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THE COHOMOLOGY THEORY OF ABSTRACT ALGEBRAIC VARIETIES

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It is less than four years since cohomological methods (i.e. methods of Homological Algebra) were introduced into Algebraic Geometry in Serre's fundamental paper[11], and it seems already certain that they are to overflow this part of mathematics in the coming years, from the foundations up to the most advanced parts. All we can do here is to sketch briefly some of the ideas and results. None of these have been published in their final form, but most of them originated in or were suggested by Serre's paper.

Let us first give an outline of the main topics of cohomological investigation in Algebraic Geometry, as they appear at present. The need of a theory of cohomology for 'abstract' algebraic varieties was first emphasized by Weil, in order to be able to give a precise meaning to his celebrated conjectures in Diophantine Geometry[20]. Therefore the initial aim was to find the '*Weil cohomology*' *of an algebraic variety*, which should have as coefficients something 'at least as good' as a field of *characteristic* 0, and have such formal properties (e.g. duality, Künneth formula) as to yield the analogue of Lefschetz's 'fixed-point formula'. Serre's general idea has been that the usual 'Zariski topology' of a variety (in which the closed sets are the algebraic subsets) is a suitable one for applying methods of Algebraic Topology. His first approach was hoped to yield at least the right Betti numbers of a variety, it being evident from the start that it could not be considered as the Weil cohomology itself, as the coefficient field for cohomology was the ground field of the variety, and therefore not in general of characteristic 0. In fact, even the hope of getting the 'true' *Betti numbers* has failed, and so have other attempts of Serre's[12] to get Weil's cohomology by taking the cohomology of the variety with values, not in the sheaf of local rings themselves, but in the sheaves of Witt-vectors constructed on the latter. He gets in this way modules over the ring $W(k)$ of infinite Witt vectors on the ground field k, and $W(k)$ is a ring of characteristic 0 even if k is of characteristic $p \neq 0$. Unfortunately, modules thus obtained over $W(k)$ may be infinitely generated, even when the variety V is an abelian variety[13]. Although interesting relations must certainly exist between these cohomology groups and the 'true ones', it seems certain now that the Weil cohomology has to be defined by a completely different approach. Such an approach was recently suggested to me by the *connections between sheaf-theoretic cohomology and cohomology of Galois groups on the one hand, and the classification of unramified coverings of a variety on the other* (as explained quite unsystematically in Serre's tentative Mexico paper[12]), and by Serre's idea that a 'reasonable' algebraic principal fiber space with structure group G, defined on a variety V, if it is not locally trivial, should become locally trivial on some covering of V *unramified* over a given point of V. This has been the starting-point of a definition of the Weil cohomology (involving both 'spatial' and Galois cohomology), which seems to be the right one, and which gives clear suggestions how Weil's conjectures may be attacked by the machinery of Homological Algebra. As I have not begun these investigations seriously as yet, and as moreover this theory has a quite distinct flavor from the one of the theory of algebraic coherent sheaves which we shall now be concerned with, we shall not dwell any longer on Weil's cohomology. Let us merely remark that the definition alluded to has already been the starting-point of a theory of cohomological dimension of fields, developed recently by Tate[18].

The second main topic for cohomological methods is the *cohomology theory of algebraic coherent sheaves*, as initiated by Serre. Although inadequate for Weil's purposes, it is at present yielding a wealth of new methods and new notions, and gives the key even for results which were not commonly thought to be concerned with sheaves, still less with cohomology, such as Zariski's theorem on 'holomorphic functions' and his 'main theorem'—which can be stated now in a more satisfactory way, as we shall see, and proved by the same uniform elementary methods. The main parts of the theory, at present, can be listed as follows:

(*a*) General finiteness and asymptotic behaviour theorems.

(*b*) Duality theorems, including (respectively identical with) a cohomological theory of residues.

(*c*) Riemann–Roch theorem, including the theory of Chern classes for algebraic coherent sheaves.

(*d*) Some special results, concerning mainly abelian varieties.

The third main topic consists in the *application of the cohomological methods to local algebra*. Initiated by Koszul and Cartan–Eilenberg in connection with Hilbert's 'theorem of syzygies', the systematic use of these methods is mainly due again to Serre. The results are the *characterization* of regular local rings as those whose global cohomological dimension is finite, the clarification of *Cohen–Macaulay's equidimensionality theorem* by means of the notion of *cohomological codimension*[23], and especially the possibility of giving (for the first time as it seems) a *theory of intersections*, really satisfactory by its algebraic simplicity and its generality. Serre's result just quoted, that regular local rings are the only ones of finite global cohomological dimension, accounts for the fact that only for such local rings does a satisfactory theory of intersections exist. I cannot give any details here on these subjects, nor on various results I have obtained by means of a *local duality theory*, which seems to be the tool which is to replace differential forms in the case of unequal characteristics, and gives, in the general context of commutative algebra, a clarification of the notion of residue, which as yet was not at all well understood. The motivation of this latter work has been the attempt to get a global theory of duality in cohomology for algebraic varieties admitting arbitrary singularities, in order to be able to develop intersection formulae for cycles with arbitrary singularities, in a non-singular algebraic variety, formulas which contain also a 'Lefschetz formula mod. p'[8]. In fact, once a proper local formalism is obtained, the global statements become almost trivial. As a general fact, it appears that, to a great extent, the 'local' results already contain a global one; more precisely, global results on varieties of dimension n can frequently be deduced from corresponding local ones for rings of Krull dimension $n+1$.

We will therefore turn now to giving some main ideas in the second topic, that is, the cohomology theory of algebraic coherent sheaves. First, I would like, however, to emphasize one point common to *all* of the topics considered (except *perhaps* for (*d*)), and in fact to all of the standard techniques in Algebraic Geometry. Namely, that the natural range of the notions dealt with, and the methods used, are not really algebraic varieties. Thus, we know that an *affine* algebraic variety with ground field k is determined by its co-ordinate ring, which is an arbitrary finitely generated k-algebra without nilpotent elements; therefore, any statement concerning affine algebraic varieties can be viewed also as a statement concerning rings A of the previous type. Now it appears that most of such statements make sense, and are true, if we assume only A to be a commutative ring with unit, provided we sometimes submit it to some mild restriction, as being noetherian, for instance. In the same way, most of the results proved for the local rings of algebraic geometry, make sense and are true for arbitrary noetherian local rings. Besides, frequently when it seemed at first sight that the statement only made sense when a ground field k was involved, as in questions in which differential forms are considered, further consideration of the matter showed that this impression was erroneous, and that a better understanding is obtained by replacing k by a ring B such that A is a finitely generated B-algebra. Geometrically, this means that instead of a single affine algebraic variety V (as defined by A) we are considering a 'regular map' or 'morphism' of V into another affine variety W, and properties of the variety V then are generalized to properties of a morphism $V \to W$ (the 'absolute' notion for V being obtained from the more general 'relative' notion by taking W reduced to a point). On the other hand, one should not prevent the rings having nilpotent elements, and by no means exclude them without serious reasons. Now just as arbitrary commutative rings can be thought of as a proper generalization of *affine* algebraic varieties, one can find a corresponding suitable generalization of arbitrary algebraic varieties (defined over an arbitrary field). This was done by Nagata[9] in a particular case, yet following the definition of schemata given by Chevalley[4] he had to stick to the irreducible case, and with no nilpotent elements involved. The principle of the right definition is again to be found in Serre's fundamental paper[11], and is as follows. If A is any commutative ring, then the set Spec(A) of all prime ideals of A can be turned into a topological space in a classical way, the closed subsets consisting of those prime ideals which contain a given subset of A. On the other hand, there is a sheaf of rings defined in a natural way on Spec (A), the fiber of this sheaf at the point p being the local ring A_p. More generally, every module M over A defines a sheaf of modules on Spec(A), the fiber of which at the point p is the localized module M_p over A_p. Now we call *pre-schema* a topological space X with a sheaf of rings $\mathcal{O}_X$ on X, called its *structure sheaf*, such that every point of X has an open neighborhood isomorphic to some Spec(A). If X and Y are two pre-schemas, a *morphism* f from X into Y is a continuous map $f: X \to Y$, together with a corresponding homomorphism $f^*: \mathcal{O}_Y \to \mathcal{O}_X$ for the structure sheaves, submitted to the one condition: if $x \in X$, $y = f(x)$, then the inverse image by $f^*: \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ of the maximal ideal in $\mathcal{O}_{X,x}$ is the maximal ideal in $\mathcal{O}_{Y,y}$. If X and Y are the prime spectra of rings A and B, then it can be shown that the morphisms from X into Y correspond exactly to the ring homomorphisms of B into A, as they should.

As was explained before, if we consider morphisms $f: X \to S$ with a fixed pre-schema S, S plays the part of a ground-field. Besides, if $S = \mathrm{Spec}(A)$, then X is a pre-schema *over* S if and only if the sheaf of rings $\mathcal{O}_X$ is a sheaf of A-algebras. In the category of all pre-schemata over a given S, there exists a product (which corresponds to the tensor product of algebras over a commutative ring A). Using this, one can define objects like pre-schemata of groups, etc., over a fixed ground-pre-schema S. One can also use products in order to introduce a mild separation condition on pre-schemata, suggested by the usual condition in the definition of algebraic varieties, so that one gets what can be called a *schema*. A schema is called *noetherian* if it is the union of a finite number of open sets isomorphic to the prime spectra of *noetherian* rings. A schema X over another S is called of *finite type over* S (or the morphism $f: X \to S$ is called of finite type) if, for every affine open set U in S, $f^{-1}(U)$ is the union of a finite number of affine open sets, corresponding to *finitely generated* algebras over the ring of U. Most of the notions and results of usual Algebraic Geometry can now be stated and proved in this new context, provided essentially that in some questions one sticks to noetherian schemata and to morphisms which are of finite type. Let us only remark here that the notion of a *complete* variety yields the notion of a *proper morphism* (which in the case of Algebraic Geometry over the complex numbers is the same as proper in the usual topological sense): the morphism $f: X \to S$ is called proper if it is of finite type, and if for every noetherian schema Y over S, the projection $X \times_S Y \to Y$ is a closed map. One can define also projective and quasi-projective morphisms, corresponding to the notions of projective and quasi-projective varieties. Many properties on general morphisms can be reduced to the case of projective or quasi-projective ones, using a suitable generalization of Chow's well-known lemma.

A sheaf of modules F on the schema X is called *quasi-coherent* if on each affine open set U in X, it can be defined by a module M over the ring A of U. If X is noetherian, then F is coherent (in the general sense of[11]) if and only if it is quasi-coherent, and the modules M are finitely generated. Quasi-coherent and coherent algebraic sheaves behave as nicely under the usual sheaf-theoretic operations (tensor products, sheaves of homomorphisms, direct and inverse images, and derived functors of the previous ones) as could possibly be expected from them.

We are now in a position to state results in the proper context. We shall limit ourselves, however, to (*a*) and (*b*). The Riemann–Roch theorem, proved independently by Washnitzer[19] and myself[2] in abstract algebraic geometry, will be exposed by Hirzebruch at this Congress. Let us only remark that the present formulation of this theorem, as suggested by Hirzebruch's formula, is substantially stronger than the latter, because as usual a statement about a *complete variety* is replaced by one about a *proper morphism*. As for the 'special results' alluded to in (*d*), due to Barsotti, Cartier, Rosenlicht and Serre, let us only state that one knows about all one wants and expects on the cohomology of an abelian variety, and a great deal of the relations between the cohomology of a variety and its Albanese variety. In particular, one gets by cohomological methods the absence of 'torsion' in an abelian variety[1, 3a, 13] and the biduality of abelian varieties[3]. As yet, there has been no question of stating such results in the general context of a schema over another, although this is certainly a reasonable one.

The main results in the cohomology theory of morphisms of schemata are the following (Theorems 1 and 2 being rather straightforward adaptations of Serre's results[11]).

Theorem 1. Let F be a quasi-coherent sheaf on the affine schema X, defined by a module M over the co-ordinate ring A of X. Then $H^i(X, F) = 0$ for $i > 0$, and $H^0(X, F) = M$.

Of course, the cohomology groups are taken in the general sense of cohomological algebra in abelian categories[5, 6]. It is important for technical reasons not to take as a *definition* of cohomology the Čech cohomology as was done in[11]; in virtue of Leray's spectral sequence for a covering, Theorem 1 implies that the cohomology groups of a quasi-coherent sheaf on a schema X can be computed by the Čech method, but this should be considered as an accidental phenomenon. There is a converse to Theorem 1[17], to the effect that if X is a noetherian pre-schema for which $H^1(X, F) = 0$ for every coherent subsheaf F of $\mathcal{O}_X$, then X is affine. This can be shown in this general case by a suitable adaptation of Serre's proof.

Let us recall that if f is a continuous map from a space X into another Y, then for every abelian sheaf F on X, one defines the direct image $f_*(F)$ of F by f to be the sheaf on Y the sections of which on an open set $U \subset Y$ are the sections of F on $f^{-1}(U)$. Taking the right derived functors of the functor f_*, we get the 'higher direct images' $R^qf_*(F)$ of F by f. The sheaf $R^qf_*(F)$ is the sheaf associated to the pre-sheaf

$$U \to H^q(f^{-1}(U), F),$$

and is well known as entering in the beginning term of Leray's spectral sequence for the continuous map f and the sheaf F. In case f is a morphism of schemata and F is an algebraic sheaf, the higher direct images are algebraic sheaves, which are quasi-coherent whenever F is quasi-coherent and f of finite type. In this case, the group of sections of $R^q f_*(F)$ on an *affine* open set U of Y is *identical* with $H^q(f^{-1}(U), F)$, as follows at once from Leray's spectral sequence, for instance. Theorem 1 is easily generalized to a property of affine morphisms $f: X \to Y$, i.e. morphisms for which the inverse image of an affine open set is affine (which in fact is a property local with respect to Y): if f is affine, then for every quasi-coherent sheaf F on X, we have $R^q f_*(F) = 0$ for $q > 0$, and the converse holds if X is noetherian.

The next theorem is concerned with *projective morphisms*. Let Y be a pre-schema, and let $\mathscr{S}$ be a quasi-coherent sheaf of graded algebras on Y. For the sake of simplicity, we suppose that $\mathscr{S}$ has only positive degrees, and is generated (as a sheaf of $\mathscr{O}_Y$-algebras) by $\mathscr{S}^1$. Then by generalizing well-known constructions, one can define a pre-schema X over Y (in fact a schema if Y is one itself), and on X an algebraic sheaf locally isomorphic to $\mathscr{O}_X$, denoted by $\mathscr{O}_X(1)$. More generally, for every quasi-coherent sheaf $\mathscr{M}$ of graded $\mathscr{S}$-modules, one defines a quasi-coherent sheaf $F(\mathscr{M})$ on X, the functor $\mathscr{M} \to F(\mathscr{M})$ being exact and compatible with the usual operations such as tensor products and $\mathscr{T}or_i$, $\mathscr{H}om$ sheaves and $\mathscr{E}xt^i$. Taking $\mathscr{M} = \mathscr{S}$, one gets $\mathscr{O}_X$, and shifting degrees by n in S one gets sheaves $\mathscr{O}_X(n)$, which can (by what we just stated) be obtained as n-fold tensor products of $\mathscr{O}_X(1)$ with itself. The definitions are such that, if f is the projection $f: X \to Y$, then we have a natural homomorphism (compatible with f) of $\mathscr{M}^0$ into $F(\mathscr{M})$, and hence also from $\mathscr{M}^n$ into $F(\mathscr{M}(n)) = F(\mathscr{M})(n)$, where for every algebraic sheaf G on X, we denote by $G(n)$ the tensor product $G \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$. The formalism just sketched, and in particular getting a pre-schema over Y from a graded quasi-coherent sheaf $\mathscr{S}$ of algebras on Y, should be considered as the natural general description of the 'blowing-up' process: this is obtained by taking a coherent subsheaf $\mathscr{J}$ of $\mathscr{O}_Y$, taking the subsheaves $\mathscr{J}^n$ of $\mathscr{O}_Y$ generated by this sheaf of ideals, and taking for $\mathscr{S}$ the direct sum of the sheaves $\mathscr{J}^n$, which is a graded sheaf of algebras in an obvious way. A pre-schema X over Y is called *projective over* Y (and the morphism $f: X \to Y$ is called a *projective morphism*) if X can be obtained, by the process alluded to above, from a sheaf $\mathscr{S}$ such that $\mathscr{S}^1$ is a finitely generated sheaf (i.e. a coherent sheaf if Y is assumed noetherian). Thus, if Y is an affine noetherian schema which is of integrity (i.e. irreducible and $\mathcal{O}_Y$ without nilpotent elements) then the schemata over Y which are projective over Y, and for which the projection $f: X \to Y$ is a birational equivalence, are those which can be obtained from Y by blowing up a coherent sheaf of ideals on Y (so that we get 'practically all' birational proper morphisms by the standard blowing up process). Let us also remark that if we take for $\mathcal{S}$ the symmetric algebra of a locally free sheaf of modules $\mathcal{V}$ on Y, then the corresponding X over Y should be looked at as the fiber space with projective spaces as fibers, which corresponds to the 'vector bundle' defined by the sheaf $\mathcal{V}'$ dual to $\mathcal{V}$. (In fact, all 'geometric' constructions of Algebraic Geometry can be carried over in the context of schemata.) The main facts about the cohomological theory of projective morphisms are stated in the following theorem:

Theorem 2. Let $f: X \to Y$ be a projective morphism, defined by a graded sheaf of algebras $\mathcal{S}$ on Y. Y is supposed noetherian, and we suppose that $\mathcal{S}^1$ is generated as a sheaf of modules by r generators (locally). Then the following statements hold for every coherent sheaf F on X:

(i) $F(n)$ is 'generated by its sections' for n large, provided we restrict to the points of X which lie above an *affine* open set of X;

(ii) $R^q f_*(F(n)) = 0$ for $q > 0$, n large;

(iii) $R^q f_*(F) = 0$ for $q > r$;

(iv) The sheaves $R^q f_*(F)$ are coherent.

The first statement (i) implies also that *every* coherent sheaf F on X can be obtained from a suitable quasi-coherent sheaf of graded modules over $\mathcal{S}$.

The next two theorems are proved by first dealing with the case of a projective morphism (N.B. projective morphisms are proper). Then the statements reduce to statements on graded modules over a polynomial ring $A[X_1, \ldots, X_n]$, with A noetherian, and can be proved by an easy *decreasing induction* on the dimension i in the cohomology. This explains why a complete statement and a simple proof of these results, even restricted to $i = 0$ (i.e. when not concerned with genuine cohomology), could not be achieved by non-cohomological methods. The case of a general morphism is then reduced to the case of a projective one by Chow's lemma, as in[7].

Theorem 3. Let $f: X \to Y$ be a proper morphism, Y being noetherian. Then, for every coherent sheaf F on X the sheaves $R^q f_*(F)$ are coherent.

Theorem 4. Let $f: X \to Y$ and F be as before, let y be a point of Y, then $R^q f_*(F)_y$ is a finitely generated module over the local ring $\mathcal{O}_y$, and the m_y-adic completion of this module is naturally isomorphic to

$$\varprojlim_k H^q(f^{-1}(y), F \otimes_{\mathcal{O}_y} (\mathcal{O}_y/m_y^k))$$

(where m_y is the maximal ideal of $\mathcal{O}_y$).

This should be considered as the complete statement of Zariski's main result on 'holomorphic functions'[21]. Here, the lim group should be considered as the 'holomorphic cohomology' of X along the fiber $f^{-1}(y)$, with coefficients in the sheaf F. From Theorem 4, one gets a result which is global with respect to Y: if Y' is a closed subset of Y, and $X' = f^{-1}(Y')$, then the holomorphic cohomology of X along X' (with coefficients in F) is the ending of a spectral sequence of cohomological type, the beginning term of which is $E_2^{p,q} = H^p(Y/Y', R^q f_*(F))$ (where the second member denotes the holomorphic cohomology of Y along Y').

Theorem 4 yields at once Zariski's connectedness theorem, in the following general form. Let $f: X \to Y$ be a proper morphism, Y noetherian. Then by Theorem 3 the direct image $f_*(\mathcal{O}_X)$ is a *coherent* sheaf of $\mathcal{O}_Y$-algebras $\mathcal{B}$ on Y. Let $y \in Y$, then $\mathcal{B}_y$ is an $\mathcal{O}_y$-algebra which is a finitely generated module. It follows at once from Theorem 4 (with $q = 0$) that the set of connected components of the fiber $f^{-1}(y)$ is in one to one correspondence with the maximal ideals of $\mathcal{B}_y$ (which of course are finite in number). If, for instance, X and Y are of integrity, f surjective, then the field K of Y is a subfield of the field L of X, and $\mathcal{B}_y$ is a subring of L which is integral over the subring $\mathcal{O}_y$ of K. If the integral closure of $\mathcal{O}_y$ in K has only one maximal ideal (we then say that y is a 'unibranch point' of Y) and if the algebraic closure of K in L is purely inseparable over K, then it follows at once that $\mathcal{B}_y$ has only one maximal ideal. Therefore, the fiber of f at y is connected. (N.B. no analytic irreducibility of $\mathcal{O}_y$ was needed.)

We can state things in a more geometric way, by using the fact that the coherent sheaf $f_*(\mathcal{O}_X) = \mathcal{B}$ of $\mathcal{O}_Y$-algebras defines in a natural way a pre-schema Y' over Y (characterized by the condition that Y' be affine over Y and the direct image of $\mathcal{O}_{Y'}$ should be $\mathcal{B}$). It follows from the Cohen–Seidenberg theorems that Y' is also proper over Y, and that the fibers in the projection $Y' \to Y$ are finite. (Conversely, it follows from Theorems 3 and 4 that any Y' over Y having these properties can be defined by a coherent sheaf of algebras on Y. We shall say that such a pre-schema Y' over Y is *integral over* Y.) Using the fact that we have an isomorphism $B \xrightarrow{\sim} f^*(\mathcal{O}_X)$, one sees that $f: X \to Y$ can be factored in a natural way as $X \xrightarrow{f'} Y' \xrightarrow{g} Y$, where now f' is such that $f'_*(\mathcal{O}_X) = \mathcal{O}_{Y'}$. Zariski's connectedness theorem can now be stated in the following way: the connected components of $f^{-1}(y)$ are in a one to one correspondence with the elements of $g^{-1}(y)$, or equivalently: the fibers of f' are connected. (This canonical factorization of f was suggested by work of Stein on analytic spaces.)

Using Theorems 3 and 4 and the connectedness theorem, one obtains also, by global and cohomological methods, the most general statement of Zariski's 'main theorem' in commutative algebra, which we will state here in the geometric language:

Theorem 5. Let $f: X \to Y$ be a quasi-projective morphism from a schema X into a noetherian schema Y. The points in X which are isolated in their fiber $f^{-1}(f(x))$ form an open set U. There exists a schema Y' integral over Y, and an isomorphism of U onto an open subset U' of Y', such that the restriction of f to U is identical to the compositum $U \to Y' \to Y$.

This statement is somewhat more general than the usual purely local statement. It should be remarked that the local rings of X and Y may contain nilpotent elements. The proof of Theorem 5 is obtained by first reducing to the case when X is proper over Y. Then Y' is the schema over Y constructed above with the sheaf $f_*(\mathcal{O}_X)$, and the fact that the canonical factorization of f induces an isomorphism from U onto an open subset of Y' is easily deduced from the connectedness theorem. The fact that there should exist cohomological proofs of Zariski's connectedness theorem and of his 'main theorem' was first conjectured by Serre. That these results hold also for general schemata, not only those of algebraic geometry, yields results such as the following (which I understand is due to Chow): let A be a noetherian local integrity domain which is normal (or more generally, 'unibranch'); then the algebraic projective space over the residue field k of A defined by the graded k-algebra $G(A) = \Sigma m^n/m^{n+1}$ (m = maximal ideal of A) is connected.

It remains to say a few words on the duality theorems, although time does not permit us to give precise statements. Let us recall first Serre's original statement[22]: if X^n is a non-singular projective variety defined over an algebraically closed field k, E a vector bundle over X, then there is a natural duality between $H^i(X, \mathcal{O}_X(E))$ and $H^{n-i}(X, \Omega^n_X(E'))$, where $\mathcal{O}_X(E)$ is the sheaf of germs of regular sections of E, and $\Omega^n_X(E')$ the sheaf of germs of regular n-differential forms of X with values in the dual vector-bundle E'. This duality is obtained by cup-product, using the fact that $H^n(X, \Omega_X^n)$ itself is naturally isomorphic to the ground-field k. Various generalizations of this result, and applications to Poincaré duality (including a Lefschetz formula) were given in [8]. However, the case of singular varieties was not taken into account, so that the statements on Poincaré duality were necessarily restricted to non-singular cycles and non-singular intersections of such cycles. Moreover, the proper algebraic foundations for the *covariant* features of the cohomology of non-singular varieties, as suggested by Poincaré duality, remained unclear. By developing a general theory of residues, this situation can now be considered as (potentially) completely clarified. To simplify matters, let us consider schemata of finite type over an arbitrary ground-field k, i.e. algebraic spaces defined over k, the local rings of X being allowed, however, to have nilpotent elements (which is technically of the highest importance). Then, on such an X, there is a canonically defined complex of algebraic quasi-coherent sheaves $\mathscr{K}_X$, with positive degrees, and a differential operator of degree -1. In dimension i, $\mathscr{K}_X$ is the direct sum of sheaves $\mathscr{D}(X/Y)$, where Y runs over the set of all closed irreducible subsets of X of dimension i. The sheaf $\mathscr{D}(X/Y)$ is the extension to X of a *constant* sheaf on Y, corresponding to a module over the local ring $\mathcal{O}_{X/Y}$ of Y in X, which should be called the *dual module* of this local ring. In general, if A is a locality, over the ground-field k (that is, a local ring of a finitely generated k-algebra) there is a well-defined A-module $D(A)$, called its dual (with respect to k), which can be obtained in the following way: take a subfield L of A separable over k such that the residue field of A be algebraic over L, then

$$D(A) = \mathrm{Hom}_L(A, \Omega^p(L)),$$

where Hom denotes the *continuous* homomorphisms, and where $\Omega^p(L)$ is the vector space (of dimension 1) over L of the highest dimensional differential forms of L with respect to k. (The fact that the second member does not depend on the choice of L is not trivial, and is obtained as a consequence of an alternative cohomological definition of $D(A)$ as a direct limit of Ext modules.) The sheaf $\mathscr{K}_X$ is an *injective* sheaf of $\mathcal{O}_X$-modules. The definition of its differential operator is a rather subtle one. The complex of sheaves $\mathscr{K}_X$ should be called the *residue complex* of X (or the complex of '*generalized Cousin data*' of X). It turns out that if X is non-singular and separable over k, then $\mathscr{K}_X$ is an (injective) *resolution* of the sheaf Ω_X^n of germs of differential forms of highest degree on X. In any case, if $\dim X = n$, then the sheaf of cycles of degree n in $\mathscr{K}_X$, denoted by $\boldsymbol{\omega}_X$, should be considered as playing the part of differential forms of degree n on X. In fact, there is a natural homomorphism (defined by the well-known Kähler process) of Ω_X^n into $\boldsymbol{\omega}_X$; if X is normal and separable over k, then $\boldsymbol{\omega}_X$ is nothing else but the sheaf of germs of differential forms on X which are regular at simple points. On the other hand, if X is Cohen–Macaulay (i.e. its local rings are Cohen–Macaulay), then $\mathscr{K}_X$ is a resolution of $\boldsymbol{\omega}_X$ (and conversely). This shows that in general the sheaves $H_i(\mathscr{K}_X)$ are not zero even for $i \neq n$ (N.B. they never are for $i = n$), however, these sheaves are always coherent (although $\mathscr{K}_X$ itself is much too big to be coherent).

Now let F be an arbitrary coherent sheaf on X, we write

$$H_i(X, F) = H_i(\mathrm{Hom}_{\mathcal{O}_X}(X; F, \mathscr{K}_X)).$$

Thus the *homology* of X with coefficients in F is defined as a *contravariant* functor in F, which moreover is a 'homological functor' because the complex $\mathscr{K}_X$ is injective, i.e. $F \to \mathrm{Hom}_{\mathcal{O}_X}(X; F, \mathscr{K}_X)$ is exact. It has all the properties one expects from homology, in particular, it is *covariant* with X with respect to *proper* morphisms (the sheaf $\mathscr{K}_X$ itself behaving covariantly with X), the cohomology of X operates on the homology by *cap-product*. Relative homology and cohomology groups of various types can also be defined. If X is Cohen–Macaulay, then the above definition gives

$$H_i(X, F) \simeq \mathrm{Ext}_{\mathcal{O}_X}^{n-i}(X; F, \boldsymbol{\omega}_X). \tag{1}$$

This relation is replaced by a spectral sequence in the general case. If X is Cohen–Macaulay and moreover F is locally free, then the above relation yields

$$H_i(X, F) \simeq H^{n-i}(X, F' \otimes \boldsymbol{\omega}_X) \tag{2}$$

so that in this case, homology can be expressed as cohomology. In the general case, this relation is replaced by a spectral sequence, analogous to Cartan's spectral sequence for an arbitrary topological space, connecting its homology and its cohomology (and yielding Poincaré duality in case the space is a variety).

Using the definitions of H_n and $\boldsymbol{\omega}_X$, we see that (if $\dim X = n$) there is a canonical element in $H_n(X, \boldsymbol{\omega}_X)$, called the *canonical homology class* of X; as homology is contravariant in the argument, it follows that there is a canonical homology class in $H_n(X, \Omega_X^n)$, and as homology is covariant with the space, it follows that for each p-cycle Z in X, there is a homology class $\gamma_X(Z)$ in $H_p(X, \Omega_X^p)$. The formation of these classes is compatible with direct images of cycles and homology classes. Moreover, if X is non-singular and separable over k, then $\gamma(Z)$ can be regarded as an element of $H^{n-p}(X, \Omega_X^{n-p})$ in virtue of (2). Using some simple facts of local character connecting the intersection and inverse images of cycles with some partially defined multiplicative structure in $\mathscr{K}_X$ (for non-singular X), we get that formation of characteristic classes is compatible with products and inverse images (which solves the difficulties encountered in [8]).

Serre's duality theorem is now generalized in the following way. Suppose F is a coherent sheaf on X with complete support, then by cap-product there is a natural pairing

$$H_i(X, F) \times H^i(X, F) \to H_0(X, \mathcal{O}_X/\mathscr{J}), \tag{3}$$

where $\mathscr{J}$ is any coherent sheaf of ideals on X such that $\mathscr{J}.F = 0$. We can take $\mathscr{J}$ such that $\mathcal{O}_X/\mathscr{J}$ has complete support, i.e. is the sheaf of local rings of an algebraic subspace Y of X. Then the second member of (3) is nothing else but $H_0(Y, \mathcal{O}_Y)$, and using the sum of residues one gets a natural k-homomorphism

$$H_0(X, \mathcal{O}_X/J) = H_0(Y, \mathcal{O}_Y) \to k. \tag{3a}$$

Using (3) and (3a), one gets a pairing

$$H_i(X, F) \times H^i(X, F) \to k. \tag{4}$$

This pairing is a duality: thus for complete supports, homology and cohomology are dual to each other. Moreover, in this duality, the direct image of homology is transposed to the inverse image of cohomology (as is needed for the formalism of Poincaré duality).

We have sketched the statement of results for schemata of finite type over a field k. More general results hold in fact, and the duality theorem can be stated for any coherent sheaf F on a noetherian schema X, such that the support of F is complete (i.e. proper over some Artin ring A, which need not be a field). Combining this with Theorem 4, one gets an equivalent statement, concerned with the homological properties of an arbitrary proper morphism (the basis Y being noetherian).

To conclude this talk, I should state some unsolved questions. In fact it is perhaps too early to do so, as the available new techniques have not been tried seriously enough to know whether or not they are able to solve these questions. The following two (which are probably related) will perhaps show some resistance. For the sake of brevity, we state them in the context of Algebraic Geometry.

Problem A. (Kodaira's vanishing theorem.) Let V be a non-singular projective variety, L a negative line bundle on V (i.e. such that some negative multiple of L defines a projective embedding of V). If V is of dimension n, is it true that $H^i(X, \mathcal{O}_X(L)) = 0$ for $0 \leqslant i < n$?

It is not difficult to show, using duality, that this problem is equivalent to the following one, which no longer involves cohomology: Let V and L be as before, and let W be a non-singular hyperplane section of V (with respect to some projective embedding of V). Then is it true that every regular $(n-1)$-form, on W, with coefficients in L, is the residue of a rational differential n-form on V, with coefficients in L, having a divisor $\geqslant -W$?

Problem B. Let f be a birational proper morphism of a non-singular variety X into another Y. Is it true that $R^q f_*(\mathcal{O}_X) = 0$ for $q > 0$?

Using Leray's spectral sequence, this is equivalent to the problem: if $f\colon X \to Y$ is as above, is it true that $H^p(X, \mathcal{O}_X) \rightleftarrows H^p(Y, \mathcal{O}_Y)$ for every p? More generally, it would follow that if E is a vector bundle on Y, F its inverse image on X, then $H^p(X, \mathcal{O}_X(F)) \rightleftarrows H^p(Y, \mathcal{O}_Y(E))$ ('birational invariance of cohomology'). It would imply that the arithmetic genus of a complete non-singular variety is a birational invariant (this is only known in the classical case when k is the field of complex numbers and X projective, by using symmetry $h^{0,p} = h^{p,0}$), more generally that (f being as before) the Todd classes of Y are the direct images by f of the Todd classes of X (as is seen by applying the Riemann–Roch formula[2]). The answer to Problem B is not known even for non-singular projective varieties over the field of complex numbers. It should be remarked that it is essential for the statement to be true, that both X and Y should be non-singular.

There are some other questions in which the cohomological methods and the heuristic insight provided by the point of view of schemata will probably prove helpful. The most important at present seem to be the following ones.

Problem C. Let X be a noetherian schema, Y a complete sub-schema. State conditions under which it is possible to 'blow down' Y to a point, i.e. to find a proper morphism of X into a schema X', mapping Y into a single point y and inducing an isomorphism of $X - Y$ onto $X' - (Y)$.

According to Grauert, this problem is closely related to problem A. Moreover, it is connected with the theory of holomorphic functions of Zariski, a necessary (yet not sufficient) condition for the blowing-down to be possible being the following (in virtue of Theorem 4): the ring of holomorphic functions of X along Y must be noetherian and of Krull dimension equal to $\dim X$.

Problem D. Let G be a schema of groups over the schema Y (G of finite type over Y), let H be a closed sub-schema of groups of G. Prove the existence of the schema G/H over X (defined by the usual universal mapping properties).

Problem E. Let X be a schema proper over another Y (Y noetherian). Prove the existence of a schema of abelian groups of Y, playing the part of a relative Picard variety with respect to the determination of locally free rank one sheaves on products $X \times_Y Z$ (Z being a variable 'parameter schema' over Y).

We shall not give here the precise definition of a 'relative Picard schema', but only remark that if this schema exists then it behaves in the simplest conceivable way with respect to change of base-space Y (which should be looked at as the analogue of a change of a base-field). Moreover, the fact that we do admit nilpotent elements in the local rings of the schemata Z provides a great amount of supplementary information on the Picard variety (especially its infinitesimal structure), which does not seem to have been obtained as yet even in the classical case. These remarks still hold if we consider the more general following situation: Given a schema of groups G over X (G of finite type over X), for every Z over Y, we consider on $X \times_Y Z$ the inverse image G^Z of G by the projection $X \times_Y Z \to X$ and the isomorphism classes of schemata over $X \times_Y Z$ which are principal under G^Z. This leads to a definition of a generalized relative Picard variety of X, with respect to G, which should be a schema over Y, and a schema of abelian groups over Y if G is abelian. It is hoped that a general existence theorem of such generalized relative Picard schema exists and will be proved in the future.

As a quite general fact, it is believed that a better insight in any part of even the most classical Algebraic Geometry will be obtained by trying to re-state all known facts and problems in the context of schemata. This work is now begun, and will be carried on in a treatise on Algebraic Geometry which, it is hoped, will be written in the following years by J. Dieudonné and myself, and which is expected to give a systematic account of all the questions touched upon in this talk.

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