

Consumer Theory

Jonathan Levin and Paul Milgrom

October 2004

1 The Consumer Problem

Consumer theory is concerned with how a rational consumer would make consumption decisions. What makes this problem worthy of separate study, apart from the general problem of choice theory, is its particular structure that allows us to derive economically meaningful results. The structure arises because the consumer's choice sets are assumed to be defined by certain prices and the consumer's income or wealth. With this in mind, we define the *consumer problem* (CP) as:

$$\begin{aligned} \max_{x \in \mathbb{R}_+^n} u(x) \\ \text{s.t. } p \cdot x \leq w \end{aligned}$$

The idea is that the consumer chooses a vector of goods $x = (x_1, \dots, x_n)$ to maximize her utility subject to a *budget constraint* that says she cannot spend more than her total wealth.

What exactly is a “good”? The answer lies in the eye of the modeler. Depending on the problem to be analyzed, goods might be very specific, like tickets to different world series games, or very aggregated like food and shelter, or consumption and leisure. The components of x might refer to quantities of different goods, as if all consumption takes place at a moment in time, or they might refer to average rates of consumption of each good over time. If we want to emphasize the roles of quality, time and place, the description of a good could be something like “Number

2 grade Red Winter Wheat in Chicago.” Of course, the way we specify goods can affect the kinds of assumptions that make sense in a model. Some assumptions implicit in this formulation will be discussed below.

Given prices p and wealth w , we can write the agent’s choice set (which was X in the general choice model) as the *budget set*:

$$B(p, w) = \{x \in \mathbb{R}_+^n : p \cdot x \leq w\}$$

The consumer’s problem is to choose the element $x \in B(p, w)$ that is most preferred or, equivalently, that has the greatest utility. If we restrict ourselves to just two goods, the budget set has a nice graphical representation, as is shown in Figure 1.

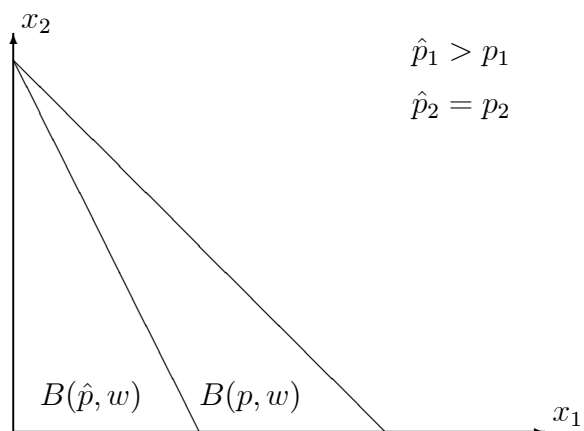


Figure 1: The budget set at different prices.

Let’s make a few observations about the model:

1. The assumption of *perfect information* is built deeply into the formulation of this choice problem, just as it is in the underlying choice theory. Some alternative models treat the consumer as rational but uncertain about the products, for example how a particular food will taste or a how well a cleaning product will perform. Some goods may be *experience goods* which the consumer can best learn about by trying (“experiencing”) the good. In that case, the consumer might want to buy some now and decide later whether to buy more. That situation would need a different formulation. Similarly,

if the agent thinks that high price goods are more likely to perform in a satisfactory way, that, too, would suggest quite a different formulation.

2. Agents are *price-takers*. The agent takes prices p as known, fixed and exogenous. This assumption excludes things like searching for better prices or bargaining for a discount.
3. Prices are *linear*. Every unit of a particular good k comes at the same price p_k . So, for instance, there are no quantity discounts (though these could be accommodated with relatively minor changes in the formulation).
4. Goods are *divisible*. Formally, this is expressed by the condition $x \in \mathbb{R}_+^n$, which means that the agent may purchase good k in any amount she can afford (e.g. 7.5 units or π units). *Notice that this divisibility assumption, by itself, does not prevent us from applying the model to situations with discrete, indivisible goods.* For example, if the commodity space includes automobile of which consumers may buy only an integer number, we can accommodate that by specifying that the consumer's utility depends only on the integer part of the number of automobiles purchased. In these notes, with the exception of the theorems that assume convex preferences, all of the results remain true even when some of the goods may be indivisible.

2 Marshallian Demand

In this section and the next, we derive some key properties of the consumer problem.

Proposition 1 (Budget Sets) *For all $\lambda > 0$, $B(\lambda p, \lambda w) = B(p, w)$. Moreover, if $p \gg 0$, then $B(p, w)$ is compact.*

Proof. For $\lambda > 0$, $B(\lambda p, \lambda w) = \{x \in \mathbb{R}_+^n \mid \lambda p \cdot x \leq \lambda w\} = \{x \in \mathbb{R}_+^n \mid p \cdot x \leq w\} = B(p, w)$. Also, if $p \gg 0$, then $B(p, w)$ is a closed and bounded subset of \mathbb{R}_+^n . Hence, it is compact. *Q.E.D.*

Proposition 2 (Existence) *If u is continuous, and $p \gg 0$, then (CP) has a solution.*

Proof. ...because a continuous function on a compact set achieves its maximum.
Q.E.D.

We call the solution to the consumer problem, $x(p, w)$, the *Marshallian (or Walrasian or uncompensated) demand*. In general, $x(p, w)$ is a set, rather than a single point. Thus $x : \mathbb{R}_+^n \times \mathbb{R}_+ \rightrightarrows \mathbb{R}_+^n$ is a *correspondence*. It maps prices $p \in \mathbb{R}_+^n$ and wealth $w \in \mathbb{R}_+$ into a *set* of possible consumption bundles. One needs more assumptions (we're getting there) to ensure that $x(p, w)$ is single-valued, so that $x(\cdot, \cdot)$ is a function.

Proposition 3 (Homogeneity) *Marshallian demand is homogeneous of degree zero: for all p, w and $\lambda > 0$, $x(\lambda p, \lambda w) = x(p, w)$.*

Proof. This one's easy. Since $B(\lambda p, \lambda w) = B(p, w)$, $x(\lambda p, \lambda w)$ and $x(p, w)$ are solutions to the *same problem!* *Q.E.D.*

The upshot of this result is that if prices go up by a factor λ , but so does wealth, the purchasing pattern of an economic agent will not change. Similarly, it does not matter whether prices and incomes are expressed in dollars, rupees, euros or yuan: demand is still the same.

Proposition 4 (Walras' Law) *If preferences are locally non-satiated, then for any (p, w) and $x \in x(p, w)$, $p \cdot x = w$.*

Proof. By contradiction. Suppose $x \in x(p, w)$ with $p \cdot x < w$. Then there is some $\varepsilon > 0$ such that for all y with $\|x - y\| < \varepsilon$, $p \cdot y < w$. But then by local non-satiation, there must be some bundle y for which $p \cdot y < w$ and $y \succ x$. Hence $x \notin x(p, w)$ — a contradiction. *Q.E.D.*

Walras' Law says that a consumer with locally non-satiated preferences will consume her entire budget. In particular, this allows us to re-express the consumer problem as:

$$\begin{aligned} & \max_{x \in \mathbb{R}_+^n} u(x) \\ & \text{s.t. } p \cdot x = w \end{aligned}$$

where the budget inequality is replaced with an equality.

The next result speaks to our earlier observation that there might be many solutions to the consumer problem.

Proposition 5 (Convexity/Uniqueness) *If preferences are convex, then $x(p, w)$ is convex-valued. If preferences are strictly convex, then the consumer optimum is always unique, that is, $x(p, w)$ is a singleton.*

Proof. Suppose preferences are convex and $x, x' \in x(p, w)$. For any $t \in [0, 1]$, $tx + (1 - t)x' \in B(p, w)$ because $p \cdot (tx + (1 - t)x') = tp \cdot x + (1 - t)p \cdot x' \leq tw + (1 - t)w = w$. Then, since $x \succeq x'$ and preferences are convex, we also have $tx + (1 - t)x' \succeq x'$. Hence, $tx + (1 - t)x' \in x(p, w)$. If preferences are strictly convex, the same construction leads to a contradiction. Suppose $x, x' \in x(p, w)$ with $x \neq x'$. Then strict convexity means that for any $t \in (0, 1)$, $tx + (1 - t)x' \succ x'$. Hence, $x' \notin x(p, w)$. *Q.E.D.*

Thus, assuming the consumer's utility is continuous and locally non-satiated, we have established four properties of the Marshallian demand function: it "exists", is insensitive to proportional increases in price and income, exhausts the consumer's budget, and is single-valued if preferences are strictly convex. The next result uses these properties to derive restrictions on the derivatives of the demand function.

Proposition 6 (Restrictions on the Derivatives of Demand) *Suppose preferences are locally non-satiated, and Marshallian demand is a differentiable function of prices and wealth. Then*

1. *A proportional change in all prices and income doesn't affect demand. For all p, w and $i = 1, \dots, n$,*

$$\sum_{j=1}^n p_j \frac{\partial}{\partial p_j} x_i(p, w) + w \frac{\partial}{\partial w} x_i(p, w) = 0$$

2. A change in the price of one good won't affect total expenditure. For all p, w and $i = 1, \dots, n$,

$$\sum_{j=1}^n p_j \frac{\partial}{\partial p_i} x_j(p, w) + x_i(p, w) = 0.$$

3. A change in income will lead to an identical change in total expenditure. For all p, w ,

$$\sum_{i=1}^n p_i \frac{\partial}{\partial w} x_i(p, w) = 1.$$

Proof. (1) This follows directly from homogeneity. For all i , $x_i(\lambda p, \lambda w) = x_i(p, w)$ by homogeneity. Now differentiate both sides by λ and evaluate at $\lambda = 1$ to obtain the result. (2) This follows from Walras' Law. For non-satiated preferences, $p \cdot x(p, w) = w$ holds for all p and w . Differentiating both sides by p_i gives the result. (3) This also follows from Walras' Law. For non-satiated preferences, $p \cdot x(p, w) = w$ for all p, w . Differentiating both sides by w gives the result. *Q.E.D.*

3 Indirect Utility

The *indirect utility function* $v(p, w)$ is defined as:

$$v(p, w) = \max u(x) \text{ subject to } p \cdot x \leq w.$$

So $v(p, w)$ is the *value* of the consumer problem, or the most utility an agent can get at prices p with wealth w .

Proposition 7 (Properties of v) Suppose u is a continuous utility function representing a locally non-satiated preference relation \succeq on \mathbb{R}_+^n . Then $v(p, w)$ is

1. homogenous of degree zero: for all p, w and $\lambda > 0$, $v(\lambda p, \lambda w) = v(p, w)$;
2. continuous on $\{(p, w) | p \gg 0, w \geq 0\}$;
3. nonincreasing in p and strictly increasing in w ;
4. quasi-convex (i.e. the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any \bar{v}).

Proof. (1) Homogeneity follows by the now-familiar argument. If we multiply both prices and wealth by a factor λ , the consumer problem is unchanged.

(2) Let $p^n \rightarrow p$ and $w^n \rightarrow w$ be sequences of prices and wealth. We must show that $\lim_{n \rightarrow \infty} v(p^n, w^n) = v(p, w)$, which we do by showing that $\liminf_n v(p^n, w^n) \geq v(p, w) \geq \limsup_n v(p^n, w^n) \geq \liminf_n v(p^n, w^n)$. The last inequality is true by definition, so we focus attention on the first two inequalities.

For the first inequality, let $x \in x(p, w)$, so that $v(p, w) = u(x)$ and let $a^n = w^n / (p^n \cdot x)$. Then, $a^n x \in B(p^n, w^n)$, so $v(p^n, w^n) \geq u(a^n x)$. By local non-satiation, $p \cdot x = w$, so $\lim_n a^n = \lim w^n / (p^n \cdot x) = w / (p \cdot x) = 1$. Hence, using the continuity of u , $\liminf_n v(p^n, w^n) \geq \liminf u(a^n x) = u(x) = v(p, w)$, which implies the first inequality.

For the second inequality, let $x^n \in x(p^n, w^n)$ so that $v(p^n, w^n) = u(x^n)$. Let n^k be a subsequence along which $\lim_{k \rightarrow \infty} u(x^{n^k}) = \limsup_n v(p^n, w^n)$. Since $p^n \gg 0$, the union of the budget sets defined by (p^{n^k}, w^{n^k}) and (p, w) is bounded above by the vector b whose i^{th} component is $b_i = (\sup w^n) / (\inf p_i^n)$. Since the sequence $\{x^{n^k}\}$ is bounded, it has some accumulation point x . Since $p^{n^k} \cdot x^{n^k} \leq w^{n^k}$, it follows by taking limits that $p \cdot x \leq w$. Thus, $v(p, w) \geq u(x) = \lim_{k \rightarrow \infty} u(x^{n^k}) = \limsup_n v(p^n, w^n)$, which implies the second inequality.

(3) For the first part, note that if $p > p'$, then $B(p, w) \subset B(p', w)$, so clearly $v(p, w) \leq v(p', w)$. For the second part, suppose $w' > w$, and let $x \in x(p, w)$. By Walras' Law, $p \cdot x = w < w'$, so by a second application of Walras' Law, $x \notin x(p, w')$. Hence, there exists some $x' \in B(p, w')$ such that $u(x') > u(x)$.

(4) Suppose that $v(p, w) \leq \bar{v}$ and $v(p', w') \leq \bar{v}$. For any $t \in [0, 1]$, consider (p^t, w^t) where $p^t = tp + (1-t)p'$ and $w^t = tw + (1-t)w'$. Let x be such that $p^t \cdot x \leq w^t$. Then, $w^t \geq p^t \cdot x = tp \cdot x + (1-t)p' \cdot x$, so either $p \cdot x \leq w$ or $p' \cdot x \leq w'$ or both. Thus, either $u(x) \leq v(p, w) \leq \bar{v}$ or $u(x) \leq v(p', w') \leq \bar{v}$, so $u(x) \leq \bar{v}$. Consequently, $v(p^t, w^t) = \max_{x: p^t \cdot x \leq w^t} u(x) \leq \bar{v}$. *Q.E.D.*

4 Demand with Derivatives

How does one actually solve for Marshallian demand, given preferences, prices and wealth? If the utility function is differentiable,¹ then explicit formulae can sometimes be derived by analyzing the Lagrangian for the consumer problem:

$$\mathcal{L}(x, \lambda, \mu; p, w) = u(x) + \lambda [w - p \cdot x] + \sum_{i=1}^n \mu_i x_i,$$

where λ is the Lagrange multiplier on the budget constraint and, for each i , μ_i is the multiplier on the constraint that $x_i \geq 0$. The “Lagrangian problem” is:

$$\min_{\lambda \geq 0, \mu \geq 0} \max_x \mathcal{L}(x, \lambda, \mu) = \min_{\lambda \geq 0, \mu \geq 0} \max_x u(x) + \lambda [w - p \cdot x] + \sum_{i=1}^n \mu_i x_i,$$

The first order conditions for the maximization problem are:

$$\frac{\partial u}{\partial x_k} = +\lambda p_k - \mu_k, \tag{1}$$

and $\lambda \geq 0$ and $\mu_i \geq 0$ for all i . The solution must also satisfy the original constraints:

$$p \cdot x \leq w \quad \text{and} \quad x \geq 0,$$

as well as the complementary slackness conditions:

$$\lambda (w - p \cdot x) = 0 \quad \text{and} \quad \mu_k x_k = 0 \text{ for } k = 1, \dots, n.$$

These conditions, taken together are called the *Kuhn-Tucker conditions*.

The Lagrangian \mathcal{L} is a linear function of the multipliers (λ, μ) , so $\max_{x \geq 0} \mathcal{L}(x, \lambda, \mu; p, w)$ is a convex function of (λ, μ) . The first order conditions for this problem are that (i) the derivative with respect to each multiplier is non-negative and (ii) that when

¹There are mixed opinions about the differentiability assumption. On one hand, there is no natural restriction on the underlying preference relation \succeq that guarantees differentiability. Purists claim that makes the assumption of dubious validity. On the other, there exists no finite set of observed choices $C(X, \succeq)$ from finite sets X that ever contradicts differentiability. Pragmatists conclude from this that differentiability of demand is empirically harmless and can be freely adopted whenever it is analytically useful.

the derivative is positive, the multiplier is zero. By the envelope theorem, the first condition is equivalent to the condition that the constraints are satisfied and the second is complementary slackness. So the Kuhn-Tucker conditions imply that (λ, μ) solves the minimization problem if x solves the maximization problem.

Suppose we find a triplet (x, λ, μ) that satisfies the Kuhn-Tucker conditions. Does x solve the maximization problem, so that $x \in x(p, w)$? Conversely, if $x \in x(p, w)$ is a solution to the consumer problem, will x also satisfy the Kuhn-Tucker conditions along with some (λ, μ) ?

The *Kuhn-Tucker Theorem* tells us that if $x \in x(p, w)$, then (subject to a certain “regularity” condition) there exist (λ, μ) such that (x, λ, μ) solve the Kuhn-Tucker conditions. Of course, there may be other solutions to the Kuhn-Tucker conditions that do not solve the consumer problem. However, if u is also quasi-concave and has an additional property, then the solutions to the consumer problem and the Kuhn-Tucker conditions coincide exactly — the Kuhn-Tucker conditions are *necessary and sufficient* for x to solve the consumer problem.

Proposition 8 (Kuhn-Tucker) *Suppose that u is continuously differentiable and that $x \in x(p, w)$ is a solution to the consumer problem. If the constraint qualification holds at x , then there exists $\lambda, \mu_1, \dots, \mu_n \geq 0$ such that (x, λ, μ) solve the Kuhn-Tucker conditions. Moreover, if u is quasi-concave and has the property that $[u(x') > u(x)] \implies [\nabla u(x) \cdot (x' - x) > 0]$, then any x that is part of a solution to the Kuhn-Tucker conditions is also a solution to the consumer problem.*

Proof. See the math review handouts.

Q.E.D.

We can use the Kuhn-Tucker conditions to characterize Marshallian demand. First, using (1), we may write:

$$\frac{\partial u}{\partial x_k} \leq \lambda p_k \text{ with equality if } x_k > 0.$$

From this, we derive the following important relationship: for all goods j and k consumed in positive quantity:

$$MRS_{jk} = \frac{\partial u / \partial x_j}{\partial u / \partial x_k} = \frac{p_j}{p_k}.$$

This says that at the consumer's maximum, *the marginal rate of substitution between j and k equals the ratio of their prices.*

Figures 2 and 3 give a graphical representation of the solution to the consumer problem. In Figure 2 both goods are consumed in positive quantities, so $\mu_1 = \mu_2 = 0$, and the marginal rate of substitution along the indifference curve equals the slope of the budget line at the optimum.

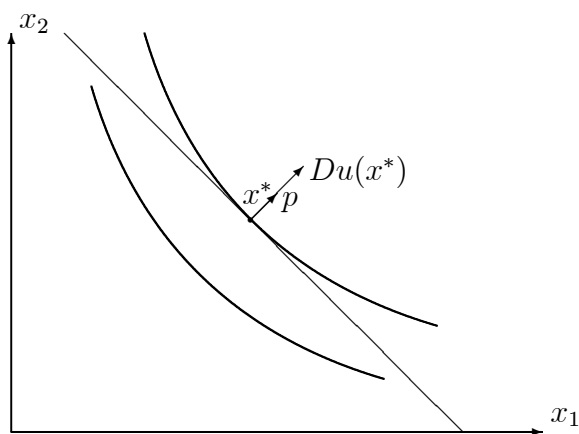


Figure 2: Marshallian demand: interior solution.

In Figure 3, we have a corner solution, so $\mu_1 > 0$ while $\mu_2 = 0$. Here the MRS does not equal the price ratio.

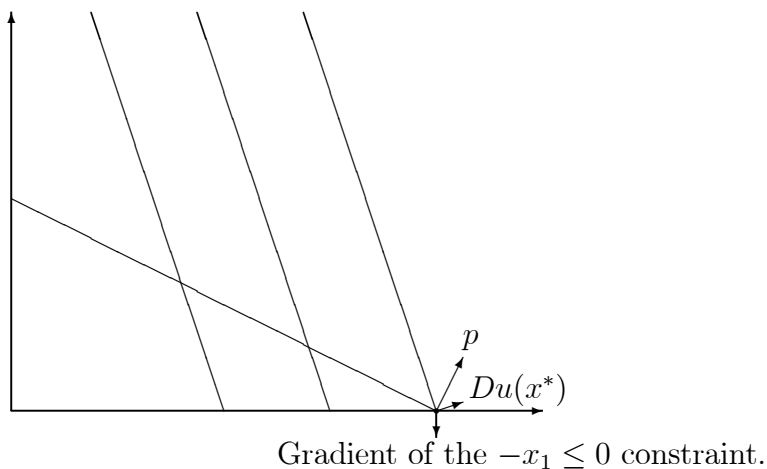


Figure 3: Marshallian demand: corner solution.

We also have a nice characterization of the Lagrange multiplier λ .

Proposition 9 (Marginal utility of income or wealth) *Suppose that u is continuous and quasi-concave, that $p \gg 0$, and that there exists a unique solution (x, λ, μ) to the Lagrangian consumer problem when prices and wealth are given by (p, w) . Then, v is differentiable at (p, w) and*

$$\frac{\partial v(p, w)}{\partial w} = \lambda \geq 0.$$

Proof. Define $\Phi(\lambda, \mu; p, w) = \max_x \mathcal{L}(x, \lambda, \mu; p, w)$. Then, $v(p, w) = \min_{\lambda \geq 0, \mu \geq 0} \Phi(\lambda, \mu; p, w)$. Since the Lagrange multipliers are unique, by the envelope theorem, $\partial v(p, w)/\partial w = \partial \Phi(\lambda, \mu; p, w)/\partial w$ evaluated at the minimum (λ, μ) . But Φ is itself a maximum value function, so another application of the envelope theorem establishes that $\partial \Phi(\lambda, \mu; p, w)/\partial w = \partial \mathcal{L}(x, \lambda, \mu; p, w)/\partial w = \lambda$. The construction itself requires that $\lambda \geq 0$. *Q.E.D.*

The Lagrange multiplier λ gives the value (in terms of utility) of having an additional unit of wealth. Because of this, λ is the sometimes called the *shadow price of wealth* or the *marginal utility of wealth (or income)*. In terms of the history of thought, the terms *marginal utility of income* or and *marginal utility of wealth* were important, because utilitarians thought that such considerations would guide the choice of public policies that redistribute of wealth or income. However, nothing in the consumer theory developed so far suggests any basis for using the marginal utility of income or wealth, as we have defined it, to guide redistribution policies.

For our next calculations, it will be useful to have $\lambda > 0$. One might think that adding an assumption of local non-satiation would imply that strict inequality, since it certainly implies that v is increasing in w . However, neither local non-satiation nor any other condition on consumer preferences \succsim is sufficient for the desired conclusion.² However, if there is everywhere at least one good j for which $\partial u/\partial x_j > 0$, then one can infer that $\partial v/\partial w \geq (\partial u/\partial x_j)/p_j > 0$.

²This is a purely technical point, but it serves to remind us that the same preferences can be represented in quite different ways. Suppose a representation u is selected so that the corresponding indirect utility satisfies $\frac{\partial v(p, w)}{\partial w} > 0$. Suppose $v(p, w) = \bar{v}$ and consider the alternative representation $\hat{u}(x) = (u(x) - \bar{v})^3$. This utility function represents the same preferences as u

Proposition 10 (Roy's Identity) *Suppose that v is differentiable at $(p, w) \gg 0$ and that $\partial v / \partial w > 0$. Then, $x(p, w)$ is a singleton and*

$$x_i(p, w) = -\frac{\partial v(p, w) / \partial p_i}{\partial v(p, w) / \partial w}.$$

Proof. Appealing again to the envelope theorem, $\partial v(p, w) / \partial p_i = -\lambda x_i$. Combining that with the previous proposition establishes the identity. *Q.E.D.*

5 Hicksian Demand

We now make what will prove to be a very useful detour in consumer theory and one that highlights the similarities between consumer theory and producer theory. In close parallel to the firm's cost minimization problem, we introduce the consumer's *expenditure minimization problem (EMP)*.

$$\begin{aligned} \min_{x \in \mathbb{R}_+^n} p \cdot x \\ \text{s.t. } u(x) \geq u, \end{aligned}$$

where $u \geq u(0)$ and $p \gg 0$. This problem finds the cheapest bundle at prices p that yields utility at least u .³

We record the following basic fact.

Proposition 11 (Existence) *If $p \gg 0$, u is continuous and there is some x such that $u(x) \geq u$, then (EMP) has a solution.*

and has corresponding indirect utility function $\hat{v}(p, w) = \hat{u}(x(p, w)) = (v(p, w) - \bar{v})^3$. Applying the chain rule leads to $\frac{\partial \hat{v}(p, w)}{\partial w} = 0$. Thus, whether the marginal utility of income is positive or zero at a point is not just a property of the preferences themselves, but is a joint property of the preferences and their representation.

³This problem is sometimes called the *dual consumer problem*, but that terminology suggests incorrectly that "duality" results always apply. In general, duality results will apply to this problem only when u is quasi-concave, but that property plays no role in most of our analysis. It is more accurate to refer to this problem as the *expenditure minimization problem*.

Proof. Let $u(\hat{x}) \geq u$. Let $S = \{x | p \cdot x \leq p \cdot \hat{x}\} \cap \{x | u(x) \geq u\}$. By construction, the first set in the intersection is non-empty and compact and by continuity of u , the second set is closed, so S is a compact set. Hence, the continuous function $p \cdot x$ achieves its minimum at some point x^* on S . By construction, for every $x \notin S$ such that $u(x) \geq u$, $p \cdot x > p \cdot \hat{x} \geq p \cdot x^*$, so x^* solve =s the *EMP*. *Q.E.D.*

The solution $x = h(p, u)$ of the expenditure minimization problem is called the *Hicksian* (or *compensated*) demand. We define the *expenditure function* to be the corresponding value function:

$$e(p, u) = \min_{x \in \mathbb{R}_+^n} p \cdot x \text{ subject to } u(x) \geq u.$$

Thus, $e(p, u)$ is the minimum expenditure required to achieve utility u at prices p , and $h(p, u)$ is the set of consumption bundles that the consumer would purchase at prices p if she wished to minimize her expenses but still achieve utility u .

What is the motivation for introducing the expenditure minimization problem, when we have already analyzed the “actual” consumer problem? We take this detour to capture two main advantages. First, we will use the expenditure function to decompose the effect of a price change on Marshallian demand into two corresponding effects. On one hand, a price reduction makes the consumer wealthier, just as if she had received a small inheritance, and that could certainly affect demand for all goods. We will call that the *wealth effect* (or *income effect*). In addition, even if the consumer were forced to disgorge her extra wealth, the price reduction would cause the optimizing consumer to substitute the newly cheaper good for more expensive ones and perhaps to make other changes as well. That is called the *substitution effect*. Because the expenditure function is the optimal value of a cost minimization problem holding the utility level constant, it is closely analogous to the cost minimization problem in producer theory (in which the level of output that is held constant). The substitution effect in consumer theory is similar to the substitution effect in the producer’s cost minimization problem.

This detour leads to a second important advantage as well: the expenditure function turns out to play a central role in welfare economics. More about that later in these notes. With these advantages lying ahead, we first introduce three

propositions to identify, respectively, the properties of Hicksian demand, those of the expenditure function, and the relationship between the two functions.

Proposition 12 (Properties of Hicksian Demand) *Suppose u is a continuous utility function representing a preference relation \succsim on \mathbb{R}_+^n . Then*

1. (Homogeneity) $h(p, u)$ is homogeneous of degree zero in p . For any p, u and $\lambda > 0$, $h(\lambda p, u) = h(p, u)$.
2. (No Excess Utility) If $u \geq u(0)$ and $p \gg 0$, then for all $x \in h(p, u)$, $u(x) = u$.
3. (Convexity/Uniqueness) If preferences are convex, then $h(p, u)$ is a convex set. If preferences are strictly convex and $p \gg 0$, then $h(p, u)$ is a singleton.

Proof. (1) Note that the constraint set, or choice set, is the same in the expenditure problem for $(\lambda p, u)$ and (p, u) . But then

$$\min_{\{x \in \mathbb{R}_+^n : u(x) \geq u\}} \lambda p \cdot x = \lambda \min_{\{x \in \mathbb{R}_+^n : u(x) \geq u\}} p \cdot x,$$

so the expenditure problem has the same solution for $(\lambda p, u)$ and (p, u) .

(2) Suppose to the contrary that there is some $x \in h(p, u)$ such that $u(x) > u \geq u(0)$. Consider a bundle $x' = tx$ with $0 < t < 1$. Then $p \cdot x' < p \cdot x$, and by the intermediate value theorem, there is some t such that $u(x') \geq u$, which contradicts the assumption that $x \in h(p, u)$.

(3) Note that $h(p, u) = \{x \in \mathbb{R}_+^n : u(x) \geq u\} \cap \{x | p \cdot x = e(p, u)\}$ is the intersection of two convex sets and hence is convex. If preferences are strictly convex and $x, x' \in h(p, u)$, then for $t \in (0, 1)$, $x'' = tx + (1 - t)x'$ satisfies $x'' \succ x$ and $p \cdot x'' = e(p, u)$, which contradicts “no excess utility”. *Q.E.D.*

Proposition 13 (Properties of the Expenditure Function) *Suppose u is a continuous utility function representing a locally non-satiated preference relation \succeq on \mathbb{R}_+^n . Then $e(p, u)$ is*

1. homogenous of degree one in p : for all p, u and $\lambda > 0$, $e(\lambda p, u) = \lambda e(p, u)$;

2. continuous in p and u ;
3. nondecreasing in p and strictly increasing in u provided $p \gg 0$;
4. concave in p .

Proof. (1) As in the above proposition, note that

$$e(\lambda p, u) = \min_{\{x \in \mathbb{R}_+^n : u(x) \geq u\}} \lambda p \cdot x = \lambda \min_{\{x \in \mathbb{R}_+^n : u(x) \geq u\}} p \cdot x = \lambda e(p, u).$$

(2) We omit this proof, which is similar to proving continuity of the indirect utility function.

(3) Let $p' > p$ and suppose $x \in h(p', u)$. Then $u(x) \geq u$, and $e(p', u) = p' \cdot x \geq p \cdot x$. It follows immediately that $e(p, u) \leq e(p', u)$.

For the second statement, suppose to the contrary that for some $u' > u$, $e(p, u') \leq e(p, u)$. Then, for some $x \in h(p, u)$, $u(x) = u' > u$, which contradicts the “no excess utility” conclusion of the previous proposition.

(4) Let $t \in (0, 1)$ and suppose $x \in h(tp + (1 - t)p')$. Then, $p \cdot x \geq e(p, u)$ and $p' \cdot x \geq e(p', u)$, so $e(tp + (1 - t)p', u) = (tp + (1 - t)p') \cdot x \geq te(p, u) + (1 - t)e(p', u)$. *Q.E.D.*

Finally, we apply the envelope theorem to recover Hicksian demands from the expenditure function.

Proposition 14 (Relating Expenditure and Demand) *Suppose that $u(\cdot)$ is a continuous utility function representing a locally non-satiated preference relation \succeq and suppose that $h(p, u)$ is a singleton. Then the expenditure function is differentiable in p , and for all $i = 1, \dots, n$,*

$$\frac{\partial e(p, u)}{\partial p_i} = h_i(p, u).$$

By analogy with the corresponding result for the firm’s cost function, some writers call this Shepard’s lemma as well. Thought question: What is $\partial e / \partial u$?

6 Hicksian Comparative Statics

Comparative statics are statements about how the solution to a problem will change with parameters. In the consumer problem, the parameters are (p, w) , so comparative statics are statements about how $x(p, w)$, or $v(p, w)$ will change with p and w . Similarly, in the expenditure problem, the parameters are (p, u) , so comparative statics are statements about how $h(p, u)$ or $e(p, u)$ will change with p and u .

Our first result gives a comparative statics statement about how a change in price changes the expenditure required to achieve a given utility level u . The “law of demand” formalizes to the idea when the price of some good increases, the (Hicksian) demand for that good decreases.

Proposition 15 (Law of Demand) *Suppose $p, p' \geq 0$ and let $x \in h(p, u)$ and $x' \in h(p', u)$. Then, $(p' - p)(x' - x) \leq 0$.*

Proof. By definition $u(x) \geq u$ and $u(x') \geq u$. So, by optimization, $p' \cdot x' \leq p' \cdot x$ and $p \cdot x \leq p \cdot x'$. We may rewrite these two inequalities as $p' \cdot (x' - x) \leq 0$ and $0 \geq -p \cdot (x' - x)$, and the result follows immediately. *Q.E.D.*

The Law of Demand can be applied to study how demand for a single good varies with its own price. Thus, suppose that the only difference between p' and p is that, for some k , $p'_k > p_k$, but $p'_i = p_i$ for all $i \neq k$. Then, with single-valued demand,

$$(p'_k - p_k) [h_k(p', u) - h_k(p, u)] \leq 0.$$

This means that $h_k(p, u)$ is *decreasing* in p_k . Or in words, *Hicksian demand curves slope downward*.

A simple way to see this graphically is to note that the change in Hicksian demand given a change in price is a shift along an indifference curve:

In contrast, Marshallian demand $x_k(p, w)$ need not be decreasing in p_k (though this is *typically* the case). To see why, consider Figure 5. We will come back to how Marshallian demand reacts to a change in price, and to the relationship between the change in Marshallian and Hicksian demand, in a minute.

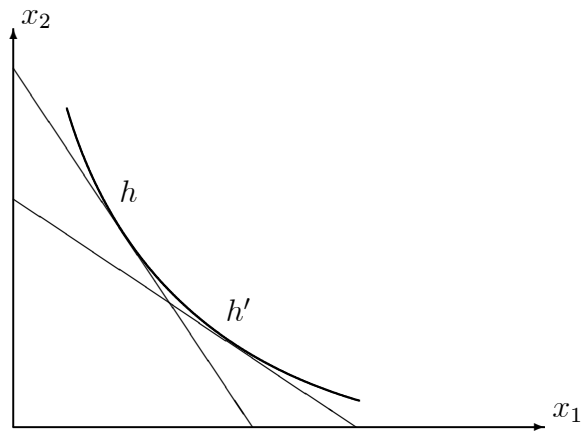


Figure 4: Hicksian Demand: Change in Price

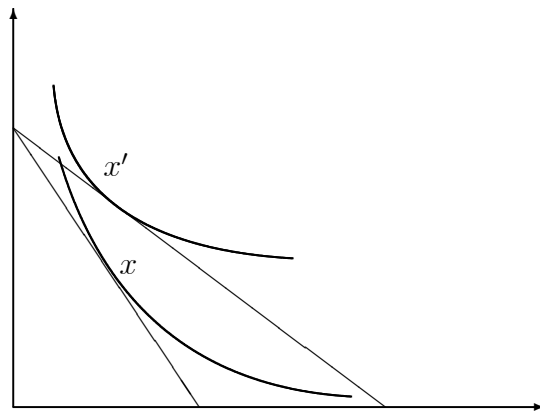


Figure 5: Marshallian demand may increase with price increase.

If the Hicksian demand function $h(p, u)$ is singleton-valued and continuously differentiable, we can use derivatives to describe how this demand responds to price changes. The next result is closely related to corresponding results that we have previously discussed concerning a firm's input demands.

Consider the matrix:

$$D_p h(p, u) = \begin{pmatrix} \frac{\partial h_1(p, u)}{\partial p_1} & \frac{\partial h_n(p, u)}{\partial p_1} \\ \dots & \dots \\ \frac{\partial h_1(p, u)}{\partial p_n} & \frac{\partial h_n(p, u)}{\partial p_n} \end{pmatrix}.$$

Recall the definition that an $n \times n$ symmetric matrix D is *negative semi-definite* if for all $z \in \mathbb{R}^n$, $z \cdot Dz \leq 0$.

Proposition 16 *Suppose that $u(\cdot)$ represents a preference relation \succsim and that $h(p, u)$ is singleton-valued and continuously differentiable at (p, u) , where $p \gg 0$. Then $D_p h(p, u)$ is symmetric and negative semi-definite.*

Proof. By Shephard's Lemma, $h_i(p, u) = \frac{\partial e(p, u)}{\partial p_i}$, so $\frac{\partial h_i(p, u)}{\partial p_j} = \frac{\partial^2 e(p, u)}{\partial p_i \partial p_j}$. We may rewrite this as:

$$D_p h(p, u) = D_p^2 e(p, u).$$

For symmetry, recall that Young's Theorem from calculus tells us that for any twice continuously differentiable function $f(x, y)$, $f_{xy} = f_{yx}$. Applying this result shows us that

$$\frac{\partial h_i(p, u)}{\partial p_j} = \frac{\partial^2 e(p, u)}{\partial p_i \partial p_j} = \frac{\partial^2 e(p, u)}{\partial p_j \partial p_i} = \frac{\partial h_j(p, u)}{\partial p_i}$$

and hence $D_p h(p, u)$ is symmetric.

For negative semi-definiteness, recall that $e(p, u)$ is a concave function of p . This implies that $D_p^2 e(p, u)$ is negative semi-definite (see the appendix of MWG for a proof. *Q.E.D.*

What is most surprising here is the symmetry of the demand matrix: the effect of a small increase in the price of good i on the quantity demanded of good j is identical to effect of a similar increase in the price of good j on the quantity

demand of good i . Thus, the derivative of the Hicksian demand for butter, say in kilograms, with respect to the price of compact disks, say in \$/disk, is the same as the derivative of the Hicksian demand for compact disks with respect to the price of butter in \$/kilogram.

The proposition also encompasses within it a differential form of the law of demand. For, the rate of change of the Hicksian demand for good j as the price p_j increases is $\partial h_j / \partial p_j = \partial^2 e(p, u) / \partial p_j^2$, which is a diagonal element of the matrix $D_p h$. The diagonal elements of a negative semi-definite matrix are always non-positive. To see why, let $z = (0, \dots, 0, 1, 0, \dots, 0)$ have a 1 only in its j^{th} place. Then since $D_p h(p, u)$ is negative semi-definite, $0 \geq z D_p h(p, u) \cdot z = \partial h_j / \partial p_j$, proving that the j^{th} diagonal element is non-positive. That is, the Hicksian demand for good j is weakly decreasing in the price of good j .

7 The Slutsky Equation

Next, we bring the theory together by relating Marshallian and Hicksian demand and using that relationship to derive the Slutsky equation, which decomposes the effect of price changes on Marshallian demand.

Proposition 17 (Relating Hicksian & Marshallian Demand) *Suppose u is a utility function representing a continuous, locally non-satiated preference relation \succeq on \mathbb{R}_+^n . Then,*

1. *For all $p \gg 0$ and $w \geq 0$, $x(p, w) = h(p, v(p, w))$ and $e(p, v(p, w)) = w$.*
2. *For all $p \gg 0$ and $u \geq u(0)$, $h(p, u) = x(p, e(p, u))$ and $v(p, e(p, u)) = u$.*

Proof. First, fix prices $p \gg 0$ and wealth $w \geq 0$ and let $x \in x(p, w)$. Since $u(x) = v(p, w)$ and $p \cdot x \leq w$, it follows that $e(p, v(p, w)) \leq w$. For the reverse inequality, we use the hypothesis of local non-satiation. It implies Walras' Law, so for any x' with $p \cdot x' < w$, it must be that $u(x') < v(p, w)$. So, $e(p, v(p, w)) \geq w$. Combining these implies that $e(p, v(p, w)) = w$ and hence that $h(p, v(p, w)) = x(p, w)$.

Next, fix prices $p \gg 0$ and target utility $u \geq u(0)$ and let $x \in h(p, u)$. Since $u(x) \geq u$, it follows that $v(p, e(p, u)) \geq u(x) \geq u$. By the "no excess utility"

proposition, for any x' with $u(x') > u$, $p \cdot x' > p \cdot x = e(p, u)$. Thus, $v(p, e(p, u)) \leq u$. So, $v(p, e(p, u)) = u$ and it follows that $x(p, e(p, u)) = h(p, u)$. *Q.E.D.*

This result is quite simple and intuitive (at least after one understands the local non-satiation condition). It says that if $v(p, w)$ is the most utility that a consumer can achieve with wealth w at prices p , then to achieve utility $v(p, w)$ will take wealth at least w . Similarly, if $e(p, u)$ is the amount of wealth required to achieve utility u , then the most utility a consumer can get with wealth $e(p, u)$ is exactly u .

Our next result is the promised decomposition of price effects in Marshallian demand.

Proposition 18 (Slutsky Equation) *Suppose u is a continuous utility function representing a locally non-satiated preference relation \succeq on \mathbb{R}_+^n and let $p \gg 0$ and $w = e(p, u)$. If $h(p, u)$ and $x(p, w)$ are singleton-valued and differentiable at (p, u, w) , then for all i, j ,*

$$\frac{\partial x_i(p, w)}{\partial p_j} = \frac{\partial h_i(p, \bar{u})}{\partial p_j} - \frac{\partial x_i(p, w)}{\partial w} x_j(p, w).$$

Proof. Starting with the identity,

$$h_i(p, \bar{u}) = x_i(p, e(p, \bar{u}))$$

letting $w = e(p, u)$ and differentiating with respect to p_j gives:

$$\frac{\partial h_i(p, \bar{u})}{\partial p_j} = \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} \frac{\partial e(p, \bar{u})}{\partial p_j}.$$

Substituting in for the last term using Shephard's lemma and the identity $h_i(p, \bar{u}) = x_i(p, w)$ gives the result. *Q.E.D.*

The Slutsky equation is interesting for two reasons. First, it gives a (fairly simple) relationship between the Hicksian and Marshallian demands. More importantly, it allows us to analyze the response of Marshallian demand to price changes, breaking it down into two distinct effects:

$$\underbrace{\frac{\partial x_i(p, w)}{\partial p_i}}_{\text{total effect}} = \underbrace{\frac{\partial h_i(p, \bar{u})}{\partial p_i}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_i(p, w)}{\partial w} x_i(p, w)}_{\text{wealth effect}}$$

An increase in p_i does two things. It causes the consumer to *substitute* away from i toward other relatively cheaper goods. And second, it makes the consumer poorer, and this *wealth effect* also changes his desired consumption — potentially in a way that counteracts the substitution effect.

Figure 6 illustrates the Slutsky equation, decomposing the demand effect of a price change into substitution and wealth effects. Fixing wealth w , when the price drops from $p = (p_1, p_2)$ to $p' = (p'_1, p_2)$ with $p'_1 < p_1$, the demand changes from x to x' . Letting $u = v(p, w)$ and $u' = v(p', w)$, note that $x = h(p, u)$, and $x' = h(p, u')$. Then the shift from x to x' can be decomposed as follows. The *substitution effect* is the consumer's shift along her indifference curve from $x = h(p, u)$ to $h(p', u)$ and a *wealth effect* or *income effect* is the consumer's shift from $h(p', u)$ to $x(p', w)$. Why is this second effect a wealth effect? Because $h(p', u) = x(p', e(p', u))$ the move corresponds to the change in demand at prices p' from increasing wealth from $e(p', u)$ to $w = e(p', u')$.

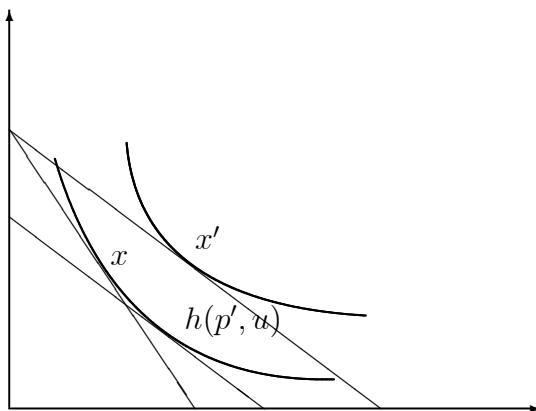


Figure 6: Wealth and Substitution Effects

The Slutsky equation contains within it a suggestion about how to test the subtlest prediction of consumer choice theory. Given enough data about $x(p, w)$, one can derive the matrix of derivatives $D_p x$ and add to each term the corresponding wealth effect to recover the matrix of substitution effects, which corresponds to $D_p h$. If consumers are maximizing, then the matrix obtained in that way must be symmetric (and negative semi-definite).

Many economists have regarded this analysis and its symmetry conclusion as a triumph for the use of formal methods in economics. The analysis does demonstrate the possibility of using theory to derive subtle, testable implications that had been invisible to researchers using traditional verbal and graphical methods. Historically, that argument was quite influential, but its influence has lessened over time. Critics typically counter it by observing that formal research has generated few such conclusions and that the maximization hypothesis on which all are based fares poorly in certain laboratory experiments.

Now consider the following definitions.

Definition 1 *Good i is a **normal good** if $x_i(p, w)$ is increasing in w . It is an **inferior good** if $x_i(p, w)$ is decreasing in w .*

Definition 2 *Good i is a **regular good** if $x_i(p, w)$ is decreasing in p_i . It is a **Giffen good** if $x_i(p, w)$ is increasing in p_i .*

Definition 3 *Good i is a **substitute** for good j if $h_i(p, u)$ is increasing in p_j . It is a **complement** if $h_i(p, u)$ is decreasing in p_j .*

Definition 4 *Good i is a **gross substitute** for good j if $x_i(p, w)$ is increasing in p_j . It is a **gross complement** if $x_i(p, w)$ is decreasing in p_j .*

Figure 7 gives a graphical depiction of what happens when income increases. As income increases from w to w' to w'' , the budget line shifts out and Marshallian demand increases from x to x' to x'' . In the figure, both goods are normal. If we plot $x(p, w)$ for each possible income level w , and connect the points, the resulting curve is called an *Engel curve* or *Income expansion curve*.⁴

Figure 8 shows what happens to Marshallian demand when prices change. Here, as the price of the first good decreases, Marshallian demand shifts from x to x' to x'' . In this picture, the first good is regular — as its price decreases, the demand

⁴An important question in development economics is how to estimate these curves empirically. The basic approach is to estimate the demands for major budget items — food, shelter, clothing — as a function of prices and income, and then ask how these demands have changed and will change as the country becomes richer.

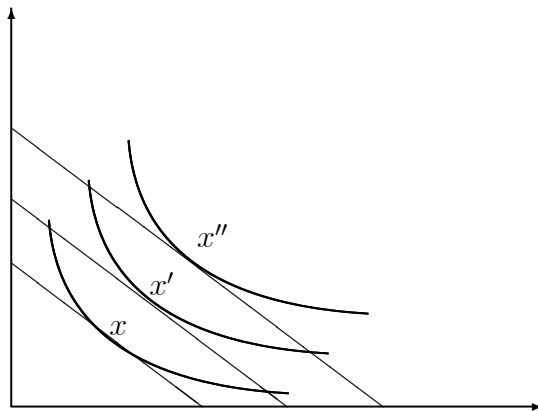


Figure 7: Engel Curve or Income Expansion Path

for it increases. Note also that as the price of good one decreases, the Marshallian demand for the second good also increases: so goods i and j are gross complements.

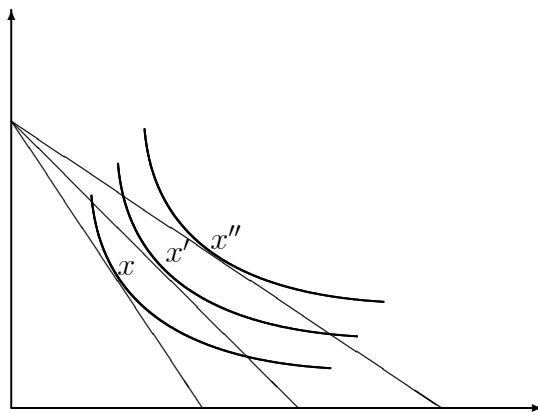


Figure 8: Offer Curves or Price Expansion Path

Traditional economics textbooks call a pair of goods substitutes or complements based on their Hicksian demands and reserve the terms “gross substitutes” and “gross complements” for the relations based on Marshallian demands. Perhaps the reason for this is that the Hicksian language is easier, because the Hicksian substitutes condition is a symmetric one, so one can simply say that “goods i and

j are substitutes” without needing to specify which is a substitute for the other. The condition is symmetric because, as previously shown:

$$\frac{\partial h_i(p, u)}{\partial p_j} = \frac{\partial h_j(p, u)}{\partial p_i}.$$

In contrast, the gross substitute condition is not generally symmetric, because the wealth effect on $x_i(p, w)$ caused by an increase in p_j is not generally the same as the wealth effect on $x_j(p, w)$ caused by an increase in p_i :

$$\frac{\partial x_i(p, w)}{\partial w} x_j(p, w) \neq \frac{\partial x_j(p, w)}{\partial w} x_i(p, w).$$

In common practice, when one says that two or more goods are “gross substitutes,” one means that each good is a gross substitute for each other good.

Keep in mind that even if goods are substitutes in one range of prices, they may still be complements for another range. When tight logical arguments are required, best practice is to describe assumptions in precise mathematical terms and to use terms like substitutes and complements as ways to describe and explicate the precise formal argument.

8 Consumer Welfare: Price Changes

We now turn to a particularly beautiful part of consumer theory: the measurement of consumer welfare. We assume throughout that consumer preferences are locally non-satiated and investigate the question: how much better or worse off is the consumer as the result of a change in prices from p to p' ?

This question is much less narrow than it may seem. For purposes of determining welfare effects, many changes in the economic environment can be viewed as price changes. Taxes and subsidies are obvious cases: they add to or subtract from the price someone pays for a good. If we want to study the welfare effects of technical change, such as the introduction of a new product, we can formulate that as a change in price from $p = \infty$ to some finite price p' .

Let (p, w) be the consumer’s status prior to the price change, and (p', w) the consumer’s status after the price change. A natural candidate for measuring the

change in welfare is to look at the change in the consumer's utility, i.e. at $v(p', w) - v(p, w)$. Of course, the problem is that this measure depends on *which* utility function we choose to represent the consumer preferences. While all give the same qualitative answer to the question of whether the consumer is better or worse off, they give different answers to the question of by how much she is better or worse off. In addition, the answer they give is in utils, which have no real meaning.

While there is no complete solution to this problem, there is an elegant partial solution. We can use the expenditure function to measure welfare changes in dollars. Essentially, we ask: *how much money is required to achieve a certain level of utility before and after the price change?* To answer this, we need to choose a level of utility as a reference point for making this comparison. There are two obvious candidates: the level of utility achieved by the consumer prior to the change and the level achieved after the change. We refer to these two measures as *compensating and equivalent variation*. Both are constructed to be positive for changes that increase welfare and negative for changes that reduce welfare.

Compensating variation specifies how much less wealth the consumer needs to achieve the same maximum utility at prices p' as she had before the price change. Letting $u = v(p, w)$ be the level of utility achieved prior to the price change,

$$\text{Compensating Variation} = e(p, u) - e(p', u) = w - e(p', u).$$

That is, if prices change from p to p' , the magnitude of compensating variation tells us how much we will have to charge or compensate our consumer to have her stay on the same indifference curve.

Equivalent variation gives the change in the expenditure that would be required at the original prices to have the same ("equivalent") effect on consumer as the price change had. Letting $u' = v(p', w)$ be the level of utility achieved after the price change.

$$\text{Equivalent Variation} = e(p, u') - e(p', u') = e(p, u') - w.$$

That is, equivalent variation tells us how much more money the consumer would have needed yesterday to be as well off as she is today.

Figure 9 illustrates compensating variation for a situation where only a single price — that of the first good — changes. In this figure, think of the second good

as a composite good (i.e. “expenditures on all other items”) measured in dollars.⁵ Prices change from p to p' where $p'_1 > p_1$ and $p'_2 = p_2 = 1$ and the budget line rotates in. To identify compensating variation, we first find the wealth required to achieve utility u at prices p' , i.e. $e(p', u)$, then find the difference between this and $w = e(p, u)$, the starting level of wealth.

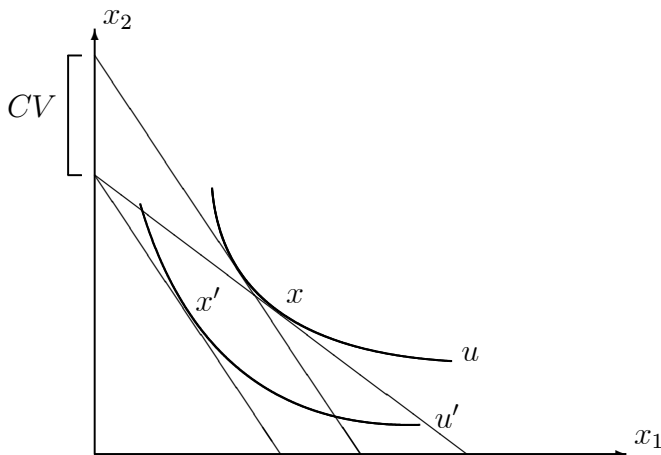


Figure 9: Compensating Variation

Figure 10 displays equivalent variation for the same hypothetical price change. Here, the first step is to find the wealth required to achieve utility u' at the original prices p .

Generally speaking, compensating and equivalent variation will not be the same, since they are the answers to different questions. There is, however, one case for which they will coincide. If preferences are quasi-linear, then equivalent and compensating variation are identical. Demonstrating this is left as a homework assignment.

If the price change affects only a single good i , we can relate equivalent and compensating variation to the Hicksian demand in this simple way:

$$CV = e(p, u) - e(p', u) = \int_{p'_i}^{p_i} \frac{\partial e(p, u)}{\partial p_i} dp_i = \int_{p'_i}^{p_i} h_i(p, u) dp_i,$$

⁵Formally, this means we are working with the two-argument utility function $\hat{u}(x_1, y) = \max_{(x_2, \dots, x_n) \in \mathbb{R}_+^{n-1}} u(x_1, \dots, x_n)$ subject to $p_2 x_2 + \dots + p_n x_n = y$.

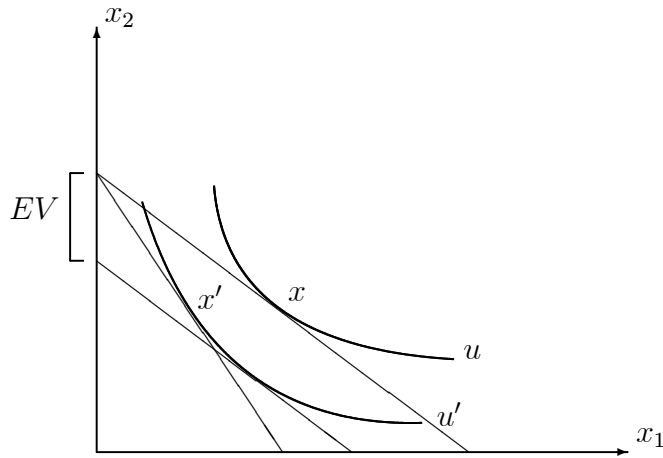


Figure 10: Equivalent Variation

and similarly

$$EV = e(p, u') - e(p', u') = \int_{p'_i}^{p_i} \frac{\partial e(p, u')}{\partial p_i} dp_i = \int_{p'_i}^{p_i} h_i(p, u') dp_i$$

Figure 11 shows the Hicksian demand curves for a single good (good one) at two utility levels $u > u'$, assuming that the good is normal. To identify CV, we need to integrate the area to the left of the $h_1(\cdot, u)$ curve between p_1 and p'_1 . Similarly, EV corresponds to the area to the left of the $h_1(\cdot, u')$ between p_1 and p'_1 .

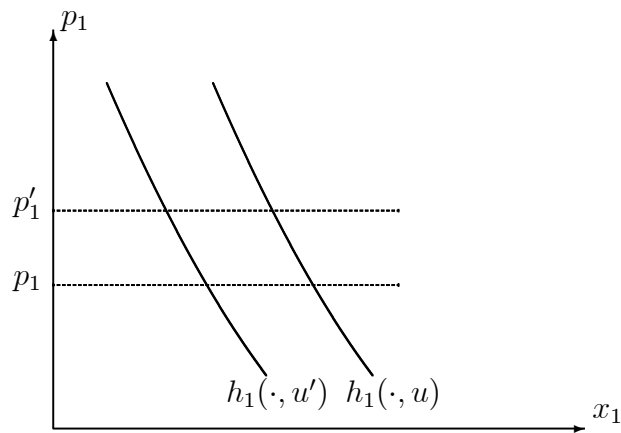


Figure 11: Relating Welfare to Demand

By construction, $w = e(p, u) = e(p', u')$. This allows us to conclude that $x(p, w) = h(p, u)$ and $x(p', w) = h(p', u')$. This relation is plotted in Figure 12.

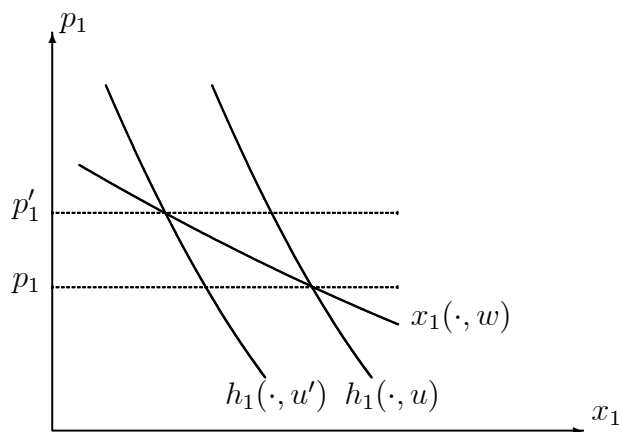


Figure 12: Relating Welfare to Demand

Figure 12 suggests that another measure of consumer welfare might be obtained by integrating to the left of the *Marshallian demand curve*. We define this — the area to the left of the Marshallian demand curve — as *consumer surplus*.

$$\text{Consumer Surplus} = \int_{p'_i}^{p_i} x_i(p, w) dp_i.$$

In empirical work, where the regressions typically provide direct estimates of a Marshallian demand curve, Marshallian consumer surplus is a very common measure of consumer welfare. There is a long-standing debate in industrial organization as to when Marshallian Consumer Surplus is a good welfare measure (with important papers by Willig (1976, *AER*) and Hausman (1981, *AER*)). Consumer surplus has an important drawback — it does not have an immediate interpretation in terms of utility theory, as do EV and CV. However, one nice feature — which is apparent in the figure — is that Consumer Surplus is typically an intermediate measure that lies *between* compensating and equivalent variation. More precisely, on any range where the good in question is either normal or inferior,⁶ we have the

⁶These are not the only logical possibilities: it is also possible that the good is normal on part of the relevant domain and inferior on another part of the domain. Only in that case can the inequality fail.

following relationship:

$$\min\{CV, EV\} \leq CS \leq \max\{CV, EV\}.$$

This typical relationship is sometimes used to justify consumer surplus as a welfare measure.

The most problematic part of using these concepts—equivalent variation, compensating variation, and consumer surplus—is the practice of simply adding up these numbers across individuals to compare overall welfare from two policies. Taken literally, this practice implies that one should be indifferent, in terms of overall welfare, between policies that redistribute benefits from the rich to the poor or from the poor to the rich. Because they omit distributional issues, these various measures can do no more than give an index of the equivalent or compensating changes in the total wealth of a society.

9 Consumer Welfare: Price Indices

In practice, perhaps the most important problem in the measurement of consumer welfare is obtaining correct measures of the growth of the economy. There are many subtleties involved in this measurement, depending on the goods that one includes in deciding about welfare. For example, there are important questions about how to measure public goods including environmental amenities, safety, and so on. One subtle issue concerns how to adjust for changes in the cost of living. That is, suppose one wants to know how much better off people are from one year to the next, given that economic growth has increased people’s incomes (that is, has increased GDP). Once we have measured this increase in income, we need to account also for any changes in prices over the same period. Thus, to measure growth in consumer welfare, we need to “adjust” nominal income by a measure of the cost of living and use this adjusted measure (of “real income”) to calculate growth.

This brings us to the topic of price indices. To define a price index (and this is essentially what the BLS does to measure inflation), one defines a “market basket” of goods — goods 1, 2, ..., n — and then compare their prices from period to period

(quarterly, yearly, whatever). Let p be the prices of these goods “before” and p' the prices “after”. There are basically two well-known ways to proceed. One way is to look at the quantities of the goods purchased in the “before” period, x , and compare the price of this basket at the two price levels. This is called a *Laspeyres index*:

$$\text{Laspeyres Index} : \frac{p' \cdot x}{p \cdot x},$$

Alternatively, one can look at the quantities of the goods purchased in the “after” period, x' . This is called a *Paasche Index*.

$$\text{Paasche Index} : \frac{p' \cdot x'}{p \cdot x'}$$

In practice, cost of living is computed using some variant of a Laspeyres or Paasche index. In theory, there is a better alternative, which is to use what is called an *ideal index*. Similar to the idea of EV or CV, an ideal index chooses some base level of utility, and asks how much more expensive it is to achieve this utility at prices p' than at prices p .

$$\text{Ideal Index} : \frac{e(p', u)}{e(p, u)},$$

where u is a “base” level of utility — typically either the utility in the “before” or “after” period.

Generally speaking, neither the Paasche or the Laspeyres Index are “ideal”. To see why, let u be the utility in the before period. Then

$$\text{Laspeyres} = \frac{p' \cdot x}{p \cdot x} = \frac{p' \cdot x}{e(p, u)} \geq \frac{e(p', u)}{e(p, u)} = \text{Ideal} (u).$$

The problem is that at prices p' , the consumer will not choose to consume x . Most likely, there is a cheaper way to get utility u . This is called the *substitution bias* because the Laspeyres index does not account for the fact that when prices change, consumers will substitute to cheaper products.

The Paasche index also suffers from substitution bias:

$$\text{Paasche} = \frac{p' \cdot x'}{p \cdot x'} = \frac{e(p', u')}{p \cdot x'} \leq \frac{e(p', u')}{e(p, u')} = \text{Ideal} (u').$$

In the last decade, the Bureau of Labor Statistics measure of inflation (the Consumer Price Index or CPI) has come under criticism. One of the main criticisms is that it suffers from substitution bias. There is also concern over the CPI for several other reasons, which include the following biases.

- “New Good Bias”. When new products are introduced, we only have a price in the “after” period, but not in the before period (or if products disappear, we have the opposite problem). The CPI deals with this by waiting 5–10 years to add these products (eg cellular telephones, rice krispies treats cereal) to the index. But these products make us better off, meaning that the CPI tends to underestimate how much better off we really are. A substantial body of recent research is focused on measuring the welfare impact of new goods.
- “Outlet Bias”. The BLS goes around and measures prices in various places, then takes an average. Over the last 20 years, people have started buying things cheaply at places like Wal-Mart and Costco. Thus, the BLS may tend to over-estimate the prices people actually pay.

Besides price indices for all goods, it is sometimes useful to construct price indices for categories of goods based solely on the prices of the goods in that category. For example, one might hope to be precise about statements like “entertainment goods have become 10% more expensive” without having to refer to non-entertainment goods like food, housing, and transportation.

Ideally, we would like our price index to stand in for more detailed information in various calculations and empirical studies, especially calculations about consumer welfare and demand studies. With those intuitive goals in mind, we turn to a formal treatment.

We divide the goods $1, \dots, n$ be divided into two groups. Let goods $1, \dots, k$ be the ones in the category of interest, which here we call “entertainment goods,” while goods $k + 1, \dots, n$ denote the other, non-entertainment goods. We make two assumptions and impose three requirements. The assumptions are (1) that consumer preferences are locally non-satiated and (2) that there exist some prices

at which the consumer prefers to make a positive expenditure on entertainment goods. The second assumption rules out trivial cases.

The first requirement is that the entertainment price index should depend only on the prices (p_1, \dots, p_k) of the entertainment goods and should be homogeneous of degree 1, so that doubling all the prices doubles the index. The second requirement is that if the prices of entertainment goods change in a way that leaves the index unchanged and if the consumer's income and the prices of non-entertainment goods remain unchanged, then consumer welfare should also remain unchanged. We formulate this as the requirement that the two conditions (1) $P(p) = P(p')$ and (2) $p_j = p'_j$ for $j = k + 1, \dots, n$ imply that for all u , $e(p; u) = e(p', u)$. Third, one should be able to compute the demands for non-entertainment goods from the prices for those goods and the price index for entertainment goods. This is formalized by the requirement that conditions (1) and (2) above should also imply $h_j(p) = h_j(p')$ for $j = k + 1, \dots, n$.

The second requirement above is a separability requirement, reminiscent of the one we analyzed in the note on choice theory. In choice theory, separability was used to decompose *choices*: we required that the decision maker's ranking of choices from one set does not depend on the choices specified from another set. Here, separability is used to decompose the price vector: we require that the welfare ranking of entertainment price vectors should not depend on non-entertainment prices. As in choice theory, this separability implies a particular structure for the ranking function—there a utility function, here the expenditure function. Separability implies that there exist two functions $P : \mathbb{R}^k \rightarrow \mathbb{R}$ and $\hat{e} : \mathbb{R}^{2+n-k} \rightarrow \mathbb{R}$, with \hat{e} increasing in its first argument, such that for all p and u ,

$$e(p, u) = \hat{e}(P(p), p_{k+1}, \dots, p_n, u).$$

We leave the proof as an exercise. By Shepard's lemma, when separability applies, we may further conclude that for $j = k + 1, \dots, n$,

$$h_j(p, u) = \frac{\partial}{\partial p_j} e(p, u) = \frac{\partial}{\partial p_j} \hat{e}(P(p), p_{k+1}, \dots, p_n, u) \equiv \hat{h}_j(P(p), p_{k+1}, \dots, p_n, u).$$

where the last equality defines the function \hat{h}_j .

Notice that the isoquants of P are the same as the “restricted” isoquants of e , where the restriction is to changes in the prices of the first k goods. If P is differentiable with non-zero derivatives, then we can characterize the slopes of the isoquants of P as follows. Applying the chain rule for any $1 \leq i < j \leq k$,

$$\frac{\partial e / \partial p_i}{\partial e / \partial p_j} = \frac{\partial \hat{e} / \partial P}{\partial \hat{e} / \partial P} \cdot \frac{\partial P / \partial p_i}{\partial P / \partial p_j} = \frac{\partial P / \partial p_i}{\partial P / \partial p_j}.$$

The construction so far guarantees the existence of a function P , but not one that is homogeneous of degree one. The next step is to convert the function P into an index \hat{P} that is homogeneous of degree 1. To that end, fix an arbitrary positive price vector prices $p \gg 0$ and normalize the index by setting $\hat{P}(p) = 100$. For any price vector $p' \gg 0$, there is a unique $\alpha > 0$ such that $P(p') = P(\alpha p)$: define $\hat{P}(p') = 100\alpha$. The index \hat{P} defined in this way is homogeneous of degree one. (We leave it as an exercise to derive the existence of such a unique α and the homogeneity of \hat{P} using our assumptions and the properties of the expenditure function.)

To summarize, a price index satisfying all the requirements set out above exists if and only if the expenditure function is separable, that is, if and only if there exist functions \hat{e} and P as described above. The separability of the expenditure function described here is different from the separability of the consumer’s utility function. Neither separability condition implies the other, and both conditions can be useful for creating tractable models for both theoretical and empirical inquiries.