \left(\frac{d}{m}\right) m^{-s}.$$

Proof. This is a lengthy exercise using the intricate but explicit procedure in [11]. We will use the specialized vocabulary developed there, giving enough details that the reader possessing a copy of [11], but unfamiliar with it, can follow along. Following [11, §7], the mass is

$$m(K) = \mathrm{std}(K) \prod_{\text{primes } p | 2d} \frac{m_p(K)}{\mathrm{std}_p(K)} \tag{2}$$

where $m_p(K)$ is the "p-mass", and $\mathrm{std}(K)$ resp. $\mathrm{std}_p(K)$ is the "standard mass" resp. "standard p-mass" of an 8-dimensional lattice of determinant d. Table 3 of [11] gives $\mathrm{std}(K)$ as $\zeta_d(4)/30240\pi^4$, and §7 of

[11] gives
$$\mathrm{std}_p(K) := \frac{1}{2(1-p^{-2})(1-p^{-4})(1-p^{-6})}.$$
For computing $m_p(K)$ we refer to §4–§5 of [11]. It is defined as
$$m_p(K) = (\text{diagonal product}) \cdot (\text{cross product}) \cdot (\text{type factor}),$$
the last term appearing only if $p = 2$. The diagonal product is the product of the "diagonal factors" $M_p(f)$, where f varies over the p-adic Jordan constituents. $M_p(f)$ is defined in [11, table 2] in terms of p and the "species" of f, which is one of the formal symbols
$$0+ \quad \text{or} \quad 1, 3, 5, 7, \ldots \quad \text{or} \quad 2+, 2-, 4+, 4-, 6+, 6-, \ldots$$
The species can be read from the Conway-Sloane symbol for f, except when $p = 2$ when it is also a function of whether f is "bound" or "free", which depends on f's neighboring Jordan constituents. See table 1 in [11]. (When $p = 2$, one sometimes also refers to the "octane value" of f when determining the species. This is an element of $\mathbb{Z}/8$ which can be read from the Conway-Sloane symbol for f. We will explain how, when we need to.) Even the 0-dimensional Jordan constituents contribute to the diagonal product, but their diagonal factors are 1 except when they are bound and $p = 2$. These exceptional constituents are called "bound love forms", which have diagonal factor $\frac{1}{2}$.

The cross product is the product of the "cross terms", one for each pair of distinct p-adic Jordan constituents. Given where $e < e'$, the cross-term for $(p^e)^{\pm n}$ and $(p^{e'})^{\pm n'}$ is $p^{nn'(e'-e)/2}$. The type factor appears only when $p = 2$, when it is $2^{n(\mathrm{I,I})-n(\mathrm{II})}$. Here $n(\mathrm{I,I})$ means the number of pairs of adjacent type I constituents, and $n(\mathrm{II})$ means the sum of the ranks of the type II constituents.

Now we begin the computations proper. We treat the case of odd $p|d$ first. The Jordan constituents $1^{(\frac{2}{p})7}$ and $\left(p^{e_p}\right)^{(\frac{2f_p}{p})1}$ have species 7 and 1 respectively. So their diagonal factors are $M_p(7) = \mathrm{std}_p(K)$ and $M_p(1) = \frac{1}{2}$. Since there are only two Jordan constituents, there is a single cross-term, namely $p^{7\cdot 1\cdot(e_p-0)/2} = p^{7e_p/2}$. By definition, $m_p(K)$ is the product of the diagonal factors and this cross-term. This gives $m_p(K)/\mathrm{std}_p(K) = \frac{1}{2}p^{7e_p/2}$.

The calculation of $m_2(K)$ is similar but more intricate, and we must treat all three possibilities for K_2 listed in lemma 5(ii). First suppose $e_2 = 0$, so $K_2 \cong 1_{\mathrm{II}}^{-8}$. The single Jordan constituent is free with type II, dimension 8 and sign $-$. The octane value of a type II constituent is 0 or 4 according to whether the sign is $+$ or $-$. Therefore the octane value of 1_{II}^{-8} is 4, so table 1 of [11] gives its species as $8-$, and then formula (5) of [11] gives its diagonal factor as

$$M_2(8-) = \frac{1}{2(1-2^{-2})(1-2^{-4})(1-2^{-6})(1+2^{-4})} = \frac{16}{17}\,\mathrm{std}_2(K).$$

There are no type I constituents (hence no bound love forms) and no cross-terms. Since type II constituents account for 8 dimensions, the type factor is 2^{-8}. So $m_2(K)/\,\mathrm{std}_2(K) = \frac{16}{17}2^{-8} = \frac{1}{272}2^{7e_2/2}$.

Now suppose $e_2 = 2$, so $K_2 \cong 1_{\mathrm{II}}^6\, 2_{f_2-1}^{(\frac{f_2}{2})2}$. The first constituent is bound and 6-dimensional of type II. So it has species 7, hence diagonal factor $M_2(7) = \mathrm{std}_2(K)$. Before analyzing the second constituent we remark that $f_2 \equiv 3 \bmod 4$. To see this, recall from the proof of lemma 5(i) that $e_2 = 2$ exactly when $k = 2l$ with l even. From $d = 4(l^2-1)$ we get $f_2 = (l+1)(l-1)$, and observe that one factor on the right is 1 mod 4 while the other is 3 mod 4. Now, the octane value of a type I constituent is its subscript, plus 4 if the constituent's sign is $-$. We have just shown that $f_2 \equiv 3$ or 7 mod 8. In either case, $2_{f_2-1}^{(\frac{f_2}{2})2}$ has octane value 6, hence species 1, hence diagonal factor $M_2(1) = \frac{1}{2}$. There is one bound love form, namely 4_{II}^{+0}, and its diagonal factor is $\frac{1}{2}$. There is one cross-term, contributing a factor $2^{6\cdot 2/2}$. There are no pairs of adjacent type I constituents, and 6 dimensions total of type II constituents, so the type factor is 2^{-6}. Therefore

$$m_2(K)/\,\mathrm{std}_2(K) = \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot 2^6\cdot 2^{-6} = \tfrac{1}{512}2^{7e_2/2}.$$

Finally, suppose $e_2 \geq 5$, so $K_2 \cong 1_{\mathrm{II}}^6\, 2_{-1}^1 \big(2^{e_2-1}\big)_{f_2}^{(\frac{f_2}{2})1}$. The constituent 1_{II}^6 is bound, hence has species 7, hence diagonal factor $M_2(7) = \mathrm{std}_2(K)$. The constituent 2_{-1}^1 is free with octane value -1, hence species $0+$, hence diagonal factor $M_2(0+) = 1$. The last constituent $\big(2^{e_2-1}\big)_{f_2}^{(\frac{f_2}{2})1}$ is free. Considering the four possibilities for f_2 mod 8 shows that the octane value is always ± 1, so this constituent also has species $0+$ and diagonal factor 1. There are three bound love forms, namely 4_{II}^{+0}, $(2^{e_2-2})_{\mathrm{II}}^{+0}$ and $(2^{e_2})_{\mathrm{II}}^{+0}$, with diagonal factors $\frac{1}{2}$. So the diagonal product is $\mathrm{std}_2(K)\cdot 1\cdot 1\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$. There is a cross-term for each pair of constituents, and the cross-product is their product, namely

$$2^{6\cdot 1/2}\cdot\big(2^{e_2-1}\big)^{6\cdot 1/2}\cdot\big(2^{e_2-2}\big)^{1\cdot 1/2} = \tfrac{1}{2}2^{7e_2/2}$$

Finally, there are no pairs of adjacent type I constituents, and 6 dimensions total of type II constituents, so the type factor is 2^{-6}. Multiplying the diagonal product, cross product and type factor together yields

$$m_2(K)/\,\mathrm{std}_2(K) = 2^{-3}\cdot\tfrac{1}{2}2^{7e_2/2}\cdot 2^{-6} = \tfrac{1}{1024}2^{7e_2/2}$$

We have now computed all the ingredients in (2), and assembling them yields the lemma. □

Proof of theorem 1. By lemma 4, $N(k)$ is at least as large as the number of $\mathrm{O}(\Lambda)$-orbits on saturated sublattices of Λ that are isometric to L. By lemma 5(*iii*), the latter quantity is at least

$$\sum_K \frac{2^{\text{number of odd primes dividing } d}}{4\,|\mathrm{O}(K)|}$$

where K varies over the genus studied there. The number of terms in the sum is the size of the genus. This is at least twice the mass given in lemma 6, because each lattice has at least two isometries. So the number of terms is at least

$$\frac{2\,d^{7/2}\,\zeta_d(4)}{1024\cdot 30240\pi^4\cdot 2^{\text{number of odd primes dividing } d}}$$

Also, replacing $\mathrm{O}(K)$ in every term by $W(E_8)$ does not increase the sum, because the largest possible order for a finite subgroup of $\mathrm{GL}_8(\mathbb{Z})$ is $|W(E_8)|$ (see [10] and its references). Therefore

$$N(k) \geq \frac{2d^{7/2}\,\zeta_d(4)}{1024\cdot 30240\pi^4\cdot 4|W(E_8)|}$$

Next we note

$$\zeta_d(4) \geq 1 - \frac{1}{2^4} - \frac{1}{3^4} - \cdots = 2 - \sum_{n=1}^{\infty} n^{-4} = 2 - \pi^4/90$$

We have shown that

$$N(k) \geq \frac{2(k^2-4)^{7/2}\bigl(2-\pi^4/90\bigr)}{1024\cdot 30240\pi^4\cdot 4|W(E_8)|} > 2.1\times 10^{-19}(k^2-4)^{7/2}$$

whenever $k \geq 3$. This almost proves our claim that the function $N(k)$ is bounded below by Ck^7 for some constant $C > 0$. What remains is to check that $N(1)$ and $N(2)$ are positive. The E_{10} root system contains an A_2 (resp. E_9) root system, so it contains a pair of roots with inner product 1 (resp. 2). Therefore $N(1)$ and $N(2)$ are positive, as desired. □

2. Other hyperbolic root lattices

The details of the previous section were E_{10}-specific, but the same philosophy looks likely to apply to the other symmetrizable hyperbolic root systems. This suggests the same enumeration-is-impracticable conclusion in rank > 3. We have not worked out the details, because for us the E_{10} result is enough to motivate the improvements to Tits' presentation that we mentioned in the introduction. But it seems valuable to give an outline of how the calculations would go.

By a hyperbolic root system we mean one arising from an irreducible Dynkin diagram that is neither affine nor spherical, but whose irreducible proper subdiagrams are. There are 238 such Dynkin diagrams, of which 142 are symmetrizable; see [6]. Symmetrizability is equivalent to the root lattice Λ possessing an inner product that is invariant under the Weyl group W. This is obviously a prerequisite to applying lattice-theoretic methods. Hyperbolicity implies that Λ has Lorentzian signature and that W has finite index in $\mathrm{O}(\Lambda)$.

The roots are the W-images of the simple roots, so there are only finitely many root norms. For each pair of such norms N, N', we can study prenilpotent pairs of roots r, r' with norms N, N'. The analogue of lemma 3 is that r, r' form a prenilpotent pair if and only if $k := r \cdot r'$ is larger than $-\sqrt{NN'}$. By taking $k > \sqrt{NN'}$ we may suppose the span L of r, r' is indefinite. We are interested in the number $N(k)$ of W-orbits of such prenilpotent pairs.

Next one studies the embeddings of L into Λ as in lemma 5, which of course depend on $d := -\det L \approx k^2$. One can follow the E_{10} argument to bound below the number of $\mathrm{O}(L)$-orbits of saturated copies of L in Λ. First one would have to work out which genera could occur as $L^\perp$. If there are any, then we fix one and and restrict attention to saturated copies of L for which $L^\perp$ lies in that genus. Then one would work out the mass of that genus. The essential part of the mass calculations in lemma 6 are the cross-terms, because they provide the $d^{7/2}$ term that yields theorem 1. The corresponding term for Λ would be $d^{(\dim\Lambda-3)/2}$. This suggests that the number of $\mathrm{O}(L)$-orbits of prenilpotent pairs (with N, N' fixed as above) grows at least as fast as a multiple of $k^{\dim\Lambda-3}$.

An obstruction to turning this into a proof is that there may be some embeddings of L into Λ that send the basis vectors to non-roots. We expect that the finiteness of $[\mathrm{O}(\Lambda) : W]$ means that this difficulty can be more or less ignored. The point is that each $\mathrm{O}(\Lambda)$-orbit of embeddings $L \to \Lambda$ splits into at most $[\mathrm{O}(\Lambda) : W]$ many W-orbits. So we expect that there is a positive constant C, such that for each k, $N(k)$ is either 0 or at least $Ck^{\dim\Lambda-3}$.

This suggests that if $\dim\Lambda > 3$ then tabulating the prenilpotent pairs is not feasible. But the $\dim\Lambda = 3$ case is borderline and may be amenable to direct attack. Indeed, Carbone and Murray [7] have studied one particular case with $\dim\Lambda = 3$. From our perspective, what is special about the $\dim \Lambda = 3$ case is that $L^\perp$ is 1-dimensional, and every 1-dimensional genus has a unique member and mass 1/2. So the main contribution to the analogue of theorem 1's $N(k)$ will be some analogue of the term

$$2^{\text{number of odd primes dividing } d} \tag{3}$$

from lemma 5(iii). As a function of k (recall $d = k^2 - 4$), this behaves irregularly. For example, if there are infinitely many primes at distance 4 from each other, then (3) takes the value 4 infinitely often, even though it also takes arbitrarily large values.

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