THE MATHEMATICAL WORK OF MORGAN WARD

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The mathematical works of Morgan Ward fall into seven categories as follows:

I	Recurring Series	(33 papers)
II	Diophantine Equations	(8 papers)
III	Abstract Arithmetic	(12 papers)
IV	Lattice Theory	(13 papers)
V	Functional Equations	(5 papers)
VI	Numerical Analysis	(4 papers)
VII	Miscellaneous	(7 papers)

These seven topics do not represent seven periods of time. Early and late works are to be found in all categories. This is true especially of recurring series, a topic in which his sustained interest is most noticeable and one which I feel better qualified to discuss than any of the others.

The topic of recurring series belongs to the mixed additive-multiplicative theory of numbers. Each term of a recurring series being a fixed linear combination of its previous k terms is additive in nature. Yet there may be unexpected divisibility and multiplicative properties to discover. For $k = 1$ of course, we have essentially the successive powers of a given base, a truly multiplicative situation. But for $k = 2$ we have the theory of Lucas' functions with their many elegant multiplicative properties and connections with cyclotomy. As with other kinds of problems, the going gets much harder when $k \geq 3$. It was one of Morgan Ward's major goals to describe the divisibility properties to be encountered, especially when $k = 3$. At the same time he worked on the problem of strengthening the results known for $k = 2$. Indeed his last paper adds to our knowledge of classes of primes dividing the ancient series of Fibonacci, see [80]. Other papers for $k = 2$ are concerned with the so-called intrinsic divisors of $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, where α, β are the roots of $X^2 - RX + Q = 0$, namely those primes p which divide U_n but not U_m for $m < n$. The theorems are to the effect that every U_n has such a prime divisor except in a finite number of specified choices (R, Q) and then for only specified values of n. The celebrated Fibonacci number $F_{12} = 144$ is an exceptional case in point.

Results like these have practical value in proving theorems on the distribution of power residues.

Another subject of study that intrigued Ward was the so-called divisibility sequences of Marshall Hall, namely sequences U_n such that $a|b$ implies that $U_a|U_b$. Many second-order recurring series are divisibility sequences as well as some recurring series of higher order. Ward [60, 62, 65, 66] found other kinds of divisibility sequences associated with the theory of the real multiplication of Jacobi elliptic functions.

Of the many results about recurrences of order $k = 3$ we select one that is easy to state. Let W_1, W_2, W_3, ... be nonperiodic and satisfy $W_{k+1} = PW_k - QW_{k-1} + RW_{k-2}$, and suppose that $X^3 - PX^2 + QX - R = (X-a)(X-b)(X-c)$, where a, b and c are coprime integers. Then $W_r = 0$ for at most 3 values of r. Since $W_n = Aa^r + Bb^r + Cc^r$, this result is something like Fermat's Last Theorem.

On Diophantine equations, two problems may be cited.

1. Multiplicative Diophantine systems. The system

$$X^9 = Y^5 = U^4 V^4 = WRST$$

has a parametric solution involving 46127626 parameters, see [18].

2. Euler's Conjecture:[1] $X^4 + Y^4 + Z^4 = W^4$ has no nontrivial solution. In [59] Ward showed that any nontrivial solution must have $W > 10^4$. This was done by hand calculation in 1945.

Ward's abstract arithmetic papers comprise a variety of notes on subjects ranging over group theory, logic, and algebraic number theory.

Morgan Ward had unbounded enthusiasm for the work of others that appealed to his sense of beauty. In many cases he proceeded to dig in these other fields and to help uncover their treasures. It was thus that he became enthused about Garrett Birkhoff's lattice theory. Of his many papers on the subject most were concerned with so-called residuated lattices.

Ward's functional equation papers include two on the problem of continuous iteration. Let $E(x) > x$ be continuous and steadily increasing. Write $E_0(x) \equiv x$, $E_1(x) = E(x)$, $E_2(x) = E(E(x))$, etc. How can you define $E_y(x)$ for nonintegral y? Ward found that the most general solution of the problem is

$$E_y(x) = \psi(\psi^{-1}(x) + y),$$

where $\psi(x) = E_{[x]}(\theta(x - [x]))$ and θ is any continuous function that increases steadily from $\theta(0) = 0$ to $\theta(1) = 1$, see [36]. Other papers deal with the properties of the coefficients of power series of functions $E(x)$ for which $E_2(x) = x$, with reversion of power series, and with special Appel polynomials.

Numerical analysis papers are mainly notes on the practical calculation of special functions and integrals. Some of the work he did during World War II gave him a chance to consider such matters. Actually almost all of his papers are full of examples of what he is talking about. He was also an exponent of the experimental approach to research. When his ingenuity was insufficient to solve all cases of a difficult problem he never was too proud to point this out.

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[1] *Editorial note.* Euler's Conjecture was recently shown to be false by Noam Elkies, *Math. Comp.* **51** (1988), 825–835. The minimal counterexample was found by Roger Frye, and is $95800^4 + 217519^4 + 414560^4 = 422481^4$.

I have mentioned his excitable and contagious enthusiasm for the fine work of many other writers. On the other hand, like E. T. Bell, he had great contempt for those who proliferate easy empty generalizations of the great classic ideas of mathematics.

RESEARCH PUBLICATIONS OF MORGAN WARD

1. *A generalization of recurrents*, Bull. Amer. Math. Soc. **33** (1927), 477–492.

2. *General arithmetic*, Proc. Nat. Acad. Sci. U.S.A. **13** (1927), 748–749.

3. *Postulates for an abstract arithmetic*, Proc. Nat. Acad. Sci. U.S.A. **14** (1928), 907–911.

4. *A simplification of certain problems in arithmetical division*, Amer. Math. Monthly **35** (1928), 9–14.

5. *On certain functional relations*, Amer. Math. Monthly **36** (1929), 426–431.

6. *Certain expansions involving doubly infinite series*, Ann. of Math. (2) **30** (1929), 578–582.

7. *A certain class of polynomials*, Ann. of Math. (2) **31** (1930), 43–51.

8. *A generalization of a familiar theorem concerning prime numbers*, J. London Math. Soc. **5** (1930), 106–107.

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10. *Postulates for the inverse operations in a group*, Trans. Amer. Math. Soc. **32** (1930), 520–526.

11. *The characteristic number of a sequence of integers satisfying a linear recursion relation*, Trans. Amer. Math. Soc. **33** (1931), 153–165.

12. *The algebra of recurring series*, Ann. of Math. (2) **32** (1931), 1–9.

13. *Some arithmetical properties of sequences satisfying a linear recursion relation*, Ann. of Math. (2) **32** (1931), 734–738.

14. *The distribution of residues in a sequence satisfying a linear recursion relation*, Trans. Amer. Math. Soc. **33** (1931), 166–190.

15. *Conditions for the solubility of the diophantine equation $x^2 - My^2 = -1$*, Trans. Amer. Math. Soc. **33** (1931), 712–718.

16. *The linear form of numbers represented by a homogeneous polynomial in any number of variables*, Ann. of Math. (2) **33** (1932), 324–326.

17. *On the behaviour of non-static models of the universe when the cosmological term is omitted*, (with R. C. Tolman), Phys. Rev. **39** (1932), 835–843.

18. *A type of multiplicative diophantine system*, Amer. J. Math. **55** (1933), 67–76.

19. *A property of recurring series*, Proc. Nat. Acad. Sci. U.S.A. **19** (1933), 914–916.

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21. *The arithmetical theory of linear recurring series*, Trans. Amer. Math. Soc. **35** (1933), 600–628.

22. *A certain class of trigonometric integrals*, Amer. Math. Monthly **40** (1933), 340–346.

23. *The representation of Stirling's numbers and Stirling's polynomials as sums of factorials*, Amer. J. Math. **56** (1934), 87–95.

24. *On the vanishing of the sum of the Nth powers of the roots of a cubic equation*, Amer. Math. Monthly **41** (1934), 313–316.

25. *Note on the period of a mark in a finite field*, Bull. Amer. Math. Soc. **40** (1934), 279–281.

26. *Note on an arithmetical property of recurring series*, Math. Z. **39** (1934), 211–214.

27. *An arithmetical property of recurring series of the second order*, Bull. Amer. Math. Soc. **40** (1934), 825–828.

28. *Note on the iteration of functions of one variable*, Bull. Amer. Math. Soc. **40** (1934), 688–690.

29. *The numerical evaluation of a class of trigonometric series*, Amer. Math. Monthly **41** (1934), 563–565.

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31. *A determination of all possible systems of strict implication*, Amer. J. Math. **57** (1935), 261–266.

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33. *On the factorization of polynomials to a prime modulus*, Ann. of Math. (2) **36** (1935), 870–874.

34. *The diophantine equation $X^2 - DY^2 = Z^m$*, Trans. Amer. Math. Soc. **38** (1935), 447–457.

35. *A calculus of sequences*, Amer. J. Math. **58** (1936), 255–266.

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37. *The null divisors of linear recurring series*, Duke Math. J. **2** (1936), 472–476.

38. *Note on divisibility sequences*, Bull. Amer. Math. Soc. **42** (1936), 843–845.

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