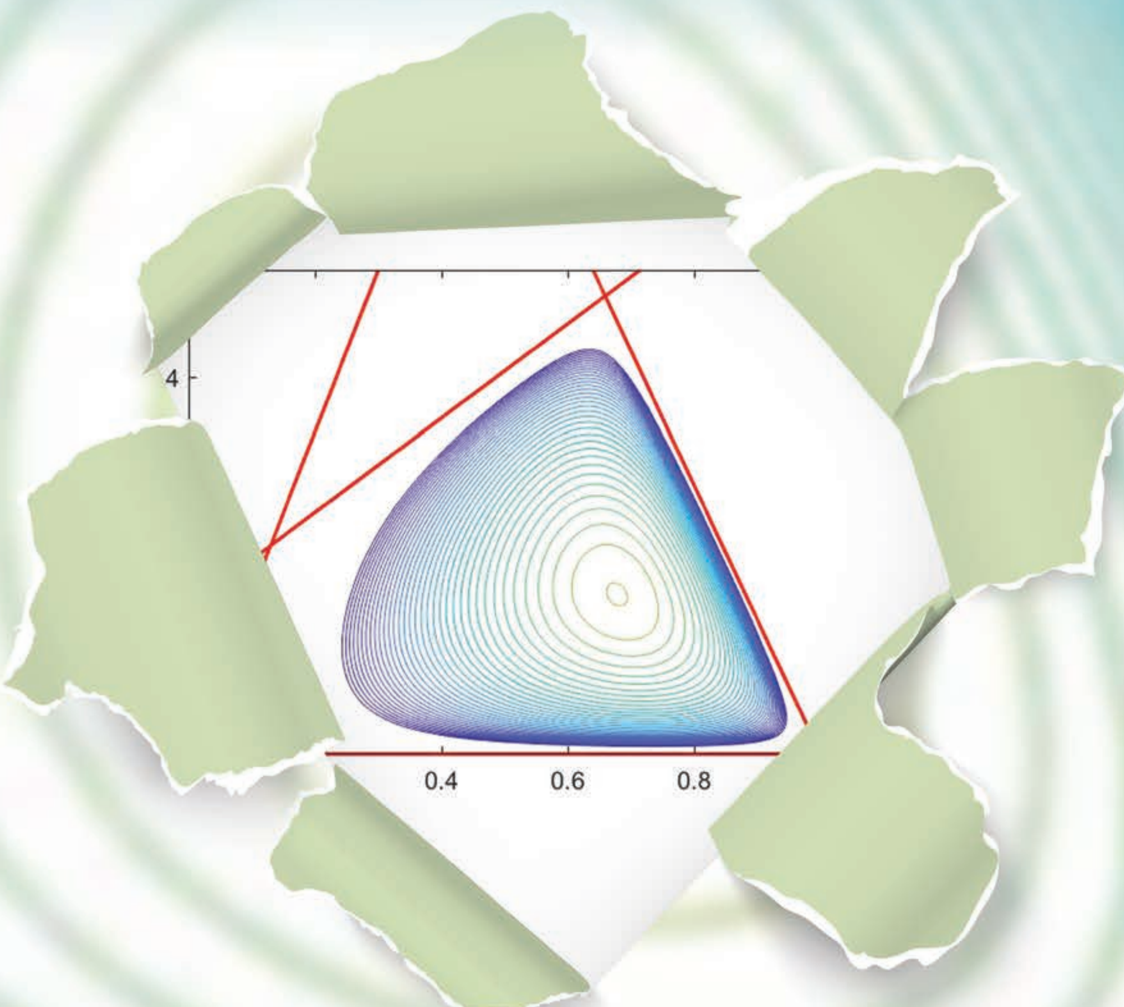
ISSN 0002-9920 (print)
ISSN 1088-9477 (online)

Notices

of the American Mathematical Society

August 2019
Volume 66, Number 7



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The cover design is based on imagery from Algebraic and Topological Tools in Linear Optimization, page 1023.

Call for Nominations

AWM–AMS NOETHER LECTURE

The Association for Women in Mathematics (AWM) established the Emmy Noether Lectures in 1980 to honor women who have made fundamental and sustained contributions to the mathematical sciences. In April 2013 this one-hour expository lecture was renamed the AWM–AMS Noether Lecture. The first jointly sponsored lecture was held in January 2015 at the Joint Mathematics Meetings (JMM) in San Antonio, Texas. Emmy Noether was one of the great mathematicians of her time, someone who worked and struggled for what she loved and believed in. Her life and work remain a tremendous inspiration.

Additional past Noether lecturers can be found at **https://awm-math.org/awards/noether-lectures.**

A nomination should include: a letter of nomination, a curriculum vitae of the candidate—not to exceed three pages, and a one-page outline of the nominee's contribution to mathematics, giving four of her most important papers and other relevant information. Nominations are accepted annually from September 1 through October 15. Nominations will remain active for an additional two years after submission.

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The nomination procedure is described here:
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If you have questions, call **401-455-4042** or email **awm@awm-math.org**.

From the AMS Secretary

ATTENTION ALL AMS MEMBERS

Voting Information for 2019 AMS Election

YOUR VOTE COUNTS!

AMS members who have chosen to vote online will receive an email message on or shortly after August 19, 2019, from the AMS Election Coordinator, Survey & Ballot Systems.

The From line will be "AMS Election Coordinator", the Sender email address will be noreply@directvote.net, and the Subject line will be "AMS 2019 Election—login information below". If you use a spam filter you may want to use the above address or subject information to configure your spam filter to ensure this email will be delivered to you.

The body of the message will provide your unique voting login information and the address (URL) of the voting website.

AMS members who have chosen to vote by paper should expect to receive their ballot by the middle of September. Unique voting login information will be printed on the ballot should you wish to vote online.

At midnight (US Eastern Time) on November 1, 2019, the website will stop accepting votes. Paper ballots received after this date will not be counted.

Additional information regarding the 2019 AMS Election is available on the AMS website: www.ams.org/election-info or by contacting the AMS: election@ams.org, 800.321.4267, extension 4128 (US & Canada), 401.455.4128 (worldwide).

Thank you and . . . please remember to vote.

Carla D. Savage

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A WORD FROM…

David Jerison, AMS Vice President

Photo is courtesy of David Jerison.

I am writing to discuss with you how our community should adapt as it grows and spreads beyond my own ability to grasp. I believe that we live in a golden age of mathematics. There are more mathematicians than ever, and rather than running out of research problems to solve or getting permanently stuck in most of them, we keep finding new connections; the outside world keeps demanding more of us.

At the same time, it has become harder for young mathematicians to progress in their careers. Given the projected number of tenure track positions relative to the number of PhDs, we can expect this trend to persist. Other issues, such as how we deliver education, professional training, access, and fairness, and how we learn about mathematics outside of our own specialties, always need the attention of the AMS. I will be concentrating on the career issue.

One virtue of the environment for mathematicians in the US is how disorganized it is. In other countries, whole cohorts of mathematicians can have their schooling and careers determined by a government initiative or the absence thereof. The US job market is too incoherent for that. There is, however, such a thing as too much chaos. The AMS has tempered some of the piecemeal aspects of the academic job market with `MathJobs.Org`. I wish there were such a system for graduate school applications. Perhaps we can cooperate to create one out of the existing AMS `MathPrograms.Org` site, where universities have begun listing programs at all levels. [See sidebar on facing page.]

While academic career services remain central to the AMS, the Society needs to develop new mechanisms and resources for the expanding mathematical job market outside academia. It can use assistance and suggestions from job seekers and employers. The April *Notices* featured several essays about jobs outside of academia in the new Early Career section that I recommend to everyone. We could use dozens more like them. I also call your attention to the blogs at the AMS site `https://blogs.ams.org`, which include discussions of academic careers.

We should be comparing notes on how each institution is dealing with the current job territory. In my department at MIT, some current graduate students have taken the initiative to advise their peers on how to get job interviews. We have invited alumni(ae) back to explain their experiences. What is happening in your department?

I am optimistic that as more mathematicians of all types work in industrial settings, those career paths will become easier to pursue. I also think that we as teachers will find it easier to explain the relevance of many mathematical ideas. Moreover, I expect that with this sociological change, even so-called pure mathematicians will be swept up in new streams of research and new perspectives on old questions.

In *The Music Man*, the stage is set with the lament that it's "different than it was" and the rejoinder "No it ain't, no it ain't, but you gotta know the territory." In my view it is, indeed, different now, and we have to learn to navigate the new territory collectively.

David Jerison is a professor of mathematics at the Massachusetts Institute of Technology and a vice president of the American Mathematical Society. His email address is `jerison@math.mit.edu`.

MathPrograms.Org

- MathPrograms.Org is a website for collecting applications to mathematical programs serving populations that range from high school students through graduate students to senior researchers. Sponsored by the AMS in cooperation with the Duke University Department of Mathematics, it is a clone of MathJobs.Org, and many of its features will be familiar to users of that system.
- MathPrograms.Org allows programs to collect all application materials at a single site. Applicants can upload any documents specified by the program and answer any set of questions the program chooses to ask. Letter-writers can upload confidential letters of recommendation, which applicants can mark for inclusion in their applications to a particular program. Each program designates the individuals who can access the applications for the purpose of reading and rating them, and confidential EEO information can be collected and made available to designated individuals. Responses to applicants can also be made through the system.
- MathPrograms.Org is convenient for applicants, who register once and then can apply to any program on the site, and for letter-writers, who need upload each letter only once, no matter how many applications it is used for. Readers can access all application materials at a single site, and application documents and data can be downloaded.
- There are currently 63 groups or programs using MathPrograms.Org, including REUs, grant programs, institutes, and honors of various kinds. Some institutions use the system for graduate admissions, since its features are well suited to that task. In 2019-2020 the fee will be $310 for one program and $590 for up to seven programs. With the university's cooperation, it is possible to upload information from MathPrograms.Org to university application systems.

For more information about MathPrograms.Org, contact Kim Kuda in the AMS Professional Programs Department at mathprograms@ams.org or 800-321-4267, ext. 4096.

Notices
of the American Mathematical Society

August 2019

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[Notices of the American Mathematical Society (ISSN 0002-9920) is published monthly except bimonthly in June/July by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2213 USA, GST No. 12189 2046 RT****. Periodicals postage paid at Providence, RI, and additional mailing offices. POSTMASTER: Send address change notices to Notices of the American Mathematical Society, PO Box 6248, Providence, RI 02904-6248 USA.] Publication here of the Society's street address and the other bracketed information is a technical requirement of the US Postal Service.

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LETTERS TO THE EDITOR

Letter to the Editor

We are writing in response to the article "On Choosing a Thesis Advisor" by Robert Lipshitz [1] in the Early Career Section of the February *Notices*. We found the advice in the article to be very helpful but have concerns about the following piece of advice:

> "Should I have more than one thesis advisor?
> "Probably not. Get advice from, and discuss mathematics with, many people, but have one designated thesis advisor who is keeping track of your progress."

Given the length of time invested in a doctoral program and the power a sole advisor has over his or her students, there are advantages for a student to having two advisors, or at least another faculty member who is sufficiently familiar with the student's work to serve as an alternate advisor if necessary. Think of it as an insurance policy for many years of work. Having two advisors provides a Plan B in case one advisor has a health problem, leaves the university, or other problems arise. Having a second advisor or additional faculty member(s) cognizant of the student's research has been recommended by the National Academies of Science, Engineering, and Medicine in response to high levels of reported sexual harassment [2], and is also helpful to prevent other forms of abusive advisor/student relationships. There are additional benefits to two advisors, such as diverse feedback on one's project and career advice. Many departments allow a student to officially have two co-advisors. If the student's department does not, he or she can stay in close contact with a second faculty member who could in an emergency supervise the completion of the thesis and provide a letter of recommendation.

This letter does not in any way comment on our own relationships with our advisors but arises rather from years of experience mentoring women. It represents the personal views of the signers, not those of their institutions nor of the AWM. It originated through the Ad Hoc Group of AWM Members to Address Sexual Harassment.

[1] Robert Lipshitz, On Choosing a Thesis Advisor. Notices of the AMS, 66 (2019): 191-193.

[2] The National Academies of Science, Engineering, Medicine. Sexual Harassment of Women: Climate, Culture, and Consequences in Academic Sciences, Engineering, and Medicine, pp. 135–6. `https://www.nap.edu/catalog/24994/sexual-harassment-of-women-climate-culture-and-consequences-in-academic`

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—Monica VanDieren,
Robert Morris University

—Oscar Vega,
California State University Fresno

(Received April 23, 2019)

**We invite readers to submit letters to the editor at* `notices-letters@ams.org`.

In Response to Dr. Grinshpon

To the Editor:

Dr. Grinshpon's letter in the Mar. 2019 *Notices* has the benefits of being clear and passionate. On the down side, there's a lot in there to be taken issue with. I could go on at quite some length about what I find objectionable or misguided, but since this is just a letter, I will confine its structure to Dr. Grinshpon's four starred points.

1. Are "politically charged" statements appropriate in the *Notices*? Earlier on, Dr. Grinshpon took the *Notices* to task for having "join[ed] the ranks of all anti-right liberally biased media in tasteless bashing of everything that can be attributed as right-wing." On the page in the *Notices* just preceding the one with his letter, it says "Opinions expressed in signed *Notices* articles are those of the authors and do not necessarily reflect opinions of the editors or policies of the [AMS]." It seems as though Dr. Grinshpon's beef is with the author of the article, and not the *Notices* or its editors. And anyway, what's Dr. Grinshpon's letter if not "politically charged?" It implies that it itself should not be published, a lovely Gödelian twist if there ever was one. Dr. Grinshpon is correct that the Dawson quote is a rarity as described; that fact might lead one to wonder, what could have moved Prof. Dawson so?
2. Prof. Dawson's reference to "an extreme right-wing establishment" initiated a discussion of "the establishment" (emphasis mine), and the assertion of "the fact that Trump is NOT part of [it]" (emphasis his). I prefer to think of the country's power structure as having different power centers, which are sometimes competing, or expressing the differing interests of various communities. It should be clear that one of these centers revolves around the man who is President and de facto leader of the Republican Party. For sure, he had limited political influence before the election. But now? Revolutionaries, like Castro, Lenin, and Hitler, were also outsiders with limited power, until they had power. But more to the point, to describe the Leader as bravely standing up to all sorts of half-amorphous dangerous powers, the stronger the better, and who always succeeds, but just barely, serves only the interests of authoritarianism.
3. To Prof. Dawson's statement that "the scientific worldview … is … under assault," Dr. Grinshpon responds with "Seriously???," and then proceeds to set up the wrong kind of strawman. He correctly points out that institutions are not being closed and academicians are not being punished, and this is good. That means that those (non-)actions are not the nature of the assault. When major power centers say that scientists are not to be believed in matters of their professional expertise, they will carry a sizable chunk of the population with them, and so be able to implement policy in contradiction to established science. This is, of course, much easier than Dr. Grinshpon's listed oppressive methods, and there is no need for them to do more than what they are already doing. It bears mention that with the stakes of climate change so high, we cannot afford even these gentler forms of manipulation.
4. I half agree with Dr. Grinshpon's feelings about comparisons between the current US administration and the Nazis, which he calls "tasteless." Through my life, I have heard US movements and politicians as diverse as Nixon and Obama called "fascist," and it had always rubbed me the wrong way. Like Dr. Grinshpon, I find such words very powerful, and they should be kept that way, by limiting their use to when it is truly deserved. Unlike Dr. Grinshpon, for the first time in my life I find their use now appropriate. I would say the Trump administration is in the fascist tradition. Of course it is not identical with the Hitler regime. And it will never be. This current brand of fascism is what fits into its time and place, just as the Hitler, Mussolini, and Franco governments were all different among themselves. I realize that such a claim needs considerable justification; this is not the place for that; I just want to make clear, I do not make such statements lightly. Be that as it may, Dr. Grinshpon finds "it's spitting at the memory of all those who perished during those truly terrible times." In contrast, I find the best way to honor their memory is to speak up when the same spirit starts to rear its ugly head again.

Yours truly,
Bob Lubarsky,
Florida Atlantic University

(Received April 26, 2019)

Rosenbaum Foundation

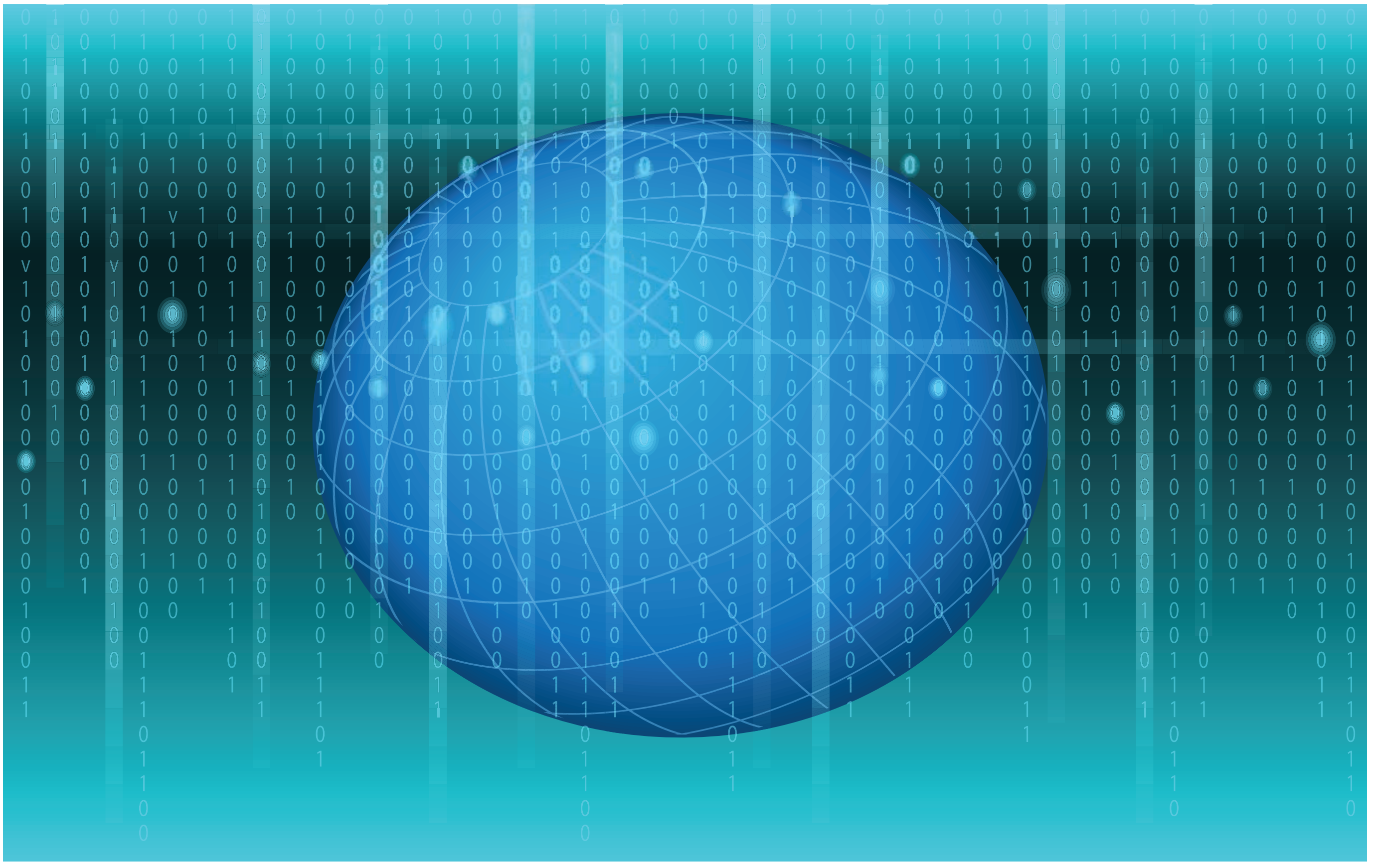
Dear Editor,

Thank you for acknowledging in the May 2019 issue of the *Notices* the Award for Impact on the Teaching and Learning of Mathematics that was bestowed upon me by the Education Committee of the AMS. On this occasion, I would like to express my sincere gratitude to Gabriella and Paul Rosenbaum Foundation and its president Madge Goldman, for providing generous grants in support of the Keystone Project at Daley College and its subsequent expansion to other universities.

Sincerely,
M. Vali Siadat
Richard J. Daley College

(Received May 9, 2019)

Computability and Randomness



Rod Downey and Denis R. Hirschfeldt

Historical Roots

Von Mises. Around 1930, Kolmogorov and others founded the theory of probability, basing it on measure theory. Probability theory is concerned with the distribution of outcomes in sample spaces. It does not seek to give any meaning to the notion of an individual object, such as a single real number or binary string, being random, but rather studies the expected values of random variables.

Rod Downey is a professor of mathematics and the deputy head of the School of Mathematics and Statistics at Victoria University Wellington. His email address is `rod.downey@vuw.ac.nz`.

Denis R. Hirschfeldt is a professor of mathematics at the University of Chicago. His email address is `drh@math.uchicago.edu`.

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DOI: https://doi.org/10.1090/noti1905

How could a binary string representing a sequence of n coin tosses be random, when all strings of length n have the same probability of 2^{-n} for a fair coin?

Less well known than the work of Kolmogorov are early attempts to answer this kind of question by providing notions of randomness for individual objects. The modern theory of *algorithmic randomness* realizes this goal. One way to develop this theory is based on the idea that an object is random if it passes all relevant "randomness tests." For example, by the law of large numbers, for a random real X, we would expect the number of 1's in the binary expansion of X to have limiting frequency $\frac{1}{2}$. (That is, writing $X(j)$ for the jth bit of this expansion, we would expect to have $\lim_{n\to\infty} \frac{|\{j<n : X(j)=1\}|}{n} = \frac{1}{2}$.) Indeed, we would expect X to be *normal* to base 2, meaning that for any binary string σ of length k, the occurrences of σ in the binary expan-

sion of X should have limiting frequency 2^{-k}. Since base representation should not affect randomness, we would expect X to be normal in this sense no matter what base it were written in, so that in base b the limiting frequency would be b^{-k} for a string σ of length k. Thus X should be what is known as *absolutely normal*.

The idea of normality, which goes back to Borel (1909), was extended by von Mises (1919), who suggested the following definition of randomness for individual binary sequences.[1] A *selection function* is an increasing function $f : \mathbb{N} \to \mathbb{N}$. We think of $f(i)$ as the ith place selected in forming a subsequence of a given sequence. (For the definition of normality above, where we consider the entire sequence, $f(i) = i$.) Von Mises suggested that a sequence $a_0 a_1 \ldots$ should be random if any selected subsequence $a_{f(0)} a_{f(1)} \ldots$ is normal.

There is, of course, an obvious problem with this approach. For any sequence X with infinitely many 1's we could let f select the positions where 1's occur, and X would fail the test determined by f. However, it does not seem reasonable to be able to choose the testing places *after* selecting an X. The question is then: What kinds of selection functions should be allowed, to capture the intuition that we ought not to be able to sample from a random sequence and get the wrong frequencies? It is reasonable to regard prediction as a computational process, and hence restrict ourselves to *computable* selection functions. Indeed, this suggestion was eventually made by Church (1940), though von Mises' work predates the definition of computable function, so he did not have a good way to make his definition mathematically precise.

As we will see, von Mises' approach had a more significant flaw, but we can build on its fundamental idea: Imagine that we are judges deciding whether a sequence X should count as random. If X passes all tests we can (in principle) devise given our computational power, then we should regard X as random since, as far as we are concerned, X has all the expected properties of a random object. We will use this intuition and the apparatus of computability and complexity theory to describe notions of *algorithmic* randomness.

Aside from the intrinsic interest of such an approach, it leads to useful mathematical tools. Many processes in mathematics are computable, and the expected behavior of such a process should align itself with the behavior obtained by providing it with an algorithmically random input. Hence, instead of having to analyze the relevant distribution and its statistics, we can simply argue about the behavior of the process on a single input. For instance, the expected number of steps of a sorting algorithm should be the same as that for a single algorithmically random input. We could also be more fine-grained and seek to understand exactly "how much" randomness is needed for certain typical behaviors to arise. (See the section on "Some Applications.")

As we will discuss, algorithmic randomness also goes hand in hand with other parts of algorithmic information theory, such as Kolmogorov complexity, and has ties with notions such as Shannon entropy and fractal dimension.

Some basic computability theory. In the 1930s, Church, Gödel, Kleene, Post, and most famously Turing (1937) gave equivalent mathematical definitions capturing the intuitive notion of a computable function, leading to the *Church-Turing Thesis*, which can be taken as asserting that a function (from $\mathbb{N}$ to $\mathbb{N}$, say) is computable if and only if it can be computed by a Turing machine. (This definition can easily be transferred to other objects of countable mathematics. For instance, we think of infinite binary sequences as functions $\mathbb{N} \to \{0, 1\}$, and identify sets of natural numbers with their characteristic functions.) Nowadays, we can equivalently regard a function as computable if we can write code to compute it in any given general-purpose programming language (assuming the language can address unlimited memory). It has also become clear that algorithms can be treated as data, and hence that there is a *universal Turing machine*, i.e., there is a listing $\Phi_0, \Phi_1, \ldots$ of all Turing machines and a single algorithm that, on input $\langle e, n\rangle$ computes the result $\Phi_e(n)$ of running Φ_e on input n.[2]

It is important to note that a Turing machine might not halt on a given input, and hence the functions computed by Turing machines are in general *partial*. Indeed, as Turing showed, the *halting problem* "Does the eth Turing machine halt on input n?" is algorithmically unsolvable. Church and Turing famously showed that Hilbert's *Entscheidungsproblem* (the decision problem for first-order logic) is unsolvable, in Turing's case by showing that the halting problem can be coded into first-order logic. Many other problems have since been shown to be algorithmically unsolvable by similar means.

We write $\Phi_e(n)\downarrow$ to mean that the machine Φ_e eventually halts on input n. Then $\emptyset' = \{\langle e, n\rangle : \Phi_e(n)\downarrow\}$ is a set representing the halting problem. This set is an example of a noncomputable *computably enumerable* (*c.e.*) set,

[1] *Due to space limitations, we omit historical citations and those found in the books Downey and Hirschfeldt* [9], *Li and Vitányi* [18], *or Nies* [24] *from the list of references. In some sections below, we also cite secondary sources where additional references can be found.*

[2] *The realization that such universal machines are possible helped lead to the development of modern computers. Previously, machines had been purpose-built for given tasks. In a 1947 lecture on his design for the Automated Computing Engine, Turing said, "The special machine may be called the universal machine; it works in the following quite simple manner. When we have decided what machine we wish to imitate we punch a description of it on the tape of the universal machine ... The universal machine has only to keep looking at this description in order to find out what it should do at each stage. Thus the complexity of the machine to be imitated is concentrated in the tape and does not appear in the universal machine proper in any way. ... [D]igital computing machines such as the ACE ... are in fact practical versions of the universal machine." From our contemporary point of view, it may be difficult to imagine how novel this idea was.*

which means that the set can be listed (not necessarily in numerical order) by some algorithm.

Another important notion is that of *Turing reducibility* (which we define for sets of natural numbers but is similarly defined for functions), where A is Turing reducible to B, written as $A \leq_T B$, if there is an algorithm for computing A when given access to B. That is, the algorithm is allowed access to answers to questions of the form "Is n in B?" during its execution. This notion can be formalized using Turing machines with oracle tapes. If $A \leq_T B$, then we regard A as no more complicated than B from a computability-theoretic perspective. We also say that A is *B-computable* or *computable relative to B*. Turing reducibility naturally leads to an equivalence relation, where A and B are *Turing equivalent* if $A \leq_T B$ and $B \leq_T A$. The *(Turing) degree* of A is its equivalence class under this notion. (There are several other notions of reducibility and resulting degree structures in computability theory, but Turing reducibility is the central one.)

In general, the process of allowing access to an oracle in our algorithms is known as *relativization*. As in the unrelativized case, we can list the Turing machines $\Phi_0^B, \Phi_1^B, \ldots$ with oracle B, and let $B' = \{\langle e, n\rangle : \Phi_e^B(n)\downarrow\}$ be the relativization of the halting problem to B. This set is called the *(Turing) jump* of B. The jump operation taking B to B' is very important in computability theory, one reason being that B' is the most complicated set that is still c.e. relative to B, i.e., B' is c.e. relative to B and every set that is c.e. relative to B is B'-computable. There are several other important classes of sets that can be defined in terms of the jump. For instance, A is *low* if $A' \leq_T \emptyset'$ and *high* if $\emptyset'' \leq_T A'$ (where $\emptyset'' = (\emptyset')'$). Low sets are in certain ways "close to computable," while high ones partake of some of the power of $\emptyset'$ as an oracle. These properties are invariant under Turing equivalence, and hence are also properties of Turing degrees.

Martin-Löf randomness. As mentioned above, Church suggested that a sequence should count as algorithmically random if it is random in the sense of von Mises with selection functions restricted to the computable ones. However, in 1939, Ville showed that von Mises' approach cannot work in its original form, no matter what countable collection of selection functions we choose. Let $X \upharpoonright n$ denote the first n bits of the binary sequence X.

Theorem 1 (Ville (1939)). *For any countable collection of selection functions, there is a sequence X that passes all von Mises tests associated with these functions, such that for every n, there are more 0's than 1's in $X \upharpoonright n$.*

Clearly, Ville's sequence cannot be regarded as random in any reasonable sense.

We could try to repair von Mises' definition by adding further tests, reflecting statistical laws beyond the law of large numbers. But which ones? Ville suggested ones reflecting the law of iterated logarithms, which would take care of his specific example. But how could we know that further examples along these lines—i.e., sequences satisfying both von Mises' and Ville's tests, yet failing to have some other property we expect of random sequences—would not arise?

The situation was finally clarified in the 1960s by Martin-Löf (1966). In probability theory, "typicality" is quantified using measure theory, leading to the intuition that random objects should avoid null sets. Martin-Löf noticed that tests like von Mises' and Ville's can be thought of as *effectively* null sets. His idea was that, instead of considering specific tests based on particular statistical laws, we should consider *all* possible tests corresponding to some precisely defined notion of effectively null set. The restriction to such a notion gets around the problem that no sequence can avoid being in *every* null set.

To give Martin-Löf's definition, we work for convenience in Cantor space 2^ω, whose elements are infinite binary sequences. (We associate a real number with its binary expansion, thought of as a sequence, so we will also obtain a definition of algorithmic randomness for reals. The choice of base is not important. For example, all of the notions of randomness we consider are enough to ensure absolute normality.) The basic open sets of Cantor space are the ones of the form $[\sigma] = \{X \in 2^\omega : X \text{ extends } \sigma\}$ for $\sigma \in 2^{<\omega}$, where $2^{<\omega}$ is the set of finite binary strings. The uniform measure λ on this space is obtained by defining $\lambda([\sigma]) = 2^{-|\sigma|}$. We say that a sequence $T_0, T_1, \ldots$ of open sets in 2^ω is *uniformly c.e.* if there is a c.e. set $G \subseteq \mathbb{N} \times 2^{<\omega}$ such that $T_n = \bigcup\{[\sigma] : (n, \sigma) \in G\}$.

Definition 2. A *Martin-Löf test* is a sequence $T_0, T_1, \ldots$ of uniformly c.e. open sets such that $\lambda(T_n) \leq 2^{-n}$. A sequence X *passes* this test if $X \notin \bigcap_n T_n$. A sequence is *Martin-Löf random* (*ML-random*) if it passes all Martin-Löf tests.

The intersection of a Martin-Löf test is our notion of effectively null set. Since there are only countably many Martin-Löf tests, and each determines a null set in the classical sense, the collection of ML-random sequences has measure 1. It can be shown that Martin-Löf tests include all the ones proposed by von Mises and Ville, in Church's computability-theoretic versions. Indeed they include all tests that are "computably performable," which avoids the problem of having to adaptively introduce more tests as more Ville-like sequences are found.

Martin-Löf's effectivization of measure theory allowed him to consider the laws a random sequence should obey from an abstract point of view, leading to a mathematically robust definition. As Jack Lutz said in a talk at the *7th Conference on Computability, Complexity, and Randomness*

(Cambridge, 2012), "Placing computability constraints on a nonconstructive theory like Lebesgue measure seems *a priori* to weaken the theory, but it may strengthen the theory for some purposes. This vision is crucial for present-day investigations of individual random sequences, dimensions of individual sequences, measure and category in complexity classes, etc."

The three approaches. ML-randomness can be thought of as the *statistician's approach* to defining algorithmic randomness, based on the intuition that random sequences should avoid having statistically rare properties. There are two other major approaches:

- The *gambler's approach*: random sequences should be unpredictable.
- The *coder's approach*: random sequences should not have regularities that allow us to compress the information they contain.

The gambler's approach may be the most immediately intuitive one to the average person. It was formalized in the computability-theoretic setting by Schnorr (1971), using the idea that we should not be able to make arbitrarily much money when betting on the bits of a random sequence. The following notion is a simple special case of the notion of a martingale from probability theory. (See [9, Section 6.3.4] for further discussion of the relationship between these concepts.)

Definition 3. A *martingale* is a function $f : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ such that

$$f(\sigma) = \frac{f(\sigma 0) + f(\sigma 1)}{2}$$

for all σ. We say that f *succeeds* on X if $\limsup_{n\to\infty} f(X \upharpoonright n) = \infty$.

We think of f as the capital we have when betting on the bits of a binary sequence according to a particular betting strategy. The displayed equation ensures that the betting is fair. Success then means that we can make arbitrarily much money when betting on X, which should not happen if X is random. By considering martingales with varying levels of effectivity, we get various notions of algorithmic randomness, including ML-randomness itself, as it turns out.

For example, X is *computably random* if no computable martingale succeeds on it, and *polynomial-time random* if no polynomial-time computable martingale succeeds on it. (For the purposes of defining these notions we can think of the martingale as rational-valued.) Schnorr (1971) showed that X is ML-random iff no left-c.e. martingale succeeds on it, where a function $f : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ is *left-c.e.* if it is computably approximable from below, i.e., there is a computable function $g : 2^{\omega} \times \mathbb{N} \to \mathbb{Q}^{\geq 0}$ such that $g(\sigma, n) \leq g(\sigma, n+1)$ for all σ and n, and $f(\sigma) = \lim_{n\to\infty} g(\sigma, n)$ for all σ. One way to think of a left-c.e. martingale is that initially we might have no idea what to bet on some string σ, but as we learn more about the universe, we might discover that σ seems more unlikely to be an initial segment of a random sequence, and are then prepared to bet more of our capital on it.

The coder's approach builds on the idea that a random string should have no short descriptions. For example, in describing 010101 … (1000 times) by the brief description "print 01 1000 times," we are using regularities in this string to compress it. For a more complicated string, say the first 2000 bits of the binary expansion of e^{π}, the regularities may be harder to perceive, but are still there and can still lead to compression. A random string should have no such exploitable regularities (i.e., regularities that are not present in most strings), so the shortest way to describe it should be basically to write it out in full. This idea can be formalized using the well-known concept of Kolmogorov complexity. We can think of a Turing machine M with inputs and outputs in $2^{<\omega}$ as a description system. If $M(\tau) = \sigma$ then τ is a description of σ relative to this description system. The *Kolmogorov complexity* $C_M(\sigma)$ of σ relative to M is the length of the shortest τ such that $M(\tau) = \sigma$. We can then take a universal Turing machine U, which emulates any given Turing machine with at most a constant increase in the size of programs, and define the *(plain) Kolmogorov complexity* of σ as $C(\sigma) = C_U(\sigma)$. The value of $C(\sigma)$ depends on U, but only up to an additive constant independent of σ. We think of a string as random if its Kolmogorov complexity is close to its length.

For an infinite sequence X, a natural guess would be that X should be considered random if every initial segment of X is incompressible in this sense, i.e., if $C(X \upharpoonright n) \geq n - O(1)$. However, plain Kolmogorov complexity is not quite the right notion here, because the information in a description τ consists not only of the bits of τ, but also its length, which can provide another $\log_2 |\tau|$ many bits of information. Indeed, Martin-Löf (see [18]) showed that it is not possible to have $C(X \upharpoonright n) \geq n - O(1)$: Given a long string ρ, we can write $\rho = \sigma\tau\nu$, where $|\tau|$ is the position of σ in the length-lexicographic ordering of $2^{<\omega}$. Consider the Turing machine M that, on input η, determines the $|\eta|$th string ξ in the length-lexicographic ordering of $2^{<\omega}$ and outputs $\xi\eta$. Then $N(\tau) = \sigma\tau$. For any sequence X and any k, this process allows us to compress some initial segment of X by more than k many bits.

There are several ways to get around this problem by modifying the definition of Kolmogorov complexity. The best-known one is to use prefix-free codes, that is, to restrict ourselves to machines M such that if $M(\tau)$ is defined (i.e., if the machine eventually halts on input τ) and μ is a proper extension of τ, then $M(\mu)$ is not defined. There are universal prefix-free machines, and we can take such a machine U and define the *prefix-free Kolmogorov complex-*

ity of σ as $K(\sigma) = C_U(\sigma)$. The roots of this notion can be found in the work of Levin, Chaitin, and Schnorr, and in a certain sense—like the notion of Kolmogorov complexity more generally—even earlier in that of Solomonoff (see [9, 18]). As shown by Schnorr (see Chaitin (1975)), it is indeed the case that X is Martin-Löf random if and only if $K(X \upharpoonright n) \geq n - O(1)$.

There are other varieties of Kolmogorov complexity, but C and K are the main ones. For applications, it often does not matter which variety is used. The following surprising result establishes a fairly precise relationship between C and K. Let $C^{(1)}(\sigma) = C(\sigma)$ and $C^{(n+1)}(\sigma) = C(C^{(n)}(\sigma))$.

Theorem 4 (Solovay (1975)). *$K(\sigma) = C(\sigma) + C^{(2)}(\sigma) \pm O(C^{(3)}(\sigma))$, and this result is tight in that we cannot extend it to $C^{(4)}(\sigma)$.*

There is a vast body of research on Kolmogorov complexity and its applications. We will discuss some of these applications below; much more on the topic can be found in Li and Vitányi [18].

Goals

There are several ways to explore the ideas introduced above. First, there are natural internal questions, such as: How do the various levels of algorithmic randomness interrelate? How do calibrations of randomness relate to the hierarchies of computability and complexity theory, and to relative computability? How should we calibrate partial randomness? Can a source of partial (algorithmic) randomness be amplified into a source that is fully random, or at least more random? The books Downey and Hirschfeldt [9] and Nies [24] cover material along these lines up to about 2010.

We can also consider applications. Mathematics has many theorems that involve "almost everywhere" behavior. Natural examples come from ergodic theory, analysis, geometric measure theory, and even combinatorics. Behavior that occurs almost everywhere should occur at sufficiently random points. Using notions from algorithmic randomness, we can explore exactly *how much* randomness is needed in a given case. For example, the set of reals at which an increasing function is differentiable is null. How complicated is this null set, and hence, what level of algorithmic randomness is necessary for a real to avoid it (assuming the function is itself computable in some sense)? Is Martin-Löf randomness the right notion here?

We can also use the idea of assigning levels of randomness to individual objects to prove new theorems or give simpler proofs of known ones. Early examples of this method tended to use Kolmogorov complexity and what is called the "incompressibility method." For instance, Chaitin (1971) (see also [17]) famously used Kolmogorov complexity to give a proof of a version of Gödel's First Incompleteness Theorem, by showing that for any sufficiently strong, computably axiomatizable, consistent theory T, there is a number c such that T cannot prove that $C(\sigma) > c$ for any given string σ (which also follows by interpreting an earlier result of Barzdins; see [18, Section 2.7]). More recently, Kritchman and Raz [17] used these methods to give a proof of the Second Incompleteness Theorem as well.[3] As we will see below, a more recent line of research has used notions of effective dimension based on partial randomness to give new proofs of classical theorems in ergodic theory and obtain new results in geometric measure theory.

Some Interactions with Computability

Halting probabilities. A first question we might ask is how to generate "natural" examples of algorithmically random reals. A classic example is Chaitin's halting probability. Let U be a universal prefix-free machine and let

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}.$$

This number is the measure of the set of sequences X such that U halts on some initial segment of X, which we can interpret as the halting probability of U, and was shown by Chaitin (1975) to be ML-random (where, as mentioned above, we identify Ω with its binary expansion, thought of as an infinite binary sequence).

For any prefix-free machine M in place of U we can similarly define a halting probability. In some ways, halting probabilities are the analogs of computably enumerable sets in the theory of algorithmic randomness. Every halting probability α is a *left-c.e. real*, meaning that there is a computable increasing sequence of rationals converging to it. Calude, Hertling, Khoussainov, and Wang (1998) showed that every left-c.e. real is the halting probability of some prefix-free machine.

We should perhaps write Ω_U instead of Ω, to stress the dependence of its particular value on the choice of universal machine, but the fundamental properties of Ω do not depend on this choice, much as those of the halting problem do not depend on the specific choice of enumeration of Turing machines. In particular, Kučera and Slaman (2001) showed that every left-c.e. real is reducible to every Ω_U up to a strong notion of reducibility known as Solovay reducibility, and hence all such Ω_U's are equivalent modulo this notion. (The situation is analogous to that of versions of the halting problem, where the relevant notion is known as 1-reducibility.)

[3] *Other recent work has explored the effect of adding axioms asserting the incompressibility of certain strings in a probabilistic way. Bienvenu, Romashchenko, Shen, Taveneaux, and Vermeeren [4] have shown that this kind of procedure does not help to prove new interesting theorems, but that the situation changes if we take into account the sizes of the proofs: randomly chosen axioms (in a sense made precise in their paper) can help to make proofs much shorter under the reasonable complexity-theoretic assumption that* NP $\neq$ PSPACE.

Left-c.e. and right-c.e. reals (those of the form $1-\alpha$ for a left-c.e. α) occur naturally in mathematics. Braverman and Yampolsky [7] showed that they arise in connection with Julia sets, and there is a striking example in symbolic dynamics: A d-dimensional *subshift of finite type* is a certain kind of collection of A-colorings of $\mathbb{Z}^d$, where A is a finite set, defined by local rules (basically saying that certain coloring patterns are illegal) invariant under the shift action

$$(S^g x)(h) = x(h+g) \text{ for } g, h \in \mathbb{Z}^d \text{ and } x \in A^{\mathbb{Z}^d}.$$

Its *(topological) entropy* is an important invariant measuring the asymptotic growth in the number of legal colorings of finite regions. It has been known for some time that entropies of subshifts of finite type for dimensions $d \geq 2$ are in general not computable, but the following result gives a precise characterization.

Theorem 5 (Hochman and Meyerovitch [15]). *The values of entropies of subshifts of finite type over $\mathbb{Z}^d$ for $d \geq 2$ are exactly the nonnegative right-c.e. reals.*

Algorithmic randomness and relative computability. Solovay reducibility is stronger than Turing reducibility, so Ω can compute the halting problem $\emptyset'$. Indeed Ω and $\emptyset'$ are Turing equivalent, and in fact Ω can be seen as a "highly compressed" version of $\emptyset'$. Other computability-theoretically powerful ML-random sequences can be obtained from the following remarkable result.

Theorem 6 (Gács (1986), Kučera (1985)). *For every X there is an ML-random Y such that $X \leq_T Y$.*

This theorem and the Turing equivalence of Ω with $\emptyset'$ do not seem to accord with our intuition that random sets should have low "useful information." This phenomenon can be explained by results showing that, for certain purposes, the benchmark set by ML-randomness is too low. A set A has *PA degree* if it can compute a $\{0,1\}$-valued function f with $f(n) \neq \Phi_n(n)$ for all n. (The reason for the name is that this property is equivalent to being able to compute a completion of Peano Arithmetic.) Such a function can be seen as a weak version of the halting problem, but while $\emptyset'$ has PA degree, there are sets of PA degree that are low, in the sense of the section on "Some basic computability theory," and hence are far less powerful than $\emptyset'$.

Theorem 7 (Stephan (2006)). *If an ML-random sequence has PA degree then it computes $\emptyset'$.*

Thus there are two kinds of ML-random sequences. Ones that are complicated enough to somehow "simulate" randomness, and "truly random" ones that are much weaker. It is known that the class of sequences that can compute $\emptyset'$ has measure 0, so almost all ML-random sequences are in the second class. One way to ensure that a sequence is in that class is to increase the complexity of our tests by relativizing them to noncomputable oracles. It turns out that iterates of the Turing jump are particularly natural oracles to use. Let $\emptyset^{(0)} = \emptyset$ and $\emptyset^{(n+1)} = (\emptyset^{(n)})'$. We say that X is *n-random* if it passes all Martin-Löf tests relativized to $\emptyset^{(n-1)}$. Thus the 1-random sequences are just the ML-random ones, while the 2-random ones are the ones that are ML-random relative to the halting problem. These sequences have low computational power in several ways. For instance, they cannot compute any noncomputable c.e. set, and in fact the following holds.

Theorem 8 (Kurtz (1981)). *If X is 2-random and Y is computable relative both to $\emptyset'$ and to X, then Y is computable.*

A precise relationship between tests and the dichotomy mentioned above was established by Franklin and Ng [11].

In general, among ML-random sequences, computational power (or "useful information") is inversely proportional to level of randomness. The following is one of many results attesting to this heuristic.

Theorem 9 (Miller and Yu (2008)). *Let $X \leq_T Y$. If X is ML-random and Y is n-random, then X is also n-random.*

There are many other interesting levels of algorithmic randomness. Schnorr (1971) argued that his martingale characterization of ML-randomness shows that this is an intrinsically *computably enumerable* rather than *computable* notion, and defined a notion now called *Schnorr randomness*, which is like the notion of computable randomness mentioned below Definition 3 but with an extra effectiveness condition on the rate of success of martingales. He also showed that X is Schnorr random iff it passes all Martin-Löf tests $T_0, T_1, \ldots$ such that the measures $\lambda(T_n)$ are uniformly computable (i.e., the function $n \mapsto \lambda(T_n)$ is computable in the sense of the section on "Analysis and ergodic theory" below). It follows immediately from their definitions in terms of martingales that ML-randomness implies computable randomness, which in turn implies Schnorr randomness. It is more difficult to prove that none of these implications can be reversed. In fact, these levels of randomness are close enough that they agree for sets that are somewhat close to computable, as shown by the following result, where highness is as defined in the section on "Some basic computability theory."

Theorem 10 (Nies, Stephan, and Terwijn (2005)). *Every high Turing degree contains a set that is computably random but not ML-random and a set that is Schnorr random but not computably random. This fact is tight, however, because every nonhigh Schnorr random set is ML-random.*

As we will discuss, various notions of algorithmic randomness arise naturally in applications.

Randomness-theoretic weakness. As mentioned above, X is ML-random iff $K(X \upharpoonright n) \geq n - O(1)$, i.e., X's initial segments have very high complexity. There are similar characterizations of other notions of algorithmic randomness, as well as of notions arising in other parts of computability theory, in terms of high initial segment complexity. For instance, Downey and Griffiths (2004) showed that X is Schnorr random iff $C_M(X \upharpoonright n) \geq n - O(1)$ for every prefix-free machine M with computable halting probability, while Kjos-Hanssen, Merkle, and Stephan (2006) showed that X can compute a diagonally noncomputable function, that is, a function h with $h(e) \neq \Phi_e(e)$ for all e, iff there is an X-computable function f such that $C(X \upharpoonright f(n)) \geq n$ for all n. But what if the initial segments of a sequence have *low* complexity? Such sequences have played an important role in the theory of algorithmic randomness, beginning with the following information-theoretic characterization of computability.

Theorem 11 (Chaitin (1976)). *$C(X \upharpoonright n) \leq C(n) + O(1)$ iff X is computable.*

It is also true that if X is computable then $K(X \upharpoonright n) \leq K(n) + O(1)$. Chaitin (1977) considered sequences with this property, which are now called *K-trivial*. He showed that every K-trivial sequence is $\emptyset'$-computable, and asked whether they are all in fact computable. Solovay (1975) answered this question by constructing a noncomputable K-trivial sequence.

The class of K-trivials has several remarkable properties. It is a naturally definable *countable* class, contained in the class of low sets (as defined in the section on "Some basic computability theory," where we identify a set with its characteristic function, thought of as a sequence), but with stronger closure properties. (In technical terms, it is what is known as a *Turing ideal*.) Post's problem asked whether there are computably enumerable sets that are neither computable nor Turing equivalent to the halting problem. Its solution in the 1950s by Friedberg and Muchnik introduced the somewhat complex priority method, which has played a central technical role in computability theory since then. Downey, Hirschfeldt, Nies, and Stephan (2003) showed that K-triviality can be used to give a simple priority-free solution to Post's problem.

Most significantly, there are many natural notions of randomness-theoretic weakness that turn out to be equivalent to K-triviality.

Theorem 12 (Nies (2005), Nies and Hirschfeldt for $(1) \rightarrow (3)$). *The following are equivalent.*

1. *A is K-trivial.*
2. *A is computable relative to some c.e. K-trivial set.*
3. *A is* low for K, *meaning that A has no compression power as an oracle. i.e., that $K^A(\sigma) \geq K(\sigma) - O(1)$, where K^A is the relativization of prefix-free Kolmogorov complexity to A.*
4. *A is* low for ML-randomness, *meaning that A does not have any derandomization power as an oracle, i.e., any ML-random set remains ML-random when this notion is relativized to A.*

There are now more than a dozen other characterizations of K-triviality. Some appear in [9, 24], and several others have emerged more recently. These have been used to solve several problems in algorithmic randomness and related areas. Lowness classes have also been found for other randomness notions. For Schnorr randomness, for instance, lowness can be characterized using notions of traceability related to concepts in set theory, as first explored by Terwijn and Zambella (2001).

Some Applications

Incompressibility and information content. This article focuses on algorithmic randomness for infinite objects, but we should mention that there have been many applications of Kolmogorov complexity under the collective title of the *incompressibility method*, based on the observation that algorithmically random strings should exhibit typical behavior for computable processes. For example, this method can be used to give average running times for sorting, by showing that if the outcome is not what we would expect then we can compress a random input. See Li and Vitányi [18, Chapter 6] for applications of this technique to areas as diverse as combinatorics, formal languages, compact routing, and circuit complexity, among others. Many results originally proved using Shannon entropy or related methods also have proofs using Kolmogorov complexity. For example, Messner and Thierauf [22] gave a constructive proof of the Lovász Local Lemma using Kolmogorov complexity.

Other applications come from the observation that in some sense Kolmogorov complexity provides an "absolute" measure of the intrinsic complexity of a string. We can define a notion of conditional Kolmogorov complexity $C(\sigma \mid \tau)$ of a string σ given another string τ. Then, for example, $C(\sigma \mid \sigma) = O(1)$, and σ is "independent of τ" if $C(\sigma \mid \tau) = C(\sigma) \pm O(1)$. Researchers comparing two sequences σ, τ representing, say, two DNA sequences, or two phylogenetic trees, or two languages, or two pieces of music, have invented many distance metrics, such as the maximum parsimony distance on phylogenetic trees, but it is also natural to use a content-neutral measure of "information distance" like $\max\{C(\sigma \mid \tau), C(\tau \mid \sigma)\}$. There have been some attempts to make this work in practice for solving classification problems, though results have so far been mixed. Of course, C is not computable, but it can be replaced in applications by measures derived from practical compression algorithms. See [18, Sections 8.3 and 8.4].

Effective dimensions. If $X = x_0x_1\ldots$ is random, then we might expect a sequence such as $x_000x_100x_200\ldots$ to be "$\frac{1}{3}$-random." Making precise sense of the idea of partial algorithmic randomness has led to significant applications. Hausdorff used work of Carathéodory on s-dimensional measures to generalize the notion of dimension to possibly nonintegral values, leading to concepts such as Hausdorff dimension and packing dimension. Much like algorithmic randomness can make sense of the idea of individual reals being random, notions of partial algorithmic randomness can be used to assign dimensions to individual reals.

The measure-theoretic approach, in which we for instance replace the uniform measure λ on 2^ω by a generalized notion assigning the value $2^{-s|\sigma|}$ to $[\sigma]$ (where $0 < s \leq 1$), was translated by Lutz (2000, 2003) into a notion of *s-gale*, where the fairness condition of a martingale is replaced by $f(\sigma) = 2^{-s}(f(\sigma 0) + f(\sigma 1))$. We can view s-gales as modeling betting in a hostile environment (an idea due to Lutz), where "inflation" is acting so that not winning means that we automatically lose money. Roughly speaking, the effective fractal dimension of a sequence is then determined by the most hostile environment in which we can still make money betting on this sequence.

Mayordomo (2002) and Athreya, Hitchcock, Lutz, and Mayordomo (2007) found equivalent formulations in terms of Kolmogorov complexity, which we take as definitions. (Here it does not matter whether we use plain or prefix-free Kolmogorov complexity.)

Definition 13. Let $X \in 2^\omega$. The *effective Hausdorff dimension* of X is

$$\dim(X) = \liminf_{n\to\infty} \frac{K(X \upharpoonright n)}{n}.$$

The *effective packing dimension* of X is

$$\mathrm{Dim}(X) = \limsup_{n\to\infty} \frac{K(X \upharpoonright n)}{n}.$$

It is not hard to extend these definitions to elements of $\mathbb{R}^n$, yielding effective dimensions between 0 and n. They can also be relativized to any oracle A to obtain the effective Hausdorff and packing dimensions $\dim^A(X)$ and $\mathrm{Dim}^A(X)$ of X relative to A.

It is of course not immediately obvious why these notions are effectivizations of Hausdorff and packing dimension, but crucial evidence of their correctness is provided by *point to set principles*, which allow us to express the dimensions of sets of reals in terms of the effective dimensions of their elements. The most recent and powerful of these is the following, where we denote the classical Hausdorff dimension of $E \subseteq \mathbb{R}^n$ by $\dim_{\mathrm{H}}(E)$, and its classical packing dimension by $\dim_{\mathrm{P}}(E)$.

Theorem 14 (Lutz and Lutz [19]).

$$\dim_{\mathrm{H}}(E) = \min_{A\subseteq\mathbb{N}} \sup_{X\in E} \dim^A(X).$$

$$\dim_{\mathrm{P}}(E) = \min_{A\subseteq\mathbb{N}} \sup_{X\in E} \mathrm{Dim}^A(X).$$

For certain well-behaved sets E, relativization is actually not needed, and the classical dimension of E is the supremum of the effective dimensions of its points. In the general case, it is of course not immediately clear that the minima mentioned in Theorem 14 should exist, but they do. Thus, for example, to prove a lower bound of α for $\dim_{\mathrm{H}}(E)$ it suffices to prove that, for each $\epsilon > 0$ and each A, the set E contains a point X with $\dim^A(X) > \alpha - \epsilon$. In several applications, this argument turns out to be easier than ones directly involving classical dimension. This fact is somewhat surprising given the need to relativize to arbitrary oracles, but in practice this issue has so far turned out not to be an obstacle.

For example, Lutz and Stull [21] obtained a new lower bound on the Hausdorff dimension of generalized sets of Furstenberg type; Lutz [20] showed that a fundamental intersection formula, due in the Borel case to Kahane and Mattila, is true for arbitrary sets; and Lutz and Lutz [19] gave a new proof of the two-dimensional case (originally proved by Davies) of the well-known Kakeya conjecture, which states that, for all $n \geq 2$, if a subset of $\mathbb{R}^n$ has lines of length 1 in all directions, then it has Hausdorff dimension n.

There had been earlier applications of effective dimension, for instance in symbolic dynamics, whose iterative processes are naturally algorithmic. For example, Simpson [25] generalized a result of Furstenberg as follows. Let A be finite and G be either $\mathbb{N}^d$ or $\mathbb{Z}^d$. A closed set $X \subseteq A^G$ is a *subshift* if it is closed under the shift action of G on A^G (see the section on "Halting probabilities").

Theorem 15 (Simpson [25]). *Let A be finite and G be either $\mathbb{N}^d$ or $\mathbb{Z}^d$. If $X \subseteq A^G$ is a subshift then the topological entropy of X is equal both to its classical Hausdorff dimension and to the supremum of the effective Hausdorff dimensions of its elements.*

In currently unpublished work, Day has used effective methods to give a new proof of the Kolmogorov-Sinai theorem on entropies of Bernoulli shifts.

There are other applications of sequences of high effective dimension, for instance ones involving the interesting class of *shift complex* sequences. While initial segments of ML-random sequences have high Kolmogorov complexity, not all segments of such sequences do. Random sequences must contain arbitrarily long strings of consecutive 0's, for example. However, for any $\epsilon > 0$ there are *ϵ-shift complex* sequences Y such that for any string σ of consecutive bits of Y, we have $K(\sigma) \geq (1-\epsilon)|\sigma| - O(1)$. These sequences can be used to create tilings with properties such as certain kinds of pattern-avoidance, and have found uses in symbolic dynamics. See for instance Durand, Levin, and Shen (2008) and Durand, Romashchenko, and Shen [10].

Randomness amplification. Many practical algorithms use random seeds. For example, the important *Polynomial Identity Testing* (*PIT*) problem takes as input a polynomial $P(x_1, \ldots, x_n)$ with coefficients from a large finite field and determines whether it is identically 0. Many practical problems can be solved using a reduction to this problem. There is a natural fast algorithm to solve it randomly: Take a random sequence of values for the variables. If the polynomial is not 0 on these values, "no" is the correct answer. Otherwise, the probability that the answer is "yes" is very high. It is conjectured that PIT has a polynomial-time deterministic algorithm,[4] but no such algorithm is known.

Thus it is important to have good sources of randomness. Some (including Turing) have believed that randomness can be obtained from physical sources, and there are now commercial devices claiming to do so. At a more theoretical level, we might ask questions such as:

1. Can a weak source of randomness always be amplified into a better one?
2. Can we in fact always recover full randomness from partial randomness?
3. Are random sources truly useful as computational resources?

In our context, we can consider precise versions of such questions by taking randomness to mean algorithmic randomness, and taking all reduction processes to be computable ones. One way to interpret the first two questions then is to think of partial randomness as having nonzero effective dimension. For example, for packing dimension, we have the following negative results.

Theorem 16 (Downey and Greenberg (2008)). *There is an X such that $\mathrm{Dim}(X) = 1$ and X computes no ML-random sequence. (This X can be built to be of* minimal degree, *which means that every X-computable set is either computable or has the same Turing degree as X. It is known that such an X cannot compute an ML-random sequence.)*

Theorem 17 (Conidis [8]). *There is an X such that $\mathrm{Dim}(X) > 0$ and X computes no Y with $\mathrm{Dim}(Y) = 1$.*

On the other hand, we also have the following strong positive result.

Theorem 18 (Fortnow, Hitchcock, Pavan, Vinochandran, and Wang (2006)). *If $\epsilon > 0$ and $\mathrm{Dim}(X) > 0$ then there is an X-computable Y such that $\mathrm{Dim}(Y) > 1 - \epsilon$. (In fact, Y can be taken to be equivalent to X via polynomial-time reductions.)*

For effective Hausdorff dimension, the situation is quite different. Typically, the way we obtain an X with $\dim(X) = \frac{1}{2}$, say, is to start with an ML-random sequence and somehow "mess it up," for example by making every other bit a 0. This kind of process is reversible, in the sense that it is easy to obtain an X-computable ML-random. However, Miller [23] showed that it is possible to obtain sequences of fractional effective Hausdorff dimension that permit no randomness amplification at all.

Theorem 19 (Miller [23]). *There is an X such that $\dim(X) = \frac{1}{2}$ and if $Y \leq_T X$ then $\dim(Y) \leq \frac{1}{2}$.*

That is, effective Hausdorff dimension *cannot* in general be amplified. (In this theorem, the specific value $\frac{1}{2}$ is only an example.) Greenberg and Miller [13] also showed that there is an X such that $\dim(X) = 1$ and X does not compute any ML-random sequences. Interestingly, Zimand (2010) showed that for *two* sequences X and Y of nonzero effective Hausdorff dimension that are in a certain technical sense sufficiently independent, X and Y together can compute a sequence of effective Hausdorff dimension 1.

In some attractive recent work, it has been shown that there is a sense in which the intuition that every sequence of effective Hausdorff dimension 1 is close to an ML-random sequence is correct. The following is a simplified version of the full statement, which quantifies how much randomness can be extracted at the cost of altering a sequence on a set of density 0. Here $A \subseteq \mathbb{N}$ has *(asymptotic) density* 0 if $\lim_{n\to\infty} \frac{|A \upharpoonright n|}{n} = 0$.

Theorem 20 (Greenberg, Miller, Shen, and Westrick [14]). *If $\dim(X) = 1$ then there is an ML-random Y such that $\{n : X(n) \neq Y(n)\}$ has density 0.*

The third question above is whether sources of randomness can be useful oracles. Here we are thinking in terms of complexity rather than just computability, so results such as Theorem 6 are not directly relevant. Allender and others have initiated a program to investigate the speedups that are possible when random sources are queried efficiently. Let R be the set of all random finite binary strings for either plain or prefix-free Kolmogorov complexity (e.g., $R = \{x : C(x) \geq |x|\}$). For a complexity class C, let C^R denote the relativization of this class to R. So, for instance, for the class P of polynomial-time computable functions, P^R is the class of functions that can be computed in polynomial time with R as an oracle. (For references to the articles in this and the following theorem, see [1].)

[4] *This conjecture comes from the fact that PIT belongs to a complexity class known as BPP, which is widely believed to equal the complexity class P of polynomial-time solvable problems, since Impagliazzo and Wigderson showed in the late 1990s that if the well-known Satisfiability problem is as hard as generally believed, then indeed* BPP = P.

Theorem 21 (Buhrman, Fortnow, Koucký, and Loff (2010); Allender, Buhrman, Koucký, van Melkebeek, and Ronneburger (2006); Allender, Buhrman, and Koucký (2006)).

1. $\mathrm{PSPACE} \subseteq \mathrm{P}^R$.
2. $\mathrm{NEXP} \subseteq \mathrm{NP}^R$.
3. $\mathrm{BPP} \subseteq \mathrm{P}^R_{tt}$ *(where the latter is the class of functions that are reducible to R in polynomial time via truth-table reductions, a more restrictive notion of reduction than Turing reduction).*

The choice of universal machine does have some effect on efficient computations, but we can quantify over all universal machines. In the result below, U ranges over universal prefix-free machines, and R_{K_U} is the set of random strings relative to Kolmogorov complexity defined using U.

Theorem 22 (Allender, Friedman, and Gasarch (2013); Cai, Downey, Epstein, Lempp, and Miller (2014)).

1. $\bigcap_U \mathrm{P}^{R_{K_U}}_{tt} \subseteq \mathrm{PSPACE}$.
2. $\bigcap_U \mathrm{NP}^{R_{K_U}} \subseteq \mathrm{EXPSPACE}$.

We can also say that sufficiently random oracles will always accelerate *some* computations in the following sense. Say that X is *low for speed* if for any computable set A and any function t such that A can be computed in time $t(n)$ using X as an oracle, there is a polynomial p such that A can be computed (with no oracle) in time bounded by $p(t(n))$. That is, X does not significantly accelerate any computation of a computable set. Bayer and Slaman (see [3]) constructed noncomputable sets that are low for speed, but these cannot be very random.

Theorem 23 (Bienvenu and Downey [3]). *If X is Schnorr random, then it is not low for speed, and this fact is witnessed by an exponential-time computable set A.*

Analysis and ergodic theory. Computable analysis is an area that has developed tools for thinking about computability of objects like real-valued functions by taking advantage of separability. Say that a sequence of rationals $q_0, q_1, \ldots$ *converges fast* to x if $|x - q_n| \leq 2^{-n}$ for all n. A function $f : \mathbb{R} \to \mathbb{R}$ is *(Type 2) computable* if there is an algorithm Φ that, for every $x \in \mathbb{R}$ and every sequence $q_0, q_1, \ldots$ that converges fast to x, if Φ is given $q_0, q_1, \ldots$ as an oracle, then it can compute a sequence that converges fast to $f(x)$. We can extend this definition to similar separable spaces. We can also relativize it, and it is then not difficult to see that a function is continuous iff it is computable relative to some oracle, basically because to define a continuous function we need only to specify its action on a countable collection of balls.

Mathematics is replete with results concerning almost everywhere behavior, and algorithmic randomness allows us to turn such results into "quantitative" ones like the following.

Theorem 24 (Brattka, Miller, and Nies [5], also Demuth (75, see [5]) for (2)).

1. *The reals at which every computable increasing function $\mathbb{R} \to \mathbb{R}$ is differentiable are exactly the computably random ones.*
2. *The reals at which every computable function $\mathbb{R} \to \mathbb{R}$ of bounded variation is differentiable are exactly the ML-random ones.*

Ergodic theory is another area that has been studied from this point of view. A measure-preserving transformation T on a probability space is *ergodic* if all measurable subsets E such that $T^{-1}(E) = E$ have measure 1 or 0. Notice that this is an "almost everywhere" definition. We can make this setting computable (and many systems arising from physics will be computable). One way to proceed is to work in Cantor space without loss of generality, since Hoyrup and Rojas [16] showed that any computable metric space with a computable probability measure is isomorphic to this space in an effective measure-theoretic sense. Then we can specify a computable transformation T as a computable limit of computable partial maps $T_n : 2^{<\omega} \to 2^{<\omega}$ with certain coherence conditions. We can also transfer definitions like that of ML-randomness to computable probability spaces other than Cantor space.

The following is an illustrative result. A classic theorem of Poincaré is that if T is measure-preserving, then for all E of positive measure and almost all x, we have $T^n(x) \in E$ for infinitely many n. For a class C of measurable subsets, x is a *Poincaré point* for T with respect to C if for every $E \in C$ of positive measure, $T^n(x) \in E$ for infinitely many n. An *effectively closed* set is one whose complement can be specified as a computably enumerable union of basic open sets.

Theorem 25 (Bienvenu, Day, Mezhirov, and Shen [2]). *Let T be a computable ergodic transformation on a computable probability space. Every ML-random element of this space is a Poincaré point for the class of effectively closed sets.*

In general, the condition that the element be ML-random is not just sufficient but necessary, even in a simple case like the shift operator on Cantor space.

The non-ergodic case has also been analyzed, by Franklin and Towsner [12], who also studied the Birkhoff ergodic theorem. In these and several other cases, similar correspondences with various notions of algorithmic randomness have been found. While many theorems of ergodic theory have been analyzed in this way, including the Birkhoff, Poincaré, and von Neumann ergodic theorems, some, like Furstenberg's ergodic theorem, have yet to be understood from this point of view.

Regarding the physical interpretation of some of the work in this area, Braverman, Grigo, and Rojas [6] have obtained results that they argue show that, while random noise makes predicting the short-term behavior of a system difficult, it may in fact allow prediction to be easier in the long term.

Normality revisited. Borel's notion of normality, with which we began our discussion, is a very weak kind of randomness. Polynomial-time randomness implies absolute normality, and Schnorr and Stimm (1971/72) showed that a sequence is normal to a given base iff it satisfies a martingale-based notion of randomness defined using certain finite state automata, a much weaker model of computation than Turing machines. Building examples of absolutely normal numbers is another matter, as Borel already noted. While it is conjectured that e, π, and all irrational algebraic numbers such as $\sqrt{2}$ are absolutely normal, *none* of these have been proved to be normal to *any* base. In his unpublished manuscript "A note on normal numbers," believed to have been written in 1938, Turing built a computable absolutely normal real, which is in a sense the closest we have come so far to obtaining an explicitly-described absolutely normal real. (His construction was not published until his Collected Works in 1992, and there was uncertainty as to its correctness until Becher, Figueira, and Picchi (2007) reconstructed and completed it, correcting minor errors.[5]) There is a sense in which Turing anticipated Martin-Löf's idea of looking at a large collection of effective tests, in this case ones sufficiently strong to ensure that a real is normal for all bases, but sufficiently weak to allow some computable sequence to pass them all. He took advantage of the correlations between blocks of digits in expansions of the same real in different bases.

This approach can also be thought of in terms of effective martingales, and its point of view has brought about a great deal of progress in our understanding of normality recently. For instance, Becher, Heiber, and Slaman (2013) showed that absolutely normal numbers can be constructed in low-level polynomial time, and Lutz and Mayordomo (`arXiv:1611.05911`) constructed them in "nearly linear" time. Much of the work along these lines has been number-theoretic, connected to various notions of well-approximability of irrational reals, such as that of a *Liouville number*, which is an irrational α such that for every natural number $n > 1$, there are $p, q \in \mathbb{N}$ for which $|\alpha - \frac{p}{q}| < q^{-n}$. For example, Becher, Heiber, and Slaman (2015) have constructed computable absolutely normal Liouville numbers. This work has also produced results in the classical theory of normal numbers, for instance by Becher, Bugeaud, and Slaman (2016).

[5] *See* `https://www-2.dc.uba.ar/staff/becher/publications.html` *for references to the papers cited here and below.*

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Rod Downey

Denis R. Hirschfeldt

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Waves, Spheres, and Tubes

A Selection of Fourier Restriction Problems, Methods, and Applications

Betsy Stovall

Introduction

The Fourier transform on $\mathbb{R}^d$

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx, \qquad \xi \in \mathbb{R}^d, \tag{1}$$

is a fundamental operator that, via the Fourier inversion formula, provides coefficients to represent nice functions as superpositions of plane waves,

$$f(x) = \tfrac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{ix\cdot\xi} d\xi. \tag{2}$$

Aside from the basic utility of decomposing functions into simpler parts, this operator is powerful in the study of differential equations because it intertwines the operations of differentiation and polynomial multiplication via the formulae

$$(\widehat{-ix_j f}) = \partial_{\xi_j} \hat{f}, \qquad (\widehat{\partial_{x_j} f}) = i\xi_j \hat{f}. \tag{3}$$

Surprisingly, given its importance, much remains unknown

Betsy Stovall is an associate professor at the University of Wisconsin–Madison. Her email address is stovall@math.wisc.edu.

This work was supported in part by NSF DMS-1653264. The author is indebted to a number of colleagues, including the referees, who provided invaluable feedback on earlier versions.

Communicated by Notices *Associate Editor Daniela De Silva.*

DOI: https://doi.org/10.1090/noti1906

about how the Fourier transform interacts with simple geometric objects.

The Fourier restriction problem for the sphere asks for which exponents $(p, q) \in [1, \infty]^2$ does there hold an inequality

$$\|\hat{f}|_{S^{d-1}}\|_{L^q(S^{d-1})} \leq C\|f\|_{L^p(\mathbb{R}^d)}, \tag{4}$$

with a constant C uniform over all smooth, compactly supported functions f. Such an *a priori* inequality would imply that the restriction operator, $f \mapsto \hat{f}|_{S^{d-1}}$, uniquely extends to all of L^p, providing a reasonable interpretation of the restriction of the Fourier transform of L^p functions to the sphere. The mismatch between the linearity of planar waves and the curvature of the sphere makes this problem highly nontrivial and leads to deep connections among seemingly disparate areas of mathematics, such as geometric measure theory, dispersive PDE, and analytic number theory.

The restriction problem originated from work of Elias M. Stein, who in the 1960s established the first positive results in unpublished work. Despite its quick resolution in two dimensions, in work by Fefferman–Stein and Zygmund, the restriction conjecture for the sphere, that (4) holds if and only if

$$p < \tfrac{2d}{d+1} \quad \text{and} \quad \tfrac{d-1}{p'} \geq \tfrac{d+1}{q}, \tag{5}$$

remains open in all dimensions $d \geq 3$. Recent developments have substantially narrowed the gap between restriction theorems and the conjecture, and these new techniques have facilitated or promise to enable progress on a vast circle of related questions. The purpose of this article is to give a brief introduction for non-experts to some of the ideas and applications surrounding Stein's restriction problem.

The article is structured around a selection of methods, which we describe, for clarity, in the context of the restriction problem for the paraboloid, a simpler model than the sphere. In the elliptic setting (which contains both the sphere and paraboloid), most of these methods have by now been surpassed, but the field has become sufficiently broad that each method is at, or even beyond, the state-of-the-art for a large number of interesting restriction problems. Giving a sense of this breadth is a particular goal of this article, so we will describe and motivate some of these related questions as we reach their record-setting methods.

The competing demands of limited space, topical breadth, and (especially) clarity for non-experts, in a time of rapid development, necessitate the neglect of a number of important results. The author apologizes for these omissions and encourages the reader to view the bibliography as merely a starting point for further reading.

Background: The Restriction and Extension Operators

Our integral definition (1) of the Fourier transform only makes sense for a small class of functions, those in $L^1(\mathbb{R}^d)$, but for applications one often needs to extend the definition to a larger class. For $f \in L^1 \cap L^2$, the Plancherel theorem states that $\|f\|_2 = \frac{1}{(2\pi)^{d/2}}\|\hat{f}\|_2$, and since $L^1 \cap L^2$ is dense in L^2, the Fourier transform extends uniquely to a bounded linear operator of L^2 onto itself. By interpolation, we obtain the Hausdorff–Young inequality, which states that for $1 \leq p \leq 2$, this extension maps L^p boundedly into $L^{p'}$ and obeys $\|\hat{f}\|_{p'} \leq C_{p,d}\|f\|_p$; here p' is the dual exponent $p' = \frac{p}{p-1}$. This range of $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ estimates is the best possible.

For $f \in L^1$, $\hat{f}$ makes sense pointwise, but for $f \in L^p$, $1 < p \leq 2$, we usually interpret $\hat{f}$ as an $L^{p'}$ limit, $\hat{f} = \lim_{n\to\infty} \hat{f}_n$, where f_n is a sequence of integrable functions converging to f in L^p. This interpretation leads to an obvious obstruction to restricting a Fourier transform to sets of Lebesgue measure zero: $L^{p'}$ convergence does not "see" sets of measure zero. Indeed, $L^{p'}$ consists of equivalence classes within which members are allowed to differ off of sets of measure zero. Thus it makes no sense to define Fourier restriction to a set of measure zero as a simple composition. This nontriviality leads to a generalization of Stein's original restriction problem. Given a set Σ of Lebesgue measure zero, for which p, q does there hold an inequality

$$\|\hat{f}|_\Sigma\|_{L^q(\Sigma)} \leq C\|f\|_p, \tag{6}$$

for smooth, compactly supported functions f? Such an *a priori* estimate would imply that one can uniquely define an extension, $\mathcal{R}_\Sigma : L^p \to L^q$, which agrees with $f \mapsto \hat{f}|_\Sigma$ on $L^1 \cap L^p$. (One needs a measure σ on Σ to define the space $L^q(\Sigma)$. Which measure to use is an interesting question, but here we will focus on the case that σ is comparable to the natural Hausdorff measure on Σ.)

Finding the optimal (smallest) constant C in (6) seems to be a very difficult problem, which has only been solved in a few cases. As such questions of sharp constants are beyond the scope of this article, we will sweep operator norms and other constants under the rug by writing $A \lesssim B$ to mean that $A \leq CB$, for a constant C depending only on admissible parameters, such as the exponents p, q and the set Σ.

It was observed early on that even when Σ has measure zero, (6) is often uninteresting. The restriction problem for $p = 1$ is trivial because the Fourier transform of an L^1 function is bounded and continuous on all of $\mathbb{R}^d$, and so restricts to any set $\Sigma \subseteq \mathbb{R}^d$ in the natural way. The Fourier transform is an L^2 isometry by Plancherel, so a function can have small L^2 norm, while being large near a given set of measure zero: the restriction problem for $p = 2$ (and hence $p \geq 2$) is impossible. Finally, for $p > 1$ and $0 \not\equiv \phi$ smooth, nonnegative, and compactly supported, the sequence $\phi_n(x) := n^{-\frac{1}{p}}\phi(x', x_d/n)$, where $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, remains bounded in L^p, but its Fourier transform, $\hat{\phi}_n(\xi) = n^{\frac{1}{p'}}\hat{\phi}(\xi', n\xi_d)$, blows up (in every L^q space) on the hyperplane $\{\xi_d = 0\}$. Similar examples show that nontrivial restriction theorems are impossible for any subset of any affine hyperplane and, moreover, that interesting sets for Fourier restriction must escape from hyperplanes in the sense of having small measure trapped inside thin planar slabs. We think of such sets as being curved (even when they have no manifold structure); for instance, elliptic hypersurfaces, such as the sphere, escape from every hyperplane at a quadratic or faster rate.

Taken together, these examples allow us to recast the restriction problem: Given a set Σ of Lebesgue measure zero that escapes from hyperplanes, what is the largest p for which (6) holds for some exponent $q \geq 1$, and, given such a p, what is the largest possible q? Raising the exponents p and q has a heuristic interpretation, to which we now turn.

The identities in (3) imply a relationship between decay of a function and smoothness of its Fourier transform, while the Lebesgue inequalities in Hausdorff–Young do as well, but in a more subtle way. Chebyshev's inequality states that $\{|f| > \lambda\}$ has measure at most $\lambda^{-p}\|f\|_{L^p}^p$. For large λ, this inequality controls the rate of blowup of f and is strongest for large p; $f \in L^p$ with p large is thus a kind of averaged smoothness condition. For small λ, Chebyshev's inequality is a kind of decay condition, as it is strongest for small p. Since p' increases as p decreases, we may reinterpret Hausdorff–Young as a version of the principle that faster decay leads to a smoother Fourier transform.

Restriction inequalities are a finer manifestation of the link between decay and smoothness. *A priori*, the restriction problem for a set Σ is to determine precisely how little L^p decay is needed to restrict the Fourier transform and then to obtain the strongest L^q averaged smoothness along Σ for Fourier transforms of functions with that decay. Heuristically, bounds on the average smoothness along a measure zero set are quite strong because the denominator used in the average is small. (This interpretation also points to an understanding of sets with nontrivial restriction theorems, such as the sphere or even the helix, as larger than hyperplanes, but such a discussion is beyond the scope of this article.)

The preceding paragraph casts restriction inequalities as tangential smoothness conditions on Fourier transforms of functions with a given decay; recently, maximal Fourier restriction inequalities, due to Müller–Ricci–Wright [17] and others, also demonstrate a kind of transverse smoothness. An application of these results is a more intrinsic interpretation of the restriction operator than one obtains by using *a priori* inequalities to approximate and take L^q limits: for certain Σ and exponents p, the restriction operator on L^p can be understood by using averages over shrinking Euclidean balls,

$$\mathcal{R}_\Sigma f(\xi) = \lim_{r \searrow 0} \frac{1}{|B_r(\xi)|}\int_{B_r(\xi)} \hat{f}(\eta)\,d\eta, \quad f \in L^p(\mathbb{R}^d), \qquad \text{a.e. } \xi \in S^{d-1}.$$

The term inside the limit always makes sense when $1 \leq p \leq 2$, but a.e. existence of the limit fails when $p > 1$ for subsets of hyperplanes, because (as we have seen) the Fourier transform of an L^p function may lack smoothness in the (fixed) direction perpendicular to the hyperplane.

Whereas the restriction operator discards a great deal of information, its dual, which is called the extension operator and is given by

$$(\mathcal{R}_\Sigma)^* g(x) =: \mathcal{E}_\Sigma g(x) = \int_\Sigma e^{ix\cdot\xi} g(\xi)\,d\sigma(\xi),$$

does not and is thus easier to work with. Every restriction theorem, $\mathcal{R}_\Sigma : L^p(\mathbb{R}^d) \to L^q(\Sigma)$, has an equivalent extension counterpart, $\mathcal{E}_\Sigma : L^{q'}(\Sigma) \to L^{p'}(\mathbb{R}^d)$, and, as of this writing, almost all restriction theorems are really proved as extension theorems. For the remainder of the article, we will consider only the extension problem, which is to prove *a priori* inequalities,

$$\|\mathcal{E}_\Sigma f\|_{L^q(dx)} \lesssim \|f\|_{L^p(\sigma)}, \tag{7}$$

with f in some appropriate dense class. Dualizing our earlier remarks, the case $q = \infty$ is trivial, the case $q \leq 2$ is impossible, and the goal is to decrease q as much as possible (show more decay), which may require increasing p (assuming more smoothness), though, optimally, we would like to increase p as little as possible.

We will focus on a specific set Σ that is easy to parametrize and visualize and for which current techniques are most advanced, the extension operator associated to the paraboloid $\mathbb{P} = \{(\xi, |\xi|^2) : \xi \in \mathbb{R}^{d-1}\}$:

$$\mathcal{E}_{\mathbb{P}} f(x) = \int_{\mathbb{R}^{d-1}} e^{ix\cdot(\xi,|\xi|^2)} f(\xi)\,d\xi,$$

initially defined for smooth functions f with support contained in some fixed ball centered at 0 in $\mathbb{R}^{d-1}$. (We will sweep under the rug the effects of the size of this ball.)

Most of the methods for bounding $\mathcal{E}_{\mathbb{P}}$ readily generalize to compact *elliptic* hypersurfaces, such as the sphere; such surfaces are locally, after an affine transformation, small perturbations of the paraboloid. However, none of these methods have been fully generalized to all "interesting" measure zero sets, nor even all hypersurfaces, as we will see.

Classical Methods and the L^2 Theory

The paraboloid has a large affine symmetry group. Since the dot product interacts well with affine transformations, so too does the extension operator $\mathcal{E}_{\mathbb{P}}$. Three particularly important symmetries are:

$$\textit{Parabolic dilations: } \mathcal{E}_{\mathbb{P}}[\varepsilon^{-(d-1)}f(\varepsilon^{-1}\xi)](x) = \mathcal{E}_{\mathbb{P}}f(\varepsilon x', \varepsilon^2 x_d); \tag{8}$$

$$\textit{Frequency translations: } \mathcal{E}_{\mathbb{P}}[f(\xi-\xi_0)](x) = e^{ix\cdot(\xi_0,|\xi_0|^2)}\mathcal{E}_{\mathbb{P}}f(x'+2x_d\xi_0, x_d); \tag{9}$$

$$\textit{Spatial translations: } \mathcal{E}_{\mathbb{P}}[e^{-ix_0(\xi,|\xi|^2)}f(\xi)](x) = \mathcal{E}_{\mathbb{P}}f(x-x_0). \tag{10}$$

The parabolic and frequency translations are associated, respectively, with the dilation $(\xi,|\xi|^2) \mapsto (\varepsilon\xi, \varepsilon^2|\xi|^2)$ and Galilean $(\xi,|\xi|^2) \mapsto (\xi+\xi_0, |\xi+\xi_0|^2)$ symmetries of $\mathbb{P}$.

We can use the symmetries to illustrate some of the geometry associated to $\mathcal{E}_{\mathbb{P}}$. Let φ be a smooth, nonnegative function with integral 1 and support in $\{|\xi| < 0.1\}$. We begin by estimating $\mathcal{E}_{\mathbb{P}}\varphi$. For $|x'|, |x_d| < 1$ the integral defining $\mathrm{Re}\,\mathcal{E}_{\mathbb{P}}\varphi$ has no cancellation, and hence $|\mathcal{E}_{\mathbb{P}}\varphi(x)|$ is nearly as large as possible:

$$|\mathcal{E}_{\mathbb{P}}\varphi(x)| \geq \int \cos(x\cdot(\xi,|\xi|^2))\varphi(\xi)\,d\xi \geq \int \cos(0.11)\varphi(\xi)\,d\xi \approx 1.$$

For large $|x|$ the integrand oscillates rapidly in ξ, leading to cancellation in the integral, and hence a small extension: $|\mathcal{E}_{\mathbb{P}}\varphi(x)| \ll 1$ for $|x| \gg 1$.

Now we apply our symmetries, first a dilation, then a frequency translation, and finally a spatial translation,

$$\varphi^{\varepsilon}_{\xi_0,x_0}(\xi) = \varepsilon^{-(d-1)}e^{ix_0\cdot(\xi,|\xi|^2)}\varphi(\varepsilon^{-1}(\xi-\xi_0)).$$

Here, due to our assumption that functions are supported in some fixed ball, we think of ε as small. The function $\varphi^{\varepsilon}_{\xi_0,x_0}$ is supported on the set $\{|\xi-\xi_0| < 0.1\varepsilon\}$, which is naturally associated to the parabolic subset

$$\kappa^{\varepsilon}_{\xi_0} := \{\zeta \in \mathbb{P} : 0 \leq (\zeta-\zeta_0)\cdot \nu_{\xi_0} < 0.01\varepsilon^2\},$$

where $\zeta_0 := (\xi_0, |\xi_0|^2)$ and ν_{ξ_0} is the upward normal $\langle -2\xi_0, 1\rangle$ to $\mathbb{P}$ at ζ_0. The set $\kappa^{\varepsilon}_{\xi_0}$ thus resembles a tilted, shallow bowl of width about ε in directions tangent to $\mathbb{P}$ and depth about ε^2 in the direction ν_{ξ_0}, the relationship between width and thickness reflecting the curvature of $\mathbb{P}$. We call both the set $\kappa^{\varepsilon}_{\xi_0}$ and the function $\varphi^{\varepsilon}_{\xi_0,0}$ ε-caps on $\mathbb{P}$ centered at ζ_0, while we call the more general $\varphi^{\varepsilon}_{\xi_0,x_0}$ a modulated ε-cap. (The comparison to headgear is more convincing for caps on $-\mathbb{P}$ or on the northern hemisphere of S^{d-1}.)

The extension of a modulated ε-cap is, per (8-10), given by

$$\mathcal{E}_{\mathbb{P}}\varphi^{\varepsilon}_{\xi_0,x_0}(x) = e^{i(x-x_0)\cdot\zeta_0}\mathcal{E}_{\mathbb{P}}\varphi(\varepsilon[(x-x_0)' + 2(x-x_0)_d\,\xi_0], \varepsilon^2(x-x_0)_d).$$

Using our estimates for $\mathcal{E}_{\mathbb{P}}\varphi$, $|\mathcal{E}_{\mathbb{P}}\varphi^{\varepsilon}_{\xi_0,x_0}|$ is comparable to 1 on the long tube

$$T^{\varepsilon}_{\xi_0,x_0} = \{x \in \mathbb{R}^d : |(x-x_0)' + 2(x-x_0)_d\,\xi_0| < \varepsilon^{-1}, |x_d - (x_0)_d| < \varepsilon^{-2}\},$$

centered at x_0 and having width $\varepsilon^{-1} > 1$ and length $\varepsilon^{-2} > \varepsilon^{-1}$ in the direction ν_{ξ_0} (this geometry again reflects the curvature of $\mathbb{P}$), and decays rapidly off of this tube. For $T = T^{\varepsilon}_{\xi_0,x_0}$ and $\varphi_T = \varphi^{\varepsilon}_{\xi_0,x_0}$, the extension $\mathcal{E}_{\mathbb{P}}\varphi_T$ is called a *wave packet* associated to T.

For any $\varepsilon \ll 1$, a partition of unity directly decomposes our original function φ as a sum of (unmodulated) ε-caps, indexed by a collection of $O(\varepsilon^{-(d-1)})$ tubes,

$$\varphi = \sum_T c_T\varphi_T, \qquad T = T^{\varepsilon}_{\xi_T,0},$$

most of whose coefficients c_T are about ε^{d-1} (the rest are smaller). The curvature of $\mathbb{P}$ means that distinct tubes T, T' have directions $\nu_T, \nu_{T'}$ separated by at least ε. Because two tubes begin to separate at the length scale $\frac{width}{angle} \gtrsim \frac{\varepsilon^{-1}}{\varepsilon}$, tubes centered at 0 are nearly pairwise disjoint on $\{|x| \sim \varepsilon^{-2}\}$. Taking $\varepsilon \sim (1+|x|)^{-\frac{1}{2}}$, we can morally approximate $\mathcal{E}_{\mathbb{P}}\varphi(x)$ using the contribution from a single tube:

$$|\mathcal{E}_{\mathbb{P}}\varphi(x)| \approx |\mathcal{E}_{\mathbb{P}}(c_T\phi_T)(x)| \approx \varepsilon^{d-1} \approx (1+|x|)^{-(d-1)/2}. \tag{11}$$

A rigorous version of (11) can be proved using the method of stationary phase.

The estimates $\|\varphi_T\|_p \approx \varepsilon^{-(d-1)/p'}$ and $\|\mathcal{E}_{\mathbb{P}}\varphi_T\|_q \gtrsim |T|^{1/q}$ (the Knapp example), and the lower bound (11), lead to the necessary conditions

$$\tfrac{d-1}{p'} \geq \tfrac{d+1}{q}, \qquad q > \tfrac{2d}{d-1} \tag{12}$$

for validity of (7). By dualizing, (12) is equivalent to the necessary condition (5) for boundedness of the restriction operator, and these are currently the only known necessary conditions. (The introduction concerned the sphere; the conjecture for the paraboloid is the same.)

Self-duality of L^p when $p = 2$ allows one to recast the extension problem for $p = 2$ as a question about the convolution operator

$$\mathscr{E}_{\mathbb{P}} \circ \mathcal{R}_{\mathbb{P}} f(x) = \int_{\mathbb{R}^{d-1}} (\mathscr{E}_{\mathbb{P}}\psi)(x-y) f(y)\, dy,$$

where ψ is smooth, compactly supported, and is identically 1 on some large ball. This formulation has the advantage that the difficult-to-understand extension operator is applied to a fixed function ψ, rather than the arbitrary function f. An argument of Stein and Tomas from the 1970s uses the decay estimate (11) together with the simple observation that an ε-cap on the paraboloid has area ε^{d-1} to show that $\mathscr{E}_{\mathbb{P}}$ maps L^2 boundedly into L^q for $q \leq \frac{2(d+1)}{d-1}$, which by (12) is the optimal L^2-based result.

The Stein–Tomas Theorem generalizes to other sets, including sets of fractional dimension, so long as decay and volume estimates are known ([1], and the references therein). However, sharp decay estimates can be quite intricate and caps much more complicated for general sets, so sharp L^2-based estimates are known in a relatively narrow collection of examples. For example, what the optimal L^2-based extension estimates for polynomial hypersurfaces are in dimension $d \geq 4$ is largely an open question, with the $d = 3$ case having been only recently resolved by Ikromov–Müller [14].

One reason the L^2 theory for the extension problem for hypersurfaces has been so well studied is that the equations (3) lead to a connection with certain PDE. For instance, for a sufficiently nice function u_0 on $\mathbb{R}^{d-1}$, the extension $\mathscr{E}_{\mathbb{P}}\hat{u}_0$ solves the linear Schrödinger equation $iu_t = -\Delta_x u$ on $\mathbb{R}^{d-1}_x \times \mathbb{R}_t$ with initial data $u(x,0) = u_0(x)$, and extension operators for higher order polynomial surfaces are associated to higher order equations. Via Plancherel, an $L^2 \to L^q$ extension estimate leads to an $L^2 \to L^q$ inequality for the data-to-solution map (known as a Strichartz estimate), and such estimates have been extremely important in the study of a variety of dispersive and wave equations.

The Wave Packet Decomposition and Kakeya

For $p > 2$, a larger range of $L^p \to L^q$ extension estimates for $\mathscr{E}_{\mathbb{P}}$ should be possible, but the problem cannot be readily reduced to a question about convolutions. Instead, one wants to directly study the extension $\mathscr{E}_{\mathbb{P}} f$, where the function f may be very rough, even on small scales. After using a partition of unity to localize $f = \sum f^{\varepsilon}_{\xi_j}$, Fourier series methods resolve each $f^{\varepsilon}_{\xi_j}$ as a superposition of modulated caps; thus

$$f = \sum_T c_T \varphi_T, \qquad T = T^{\varepsilon}_{\xi_T, x_T}. \tag{13}$$

This *wave packet decomposition* lies at the heart of modern approaches to the subject, to which we turn in the coming sections. It first appeared in work of Bourgain in 1991, though it is closely connected with the earlier combinatorial approach of Fefferman and Córdoba. (A detailed implementation and more precise references may be found in e.g. [12].)

When $p = 2$, different modulations of the same cap are mutually orthogonal, and we can approximate

$$\|f\|_2 \approx (\sum_T \|c_T \varphi_T\|_2^2)^{\frac{1}{2}} \approx \varepsilon^{-(d-1)/2} (\sum_T |c_T|^2)^{\frac{1}{2}}; \tag{14}$$

this is analogous to the Plancherel theorem for the Fourier transform. For $p \neq 2$, however, the analogue to (14) is valid if and only if each cap is associated to at most one tube (instead of many parallel translates).

Let $\mathcal{T}^{\varepsilon}$ be a collection of length ε^{-2}, width ε^{-1} tubes in one-to-one correspondence with the set of ε caps. Understanding how these tubes can pile up and interact with one another is a very difficult geometric problem, with implications for bounds on $\mathscr{E} \sum_{T \in \mathcal{T}^{\varepsilon}} \varphi_T$. To illustrate this, we adapt an argument from [2].

In the previous section, we used the approximation $|\mathscr{E}_{\mathbb{P}}\varphi_T| \approx \chi_T$, but this ignores oscillatory factors, which might lead to cancellation (and hence a smaller sum) if many tubes overlap. Nevertheless, randomization (Khintchine's inequality) allows us to obtain a lower bound

$$\|\sum_{T \in \mathcal{T}^{\varepsilon}} \chi_T\|_{q/2}^{1/2} = \|(\sum_{T \in \mathcal{T}^{\varepsilon}} |\chi_T|^2)^{1/2}\|_q \lesssim \|\mathscr{E}_{\mathbb{P}} f\|_q, \tag{15}$$

for some $f := \sum_{T \in \mathcal{T}^{\varepsilon}} \pm \phi_T$. The $L^{q/2}$ norm on the left of (15) is closely connected, via Hölder's inequality, with the volume of the union of tubes:

$$\sum_{T \in \mathcal{T}^{\varepsilon}} |T| = \|\sum_{T \in \mathcal{T}^{\varepsilon}} \chi_T\|_{L^1} \leq |\bigcup_{T \in \mathcal{T}^{\varepsilon}} T|^{1-\frac{2}{q}} \|\sum_{T \in \mathcal{T}^{\varepsilon}} \chi_T\|_{q/2}. \tag{16}$$

Meanwhile, validity of (7) would allow us to bound the right-hand side of (15) by $\|f\|_p \approx \varepsilon^{-(d-1)}$. After a bit of arithmetic, (7), (15), and (16) thus imply

$$\varepsilon^{\alpha} \sum_{T \in \mathcal{T}^{\varepsilon}} |T| \lesssim |\bigcup_{T \in \mathcal{T}^{\varepsilon}} T|, \quad \alpha := \frac{2[q(d-1)-2d]}{q-2}, \tag{17}$$

with implicit constants uniform in $\varepsilon \ll 1$.

The case $\alpha = 0$ of (17) corresponds to the case $q = \frac{2d}{d-1}$, the conjectured endpoint for the extension problem. The brief remainder of this section will be devoted to some geometric heuristics around (17), which is easier to visualize in the (equivalent) rescaled version

$$\varepsilon^{\alpha} \lesssim |\bigcup_{T \in \mathcal{T}^{\varepsilon}} \tilde{T}|, \tag{18}$$

in which tubes $\tilde{T} := \varepsilon^2 T$ have length 1 and width ε, and the sum of their volumes is about 1.

Inequality (18) is clearly false when $\alpha < 0$: a union cannot be larger than the sum of the volumes. Overly simplistic reasoning suggests that (18) might be true for $\alpha = 0$. Indeed, since two distinct tubes separate (as we have seen), it seems unlikely that a collection $\mathcal{T}^\varepsilon$ could possess sufficient overlap for the union to have much smaller measure than the sum of the volumes. However, a surprising 1920s example of Besicovitch showed that for all $d \geq 2$, there are configurations of tubes with such strong overlap that

$$|\bigcup_{T\in\mathcal{T}^\varepsilon} \widetilde{T}| = o_\varepsilon(1), \quad \text{as} \quad \varepsilon \searrow 0.$$

As the tubes $\widetilde{T}$, with $T \in \mathcal{T}^\varepsilon$, together contain length-one line segments pointing in (a large portion of) all possible directions, Besicovitch's example leads to the conclusion that there exist compact sets of measure zero containing a unit line segment in each direction. Such sets are called Besicovitch sets (and sets containing a unit line segment in every direction are called Kakeya sets).

One version of the Kakeya Conjecture is that for all $d \geq 2$, Besicovitch sets, though measure zero, are not too small, in the sense that (18) holds for all configurations $\mathcal{T}^\varepsilon$ and all $\alpha > 0$. This conjecture is only settled in dimension $d = 2$, and there is good evidence that the remaining cases are extremely difficult, with the current record for $d = 3$ only slightly better than $\alpha = \frac{1}{2}$. (The current record for α, which is due either to Katz–Łaba–Tao or to Katz–Zahl, has not been precisely computed.)

Capitalizing on Transversality: Multilinear Methods

Let $\tau_1, \dots, \tau_k$, $2 \leq k \leq d$ be balls in $\{|\xi| < 1\}$, of diameter comparable to 1, whose corresponding parabolic caps are k-transversal in the sense that $\|\nu_1 \wedge \cdots \wedge \nu_k\| \sim 1$ whenever each ν_j is normal to the portion of $\mathbb{P}$ above τ_j. We can define a multilinear extension operator (somewhat confusingly called a multilinear restriction operator in the literature)

$$\mathcal{M}_{\mathbb{P}}(f_1, \dots, f_k)(x) := \prod_{j=1}^{k} \mathcal{E}_{\mathbb{P}}(f_j \chi_{\tau_j})(x), \tag{19}$$

and ask for inequalities of the form

$$\|\mathcal{M}_{\mathbb{P}}(f_1, \dots, f_k)\|_{\frac{q}{k}} \lesssim \prod_{j=1}^{k} \|f_j\|_p. \tag{20}$$

If (7) holds for p and q, then (20) is an immediate consequence of Hölder's inequality. Thus the interesting case is when the left side exhibits better-than-expected decay, $q < \frac{d+1}{d-1}p'$. Earlier, we sketched a heuristic argument that slight differences in the directions of tubes lead to separation and consequently the decay estimate (11), which in turn leads to the Stein–Tomas theorem. Transverse tubes separate more rapidly, and the relatively small size of their intersection makes improved decay plausible in (20). Improved decay is extremely useful because it makes linear inequalities that improve on Stein–Tomas accessible to L^2-based wave packet methods.

Variations on the multilinear idea have been around since early in the development of the subject, having played a role in the L^4 theory of Fefferman, Sjölin, and Carleson in the 1970s, and the work of Prestini, Drury, and Christ on restriction to curves in the 1980s. Bourgain, in the 1990s, was the first to improve on Stein–Tomas in higher dimensions by using bilinear estimates. These ideas were further developed by Moyua–Vargas–Vega and Tao–Vargas–Vega, culminating in near-optimal bilinear restriction inequalities for the cone (Wolff) and elliptic hypersurfaces (Tao) in the early 2000s. Since that time, there has been significant progress toward a bilinear theory for more general hypersurfaces, though major open questions remain. Somewhat better known are the developments on multilinear restriction estimates for elliptic surfaces, to which we devote the majority of this section. We will describe some challenges for more general hypersurfaces at the end of the section.

In the special case $p = 2$, it is conjectured [3] that (20) should hold for $q > \frac{2(d+k)}{d+k-2}$. Below that endpoint, the inequality fails, as can be seen by considering functions f_j formed by summing ε-caps with normal directions lying within ε of a fixed k-plane. There are $\varepsilon^{-(k-1)}$ such caps, each with norm roughly $\varepsilon^{-\frac{d-1}{2}}$, so

$$\prod_j \|f_j\|_2 \sim (\varepsilon^{-\frac{d+k-2}{2}})^k.$$

The extension of each f_j is comparable to 1 on the ε^{-1} neighborhood of an ε^{-2}-ball in the k-plane. As this (shared) cylinder has volume $(\varepsilon^{-2})^k(\varepsilon^{-1})^{d-k}$,

$$\|\mathcal{M}(f_1, \dots, f_k)\|_{\frac{q}{k}} \sim \varepsilon^{-(d+k)\frac{k}{q}},$$

and sending $\varepsilon \searrow 0$ in (20) leads to the claimed necessary condition.

The above-described multilinear restriction conjecture for $\mathbb{P}$ is settled when $k = 1$ (Stein–Tomas), when $k = 2$ (Tao), and when $k = d$ (Bennett–Carbery–Tao, [4]). The latter result also gives a nontrivial improvement for all $k \geq 3$, which we now state.

Theorem 1 ([4]). *Inequality* (20) *holds when* $p = 2$ *and* $q > \frac{2k}{k-1}$.

Of particular note is the fact that for $k = d$, the endpoint $\frac{2k}{k-1}$ matches the endpoint in the extension conjecture (12). Theorem 1 is based only on the transversality of the surface patches, not their curvature, and hence in its full generality (for which (19) is a special case) it is optimal.

We turn now to a heuristic description of the process of extracting linear inequalities from the bi- and multi-linear ones. The "bilinear-to-linear" argument of [19] expresses $|\mathcal{E}f|^q = |(\mathcal{E}f)^2|^{q/2}$ as the $\frac{q}{2}$ power of a double integral over $\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$. The diagonal in this space has measure zero, so the double integral may be taken over (ξ, η) with $\xi \neq \eta$. Each such pair is associated with a pair of balls of radii about $|\xi - \eta|$, separated from one another by a distance about $|\xi - \eta|$, one containing ξ and one containing η. Separated balls have transverse normals because the gradient map is a diffeomorphism (thanks to the curvature). The bilinear theory applies to such pairs of balls after rescaling. Orthogonality and convexity arguments then make it possible to sum.

The geometry described above does not readily generalize to (for instance) the d-linear case, because d distinct points on $\mathbb{P}$ could have normals all transverse to one another, all arranged along some k-plane, or lying in some intermediate configuration. Because of the variety in the possible configurations, it took some years before multilinear restriction was used to obtain new linear estimates by Bourgain–Guth [7].

The Bourgain–Guth argument is via a transverse vs. linearly dependent dichotomy and induction on scales, for which we give a brief cartoon (which owes much to the exposition in [16]). The extension $\mathcal{E}f(x)$ may be thought of as a superposition of many tubes though x. There is either a significant contribution from d transverse tubes, or the bulk of the tubes lie near some hyperplane passing through x. The d-transversal case is favorable because d-linear extension estimates come with an exponent $q = \frac{2d}{d-1}$, which matches the extension conjecture. The hyperplane case is refined further to determine the right dimension of transversality for the tubes contributing to $\mathcal{E}f(x)$. We have already seen that one extreme, the d-transversal case, is favorable. At the other extreme, $\mathcal{E}f(x)$ is 1-transversal, i.e., parallel: it primarily comes from tubes whose corresponding caps lie within some small ball. This case is favorable as well, because of an argument known as induction on scales: parabolic rescaling expands the small ball to have radius 1 (so we nearly have the same problem again), but introduces an additional favorable term. This term arises because the method aims for $L^p \to L^q$ inequalities that do not scale (meaning that $\|\mathcal{E}_{\mathbb{P}} f\|_q$ and $\|f\|_p$ behave differently under rescaling).

The k-transversal case is intermediate between the two extremes. On the one hand, k-linear restriction comes with a minimal exponent of $q = \frac{2k}{k-1}$, which is smaller, hence better (closer to the conjectured $q > \frac{2d}{d-1}$) for large k; for small k, the exponent must be reduced further, leading to a loss. On the other hand, the tubes correspond to caps lying near a k-plane, and the k-plane can be covered by small balls, which can be parabolically rescaled. For this estimate, smaller k corresponds to fewer small balls in the cover, and hence a more favorable term from parabolic rescaling. For a certain range of exponents, the gains compensate for the losses, leading to new (at the time, but by now surpassed) estimates.

Among hypersurfaces, the elliptic case is generally the best studied, and there are currently few linear results beyond Stein–Tomas for hyperbolic surfaces and surfaces whose curvature varies. (As already noted, even the L^2 theory is not fully developed for such surfaces.) We give two examples to illustrate some of the challenges.

Hyperbolic surfaces may be ruled (expressible as unions of lines, such as the one-sheeted hyperboloid or the hyperbolic paraboloid), or have high order contact with certain lines. It is easy to produce lots of overlap among tubes with directions normal to a surface along some line within the surface, necessitating additional separation hypotheses for nontrivial bilinear restriction. In particular, in the case $k = 2$ of (20), to obtain a result matching that for the paraboloid, the images of the balls τ_1, τ_2 on the surface cannot be nearly collinear. Stronger hypotheses make for weaker theorems, leading to a more difficult bilinear-to-linear deduction; indeed, the natural analogue of the Tao–Vargas–Vega deduction has only been fully carried out for the hyperbolic paraboloid in $\mathbb{R}^3$. Moreover, the hyperbolic case does not perturb as well as the elliptic case, because the natural caps for the hyperbolic paraboloid are rectangles, which may be long and thin. Whereas perturbed balls (such as arise in the elliptic case) still look roughly like balls, perturbed rectangles may curve, leading to new difficulties. Some of these issues are described in [8] and the references therein.

An advantage in the hyperbolic case is that the multilinear extension result of Bennett–Carbery–Tao, Theorem 1, depends only on transversality, and hence readily generalizes to any surface (or subset thereof) on which the unit normal map is a local diffeomorphism. This property does not hold for surfaces on which the Gaussian curvature may vanish, and, moreover, variable curvature can lead to very different behavior on different parts of the surface. Even for convex polynomial hypersurfaces in dimension $d \geq 3$, how to deal with these issues to achieve results matching the elliptic case is a largely open question.

Discretization Via Decoupling

An important application of the multilinear perspective is in proving decoupling inequalities. These inequalities were introduced by Wolff in his work on the L^p regularity properties of the wave equation (the local smoothing problem), and a recent result of Bourgain–Demeter [5] established a stronger version of Wolff's inequalities in the optimal range. In this section, we will state and provide some background on the Bourgain–Demeter result and mention two recent applications of the method, one in analytic number theory, and one for restriction to fractal sets; an application to PDE is discussed in the final section.

Let f be a function on $\mathbb{R}^d$ with $\widehat{f}$ having compact support contained in the δ-neighborhood of $\mathbb{P}$, $N_\delta(\mathbb{P})$. We can discretize $N_\delta(\mathbb{P}) = \bigcup_{\theta \in \mathcal{P}_\delta} \theta$, where the θ's are finitely overlapping caps of thickness δ and width $\delta^{\frac{1}{2}}$. A partition of unity decomposes $f = \sum_\theta f_\theta$, with each $\widehat{f}_\theta$ supported on θ. By the triangle inequality, the simple estimate $\#\mathcal{P}_\delta \approx \delta^{-(d-1)/2}$, and Cauchy–Schwarz we have

$$\|f\|_\infty \le \delta^{-(d-1)/4} (\sum_\theta \|f_\theta\|_\infty^2)^{\frac{1}{2}}, \qquad (21)$$

while Plancherel and finite overlap of the caps immediately implies

$$\|f\|_2 \lesssim (\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_2^2)^{\frac{1}{2}}. \qquad (22)$$

Both of these results are optimal, and an interpolation argument yields

$$\|f\|_p \lesssim \delta^{-\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})} (\sum_\theta \|f_\theta\|_p^2)^{\frac{1}{2}}, \quad 2 \le p \le \infty. \quad (23)$$

Inequality (23), it turns out, is not optimal, due to the curvature of $\mathbb{P}$.

The ℓ^2 decoupling theorem of Bourgain–Demeter [5] states that for f as above, the inequality

$$\|f\|_p \lesssim \delta^{-\frac{d-1}{4}+\frac{d+1}{2p}-\varepsilon} (\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_p^2)^{\frac{1}{2}} \qquad (24)$$

holds for all $p \ge \frac{2(d+1)}{d-1}$ and $\varepsilon > 0$, which nearly eliminates the negative power of δ in (23) when $p = \frac{2(d+1)}{d-1}$.

It is no coincidence that the critical exponent $p = \frac{2(d+1)}{d-1}$ for (24) matches that for the Stein–Tomas theorem (i.e. the optimal L^2-based extension theorem), and an important application of (24) was to prove the discrete analogue of the Stein–Tomas theorem (which has applications to the periodic Schrödinger equation). Discrete versions of inequalities in harmonic analysis tend to be harder than the continuous versions, because delicate number theoretic issues come into play. (Compare, for instance, the difficulty in finding the measure of the real solution set of $x^a + y^a = z^a$ with finding the cardinality of its integer solution set.) These issues are somewhat easier to see in cases wherein the exponents involved are integers, so we now turn to a related example.

The Main Conjecture in Vinogradov's Mean Value Theorem is (was) the upper bound $\#J_{s,d}(N) \le C_{\varepsilon,s,d} N^{s+\varepsilon} + N^{2s-\frac{d(d+1)}{2}+\varepsilon}$ on the number, $J_{s,d}(N)$, of solutions to the system of equations

$$j_1^i + \cdots + j_s^i = j_{s+1}^i + \cdots + j_{2s}^i, \quad 1 \le i \le d, \quad j \in [1,N]^{2s}.$$

This was a classical result for $d = 1, 2$, and was proved by Wooley in the case $d = 3$, with partial progress in higher dimensions, via efficient congruencing. However, a different approach, via decoupling associated to the curve $(t, t^2, \dots, t^d)$, was used by Bourgain–Demeter–Guth in [6] to affirmatively resolve the conjecture in all dimensions.

The connection between the number theoretic problem and restriction can be seen by direct computation. Expanding

$$J_{s,d}(N) = \int_{[0,1]^d} |\sum_{j=1}^N \exp[2\pi i(x_1 j + x_2 j^2 + \cdots + x_d j^d)]|^{2s}\, dx,$$

and realizing the right as an L^q norm of the discrete extension operator

$$\mathcal{E}_\gamma a(x) := \sum_{j=1}^N \exp[2\pi i (x \cdot \gamma(j))] a_j,$$
$$\gamma(j) := (j, j^2, \dots, j^d),\ a \in \mathbb{C}^N,$$

the conjectured bound on $J_{s,d}(N)$ is equivalent to the estimate

$$\|\mathcal{E}_\gamma a\|_{L^p([0,1]^d)} \lesssim (N^\varepsilon + N^{\frac{1}{2}-\frac{d(d+1)}{2p}+\varepsilon}) \|a\|_{\ell^2([1,N])},$$
$$p = 2s \ge 2, \quad (25)$$

where $a := (1, \dots, 1)$. The estimate (25) in turn follows from decoupling.

Ideas from decoupling and work of Bourgain on $\Lambda(p)$ sets have recently been used by Łaba–Wang [15] to construct examples of fractal sets in $\mathbb{R}$, having arbitrary Hausdorff dimension in the interval $(0,1)$, that obey extension inequalities in a range inaccessible to the generalized Stein–Tomas theorem from [1]. These examples are Cantor-like sets, wherein the deletion process in the standard Cantor construction is randomized. Randomization has been used in the past to construct interesting sets for Fourier restriction (further references are in [15]); the advantage of decoupling is that it helps to separate out the different pieces of the set at each scale.

Lines, Curves, and Varieties: Polynomial and Kakeya Methods

In this final section, we discuss some techniques that comprise the current state of the art for linear restriction to the paraboloid. These techniques use, in various ways, the fact that a tube is a thickened version of an algebraic object (a line) and hence should not intersect a given low-degree algebraic variety too many times. One reason algebraic varieties are relevant is that, as noted earlier, tubes lying close to k-planes provide the conjectured sharp examples for multilinear extension inequalities in the presence of curvature; controlling the extension of a function on lower dimensional sets is also relevant in PDE, as described at the end of this section.

The current best (smallest q) results for $\mathcal{E}_\mathbb{P}$ involve bounds for quantities called k-broad norms, which were introduced by Guth [11, 12]. These linearize the transversality utilized in multilinear estimates. We will not give the precise definition, but roughly, the k-broad norm counts only the part of $\mathscr{E}_{\mathbb{P}}f$ that genuinely comes from k-transversal parts of $\mathbb{P}$, and thus is closely related to the k-linear extension operator (19). Bounding these norms involves counting intersections of transversal tubes (a combinatorial perspective that traces back to the work of Bourgain and Wolff), and optimal estimates were proved for these norms by Guth in [11, 12] using a technique called polynomial partitioning.

Polynomial partitioning is a divide-and-conquer strategy for bounding the number of certain types of incidences (such as k-transversal intersections of tubes). Very roughly, the "Polynomial Ham Sandwich Theorem" allows one to find a low-ish degree algebraic variety whose complement is a union of cells that each contain roughly the same number of incidences. The cells are good objects for induction on the number of tubes: each tube is like a line and so by the Fundamental Theorem of Algebra only enters a few cells; therefore some cell meets only a small number of tubes, yielding a simpler problem. The variety contains two types of incidences: those occurring exclusively among tubes that are tangent to the variety and those involving at least one tube that intersects the variety transversally. The former situation sets up an induction on dimension, while transversal intersections are not too common because any given tube can only cross the variety a few times. The polynomial partitioning method originated in work of Guth–Katz, with important precursors in the works of Dvir and Clarkson–Edelsbrunner–Guibas–Sharir–Welzl; a detailed history of the method may be found in the introduction of [11].

The estimates for the k-broad norms in [11, 12] correspond to the conjectured optimal range for multilinear restriction to $\mathbb{P}$, and yield the same bounds for $\mathscr{E}_{\mathbb{P}}$ as would have been achieved via the methods of [7] using the optimal multilinear restriction conjecture. These bounds on $\mathscr{E}_{\mathbb{P}}$ have been improved in all except a handful of dimensions ([13, 20]), and are thought not to be optimal in any dimension, but their analogues are optimal for a variable coefficient generalization recently obtained by Guth–Hickman–Iliopoulou. In the variable coefficient generalization, tubes are replaced by neighborhoods of curves that can compress into low-dimensional sets in ways that lines cannot, as observed in [7] and earlier work of Wisewell.

Improvements over [12] have used polynomial partitioning in conjunction with additional geometric arguments that further limit the ability of straight-line tubes to compress into small sets. These include an argument of Wang in [20] that organizes the tubes lying along a hypersurface in $\mathbb{R}^3$ into "brooms," and one of Hickman–Rogers in higher dimensions [13] that utilizes a theorem of Katz–Rogers verifying what are called "polynomial Wolff axioms," which bound from below the size of a sub-algebraic set containing a large number of tubes.

The extension problem for the cone $\{(\xi, |\xi|) : 1 \leq |\xi| \leq 2\}$ is almost as well studied as that for the paraboloid, and the former problem is closely connected with the wave equation. The cone and the cylinder $[1,2] \times S^{d-2} \subseteq \mathbb{R}^d$ both have $d-2$ nonvanishing principal curvatures, and a variation on the Knapp example suggests the same restriction conjecture for these surfaces as for the sphere of one lower dimension. In fact, the product structure makes the extension problem for the cylinder exactly the same as the sphere of one lower dimension. However, the changing slope of the flat direction on the cone has facilitated much better progress on its restriction theory than that of the lower dimensional sphere, and the cone restriction problem has also been completely solved in dimensions $d = 3$ (Barceló, classical), $d = 4$ (Wolff, using the bilinear theory), and $d = 5$ (Ou–Wang, [18]). The latter result used polynomial partitioning to reduce q to the optimal range and the bilinear theory to obtain the optimal p range.

We close by describing the recent resolution of a question in PDE in which the size of a Fourier extension near some lower-dimensional set plays an important role. The solution to the linear Schrödinger equation $iu_t + \Delta u = 0$, with initial data $u(0) = u_0 \in L^2(\mathbb{R}^{d-1})$, is simply the extension $u(x,t) = (\mathscr{E}_{\mathbb{P}}\hat{u}_0)(x,t)$. Here, u_0 is the initial data in the sense that $\lim_{t\to 0} \|u(t) - u_0\|_{L^2} = 0$. Carleson's problem, posed in the late 1970s, asks for the precise number of derivatives needed to ensure that $\lim_{t\to 0} u(t,x) = u_0(x)$, for almost every x. In dimension $d = 2$, the problem was resolved relatively quickly by Carleson himself ($\frac{1}{4}$ derivative is sufficient) and Kenig–Dahlberg ($\frac{1}{4}$ derivative is necessary), while its (almost) complete resolution took nearly 40 years longer.

A.e. convergence follows from bounding the maximal operator

$$f \mapsto \sup_{0<t\leq 1} |\mathscr{E}_{\mathbb{P}}\hat{f}(x,t)|.$$

This maximal operator can be linearized as $\mathscr{E}_{\mathbb{P}}\hat{f}(x, t(x))$, which looks like the (functional) restriction of $\mathscr{E}_{\mathbb{P}}\hat{f}$ to a $(d-1)$-dimensional set. Du–Guth–Li [9] proved in dimension $d = 2$ (using polynomial partitioning), and Du–Zhang [10] proved in dimension $d \geq 3$ (using decoupling and multilinear restriction) that having more than $\frac{d-1}{2d}$ L^2 derivatives (in other words for $u_0 \in H^s$, $s > \frac{d-1}{2d}$), suffices for a.e. convergence of u to u_0. An earlier result of Bourgain showed that $s \geq \frac{d-1}{2d}$ derivatives is necessary, so Carleson's problem is resolved except at the critical value.

ACKNOWLEDGMENT. Jean Bourgain and Elias Stein passed away shortly before this article was submitted, and the author would like to express her deep gratitude and admiration for their beautiful mathematics and the subsequent work that it inspired.

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Algebraic and Topological Tools in Linear Optimization

Jesús A. De Loera

The Linear Optimization Problem

This is a story about the significance of diverse viewpoints in mathematical research. I will discuss how the analysis of the linear optimization problem connects in elegant ways to algebra and topology. My presentation has two sections, grouped under the guiding light of these areas. These mathematical areas, while often considered to be pure mathematics, in fact connect deeply to many other important computational problems besides linear optimization (see [5] and the many references therein). Clearly, the famous quest for the "most efficient" algorithm possible to solve the linear optimization problem shapes my story, too; this is a challenge that many consider to be one of the top mathematical challenges of the century (see, e.g., [14]). My narrative is informal, thus I will not give all details, but I hope enough intuition will entice others to learn more about these methods. My target reader is a non-expert mathematician, say a graduate student curious about how algebra connects to optimization, but I also hope to give experts quick pointers to remarkable new activity since 2010. For the sake of space I was asked to leave out the majority of references, but the interested reader can obtain a longer version, with all missing references, by contacting me.

Jesús A. De Loera is a professor of mathematics at the University of California, Davis. His email address is `deloera@math.ucdavis.edu`.

The author gratefully acknowledges support from the NSF grant DMS-1818969.

Communicated by Notices *Associate Editor Emilie Purvine.*

DOI: https://doi.org/10.1090/noti1907

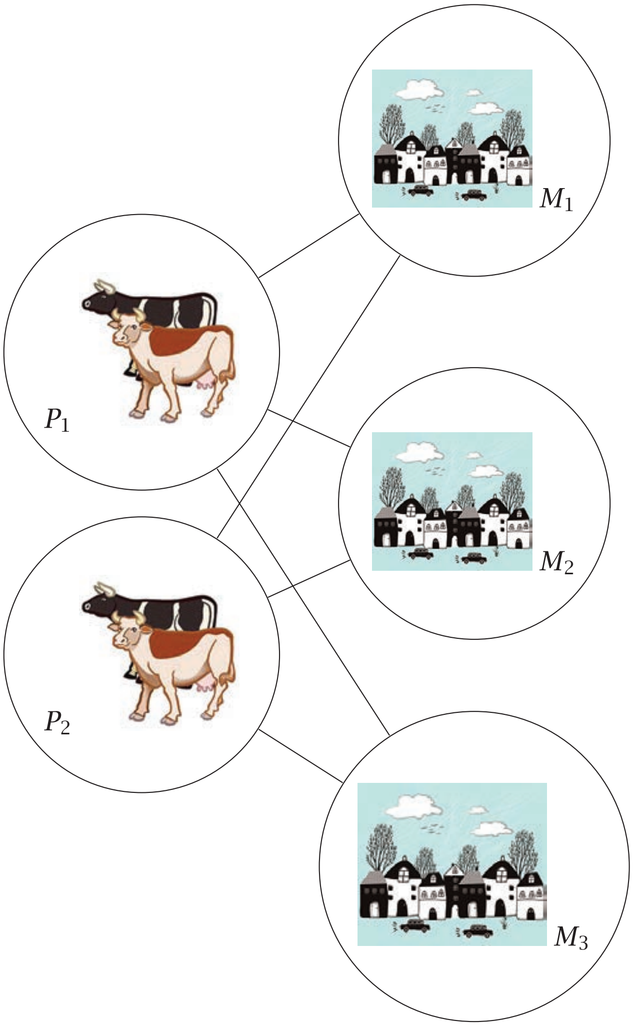

Figure 1. Two milk production plants supply three cities.

The hero of my story is the *linear optimization problem*. This is the basic, but fundamental, computational problem of maximizing or minimizing a linear function subject to the condition that the solution must satisfy a given set of linear inequalities and equations. One wishes to find a vector $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ such that we maximize $c_1x_1 + c_2x_2 + \ldots c_dx_d$ subject to

$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,d}x_d \leq b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,d}x_d \leq b_2, \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \ldots + a_{n,d}x_d \leq b_n. \end{array}$$

Here $a_{i,j}, b_i, c_j$ are assumed to be integers. The same problem presented in matrix-vector notation is summarized as

$$\max\{\mathbf{c}^{\mathcal{T}}\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \in \mathbb{R}^n\}.$$

As a very concrete example, imagine there are two milk production plants, P_1 and P_2, that supply three cities, M_1, M_2, and M_3, with fresh milk. Facility P_i bottles s_i gallons of milk, and city M_j consumes r_j gallons. There is a cost C_{ij} for transporting one gallon of milk from plant P_i to city M_j. See Figure 1 for an illustration of the transportation problem. The problem is to meet the market requirements, i.e., satisfy demand with the available supply, doing so at a minimum transportation cost. If the amount of milk to be shipped from P_i to M_j is given by x_{ij}, then we can write a *linear program* modeling the optimization challenge of minimizing $\sum_{i=1}^{2}\sum_{j=1}^{3} C_{ij}x_{ij}$ subject to

$$\begin{array}{rl} \sum_{i=1}^{2} x_{ij} \geq r_j & \text{for each city,} \\ \sum_{j=1}^{3} x_{ij} \leq s_i & \text{for each milk plant,} \\ x_{ij} \geq 0. & \end{array}$$

Research on linear optimization can be traced back at least to Fourier, but the subject really starts developing at full speed around the time of World War II. In fact, it is around 1939–1941 that Kantorovich and Koopmans investigated the simple type of transportation problem we saw earlier. They later received the Nobel Prize in Economics for it. Dantzig (see Figure 2), von Neumann, Gale, Kuhn, and Tucker were crucial in the first developments of the subject in the late forties. The name *linear program* is old, and the word *programming* was not about computer programming, but was used as a synonym for planning. Following that old tradition I will call *linear programs*, or LPs for short, the instances of the linear optimization problem.

If distributing milk efficiently is not your favorite way to engage with mathematics, consider the following way to think about linear programs: With a simple reformulation adding auxiliary variables, the inequality system can always be rewritten as the solutions of a system of linear *equations* over the *non-negative* real numbers, $\max\{\mathbf{c}^{\mathcal{T}}\mathbf{x} : A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq 0, \ \mathbf{x} \in \mathbb{R}^n\}$. Thus linear optimization can be thought of as *linear algebra with non-negative variables*. In fact, Fourier's initial algorithm was a process of variable elimination quite similar to Gaussian elimination. Today we know it as the Fourier–Motzkin elimination algorithm (it was rediscovered by T. S. Motzkin).

Why do LPs matter? Well, LPs are simple yet very expressive models, which include as special cases several important problems: for instance, the challenges of finding the shortest paths on a graph, the maximum flow on a network, the minimum weight spanning tree or matching on a graph, and all two-player zero-sum games can be formulated directly as linear programs. More importantly, all other parts of optimization rely heavily on linear optimization as a pillar for computation and theory. For example, optimization problems with discrete variables are most often reduced via branching to the repeated use of linear programming. Linear programs are also used in various approximation schemes for combinatorial and non-linear optimization. But the impact of linear optimization goes well beyond optimization itself and reaches other areas of mathematical research: e.g., in combinatorics and graph theory and in discrete geometry, for the solution of Kepler's conjecture. Exciting new applications continue to appear, and the impact of linear optimization is also palpable in practical ventures (e.g., airlines, oil industry, etc.) because there are very fast software packages that can solve concrete problems with millions of variables in reasonable time. For those wishing to learn more, there are several excellent books ([10, 13]) and surveys (such as [8, 11, 15]) covering the theory of linear optimization.

Before we start, a word about the geometry behind linear programs. The *feasibility region* of a linear program, i.e., the set of possible solutions, is always a *convex polyhedron*. Polyhedra are beautiful jewel-like objects that have attracted mathematicians for centuries. Here a polyhedron $P \subset \mathbb{R}^d$ is the set of solutions of a system of linear inequalities of the form $A\mathbf{x} \leq \mathbf{b}$, where A is an integer matrix with d-dimensional row vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and $\mathbf{b}$ is an integer vector. In this way, the *input size* is given by d variables, n constraints, and the maximum binary size L of any coefficient of the data. We assume that row vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ span $\mathbb{R}^d$. Thus $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq b\}$. When P is bounded, we call it a *polytope*. Two important types of polytopes are *simple* and *simplicial* polytopes. Simple polytopes are those where at every vertex, d inequalities meet with equality, for example a cube. *Simplices* are d-dimensional polytopes with exactly $d+1$ inequalities. A *simplicial* polytope is one whose faces are all simplices, e.g., an octahedron or an icosahedron. Not all polytopes fit in these two types (e.g., the polytope in Figure 3), but most arguments in linear optimization go through them. For example, simple polytopes correspond to non-degenerate linear programs, which are typically run in computation. The beautiful geometry of polytopes is clearly presented in [17]. We now begin the story.

A Topological Point of View

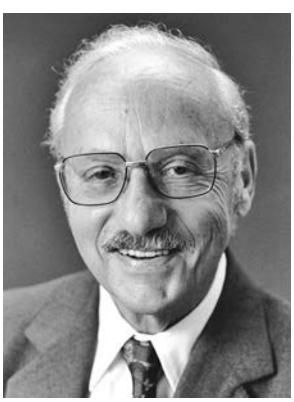

Figure 2. George Dantzig (1914–2005).

Dantzig's simplex method from 1947 [4] is one of the most common algorithms for solving linear programs. It can be viewed as a family of combinatorial local search algorithms on the graph of a convex polyhedron. More precisely, the search is done over a finite graph, the one-skeleton of the polyhedron or *graph of the polyhedron*, which is composed of the zero- and one-dimensional faces of the feasible region (called *vertices* and *edges*). The search moves from a vertex of the one-skeleton to a better neighboring vertex according to some *pivoting rule* that selects an improving neighbor. The operation of moving from one vertex to the next is called a *pivot step* or simply a *pivot*. Geometrically the simplex method traces a finite path on the graph of the polytope. See Figure 3.

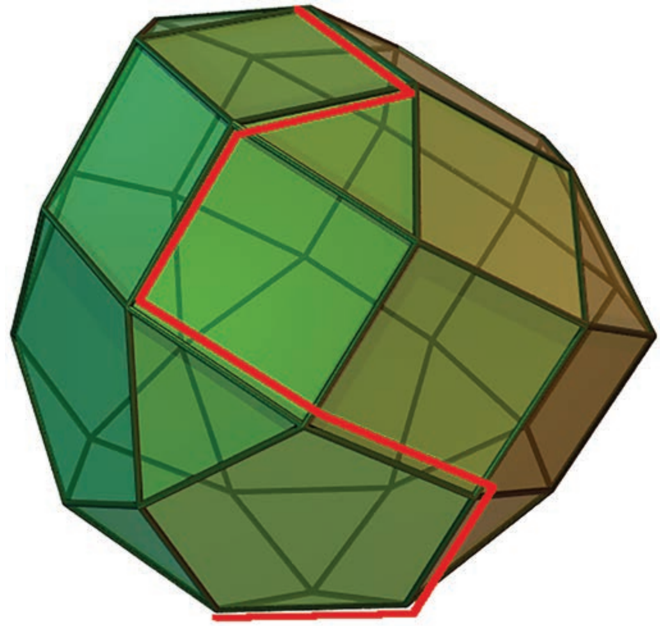

Figure 3. The simplex algorithmic steps trace a path on the graph of a polyhedron.

Today, after sixty years of use, and despite competition from other algorithms, the simplex method is still widely popular and useful. The simplex method was even named as one of the most influential algorithms in the twentieth century, but we still do not completely understand its theoretical performance sufficiently well to explain its practical performance. *Is there a polynomial-time version of the simplex method?* Such an algorithm would allow the solution of a linear program with a number of pivot steps that is a polynomial in d, n, and L. This is a very famous problem that has received a lot of attention. Suggested by this complexity question, there is a related geometric puzzle about the *diameter* of the graph of a polytope used to make the simplex walk. The diameter is the length of the longest shortest path among all possible pairs of vertices; e.g., for a three-dimensional cube the diameter is three. It remains a well-known open problem whether there is always a polynomial bound on the diameter. If a counterexample exists, then a polynomial simplex method would be impossible. So what are current bounds for the diameter?

The best upper bounds for the diameter of polytopes, valid for all polytopes, originated in a groundbreaking paper by Kalai and Kleitman. If we denote by n the number of facets of a polytope, and d its dimension, then their result asserts a bound of $n^{\log(d)+1}$. This was improved by Todd and then, most recently, Sukegawa with the current

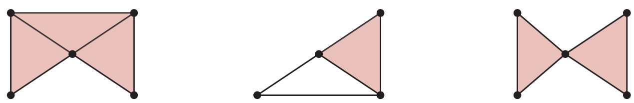

Figure 4. Some simplicial complexes (from left to right): a pure strongly connected complex, a non-pure complex, a non-strongly connected complex.

record of $(n-d)^{\log O(d/\log(d))}$. In a different influential approach Larman proved that the diameter is no more than $2^{d-3}n$. This bound was improved by Barnette to $\frac{2^{d-3}}{3}(n-d+5/2)$. Remarkably this bound shows that, in fixed dimension d, the diameter must be *linear* in the number of facets. Could the diameter of all polytopes be in fact linear in terms of n, d? We will come back to this later!

A key point I would like to stress is that the proofs of these general upper bounds use only very limited (metric) geometric properties of polytopes; the coordinates and equations defining an LP do not play a big role. The bounds hold for more abstract combinatorial objects called *simplicial complexes.*

A simplicial complex K is a finite collection of simplices that are glued to each other in a structured manner. If $\sigma \in K$, then all its faces (which are smaller simplices too!) are also in K. Thus a simplicial complex must be closed under containment. The intersection of any two elements of K is another element of K. The *dimension* of a simplex is equal to the number of vertices minus one (e.g., a triangle is two-dimensional). A simplex of K that is not a face of another simplex is called *maximal.* A simplicial complex K is called *pure* if all of its maximal faces are of the same dimension. Maximal faces of a pure simplicial complex are called *facets*. For a pure simplicial complex K, one may form its *dual graph*, which is a graph whose vertex set is given by the facets of K and whose edge set is given by pairs of facets in K that differ in a single element. A simplicial complex is *strongly connected* if its dual graph is a connected graph. See Figure 4.

Pure simplicial complexes are excellent topological abstractions of polytopes because the boundary complex of a simplicial $d+1$-polytope is always a pure simplicial complex of dimension d, and there is a direct way to go from non-degenerate linear optimization problems (known to have the largest diameters anyway) to pure simplicial complexes by using the *polarity* operation. This is illustrated in Figure 5, where facets turn into vertices and vertices into facets under polarity to obtain an octahedron that is simplicial (all faces are now triangles). Note that the n facets of a simple polytope turn into n vertices of a simplicial complex.

A path on the edge of the cube becomes a path on the (triangles) simplices of an octahedron. In higher dimensions the distance between two facets of the simplicial complex, F_1, F_2, is the length s of the shortest simplicial path $F_1 = f_0, f_1, \dots, f_s = F_2$. Consecutive d-dimensional simplices in the path must share a common $(d-1)$-dimensional face. The diameter of a simplicial complex is the maximum over all distances between all pairs of facets. Note that the diameter of K equals the graph theoretic diameter of its dual graph.

The *topological* approach to finding the diameter of simple polytopes instead explores the diameter of their corresponding (polar) simplicial complexes. We remind the reader that now n refers both to the number of facets of the simple polytope and to the number of vertices of its (polar) simplicial complex. This idea has a long history, starting with the introduction of *abstract polytopes* by Adler and Dantzig [1]. Mani and Walkup, Kalai, Billera and Provan, and Klee and Kleinschmidt were some of the pioneers. The message is that special simplicial complexes have nicely bounded diameter. Coordinates and coefficients do not matter; distances and angles do not matter. We present here a "taste" of topological results about diameter simplicial complexes.

First of all, topological abstraction is justified given our current bounds. Eisenbrand, Hähnle, Razborov, and Rothvoss worked with a class of simplicial complexes called *normal complexes* which again include all those coming from linear programs. They proved that both the Larman-style bounds and the Kalai–Kleitman-style bound hold in fact for normal simplicial complexes. Their work led to the following tantalizing conjecture:

Conjecture 1 (Hähnle, 2014). *The diameter of every normal $(d-1)$-complex with n vertices, including simplicial d-polytopes with n vertices, is at most $(n-1)d$.*

Overall, nice topological or combinatorial conditions give rise to good diameter bounds. Consider for example the following property: Peeling off a simplicial complex, piece by piece, is an important tool in combinatorial topology; e.g., a *shelling* of a pure simplicial complex is an ordering of its facets in which, at each step, the ith facet intersects nicely with the union of the other previous facets. Shellings do not exist for all pure complexes, but remarkable results have been shown for shellable complexes. For our purposes *vertex decomposability* is very important.

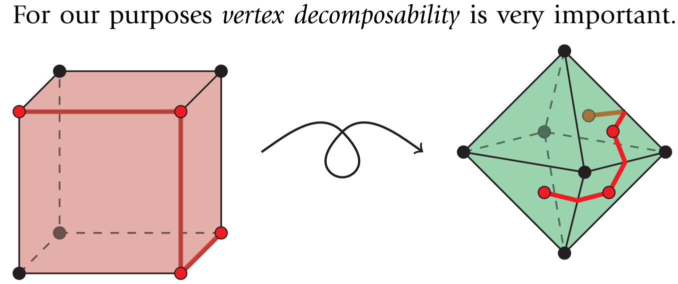

Figure 5. An example of the polarity operation: The polar of a cube is an octahedron and vice versa.

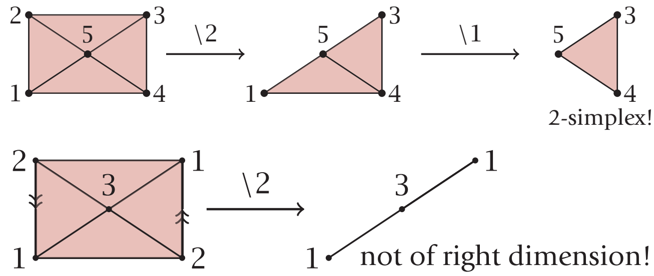

Figure 6. An example of a weakly 0-decomposable complex (on top) and a Möbius band, which is not weakly 0-decomposable.

Billera and Provan conceived a natural way to prove a linear bound on the diameter that relies on vertex decomposition properties of complexes. They introduced the notion of a *weakly k-decomposable complex*. A d-dimensional simplicial complex Δ is weakly k-decomposable if it is pure of the same dimension, and either Δ is a single d-simplex, or there exists a face τ of Δ (called a *shedding face*) such that $\dim(\tau) \le k$ and $\Delta \backslash \tau$ is d-dimensional and weakly k-decomposable. So one recursively "peels off" the simplicial complex using a sequence of faces so that finally we arrive at a (full-dimensional) simplex. In Figure 6 we show an example of a weakly 0-decomposable complex through a possible shedding order of vertices (in three steps), but when the same simplicial complex is made into a Möbius band identifying one pair of opposite sides, the complex is not anymore weakly 0-decomposable.

The reason why weak-decomposability is so interesting for bounding diameters is the following theorem:

Theorem 2 (Billera, Provan, 1980). *If Δ is a weakly k-decomposable simplicial complex, $0 \le k \le d$, then*

$$\operatorname{diam}(\Delta) \le 2 f_k(\Delta),$$

where $f_k(\Delta)$ is the number of k-faces of Δ. In the case of weakly 0-decomposable, we have the following linear *bound ($n = f_0(\Delta)$):*

$$\operatorname{diam}(\Delta) \le 2 f_0(\Delta) = 2n.$$

Note that if all simplicial polytopes were weakly 0-decomposable, then the diameter would be linear, being no more than twice the number of facets of the polar simple polytope. All simplicial d-dimensional polytopes are weakly d-decomposable because they are *shellable* (see, e.g., [17] for an introduction). The question is then, which simplicial polytopes are weakly 0-decomposable? More strongly, is there a fixed constant $k < d$ for which all simplicial polytopes are weakly k-decomposable? If this were true for $k = 0$ (weakly 0-decomposable), then the desired linear bound would be achieved! These challenges have been settled. De Loera and Klee constructed examples of simplicial polytopes that are not weakly 0-decomposable disproving this method as an approach for a linear bound of the diameter for polytopes. Interestingly, the counterexamples are explicit from the transportation problems, like those in the introduction, with two milk factories and at least five cities to service.

As a word of warning, we note that going into too much topological generality will certainly not give a useful bound for the diameter, at least not one sufficiently good for linear optimization. This is evident from the pioneering work by Santos, and more recently Criado and Santos; they showed

Theorem 3 (Criado, Santos 2017). *If $H_s(n,d)$ denotes the maximum diameter of a pure strongly connected $(d-1)$-complex with n vertices, then*

$$\frac{n^{d-1}}{(d+2)^{d-1}} - 3 \le H_s(n,d) \le \frac{1}{d-1}\binom{n}{d-1} \simeq \frac{n^{d-1}}{d!}.$$

Thus the diameter of many complexes grows as $c_d n^{d-1}$ for a constant c_d depending only on d.

They also showed that similar exponential behavior appears for simplicial *pseudo-manifolds* (to be a pseudo-manifold every $(d-2)$-dimensional simplex of the complex is contained in exactly two maximal d-dimensional ones). On the other hand, linear programs are associated to simplicial polytopes, which are simplicial spheres, a much more restricted type of simplicial complex.

Some constructions of spheres and balls with "large" diameter were presented by Mani and Walkup with exciting new improvements by Criado and Santos. Nevertheless, today all known simplicial spheres and balls have diameter bounded by only $1.25n$. Regarding construction of polytopes with "large" diameter, Warren Hirsch conjectured in 1957 that the diameter of the graph of a polyhedron defined by n inequalities in d dimensions is at most $n - d$. Dantzig popularized the conjecture in his classic book, and it became known as the *Hirsch conjecture*. Counterexamples in the *unbounded* case were found quickly by Klee and Walkup, but it took fifty-three years of hard work to build a counterexample to the *Hirsch conjecture* for polytopes. In his historic paper [12] Francisco Santos showed

Theorem 4 (Santos, 2010).

- *There is a 43-dimensional polytope with 86 facets and of diameter at least 44 (this result has now been improved).*
- *There is an infinite family of non-Hirsch polytopes with n facets and diameter $\sim (1+\epsilon)n$ for some $\epsilon > 0$. This holds true even in fixed dimension.*

A key observation of Santos's construction was an extension of a well-known result of Klee and Walkup. They showed that the Hirsch conjecture could be proved true from just the case when $n = 2d$. In that case the problem is to prove that given two vertices u and v that have no facet in common, one can pivot from one to the other in d steps so that at each pivot we abandon a facet containing u and enter a facet containing v. This was named the *d-step conjecture* (see [8]).

The construction of Santos's counterexample uses a variation of this result for a family of polytopes called *spindles*. Spindles are polytopes with two distinguished vertices u, v such that every facet contains either u or v but not both. Examples of a spindle include the cross polytopes, the cube, and the polytope in Figure 7. Spindles can be seen as the overlap of two pointed cones (as shown in Figure 7). The *length of a spindle* is the distance between this special pair of vertices.

Santos's strong d-step theorem for spindles says that from a spindle P of dimension d with $n > 2d$ facets and length λ one can construct another spindle P' of dimension $d+1$ with $n+1$ facets and length $\lambda+1$. Since one can repeat this construction again and again, each time increasing the dimension, length, and number of facets of the new spindle by one unit, we can repeat this process until we have $n = 2d$ (number of facets is twice the dimension). In particular, if a spindle P of dimension d with n facets has length greater than d, then there is another spindle P' of dimension $n-d$ with $2n-2d$ facets and length greater than $n-d$ that violates the Hirsch conjecture. Santos constructed such a five-dimensional spindle. As of today the work on constructing lower bounds still leaves open the possibility of a linear diameter for polyhedra.

An Algebraic Point of View

In 1984 Narendra Karmarkar presented an algorithm to solve LPs that used a different principle from the simplex method. At each iteration of the algorithm a point in the interior of the polytope will be generated. His paper [7] started the revolution of *interior-point methods* and gave an alternative proof of polynomiality of linear programming. Originally Karmarkar presented his work in terms of projective transformations, but later it was shown his algorithm was equivalent to an earlier idea. One replaces the linear objective function with something more complicated. A *barrier function* is added to the original linear objective function. A barrier function has a singularity at the boundary of the polyhedron and thus prevents the points at each iteration from leaving the feasible region. Barrier functions originated in non-linear programming during the 1960s when Fiacco and McCormick showed they defined a smooth curve, the barrier trajectory or the *central path*, that converges to the solution of the constrained problem from the strict interior of the feasible region, completely avoiding the boundary. Interior-point methods have had a profound impact in modern optimization. Interior-point methods are used not just for linear

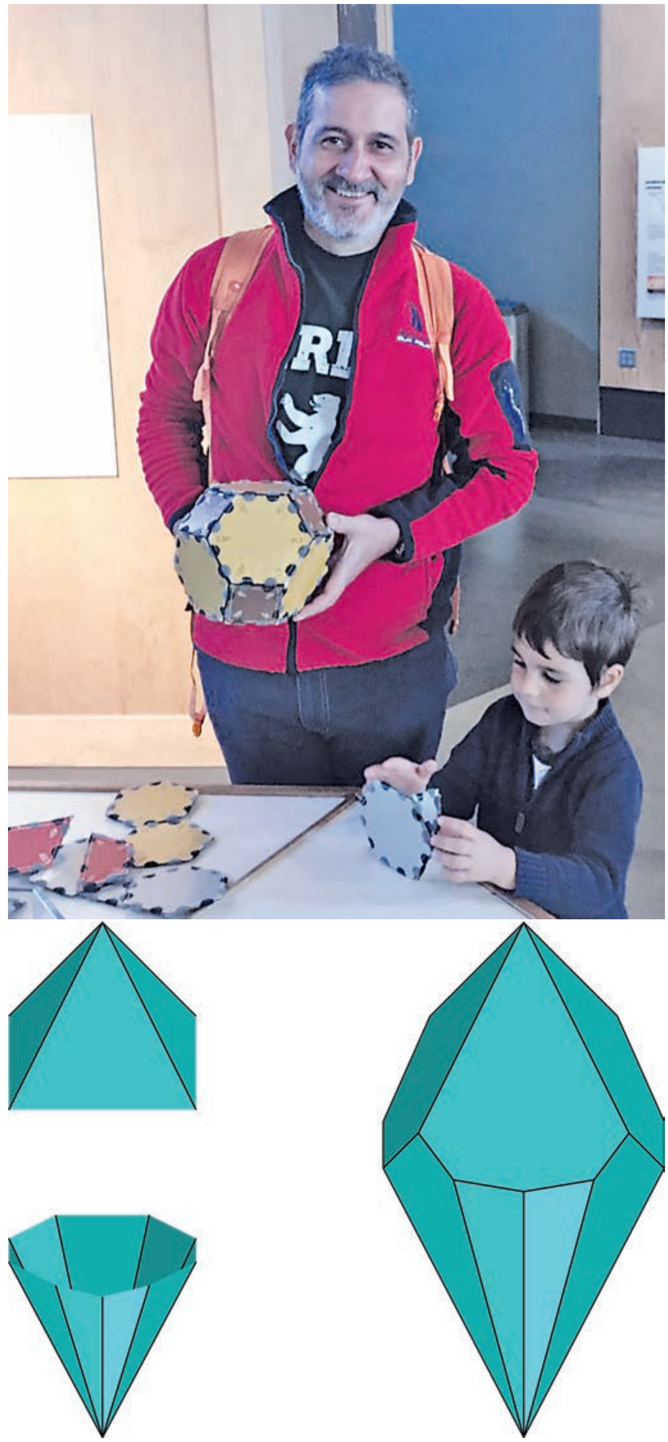

Figure 7. Francisco Santos and a younger geometer (top) and a three-dimensional spindle (bottom).

programming, but for non-linear optimization and other forms of convex optimization, e.g., semidefinite programming (see, e.g., [16] and the references therein).

A logarithm function is a classical choice to use as the barrier function. For a concrete example, consider the problem to maximize $\mathbf{c}^T\mathbf{x}$ subject to the conditions

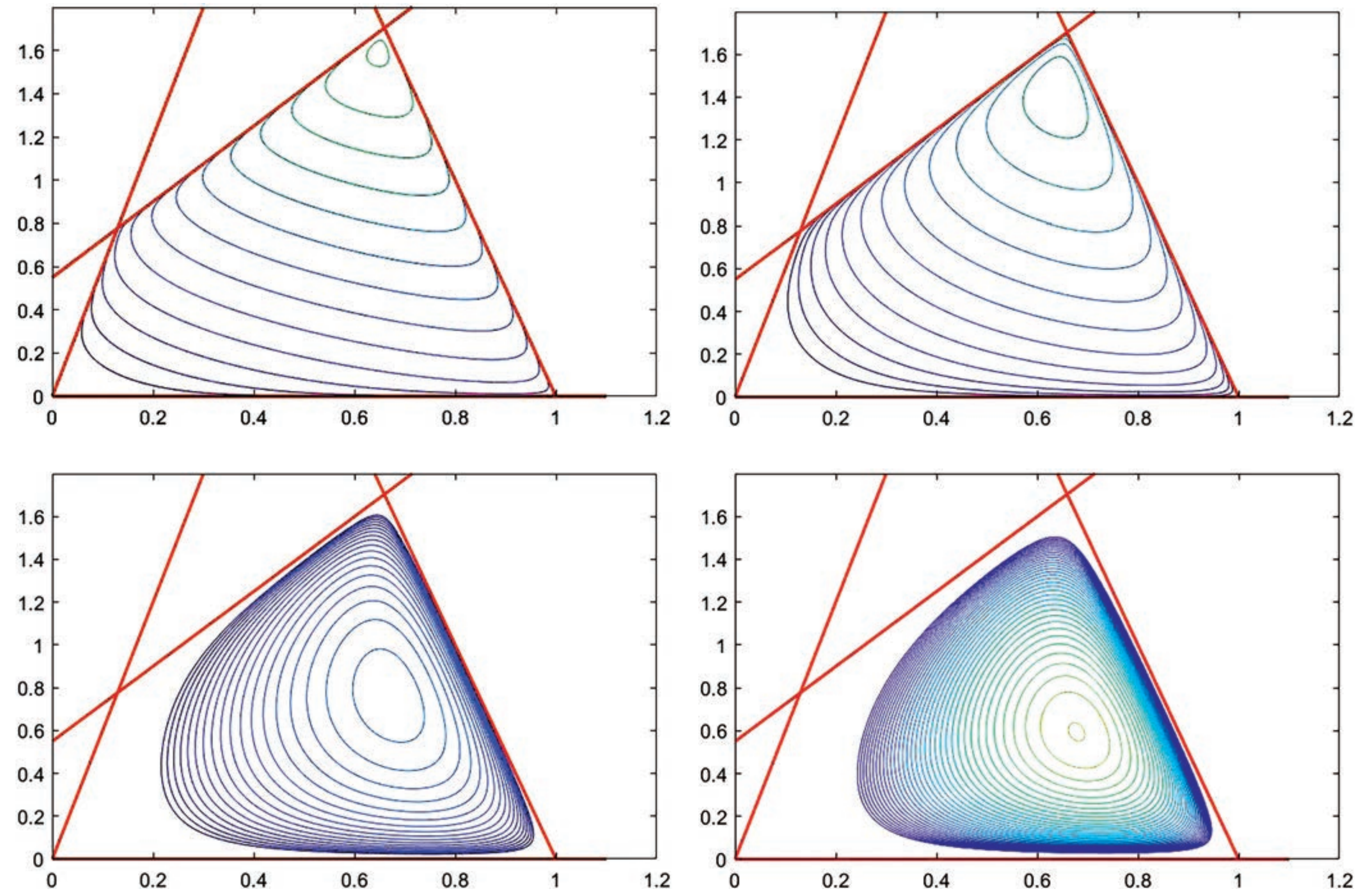

Figure 8. The level sets of a logarithmic barrier function f_λ at four different values of λ. Top left: λ is close to zero, thus the optimum of f_λ is near the LP optimum. Bottom right: λ is a large positive number, thus the optimum of f_λ is near the analytic center of the polyhedron.

$A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$; the *logarithmic barrier function* for this problem will be $f_\lambda(\mathbf{x}) := \mathbf{c}^T\mathbf{x} + \lambda \sum_{i=1}^{n} \log x_i$. It is natural to use Newton's method to find unconstrained maximizers of the logarithmic barrier function for a sequence of positive values of λ converging to zero. This numerical approach yields, for finitely many values of λ, a piece-wise approximation to the central path. See Figure 8.

Let me show you now an amazing mathematical connection to algebra! This discussion will be motivated by another longstanding open problem. With the existence of polynomial-time algorithms for linear programming, we can be more ambitious and ask, *is there a strongly polynomial-time algorithm that decides the feasibility of the linear system of inequalities $Ax \leq b$?* Strongly polynomial-time algorithms take a number of steps bounded only by a polynomial function of the number of variables and constraints. In particular, the size of the coefficients would not matter. Some instances of linear optimization in which we know we can do that include network type LPs, LPs with at most two variables per inequality, and combinatorial LPs (those with bounded maximum sub-determinant).

It is natural to ask whether interior-point methods can be adjusted or adapted to work as strongly polynomial-time algorithms for all LPs. We will see here a partial negative answer of this question for interior-point methods. Some fascinating properties on the differential geometry of the central paths will become relevant, and to answer them the methods of tropical algebraic geometry will be key. While typically interior-point methods are seen as a family of numerical-approximation algorithms, the progress came from looking at the intrinsic algebraic and symbolic nature of interior-point methods and new combinatorial tools, such as tropical geometry, to analyze systems of polynomial equations.

To set up the story, recall the fundamental fact that all linear optimization problems come in pairs, the pair of linear programming problems in *primal* and *dual* formulations (with slack variables):

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0; \quad (1)$$

$$\text{Minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} - \mathbf{s} = \mathbf{c} \text{ and } \mathbf{s} \geq 0. \quad (2)$$

As before, here A is an $n \times d$ matrix. The fundamental results of duality theory, available in all good books on the subject (e.g., [13]), allow us to set up the basic algebraic formulation of the *primal-dual* interior-point methods:

$$A\mathbf{x} = \mathbf{b}\,,\ A^T\mathbf{y} - \mathbf{s} = \mathbf{c},\ \text{ and } x_i s_i = \lambda \quad \text{for } i = 1, 2, \ldots, d. \quad (3)$$

Perhaps the key observation for this section is that *this is a system of polynomial equations (quadratic and linear), and thus its solutions can be studied as an algebraic variety*. This algebraic variety is the *central curve*. Algebraic thinking about the central curve goes back to pioneering work by Bayer and Lagarias in the early days of interior-point methods. Today we are taking this further.

It is known that for each value of $\lambda > 0$, the system of polynomial equations (3) (with interior feasible points) has a unique real solution $(\mathbf{x}^*(\lambda), \mathbf{y}^*(\lambda), \mathbf{s}^*(\lambda))$ with the properties $\mathbf{x}^*(\lambda) > 0$ and $\mathbf{s}^*(\lambda) > 0$. The point $\mathbf{x}^*(\lambda)$ is precisely the optimal solution of the logarithmic barrier function $f_\lambda(\mathbf{x})$ we saw earlier for problem (1).

The parametrized set of solutions, given by the changing parameter λ, is the *central path*. These solutions for $\lambda \to 0$ have a limit point $(\mathbf{x}^*(0), \mathbf{y}^*(0), \mathbf{s}^*(0))$ which satisfies equation (3) when $\lambda = 0$, which defines optimal solutions, and thus in the limit we reach an optimum point of the linear program. Traditionally the central path is only followed approximately: discrete incremental steps that follow the path are generated by applying a Newton method to the equations. Similarly, tradition dictates the central path only connects the optimal solution of the linear programs in question with its *analytic center* within *one* single cell, with $s_i \geq 0$. Our plan now is to break with tradition: we look at the algebraic curve with all *exact* solutions defined by the system of equations (3).

In Figure 9 we see a depiction of one difference between the numeric point of view and the algebraic point of view of the central path. The central path is just a small portion of the entire explicit central curve that in reality extends beyond a single feasibility region (given by different sign constraint choices on variables). The central (algebraic) curve passes through all the vertices of a hyperplane arrangement defined by the LP constraints.

One way to estimate the number of Newton steps needed to reach the optimal solution is to bound the *curvature* of the central path. The intuition is that curves with small curvature are easier to approximate with fewer line segments. This idea has been investigated by various authors, and it has yielded interesting results. For example, S. Vavasis and Y. Ye found that the central path contains no more than n^2 crossover events (turns of a special type). This finding led to an interior-point algorithm whose running time depends only on the constraint matrix A. The notion of curvature we need is the *total curvature* of the central curve. It is defined as the degree of the map to the unit sphere assigning to each point of the curve the unit tangent vector at that point. This assignment is called the *Gauss map*. For a curve in the plane this is the sum of arc lengths on the unit circle. In Figure 10 we illustrate the tracing of an arc on the unit circle for one curve segment in the sequence of tangent vectors.

Dedieu, Malajovich, and Shub showed that the *average* total curvature of the primal, the dual, and the primal-dual central paths of the strictly feasible polytopes defined by (A, b) is at most $2\pi(d-1)$ (primal), at most $2\pi d$ (dual), and at most $2\pi d$ (primal-dual). In particular, it is independent of the number n of constraints. Later De Loera, Sturmfels, and Vinzant obtained bounds for the total curvature in terms of the degree of the Gauss maps of the curve. They also computed the degree, arithmetic genus, and defining prime ideal of the central curve and their primal dual projections. Their techniques used classical formulas from algebraic geometry. Unfortunately, for practical applications, the more relevant quantity is not the total curvature of the entire algebraic curve but rather the total curvature in and around the usual portion of the central path within a specific polytope region, going from the analytic center to an optimum. A. Deza, T. Terlaky, and Y. Zinchenko investigated the total curvature in a series of interesting papers. They conjectured that the largest possible total curvature of the associated central path with respect to all cost vectors is no more than $2\pi n$, where n is the number of facets of the polytope.

In an exciting 2018 paper [2], X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig disproved the Deza–Terlaky–Zinchenko conjecture, and they also showed that (certain) logarithmic-barrier interior-point methods can never be strongly polynomial for linear programming!

Theorem 5 (Allamigeon et al. 2018 [2]).

- *There is a parametric family of linear programs in $2d$ variables with $3d+1$ constraints, depending on a parameter $t > 1$, such that the number of iterations of any primal-dual path-following interior-point algorithm with a log-barrier function that iterates in the wide neighborhood of the central path is exponential*

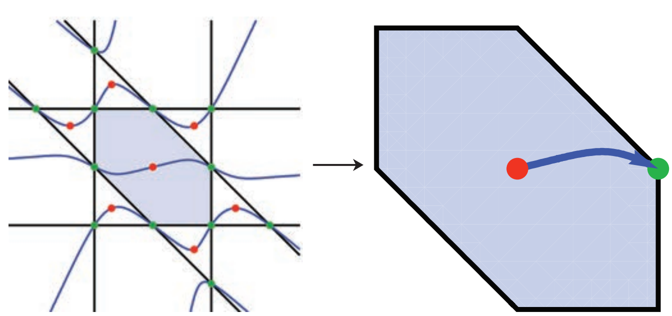

Figure 9. One difference from the central path (left) to the central curve (right) is more points of solution exist.

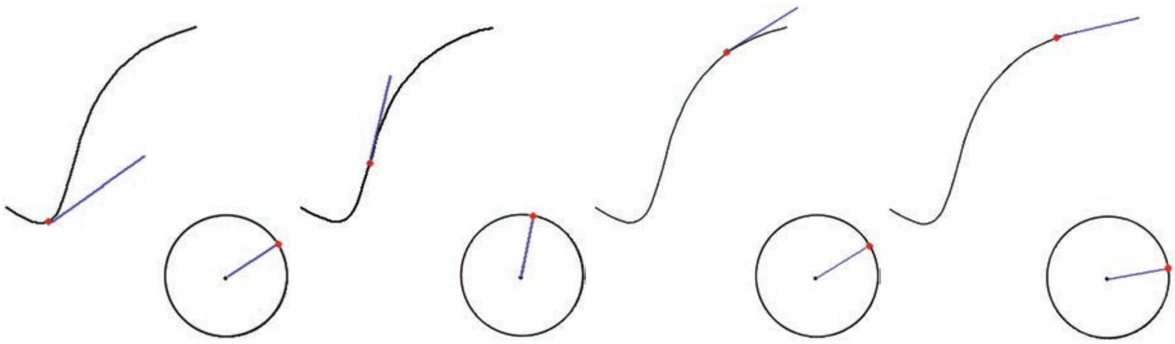

Figure 10. Illustration of the total curvature as the total traversed arc length of the unit tangent vectors of a curve.

in d for all t sufficiently large.

- *There exists an explicit family of linear programs with $3d+1$ inequalities in $2d$ variables where the central path has a total curvature that is exponential in d.*

This is exciting news, but even more surprising, these remarkable counterexamples were produced using new tools from *tropical geometry*. Tropical methods have been used before in optimization, game theory, and control theory. Tropical geometry is today a very active field of study, and I can only scratch its surface here. For more details see [9].

Tropical geometry is algebraic geometry over the *max-plus* semiring $(\mathbb{R}_{\max}; \oplus, \odot)$ where the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ is endowed with the operations $\mathbf{a} \oplus \mathbf{b} = \max(\mathbf{a}, \mathbf{b})$ and $\mathbf{a} \odot \mathbf{b} = \mathbf{a} + \mathbf{b}$. The max-plus semirings are also called *tropical semirings*. In this way we have some funny arithmetic, e.g., $1 \oplus 3 = 3$ and $5 \odot 0 = 5$. Tropical arithmetic is associative, distributive, the additive identity is $-\infty$, and the multiplicative identity is 0. Tropical arithmetic extends of course to matrices and polynomials, and so one can define tropical varieties, tropical polyhedra, and much more. For example, a *tropical halfspace* is the set of vectors x satisfying a "linear" inequality in the tropical semiring; this translates to those vectors that satisfy $\max(\alpha_1^+ + x_1, \ldots, \alpha_n^+ + x_n, \beta^+) \geq \max(\alpha_1^- + x_1, \ldots, \alpha_n^- + x_n, \beta^-)$. Just like convex polyhedra are the feasible sets of linear optimization, now tropical polyhedra will play a role for feasible solution sets. A *tropical polyhedron* is simply a finite intersection of tropical halfspaces. Figure 11 shows an example with five halfspaces (marked by colors); a tropical pentagon is indicated in gray.

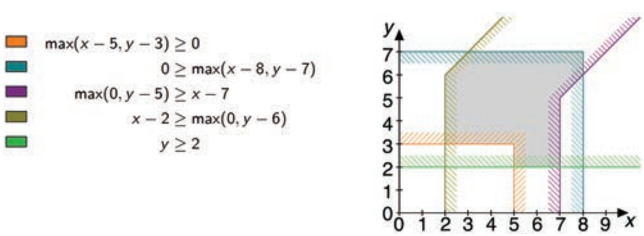

Figure 11. An example of a tropical polyhedron.

It turns out that for every algebraic variety, there is a way to obtain a tropical variety. The *tropicalization* of an algebraic variety V is a process that yields a polyhedral complex $T(V)$ in $\mathbb{R}^d$. The combinatorial complex $T(V)$ has many of the key properties of the original algebraic variety. For instance, if V is a planar algebraic curve over an algebraically closed field, then $T(V)$ is a planar graph. Several key features of the algebraic variety V are easier to see in its tropicalization $T(V)$. For example, if V is irreducible, then $T(V)$ is connected. Figure 12 shows one example of a genus three Riemann surface (it is given by a smooth degree four homogeneous polynomial equation on three complex variables). Its tropicalization preserves the homological information (number of holes).

For our purposes the key idea is that any tropical polyhedron is actually the tropicalization of a linear program with coefficients over a field of absolutely convergent real-valued Puiseux series K (these are power series that allow for negative and fractional exponents). As K is an ordered real-closed field, the basic results of linear programming (Farkas' lemma, strong duality, etc.) still hold true, and the central path of such a linear program is well-defined. The elements of K are real-valued functions. The key point is: a linear program over the field K encodes an entire parametric family of traditional linear programs over the reals $\mathbb{R}$, and the central path on K describes all central paths of this parametric family. The tropicalization of the K-linear programs allows us to see the behavior of the central path, now shown as a piece-wise linear path inside a tropical polyhedron. This piece-wise linear path is the *tropical central path*, defined as the image under the tropicalization. Consider the following example. The Puiseux polyhedron $\mathcal{P} \subset K^2$ is defined by five inequalities:

$$\begin{aligned} \mathbf{x}_1 + \mathbf{x}_2 &\leq 2 \\ t\mathbf{x}_1 &\leq 1 + t^2\mathbf{x}_2 \\ t\mathbf{x}_2 &\leq 1 + t^3\mathbf{x}_1 \\ \mathbf{x}_1 &\leq t^2\mathbf{x}_2 \\ \mathbf{x}_1, \mathbf{x}_2 &\geq 0 \,. \end{aligned} \tag{4}$$

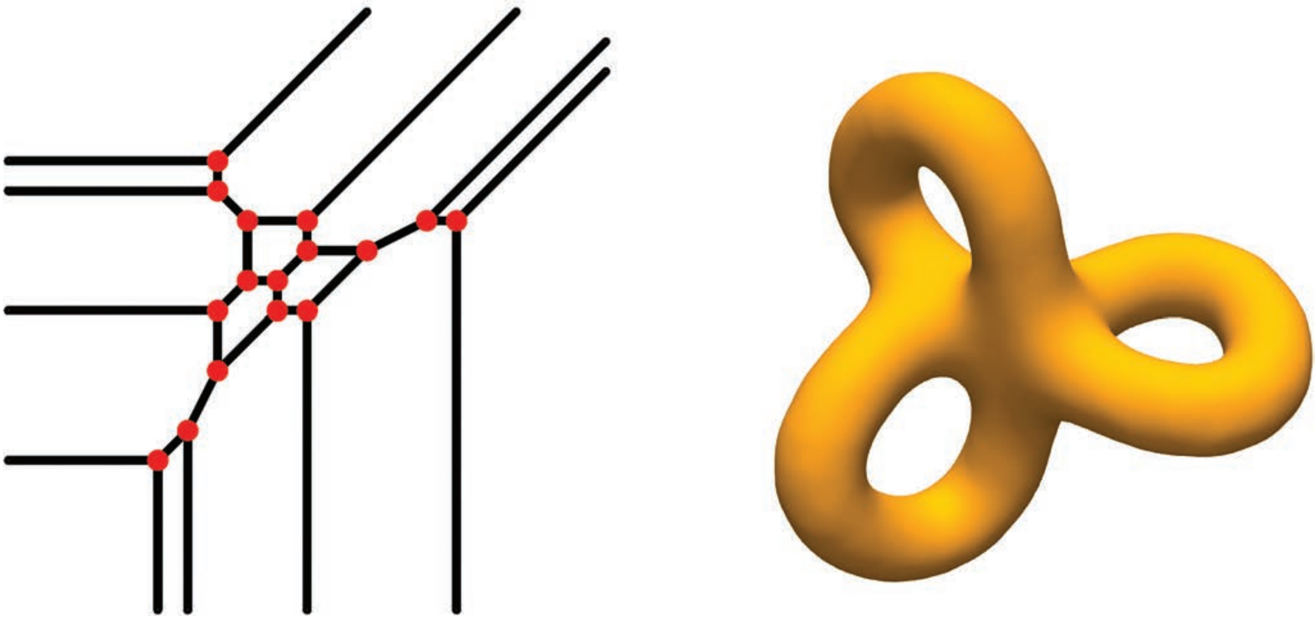

Figure 12. The picture on the left shows a smooth plane tropical curve of degree 4 and genus 3. This arises as the tropicalization of a smooth complex algebraic curve of the same degree and the same genus. As a manifold that curve is a smooth surface of genus 3, and this is the picture on the right. The pictures illustrate that the graph-theoretic genus corresponds to the topological genus.

The tropicalization of $\mathcal{P}$ is described by five *tropical* linear inequalities:

$$\begin{aligned} \max(x_1, x_2) &\leq 0 \\ 1 + x_1 &\leq \max(0, 2 + x_2) \\ 1 + x_2 &\leq \max(0, 3 + x_1) \\ x_1 &\leq 2 + x_2 \,. \end{aligned} \tag{5}$$

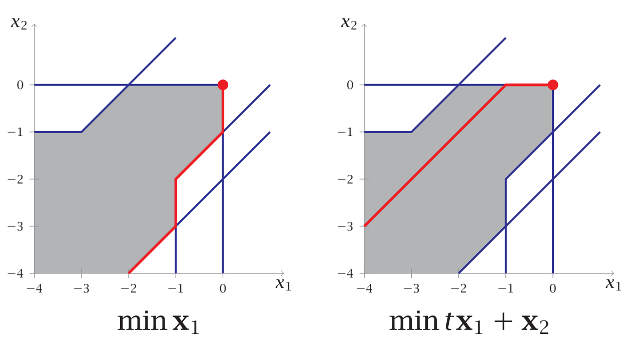

Figure 13. Two tropical central paths corresponding to two different objective functions.

Figure 13 shows the tropicalization of the Puiseux polyhedron from equation (4) (the shaded region), but it also shows two different tropical central paths for two different objective functions. One can see that the tropical central path on the left degenerated to a vertex-edge path, akin to the simplex method moving through the boundary of the polyhedron. The tropicalization allows for simpler calculations. The total curvature of an algebraic curve becomes, under the tropicalization, a sum of piece-wise linear angular turns. The problem of computing total curvature becomes a problem of adding polygonal angles.

Conclusions

I hope the reader has seen the power of algebraic and topological tools in the analysis and creation of linear optimization algorithms. While here I focused on the theory of linear optimization, algebraic-geometric-topological techniques have had impact in other parts of optimization, too. For example, real algebraic geometry has been decisive for global optimization problems with polynomial constraints. Through the theory of sums of squares and semidefinite programming one can compute convex optimization approximations to difficult highly non-convex problems [3]. Algebraic methods were used in integer programming through Graver bases tools [6].

If you can see the benefits of having a diversity of mathematical perspectives, can you imagine the result of having a larger, more diverse group of mathematicians working on solving problems and on finding new applications of mathematics?

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Jesús A. De Loera

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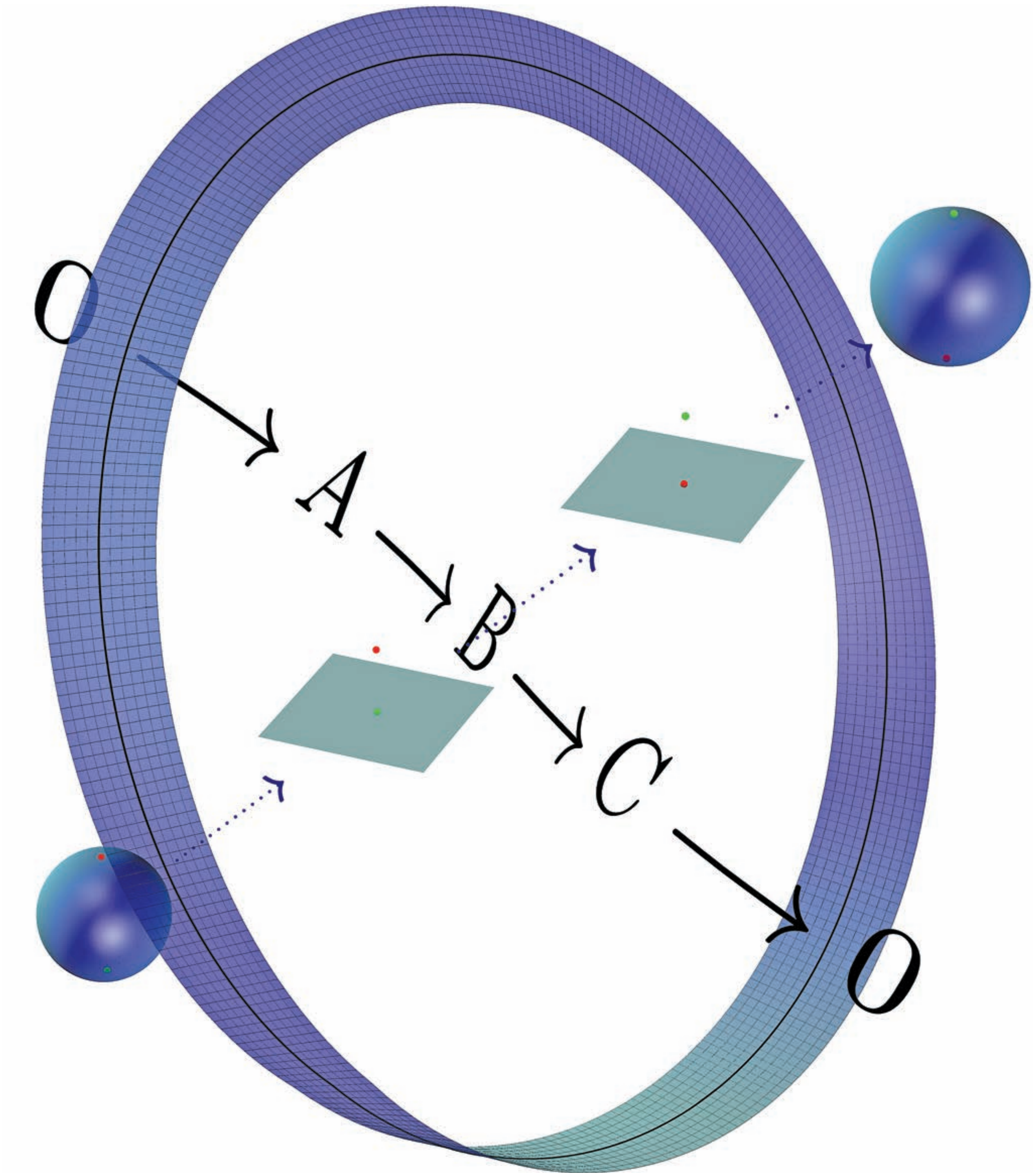
Attitudes of K-theory

Topological, Algebraic, Combinatorial



Inna Zakharevich

Introduction: Divide and Conquer

In many areas of mathematics we use similar methods for analyzing problems. One of the most common is known as "divide and conquer": split up a problem into smaller problems, solve each of the smaller problems, then glue the solutions back together into a solution to the whole thing. Often, at the end, there is a frustrating question left over:

> Is the constructed solution the only one possible? In other words, do the parts of the solution uniquely determine the solution to the whole?

One approach to solving this question is that of considering only locally defined objects. For example, if two functions agree at all points then they are equal; thus the answer to the question is always "yes." Sometimes, however, this is not possible, and it is necessary to analyze all of the different ways that pieces can be put together. For example, even if it is known that an abelian group has a filtration with associated graded $\mathbf{Z}/2 \oplus \mathbf{Z}/2$, there are multiple possibilities for what the group could be: $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ or $\mathbf{Z}/4$. This is where K-theory comes in.

K-theory is the study of invariants of assembly problems. Topological K-theory studies how vector bundles are assembled; algebraic K-theory studies how modules over a ring are assembled; other kinds of K-theory study how other kinds of objects can be assembled. The general approach is to define a group K_0 with a generator for each object of study and a relation for each possible "assembly." Because different areas of mathematics have different tools and approaches, the question of classifying different forms of K-theory varies drastically in difficulty from field to field.

In this article we give a brief overview of three different kinds of K-theories: topological, algebraic, and combinatorial. Between the sections we give some context and background to explain how the modern perspective on K-theory developed, and some of the motivation for the theory behind it. Although this article is not (and cannot) be a comprehensive history of or motivation for the topic, we hope that it will be an interesting perspective on the development of this field.

Topological K-Theory

The first example of K-theory that most people see is topological K-theory. The idea behind topological K-theory is to study a (compact Hausdorff) space by examining the ways that vector bundles on that space behave. A *vector bundle* on a space X is a continuous family of vector spaces indexed by X. Somewhat more precisely, it is a space E together with a map $p : E \longrightarrow X$, such that for any $x \in X$, $p^{-1}(x)$ is a vector space (and all of these fit together nicely). We call $p^{-1}(x)$ the *fiber over* x, and denote it E_x. These E_x are exactly the vector spaces in our continuous family.

We can also think of vector bundles more locally. An example of a vector bundle on X is the "trivial bundle" $X \times \mathbf{R}^n$. A general vector bundle looks like a trivial bundle in a neighborhood of any point. Thus we can assemble a bundle by taking several of these and gluing them together. More precisely, we can cover X by a family of open subsets $\{U_\alpha\}_{\alpha \in A}$, and think of the vector bundle as looking like $U_\alpha \times \mathbf{R}^n$ over each U_α. We then "glue together" these bundles into a bundle on X similarly to how manifolds are glued together: by giving gluing data that tell us how to identify the part of $U_\alpha \times \mathbf{R}^n$ that sits over $U_\alpha \cap U_\beta$ to the part of $U_\beta \times \mathbf{R}^n$ that sits over $U_\alpha \cap U_\beta$. One requires that this gluing data should restrict to a linear isomorphism from the copy of $\mathbf{R}^n$ over x in U_α to the copy of $\mathbf{R}^n$ over x in U_β; this gives an element of $\mathrm{GL}_n(\mathbf{R})$ for every $x \in U_\alpha \cap U_\beta$. This gluing data should be continuous; this means that we want continuous maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_n(\mathbf{R})$ for all α, β. Assuming that these satisfy some standard relations (for example, that $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$ for all $x \in U_\alpha \cap U_\beta$) this gives enough data to construct the vector bundle E. (For a more precise treatment, see for example [11, Chapter 1].)

Two vector bundles $p : E \longrightarrow X$ and $p' : E' \longrightarrow X$ are considered isomorphic if there is a homeomorphism $f : E \longrightarrow E'$ such that for all $x \in X$, the restriction of f to E_x gives a linear isomorphism to E'_x. In other words, f must satisfy $p' \circ f = p$.

Let us consider an example. Suppose that $X = S^1$ and $n = 1$. To each point in S^1 we attach a line, in such a way that these lines form a continuous family; thus we can think of this as attaching an infinitely wide strip of paper to a circle. Starting at a point in the circle, let us walk around clockwise and try to make sure that the paper is standing up "vertically" (compared to the plane of the circle). When we get back to the beginning one of two things can happen: either the entire strip is standing up vertically (in which case we have the bundle $S^1 \times \mathbf{R}^1$) or we failed: there is a twist in the paper, so that we have a Möbius strip. These are the two possibilities for one-dimensional vector bundles on a circle.

A vector bundle is fundamentally a geometric object, but it turns out to be classified by *homotopical* information. For a concrete example, let us consider $X = S^n$ and cover it by the family $\{S^n \setminus \{\text{north pole}\}, S^n \setminus \{\text{south pole}\}\}$.

Inna Zakharevich is an assistant professor of mathematics at Cornell University. Her email address is zakh@math.cornell.edu.

The author was supported in part by NSF grant DMS-1654522.

Communicated by Notices *Associate Editor Daniel Krashen.*

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DOI: https://doi.org/10.1090/noti1908

An n-dimensional vector bundle on X is then uniquely determined by a single function $S^n \setminus \{\text{poles}\} \longrightarrow \mathrm{GL}_n(\mathbf{R})$. It turns out that the isomorphism class of the vector bundle depends only on the *homotopy class* of this map, not on the actual map; in particular, we can consider it to be a map from $S^{n-1} \longrightarrow \mathrm{GL}_n(\mathbf{R})$ (see [11, Section 1.2]). For example, applying this to the circle case above we see that we have four homotopy classes of maps $S^0 = \{\pm 1\} \longrightarrow \mathrm{GL}_1(\mathbf{R})$, depending only on the sign of the image of each of the two points. By changing the orientation on one of the two patches, we can assume that the point 1 is mapped to the identity transformation, which gives us two possibilities: if the image of -1 is negative in $\mathrm{GL}_1(\mathbf{R})$ then we get a Möbius strip, and otherwise we get the trivial bundle. This confirms our above intuitive description of the possible bundles on S^1.

With this perspective we see that the *geometric* data of the vector bundle is controlled by the *homotopical* data of the gluing maps. In fact, this kind of construction works more generally. Let $\mathrm{Vect}^n(X)$ be the set of isomorphism classes of vector bundles on X.

Theorem 1 ([11, Theorem 1.16]). *There exists a space called $BO(n)$ such that for finite CW complexes*[1] *X,*

$$\mathrm{Vect}^n(X) = [X, BO(n)]$$

where the right-hand side denotes the homotopy classes of maps $X \longrightarrow BO(n)$.

One way of understanding this theorem is to say, as we did above, that the geometric data of vector bundles is controlled by homotopical data; another way is to say that the geometric data of $\mathrm{Vect}^n(X)$ contains homotopical information about X.

We can develop this idea into a very powerful theory called *topological K-theory*.

Let us consider the set of isomorphism classes of vector bundles on X, which we write $\mathrm{Vect}(X)$. This set has two operations on it: $\oplus$ and $\otimes$. For any two vector bundles E, E' over X we can construct a vector bundle $E \oplus E'$ where we set $(E \oplus E')_x = E_x \oplus E'_x$, or a vector bundle $E \otimes E'$ by $(E \otimes E')_x = E_x \otimes E'_x$. These two operations are unital: the 0-dimensional bundle $X \times \mathbf{R}^0$ is the unit for $\oplus$, and the 1-dimensional bundle $X \times \mathbf{R}^1$ is the unit for $\otimes$. Thus the only thing stopping us from having a ring of vector bundles on X is that we have no additive inverses.

The solution to this is to add them formally, the same way as we build the rational numbers from the integers. Consider a pair $(E, E') \in \mathrm{Vect}(X)^2$, which we think of as a "formal difference" $E - E'$. We define an equivalence relation $(E, E') \sim (F, F')$ if $E \oplus F' \cong E' \oplus F$, exactly analogously to how two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal if $ad = bc$.

[1] *Actually, all that is required is that X is paracompact.*

We can then define addition by

$$(E, E') + (F, F') = (E \oplus F, E' \oplus F')$$

and multiplication by

$$(E, E')(F, F') = ((E \otimes F) \oplus (E' \otimes F'), (E \otimes F') \oplus (E' \otimes F).$$

These operations define a ring structure on

$$KO^0(X) \overset{\mathrm{def}}{=} \mathrm{Vect}(X)^2 / \sim .$$

(The group $K^0(X)$ is defined analogously using complex vector bundles.)

It turns out [11, Chapter 2] that $KO^0(X)$ is a homotopy invariant. Moreover, if we define

$$KO^{-n}(X) \overset{\mathrm{def}}{=} K^0(\Sigma^n(X_+)) \qquad n \geq 0$$

(where $\Sigma^n(X_+)$ is the reduced suspension of X with a disjoint basepoint added [12, Example 0.10]) it turns out that these groups have an 8-fold periodicity. (This is called the *Bott periodicity theorem* [5, 1.15].) This allows us to extend this definition to all integers n. With these definitions $KO^*(X)$ has been used to great advantage to solve various geometric problems. The two most famous are the following:

- **Hopf invariant** 1: There are some standard examples of spheres that have unital multiplications, starting with S^0 and S^1, which both have abelian group structures. By noting that S^0 is the elements of norm 1 in $\mathbf{R}$ and S^1 is the elements of norm 1 in $\mathbf{C}$ we can construct similar multiplications on S^3 and S^7, viewing them as the units in the quaternions and octonions. It turns out that these are the *only* examples of spheres with unital multiplication; this was originally proved by Adams in [1] using the Steenrod algebra, but a much simpler and cleaner proof was discovered by Adams and Atiyah [3] using topological K-theory.

  This problem is closely related to other classical questions, including the existence of normed division algebra structures on $\mathbf{R}^n$, the parallelizability of S^{n-1}, and the existence of spaces with (graded commutative) polynomial cohomology.
- **Counting vector fields on spheres:** The hairy ball theorem states that there is no nonvanishing continuous vector field on S^2. However, for odd-dimensional spheres such fields exist, and so the more complicated question becomes: what is the maximum number of everywhere-linearly-independent vector fields that can be put on a sphere? Adams [2] showed, using topological K-theory, that there are $\rho(n) - 1$ such fields on S^{n-1}, where $\rho(n)$ is the Radon–Hurwitz number. (Prior to Adams it was known that there are at least that many fields; using the structure of topological K-theory Adams was able to show that this is a sharp bound.)

It turns out that topological K-theory is a *generalized cohomology theory*: a way of assigning groups to spaces that behaves just like singular cohomology, but which does not satisfy the dimension axiom.[2] Thus all of the standard computational tools that work for ordinary cohomology theory work for topological K-theory. For example, for a sub-CW-complex Y of a finite CW-complex X there exists a long exact sequence

$$\cdots \longrightarrow KO^n(X/Y) \longrightarrow KO^n(X) \longrightarrow KO^n(Y) \longrightarrow KO^{n+1}(X/Y) \longrightarrow \cdots.$$

These new invariants, together with their geometric underpinnings, are what give topological K-theory its power.

Segue: Representable Cohomology Theories

The group $KO^0(X)$ was defined algebraically in the previous section, but it is in fact possible to define it completely homotopically. If we write $[X, Y]$ for the set of (basepointed) homotopy classes of maps from X to Y, it turns out that

$$KO^0(X) \cong [X_+, \mathbf{Z} \times BO],$$

where X_+ is X with a disjoint basepoint added, and BO is the delooping[3] of the infinite orthogonal group (in which elements are orthogonal matrices that differ from the identity in only a finite number of spaces) [11, Section 1.2]. The space BO is closely related to the spaces $BO(n)$ mentioned above; the difference is that previously we were considering vector bundles of a fixed dimension, while here we consider all vector bundles at once. Using the above expression we can then see that

$$KO^{-n}(X) \cong [\Sigma^n(X_+), \mathbf{Z} \times BO] \cong [X_+, \Omega^n(\mathbf{Z} \times BO)],$$

where ΩY is the *loop space* of Y, whose points are (basepointed) maps $S^1 \longrightarrow Y$. Bott periodicity simply states that

$$\Omega^8(\mathbf{Z} \times BO) \simeq \mathbf{Z} \times BO.$$

Using these results we can construct a sequence of spaces

$$\ldots, K_{-2}, K_{-1}, K_0, K_1, K_2, \ldots$$

such that $KO^n(X) \cong [X_+, K_n]$.

A similar statement can be made for ordinary cohomology theory. Letting $E_n = K(\mathbf{Z}, n)$ be the n-th *Eilenberg–MacLane* space of $\mathbf{Z}$ [12, Section 4.2], and setting $E_n = *$ for $n < 0$ we have

$$H^n(X; \mathbf{Z}) \cong [X_+, E_n].$$

[2] *The dimension axiom states that all cohomology groups of a point other than the 0-th one must be 0. This is what allows us to read off the "dimension" of a space from its cohomology. Topological K-theory doesn't work this way: the topological K-theory of a point has infinitely many nonzero groups.*

[3] *The delooping of a space Y is a space Z such that $Y \simeq \Omega Z$. The delooping of a discrete group is just the classifying space of the group; however, since O has a nondiscrete topology a bit more care is necessary.*

In fact, the Brown Representability Theorem [6] states that such a construction is possible for *any* generalized cohomology theory. It thus makes sense to think of a cohomology theory not as a sequence of functors, but instead as a sequence of spaces. The only extra data necessary to produce a cohomology theory from a sequence of spaces is *structure maps*, which are weak equivalences $X_n \xrightarrow{\sim} \Omega X_{n+1}$. Given this extra data, it makes sense to drop all spaces with negative indices and assume that

$$X_{-n} \simeq \Omega^n X_0 \qquad \text{for } n > 0.$$

A space X_0 that can appear at the beginning of such a sequence is called an *infinite-loop space*, since it is weakly equivalent to an n-th loop space for all n. In this case, the groups $\pi_n X_0$ are exactly the cohomology groups of the sphere in the cohomology theory represented by this sequence.

It is common, therefore, to shift perspective from constructing a cohomology theory to constructing a space (which turns out to be an infinite-loop space) whose homotopy groups are exactly the cohomology groups of the sphere (for the cohomology theory in question). Such a space contains all of the information of the cohomology theory, but has much more structure to work with.

Algebraic K-Theory

The next step is to construct a version of K-theory for rings, rather than spaces. It is important to note that topological and algebraic K-theory are completely distinct, both in approach and results: topological K-theory starts with a space and constructs a ring, while algebraic K-theory starts with a ring and constructs a space. The connection between them is the spirit of the Serre–Swan theorem:

Theorem 2 (Serre–Swan). *The category of real vector bundles over a compact X is equivalent to the category of finitely generated projective modules over the ring $C(X)$ of continuous real-valued functions on X.*

This theorem is motivation for starting with the following analogy: vector bundles over a space are analogous to finitely generated projective modules over a ring.

We can rewrite the definition of $KO^0(X)$ to be the free abelian group generated by vector bundles over X, modulo the relation that $[E \oplus E'] = [E] + [E']$. Analogously, we define the group $K_0(R)$ to be the group generated by isomorphism classes of finitely generated projective R-modules, modulo the relation that $[A \oplus B] = [A] + [B]$. At this point, when working with topological K-theory, we were done: we could use suspension to define the negative K-groups, and Bott periodicity handled the rest. Unfortunately, an analogous definition generally produces no new useful groups, and no Bott periodicity; thus a different approach is needed in this situation.

The Serre–Swan theorem also hints that we must change our grading, from cohomological to homological. Given a map $Y \longrightarrow X$ of spaces, the induced map on rings of functions goes in the opposite direction: $C(X) \longrightarrow C(Y)$. The same will hold for our K-theories: previously, if we had a map $Y \longrightarrow X$ it induced a map $K^0(X) \longrightarrow K^0(Y)$ by pulling back vector bundles. Now, on the other hand, we have the opposite map: given a homomorphism $S \longrightarrow R$ of rings we get a map $K_0(S) \longrightarrow K_0(R)$ by tensoring an S-module up to an R-module. In addition, this switch reverses the signs of the K-groups, with $K_1(R)$ playing an analogous role to $KO^{-1}(X)$.

Topological K-theory is a generalized cohomology theory, whose defining property is the existence of a long exact sequence. We are going to keep this, as well as the definition of K_0. The imagined leap of faith is the observation that long exact sequences appear naturally for the homotopy groups of topological spaces. This is what we are going to construct: given R, we want to have a topological space $\mathbf{K}(R)$ and define $K_i(R) \stackrel{\text{def}}{=} \pi_i\mathbf{K}(R)$. This space should have the property that $\pi_0\mathbf{K}(R) \cong K_0(R)$, and such that its higher homotopy groups are meaningful algebraic invariants; in particular, at $K_1(R)$ and $K_2(R)$ they should agree with classically-defined invariants (for more on this, see [22, Chapters II, III]).

One can see that $K_0(R)$ encodes R-modules up to certain transformations: isomorphism and stability. Thus $\pi_1\mathbf{K}(R)$ should encode loops of such transformations. Let us consider isomorphisms first. If we think of a path between modules as an isomorphism between them, then a loop of such transformations is simply an automorphism of a module. To make this into a group we need to think of the automorphisms of R-modules as all living in the same place; fortunately, all finitely generated projective modules can be thought of as summands of the same module, R^∞. An automorphism of a module M sitting inside $M \oplus N$ can be thought of as an automorphism of $M \oplus N$ preserving the summands and trivial on N. Thus the group of automorphisms we are concerned with is $GL(R)$, the group of infinite invertible matrices with entries in R which are not equal to the identity only within a finite region. This also partially captures the necessary stability.

Since composition of paths of isomorphisms matches multiplication in $GL(R)$, we have a homomorphism $p : GL(R) \longrightarrow K_1(R)$. It turns out that stability also implies that p factors through $GL(R)^{\text{ab}}$. Indeed, an element in $GL(R)$ can be represented by a finite matrix A by picking a preimage under an inclusion $GL_n(R) \longrightarrow GL(R)$. Because of stability, for a given finite matrix A, we must have $p(A \oplus I) = p(A) = p(I \oplus A)$. Thus we must have

$$p(A)p(B) = p(A \oplus I)p(I \oplus B) = p((A \oplus I)(I \oplus B)) = p(\text{diag}(A, B))$$

and analogously $p(A)p(B) = p(\text{diag}(B, A))$. But by this same argument, $p(B)p(A) = p(\text{diag}(A, B))$, so $K_1(R)$ must be *abelian*. It turns out that defining

$$K_1(R) \stackrel{\text{def}}{=} GL(R)^{\text{ab}}$$

produces a stable invariant. When R is a field this is just the group of invertible elements in R, and the map $GL(R) \longrightarrow K_1(R)$ takes the determinant of a matrix.

Example. To see that this is a reasonable definition, at least for fields, let us consider the example when R is a Dedekind domain with fraction field F. In this case [22, Corollary II.2.6.3] we have $K_0(R) \cong \mathbf{Z} \oplus \text{Cl}(R)$, where $\text{Cl}(R)$ is the class group of R. Since $K_0(F) \cong \mathbf{Z}$, the definition of $\text{Cl}(R)$ leads to an exact sequence

$$R^\times \longrightarrow F^\times \longrightarrow \bigoplus_{\mathfrak{p}\text{ prime}} \mathbf{Z} \longrightarrow K_0(R) \longrightarrow K_0(F) \longrightarrow 0.$$

Note that each term in the infinite sum is actually $K_0(R/\mathfrak{p})$. The determinant gives a surjective map $GL(R) \longrightarrow R^\times$, so in fact there is a surjective map $K_1(R) \longrightarrow R^\times$. We can therefore rewrite the above sequence as

$$K_1(R) \longrightarrow K_1(F) \longrightarrow \bigoplus_{\mathfrak{p}\text{ prime}} K_0(R/\mathfrak{p}) \longrightarrow K_0(R) \longrightarrow K_0(F) \longrightarrow 0.$$

This may be considered another confirmation of the correctness of our definition of $K_1(R)$.[4]

We now turn to a construction of $\mathbf{K}(R)$. A naive way to begin is to try and construct a CW complex whose points are represented by finitely generated projective R-modules, with higher cells matching relations on K_0. However, this is difficult because of the nature of the relation on K_0: it is a three-term relation. Adding higher cells between 0-cells can impose two-term relations, setting things equal to one another, but it is much more difficult to impose a three-term relation. One way to attempt this is to add points labelled by formal sums of projective R-modules. However, the more fruitful approach is to shift perspective: a three-term relation can be represented by a triangle; thus constructing a space where *arcs* represent projective R-modules and *triangles* represent relations is a more natural approach. As this produces a space whose π_1 is $K_0(R)$, we end our construction by taking a loop space. One extra advantage of this approach is that we no longer need to worry

[4] *In fact, historically speaking, we are doing this backwards. The definitions of K_0, K_1, and K_2 came first; they were then observed to fit into such an exact sequence, and the idea of thinking of them as the homotopy groups of a space came later.*

about constructing "negatives" of R-modules: since each module is represented by a loop, its negative is simply the same loop, traversed backwards.

We begin the construction with a single point. We then add an arc for every finitely generated projective R-module. We then add a cell for each relation. It is important to note that we need a relation for *every way* of assembling $A \oplus B$ from A and B; the best way to keep track of these is to label relations by exact sequences

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0.$$

We now have the correct π_1, but our higher homotopy groups are far too large: many of these exact sequences have relations between them. We just need to figure out what such relations may look like.

Already the case when R is a field is highly nontrivial. For example, $K_*(\mathbf{C})$ is known completely only for finite coefficients [22, Section VI.1]. Suppose that we are given a vector space V of dimension $a+b+c$. We wish to decompose it into spaces of dimensions a, b, and c. We can do this by selecting a subspace of dimension a, then quotienting to get a space of dimension $b+c$, then picking a subspace of that of dimension b and quotienting for the space of dimension c. Alternately, we could pick a subspace of dimension $a+b$, quotient to get a subspace of dimension c, and then decompose the subspace of dimension $a+b$. These ways of getting a subspace decomposition should, in some real sense, be equivalent.

There are several ways of correctly encoding such equivalences. The most general is to shift our perspective on exact sequences: instead of thinking of an exact sequence as a way of decomposing $A \oplus B$ into A and B, we instead think of it as a filtered[5] R-module $A \oplus B$ with only two filtered layers. The three-term relation then says that the whole module is equal to the first filtration plus the quotient. This gives us a method for inserting new relations: extending the length of the filtration. Given a filtered R-module $A_1 \subseteq A_2 \subseteq A_3$, the three-term relation can break down A_3 in two different ways. First, we could break down A_3 as A_2 and A_3/A_2 and break down A_2 as A_2/A_1 and A_1. Alternately, we could break down A_3 into A_3/A_1 and A_1, and then further break A_3/A_1 as $(A_3/A_1)/(A_2/A_1) \cong A_3/A_2$ and A_2/A_1. These two ways of decomposing A_3 should be equivalent, as both decompose A_3 into A_1, A_2/A_1, and A_3/A_2. Such a filtered module should therefore impose a four-term relation

$$\begin{aligned}[A_1 \subseteq A_2] + [A_2 \subseteq A_3] \\ = [A_2/A_1 \subseteq A_3/A_1] + [A_1 \subseteq A_3].\end{aligned}$$

We can add a 3-cell labelled by each triple of projective R-modules via a gluing map corresponding to this equation. There are now relations that are imposed by four-term filtered R-modules, and five-term filtered R-modules, and so on; we thus produce an infinite-dimensional CW complex that is called $|s_\bullet \mathbf{P}_R|$; as we shall see in the next section, this is a special case of a construction called the *Waldhausen $s_\bullet$-construction* [20, Section 1.4].[6] To get the K-theory with the correct π_0 we take its loop space, shifting the homotopy groups back down a dimension:

$$\mathbf{K}(R) \stackrel{\text{def}}{=} \Omega|s_\bullet \mathbf{P}_R|.$$

To get the higher K-groups we define

$$K_n(R) \stackrel{\text{def}}{=} \pi_n \mathbf{K}(R).$$

By construction this gives the correct K_0. Proving that this gives $\mathrm{GL}(R)^{\mathrm{ab}}$ on π_1 is more difficult; for details on this see [22, Exact Categories IV.8.6, Corollary IV.7.2].

Algebraic K-theory is very mysterious: it is famously difficult to compute and is related to many important invariants. For example, the Quillen–Lichtenbaum conjecture (proved by Vladimir Voevodsky) states that there should be a spectral sequence beginning at étale cohomology of $\operatorname{Spec} R$ and converging to the K-theory of R. Assuming the Kummer–Vandiver conjecture (which is a statement about the class numbers of cyclotomic fields) [21, Theorem 1] the K-theory of the integers was computed to be $K_0(\mathbf{Z}) \cong \mathbf{Z}$, $K_1(\mathbf{Z}) \cong \mathbf{Z}/2$, and

$$K_n(\mathbf{Z}) \cong \begin{cases} 0 & \text{if } n > 0 \text{ and } n \equiv 0 \pmod 4 \\ \mathbf{Z} \oplus \mathbf{Z}/2 & \text{if } n \equiv 1 \pmod 8 \text{ and } n > 1 \\ \mathbf{Z}/c_k \oplus \mathbf{Z}/2 & \text{if } n \equiv 2 \pmod 8 \\ \mathbf{Z}/8d_k & \text{if } n \equiv 3 \pmod 8 \\ \mathbf{Z} & \text{if } n \equiv 5 \pmod 8 \\ \mathbf{Z}/c_k & \text{if } n \equiv 6 \pmod 8 \\ \mathbf{Z}/4d_k & \text{if } n \equiv 7 \pmod 8), \end{cases}$$

where c_k/d_k is the Bernoulli number B_{2k}/k in lowest terms and $n = 4k-1$ or $4k-2$. The most famous computation of algebraic K-theory is still Quillen's original computation of the K-theory of finite fields [14], which has since strongly resisted generalizations. More modern computational techniques use close approximations to K-theory, such as THH and TC; for more on this see for example [9].

[5] *We need a little bit more structure to make this work correctly; instead of taking just a filtration $M_0 \subseteq \cdots \subseteq M_n$ we must take a sequence $M_0 \hookrightarrow \cdots \hookrightarrow M_n$ of inclusions. In particular, isomorphisms $M_0 \xhookrightarrow{\cong} M_1$ are allowed, and different choices of isomorphism are assigned to different cells.*

[6] *This construction is one of two constructions in Waldhausen's paper, called the $s_\bullet$-construction and the $S_\bullet$-construction. The difference between them is that the $S_\bullet$-construction can also incorporate a notion of "weak equivalence."*

Intermezzo: Categories Representing Spaces

The close connection between algebraic topology and category theory is often presented historically: many of the fundamentals of category theory were developed by homotopy theorists in connection with homology and cohomology. Here we pursue a different approach illustrating the importance of categories in homotopy theory, with the following slogan:

> Categories give algebraic models for homotopy types of spaces, analogously to the way that rings of functions give algebraic models for geometric objects.

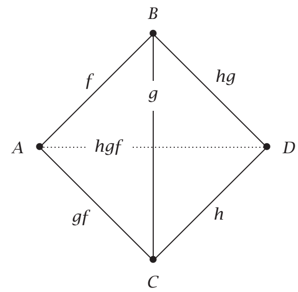
This point of view clarifies why category theory is so intimately intertwined with homotopy theory. To substantiate the claims above, let us explore the basic constructions. This is a very informal introduction; for a more complete treatment see, for example, [16].

A *category* is a collection of *objects*, together with a collection of *morphisms* between objects. A morphism has a source and a target; a morphism with source A and target B is written $f : A \longrightarrow B$. Morphisms have an operation of *composition*: given a morphism $f : A \longrightarrow B$ and a morphism $g : B \longrightarrow C$ there is a morphism $g \circ f : A \longrightarrow C$.

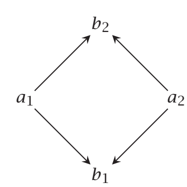
Some examples of categories:

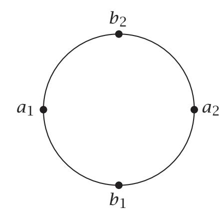
1. The category **Set** has sets as objects and functions as morphisms between sets (with the usual composition). One can similarly define categories of groups and group homomorphisms, topological spaces and continuous maps, smooth manifolds and smooth maps, etc.
2. Suppose that we are given a partial order $(P, \leq)$. We can define a category with objects the elements of P, and with exactly one morphism $a \longrightarrow b$ if $a \leq b$. Composition is well-defined by transitivity of $\leq$.
3. Let G be a group. We define a category with one object $*$, and with the set of morphisms $* \longrightarrow *$ given by the elements of G. Composition is the multiplication in G.

Given a category C for which objects form a set, we can define the *classifying space* $|C|$. Take a 0-cell for every object in the category. For every morphism $A \longrightarrow B$ attach a 1-cell between the 0-cell for A and the 0-cell for B. For every pair of composable morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ attach a 2-cell along f, then g, and then backwards along $g \circ f$. For every triple of composable morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$, $h : C \longrightarrow D$ attach a 3-cell by thinking of it as a tetrahedron and gluing it in along the triangles shown below:

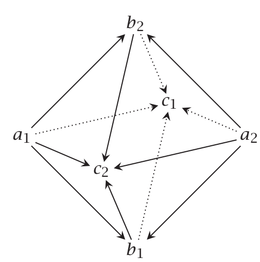


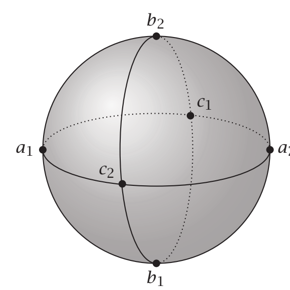
Continue likewise for all n-tuples of composable morphisms.[7] In the case when the category is obtained from a group G this construction produces the usual classifying space BG of the group. In the case when the category comes from a finite partial order $(P, \leq)$, this produces the abstract simplicial complex given by the set of finite totally ordered subsets of P. Thus, for example, if $P = \{a_1, a_2, b_1, b_2\}$ with the ordering given by $a_i \leq b_j$ for $i, j = 1, 2$, the category obtained from P is drawn on the left, and its classifying space is the circle on the right:





In the slightly more complicated case when $P = \{a_1, a_2, b_1, b_2, c_1, c_2\}$ with the ordering given by $a_i \leq b_j \leq c_k$ for $i, j, k = 1, 2$ the category obtained from P is drawn on the left and its classifying space is the sphere on the right:





Comparing this to the construction of $|s_\bullet \mathbf{P}_R|$ in the previous section, we see that it is almost directly analogous to the construction of the classifying space of a category. However, the $s_\bullet$-construction has n-simplices labelled by compositions with indexing 1, rather than 0; for example, a morphism gives a 1-simplex in the classifying space of a category, but a 2-simplex in the $s_\bullet$-construction. We thus need an extra face to each simplex; this is given by taking $A_1 \hookrightarrow \cdots \hookrightarrow A_n$ to $A_2/A_1 \hookrightarrow \cdots \hookrightarrow A_n/A_1$.

[7] *This is somewhat of an oversimplification, as we must deal with identity maps (and morphisms that compose to identity maps) in a special way. This construction is the geometric realization of the nerve of the category; for a more detailed introduction see* [16, Chapter 6], *as well as* [10, Section 1.1].

Before we end this section, let us spend a little time considering how the categorical structure can help model the homotopy class of a space. When working with categories, we do not work with categories up to isomorphism; instead, we work up to *equivalence*. The idea here is that it should not matter how many different isomorphic models of the same object exist in a category. For example, we can consider the following three categories:

1. The category whose objects are finite subsets of $\mathbf{R}$, with functions as its morphisms.
2. The category whose objects are finite subsets of $\mathbf{Z}$, with functions as its morphisms.
3. The category whose objects are the sets $\{1, \dots, n\}$ for $n \geq 0$, with functions as its morphisms.

All of these should model "the category of finite sets," and thus should all behave the same way, despite the fact that each one restricts to a subset of the objects of the previous ones. This is the notion of equivalence of categories, and it behaves analogously to the way that homotopy equivalences can "thicken" spaces by, for example, replacing a circle with a solid torus.

We can also use the algebraic structure of a category to understand the homotopy type of its classifying space. For example, a category with an initial object (an object with a unique morphism to every other object in the category, such as the empty set in $\mathbf{Set}$) has a contractible classifying space: the initial object behaves like a "cone point," connecting uniquely to everything else, and allows us to retract every point in the classifying space to it. For those readers familiar with adjoint functors, we state the following theorem, which is a generalization of this observation:

Theorem 3. *Given a pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ the induced maps on the classifying spaces give mutually inverse homotopy equivalences.*[8]

In fact, it turns out that *all* homotopical behaviors of topological spaces can be modeled by categories; this observation is due to Thomason [19].

Combinatorial K-Theory

When we constructed algebraic K-theory we used almost none of the algebraic information about R; all we used is data about how to include and quotient R-modules. Thus this same kind of construction can be used with more geometric objects—such as varieties or polytopes—replacing modules to construct a K-theory of a more combinatorial nature.

Let us consider the $s_\bullet$-construction again and attempt to create a K-theory for finite sets. Finite sets are made up of their subsets, in the same way that modules are made up by submodules (and quotients); thus we begin by considering the disjoint union of finite sets. By analogous logic to the section "Algebraic K-theory," this leads to $K_0(\mathbf{FinSet}) \cong \mathbf{Z}$, with the element n representing a set of size n, and $K_1(\mathbf{FinSet}) \cong \mathbf{Z}/2$ representing the abelianization of the group of permutations: the sign. Our hope is that the $s_\bullet$-construction should give a construction of a K-theory space that agrees with these computations.

We consider the simplices in the $s_\bullet$-construction. We take an n-cell for each filtered finite set $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$ and, thinking of it as an n-simplex, want to attach it to the cells labeled $A_1 \subseteq \cdots \subseteq \widehat{A_i} \subseteq \cdots \subseteq A_n$, as well as $(A_2/A_1) \subseteq \cdots \subseteq (A_n/A_1)$—but what should we take instead of $/$? The analog of taking quotients of modules is takings quotients of sets. In order for our K_0 to work correctly, we need the 2-cell labeled by $A \overset{=}{\subseteq} A$ to be attached to A, A, and $\varnothing$, to represent the relation $[A] = [A] + [\varnothing]$. However, if we take the quotient of finite sets, we get $[A] = [A] + [\text{singleton}]$, which would imply the relation $|A| = |A| + 1$ inside $\mathbf{Z}$; clearly not what is intended.

To solve this problem we take a very simple approach: we simply take complements instead of quotients. We label the last face by

$$(A_2 \smallsetminus A_1) \subseteq \cdots \subseteq (A_n \smallsetminus A_1),$$

which now gives the correct relations. We call this construction the $\tilde{s}_\bullet$-construction. For a more detailed exploration of this construction, see [7]. We can then define

$$\mathbf{K}(\mathbf{FinSet}) \overset{\text{def}}{=} \Omega|\tilde{s}_\bullet \mathbf{FinSet}|.$$

By the Barratt–Priddy–Quillen Theorem, the K-theory of finite sets is the space QS^0 (the 0-space of the sphere spectrum), whose homotopy groups are the stable homotopy groups of spheres (the calculation of which is one of the fundamental unsolved problems in algebraic topology today).

This sets up an interesting parallel. The integers are defined as a group completion of finite sets, with multiplication induced by Cartesian product of sets. The ring of integers is the fundamental object in number theory and commutative algebra. Under K-theory, the finite sets (with a multiplicative structure induced by Cartesian product) are related to the sphere spectrum, which is a fundamental object in homotopy theory and derived algebraic constructions. This illustrates that K-theory is intrinsically intertwined with the foundations of algebraic topology, providing further evidence that our construction was well chosen.

We can apply the $\tilde{s}_\bullet$-construction more generally by looking at other types of objects that have complements. For example, we can consider finite G-sets (or almost-finite G-sets for a profinite group G), polytopes in a uniform

[8] *This theorem can be proved as an application of* [15, Theorem A]. *It can also be proved directly from the definition of the nerve; for a quick introduction to simplicial sets and nerves, see for example* [17].

geometry, O-minimal structures, or varieties. In each of these the construction produces a K-theory in which elements of K_0 classify (stable) scissors congruence invariants. (For a more in-depth discussion of such examples and their K-theory, see [7, 24].)

Let us investigate the case of varieties more closely. We take the category whose objects are varieties over a field k, with closed immersions as morphisms. The $\tilde{s}_\bullet$-construction gives a space $\mathbf{K}(\mathbf{Var}_k)$ where $K_0(\mathbf{Var}_k)$ is the free abelian group generated by varieties, modulo the relation that

$$[X] = [Y] + [X \setminus Y]$$

whenever Y is a closed subvariety of X. This is the *Grothendieck ring of varieties*. To construct $\mathbf{K}(\mathbf{Var}_k)$ we can again use the Waldhausen $\tilde{s}_\bullet$-construction. For the inclusions we use closed embeddings; for the complements we use the complement of the image of the embedding.[9] Analogously to the case of finite sets defined above, this produces a space and we can define

$$\mathbf{K}(\mathbf{Var}_k) \stackrel{\text{def}}{=} \Omega|\tilde{s}_\bullet \mathbf{Var}_k|.$$

The higher K-groups encode invariants of piecewise-automorphisms.

In previous examples, the calculation of K_0 was simple, and it was the higher K-groups that gave interesting invariants related to automorphisms. However, for varieties, even the calculation of K_0 is extremely nontrivial; it turns out the higher homotopy of $\mathbf{K}(\mathbf{Var}_k)$ can create new tools for the analysis of K_0.

Consider the filtration by dimension on the set of varieties. For any dimension n, we can construct a group $K_0(\mathbf{Var}_k^{\leq n})$ generated by varieties of dimension at most n (with the same relation as above). We can then filter $K_0(\mathbf{Var}_k)$ by setting the n-th filtration to be the image of the natural map $\iota_n : K_0(\mathbf{Var}_k^{\leq n}) \longrightarrow K_0(\mathbf{Var}_k)$. Unfortunately, since we do not know that ι_n is injective, we cannot use information about $K_0(\mathbf{Var}_k^{\leq n})$ to learn about $K_0(\mathbf{Var}_k)$.

Here is where having a space, rather than a set, is crucial. A filtration on a space does not need to produce a filtration on the connected components. Consider, for example, the two-step filtration $\mathbf{Z} \subseteq \mathbf{R}$ of the integers sitting inside the reals; the integers have infinitely many connected components, while the reals only have one, but the filtration is topologically well-defined. In fact, the filtration quotient is a bouquet of circles, which in its fundamental group encodes the data that the filtration joins infinitely many connected components into one. We can therefore hope that, if we can compute the associated grade of the dimension filtration on the space $\mathbf{K}(\mathbf{Var}_k)$, we can extract information about the filtration on $K_0(\mathbf{Var}_k)$.

From our construction we see that the map $\mathbf{K}(\mathbf{Var}_k^{\leq n-1}) \longrightarrow \mathbf{K}(\mathbf{Var}_k^{\leq n})$ induces an inclusion of spaces, which allows us to compute the cofiber of this map as a spectrum.[10] (For a much more detailed discussion of this, see [8, 23, 24].) In fact, we have the following:

Theorem 4 ([23, Theorem A]). *The cofiber of the inclusion $\mathbf{K}(\mathbf{Var}_k^{\leq n}) \hookrightarrow \mathbf{K}(\mathbf{Var}_k^{\leq n-1})$ is equivalent (as a spectrum) to $\bigvee_{[X]\in B_n} \Sigma^\infty(B\mathrm{BiratAut}(X)_+)$. Here, B_n is the set of birational isomorphism classes of varieties of dimension n, $\mathrm{BiratAut}(X)$ is the group of birational automorphisms of X, Σ^∞ takes the suspension spectrum associated to a space, and $\cdot_+$ adds a disjoint basepoint.*

In particular, this means that, morally speaking, the spectrum $\mathbf{K}(\mathbf{Var}_k)$ is assembled out of classifying spaces of birational automorphism groups. As a consequence of this we can conclude that all elements in the kernel of ι_n must be in the form $[X \setminus U] - [X \setminus V]$ for some birational automorphism $\varphi : X \dashrightarrow X$ represented by an isomorphism $U \longrightarrow V$. In this way the automorphism invariants living in K_1 control what can happen in $K_0(\mathbf{Var}_k)$; the higher structure of $\mathbf{K}(\mathbf{Var}_k)$ is fundamentally intertwined with its 0-level structure.

Coda: The Birds-Eye Perspective

The focus of this article has been on problems arising in algebraic or geometric contexts: vector bundles, projective modules, polytopes, etc. In each context, the moral of the story was that for each such problem there is a K-theory space, whose homotopy groups classify higher invariants.

In fact, there is an even more general approach to K-theory. Any time that a problem can be modeled by a category C with an operation[11] (such as finite sets with isomorphisms as morphisms, and disjoint union as the operation) we can construct a topological monoid, since the classifying space $|C|$ of the category inherits an operation from C. The K-theory is, morally speaking, the group completion of this monoid. There is an interesting theorem of McDuff–Segal:

Theorem 5 ([13]). *For a topological monoid M, the topologically correct group completion of M is ΩBM; here, BM is a generalization of the delooping of M.*

For a category C with operation, it therefore makes sense to define

$$\mathbf{K}(C) \stackrel{\text{def}}{=} \Omega B|C|.$$

[9] *As in the section "Algebraic K-theory," we need to remember the particular immersion that was chosen for each cell in the construction.*

[10] *For those unfamiliar with spectra, it is possible to mentally replace "spectrum" with "space" in the following discussion. This will not produce correct mathematics, but will be close enough to explain the narrative.*

[11] *More formally, a symmetric monoidal structure.*

This formula describes $\mathbf{K}(C)$ as a group completion, which is defined by its universal properties. Likewise, more general K-theory can be defined via a universal property (see [4]).

Unfortunately, this perspective comes with a severe drawback: it is almost impossible to work with. Generally, when constructing K-theories, we wish to be able to prove things about them: to compute the higher K-groups, to relate them to other forms of K-theory, to compute the homotopy fibers or cofibers of maps between K-theories, and so on. The definition of K-theory given by the McDuff–Segal theorem was not designed for such functionality, and thus (unsurprisingly) is not well-suited for such analysis. This is because the B functor in the definition is extremely difficult to work with, and is particularly unsuitable for calculations.

The solution to this is often to find another construction—such as the $s_\bullet$-construction [20], Quillen's Q-construction [15], or the B-construction for Γ-spaces [18]—for the K-theory space. While the McDuff–Segal theorem gives a clean and simple definition, these constructions come with many tools that one can use to analyze their results. Thus the study of K-theory often sits awkwardly between the formal and the practical, trading off simplicity for usefulness and looking for beauty in between.

ACKNOWLEDGMENT. The author would like to thank Ilya Zakharevich for extensive edits and suggestions which massively improved the quality of exposition. The author would also like to thank the two anonymous referees for their time and assistance in tightening up the exposition even further.

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Inna Zakharevich

Credits

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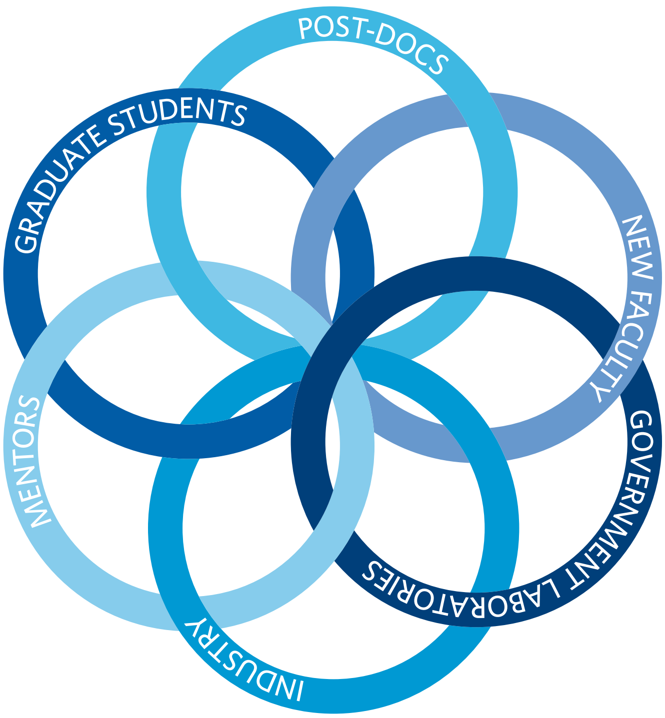
The Early Career Section is a compilation of articles that provide information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section. Next month's theme will be mentoring.



Teaching

Why Do We Teach?[1]

I'm serious. Let's have a talk about what we hope students get out of the classroom experience. Of course, it's a bit of a one-sided talk since I am doing the writing and you're doing the reading, but I do hope you will ask yourself this question and think about it as you continue to read.

Most often, faculty say, "I have a body of material that I want them to learn. My job is to get as much of it across to them as possible." Maybe it's multivariable calculus, and the goal is to teach them partial derivatives and multiple integrals. Maybe it's a first class on proofs, and the goal is to get them proficient in proof-writing. Maybe it's algebraic topology, and the goal is to teach them homotopy and homology. And teaching the material certainly is and should be a goal of every course.

But there is a second goal that to my mind is at least as important as the first. We want to impart a love of mathematics. We want to get across the power and beauty of the field. We want students to feel the same awe that we feel, for the "unreasonable effectiveness of mathematics." If we impart that to them, in whatever measure possible, then not only do they learn the material from the course, but they go on to take another course. And then hopefully another course. And their mathematical odyssey becomes a lifelong endeavor. They learn much more mathematics than just what was in that one course.

Steven Ceci is a psychologist at Cornell University. Having taught the same developmental psychology course every semester for twenty years, he decided to do an experiment. In the fall, he taught the course as he usually did, to a total of 243 students, only this time audiotaping the lectures. Then, over the winter break, he took a teaching skills workshop taught by a professional media consultant. The instructor worked with the participants on methods of conveying enthusiasm and excitement for the material: varying voice pitch, using hand gestures, trying to exude

[1]*Colin Adams is the Thomas T. Read Professor of Mathematics at Williams College. His email address is* cadams@williams.edu.

DOI: https://dx.doi.org/10.1090/noti1921

enthusiasm and connect with the audience through body language.

Then in the spring semester, Ceci taught the same course to 229 students. He did his best to replicate exactly what he had done in the fall, using the same book, same syllabus, same room, same time of day, same teaching aids, and the same exams and quizzes (which were not returned to the students in the fall semester). Before each class, he would listen to the audiotape of the corresponding lecture from the fall, and do his best to memorize it, which, having taught it for twenty years, was not that difficult. Then, when he presented the lecture, his only change was to follow the recommendations of the consultant and to show more enthusiasm.

Then Ceci and his co-author Wendy Williams examined the student course survey results for the two semesters. Questions were on a five-point scale, with 5.00 being the highest.

The most important measure of the success of the experiment was the question on the enthusiasm of the professor. In the first semester, Ceci received a mean score of 2.14, whereas in the second semester, he received a 4.21. So, the goal of conveying enthusiasm was definitely achieved.

But now let's look at the results for some other questions. On the question of whether the instructor was knowledgeable, he received a 3.61 in the fall and a 4.05 in the spring, which is kind of funny given that he was no more knowledgeable in the spring than he had been in the fall. But that's not such a huge difference after all.

On the question of whether he was accessible, he received a 2.99 in the fall and a 4.06 in the spring. That's in spite of the fact that Ceci states unequivocally that he was equally accessible in the fall and spring. For organization, he received a 3.18 in the fall and a 4.09 in the spring.

On the question of how much the students had learned, the average score was a 2.93 in the fall and a 4.05 in the spring. In fact, although the students believed they had learned more in the spring, the total number of points garnered on the exams averaged over each of the two semesters were nearly identical. Students in the second semester had not learned more.

Here's my favorite. The students were asked to rate the textbook. In the fall, they gave it a rating of 2.06 versus a spring rating of 2.98. The professor's more enthusiastic style of teaching influenced their appraisal of the book!

And perhaps the most important question in terms of the evaluation of faculty is the overall rating of the course. In the fall, the score was 2.50 and in the spring it was 3.91, a dramatic difference.

And finally, the last question, which is certainly also a critical one, is whether the student would recommend the course to others. In the fall, the score was a 2.36 whereas in the spring, it was 2.81.

Now, before reading on, you should ask yourself, what is this study telling us? What is the conclusion we should draw? Go ahead, ask yourself, I'll go make myself a sandwich.... What? Done already? Well, then, I'll eat the sandwich later.

Let's first see what the authors say. Their conclusion is that the study is a clear indictment of the student course surveys and their use in tenure and promotion decisions. The fact that such minor changes in teaching style could so substantially impact the scores indicates that student course survey data is easily influenced by factors that do not translate into better learning, and therefore colleges should not use this kind of data in their decision-making processes. The scores are not measuring how effectively the faculty are imparting information.

But my interpretation is entirely different. Of these two groups of students, one in the fall and one in the spring, which are more likely to take a follow-up course? Which are more likely to want to continue their studies in psychology? Which are more likely to become lifelong learners in the field?

Unfortunately, Ceci and Williams did not collect that information. But I think we can both make an educated guess to the answer. When Ceci showed enthusiasm by any means possible, the students had a better experience in the class. They were more engaged. And one must suspect, they were much more likely to continue on to a subsequent course in psychology.

Quite a while ago, at the MAA MathFest conference in Seattle, Ed Burger and I put on a teaching workshop. The idea was to experiment with the latitude that we have in our teaching styles in the classroom. Too often, faculty are convinced there is a particular manner in which we should behave in the classroom, a "professorial demeanor" we should take on. We wanted participants to explore the bounds, experiment with the reality of how much leeway we actually have.

To that end, we asked all the participants to come prepared with a three-minute presentation on any topic in mathematics they wanted. Then before they began, we had them pick a small piece of paper out of a hat. On the pieces of paper were written instructions for how they should present the material. Some examples are:

- You are in a Broadway show, sing and dance it
- Your hands/feet are tied
- You HATE this math
- You're a used car salesperson
- You just won the Fields Medal and are very pleased with yourself
- You have twenty extra minutes, and must stretch the material out
- You really have to go to the bathroom

The presentations were amazing, and incredibly entertaining. We all laughed until we cried. But the point was not that we should now all act as if we were in a Broadway show. That could be ugly, at least in my case. Rather, the point was that we all have immense leeway in our teaching style. We do not need to restrict ourselves and put on a professorial demeanor. In fact, that's exactly what we shouldn't do. We shouldn't teach the way Ceci taught that fall and the way he had been teaching for the last twenty years. Let your personality out. Let your enthusiasm out. You obviously went into math because you love it (you certainly didn't do it for the money). Let the students see your love of mathematics.

This is true if you lecture, if you flip your class, if you use active learning or inquiry-based learning. I will always believe that there is no one correct method for teaching mathematics. New ways will come and go, some good, some not so good. I urge you to ignore the people who tell you otherwise. It all depends on who you are and who the students are. Most important is to figure out what works for you.

I remember seeing a lecture by a professor who had won a variety of teaching awards. As I always am, I was curious to see what it was he did that made him so successful in the classroom. He gave a relatively standard lecture, but definitely well organized and clear. Okay, that's fine, but it's not going to win you prizes. There was only one thing he did differently that really stood out. During the entire lecture, he was grinning from ear-to-ear. It was clear for all to see how much he enjoyed the mathematics. He was in his element and everyone in the room could see and feel that. Of course, this doesn't mean that you should lecture with a grin plastered across your face. If it is not naturally you, students can tell, and you end up looking like a raving lunatic.

But we can each find our own ways to get across our continual amazement at the power of mathematics. And by imparting it to our students, we can create the next generation of mathematicians.

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Colin Adams

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Author photo is courtesy of Colin Adams.

Thoughts on Helping Students to Feel Included[2]

In this note I'll say a little bit about how a campus initiative at my institution, Mount Holyoke College, helped me understand issues that make students feel unwelcome in the mathematical community and what I changed as a result.

In the spring of 2017 the college canceled classes for a daylong conference dedicated to working together as a community to listen, brainstorm, and discuss diversity on campus. My colleagues KC Haydon (Psychology), Kate Ballentine (Environmental Studies), and Gary Gillis (Biology and Associate Dean of the Faculty) organized a session to discuss the experience of people of color in STEM courses on campus. Students and faculty broke out in small groups and talked and listened for a sustained amount of time. Responses to several prompts were returned to the organizers on Post-its, and a full group discussion followed.

I learned a lot by listening to the students. For example, it is important to our department that we provide lots of resources for students to get help. We have an evening help system where TAs assigned to specific classes hold nightly sessions where students are encouraged to work together. This is a free resource that was available to all students, but I hadn't realized that it was making some students feel even more isolated and alone. What I heard was that some students went with well-established working groups and other students came by themselves. The large groups tended to attract more TA help than isolated students. Even worse, walking into a room alone and seeing lots of students who already have working relationships was only reinforcing student doubts about whether they truly belonged in the class.

As a consequence of this I made two changes, one at the department level, and one in my own courses. At the department level, we made sure that our TAs are aware of this dynamic and how harmful it can be. We also implemented on-demand individual tutoring sessions for students. The TAs who are offering individual tutoring sessions work with the course instructor to determine times that make sense. Then the student TA sets up appointment slots on a Google calendar that the instructor shares with the class. Students can sign up for these slots, no questions asked. This allows students to get help when they need it without going through the instructor, which may feel intimidating or stigmatizing.

In my own courses I've become much more transparent with students about the pedagogical choices that I am making. For example, I used to do group work partly so that students would meet one another. Now I tell them explicitly that this is one of the reasons I have them work

[2]*Jessica Sidman is the John S. Kennedy Professor of Mathematics at Mount Holyoke College. Her email address is* `jsidman@mtholyoke.edu`.

DOI: https://dx.doi.org/10.1090/noti1914

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in groups. Instead of telling the evening help TAs to invite students who seem isolated to work with others, I now tell my students that if they go to evening help they should be thinking about including students who come in alone to work in their groups and that TAs will remind them to do this.

From the first day I am explicit about how I want everyone to feel included and that we all have to work together to make that happen. Just as with anything else we want students to learn, this bears repeating often. One way that I drive this home in the first few classes is by drawing them into my attempts to learn names. I spend the five minutes before class greeting the students who come early, adding new students to my repertoire as they arrive. Then I make a game of asking the students to try to name all of the people in their row, or the people sitting next to them, or all of the students in another row in the first minute or two of class. I also do group work in the very first class, and make sure to exhort the students to introduce themselves. I have found that they engage with one another more if they work standing up at the boards rather than at tables, and I ask them to rotate who serves as the group scribe. In later weeks, as I stroll around to check in on different groups I give lighthearted "pop quizzes" where I ask them to name everyone in their group to reemphasize that this is still a priority.

Though canceling classes for a full day is not on the table for most of us, a department event that allows faculty to listen to students and better understand the student experience followed by a commitment to make changes could go a long way. We can learn so much by listening to students, and this is something that we can do inexpensively if we set aside some time and make an inviting space. Of course there are some changes, like diversifying the faculty, that will take significant institutional resources and will. However, there are things that we can do to make our classes more inclusive that are more on the order of small tweaks that any of us can implement that can have immediate impact. The conversations around inclusive teaching on my campus have really changed me, and I think they are well worth having.

Jessica Sidman

Credits

Author photo is by Jordan Tirrell.

Teaching at a Community College[3]

It is a bit strange to think that this is my sixth year teaching at a community college. I am not sure how I feel about that. In some ways, I still feel as if I've just started. Each semester is a different experience, even if I've taught the same courses many times. In other ways, I feel very experienced. I have notes and lecture schedules prepped for a majority of the math courses offered at my college. One thing that I am sure about, though, is that I am a completely different professor now than I was my first year.

When I was in graduate school, teaching was not as much of a priority for me. I was in a research-oriented environment. My main focuses (and that of my professors and peers) were working on unsolved math problems and publishing and sharing the results. This was, of course, expected, as I was obtaining my mathematics PhD at a university.

My department did require graduate students to take a teaching seminar before we became teaching assistants and instructors, which was helpful for me. However, my teaching experience at the university was extremely different from my current experience at a community college. Many of my students at the university had honed their study skills, were confident in their prerequisite knowledge, and only had to focus on their education. In contrast, students at a community college may not have developed their study skills and mathematical foundations. They also sometimes work full-time jobs (or several part-time jobs) and regularly deal with personal issues that prevent them from being able to focus solely on their education.

Toward the end of graduate school, I realized that I didn't see myself in a research-focused career. So, I decided to apply to jobs in both industry and academia (mostly at liberal arts colleges that were more focused on teaching). I ended up with job offers in both industry and academia. Since I was always curious if I could survive in industry, I chose industry. However, after just a year, I realized that my true calling was to be in the classroom. I wanted to be focused on teaching, so I applied to community college teaching positions. Six years later, I am still glad that I made the switch.

I recently sat on my first hiring panel, and I realized that I was very lucky to be offered a position the first time that I went through the hiring process. From my experience, I've learned that even the same math department can be looking for different qualities in candidates depending on the year and the people on a particular hiring panel.

[3] *Jasmine Ng is an associate professor of mathematics at El Camino College. Her email address is* `jng@elcamino.edu`.

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There are some qualities that all successful candidates tend to have in common, though. Having teaching experience at a community college is preferable. However, I don't think it is as important as most people would think. I favor candidates who show strong knowledge in a variety of math courses (from elementary algebra to calculus) during the in-person interview. Knowing statistics is a bonus. I also pay special attention to teaching demonstrations during interviews, since being able to convey mathematical concepts clearly is an integral part of a professor's job.

One of the most important qualities that I look for, though, is experience with a diverse student population. Community college students tend to come from extremely varied socio-economic backgrounds and are in various stages in life (some have just finished high school and some are coming back to school after thirty years). Teaching students with such a wide range of educational and life experiences is a challenge, to say the least. So, it's nice to know that a potential hire has experience with students with varied math and personal backgrounds.

At my college, a typical teaching load per semester is twenty contact hours per week (fifteen hours in the classroom and five office hours). Besides teaching, faculty are expected to participate in department committees that develop and review course curricula. There are also campus-wide committees, such as the Academic Senate or the Distance Education Committee, that faculty may join if they wish. So, there are plenty of opportunities to be involved with projects outside of the classroom if one so chooses.

Over the years, the aspect of my job that I've enjoyed the most is getting to know my students personally, listening to their stories, and being able to help them achieve their goals. I've learned about my students' struggles as immigrants who have had to learn English and work full-time to support their families while attending school. I've heard about their dreams to be the first to graduate college in their families. I've listened to veterans who are readjusting to student life after coming back from serving our country. I've encouraged students not to give up because they don't remember basic math concepts right away after decades away from school. I've pushed students to stop thinking they are not "smart enough" to do well at math just because they have been told or felt that at some point in school.

The most rewarding conversations, though, are the ones in which former students tell me that they are going to graduate and what their futures entail, like plans to transfer to a four-year college or start a new job. These conversations have always kept my motivation tank full and made me grateful to be part of their journeys.

When I first started, I was very focused on teaching math and not as much on getting to know my students. However, I've since realized that the charm of teaching at a community college is, well, the community. I'm not just there to impart mathematical knowledge. I'm there to listen to and understand my students' struggles, and guide them in their journeys to achieve their educational and life goals. I'm continually evolving as a professor, and always in awe of what my students can achieve despite life's obstacles. Teaching at a community college is a challenging, but rewarding experience, and I look forward to many more years of it.

Jasmine Ng

Credits

Author photo is courtesy of Jasmine Ng.

Inquiry Based Learning[4]

What is Inquiry Based Learning?

This is one of those obvious questions that is difficult to answer succinctly. The community of college-level math educators that has formed around the IBL label has intentionally taken a "big tent" approach to defining the term so that it's more of a philosophy than a specific teaching method. One large study of IBL in math courses [4] identifies two key features (sometimes called the "twin pillars"). If you're teaching an IBL course, your students should:

- actively engage with rich mathematical tasks during class, and
- collaborate and practice communicating mathematical ideas.

The details of how you do this will depend on what works best for you, your students, the course content, and other logistical considerations like the size of the class and the layout of the classroom.

One approach to IBL is to give students problem sets that have been carefully designed to lead students towards important ideas. The problems should generally be challenging but accessible for the students in the course. Class time can be spent in some mixture of having students work in small groups to solve these problems while you circulate, and having students present their solutions, followed by a class discussion of these solutions.

It is also possible to make small changes to a primarily lecture-style course to encourage students to engage with math during class. One simple, very effective strategy is known as think-pair-share. Give students a problem to work on in class, and then have them:

1. start by thinking about the problem on their own,
2. pair up with someone nearby to discuss their approach, and
3. share their ideas with the whole class

This approach gives students time to make sense of the problem, opportunities to learn more about what makes the problem interesting or difficult, and more reason to be interested and invested in ultimately seeing the problem solved.

For a more detailed account of what IBL is and some examples of implementation, see [1].

Why teach using IBL?

Students learn better from IBL. There is overwhelming evidence of this: a meta-study [2] found that active learning in STEM classes had a number of positive effects on student learning, and an extensive study of IBL [4] found that IBL math courses were particularly helpful in closing the achievement gap for women and for lower-performing students, especially when encountering IBL early in their mathematics career.

This makes sense. People learn new skills, whether it's playing a musical instrument, competing in a sport, or doing math, by practicing them, so we should prioritize giving our students opportunities to practice math while we're there to provide guidance.

You get to watch your students doing mathematics. This is an obvious consequence of having students do math during class, but it is worth emphasizing because witnessing this is one of the joys of teaching an IBL class. You know that moment that happens sometimes in office hours, when a student finally makes sense of a topic they've been struggling with? In an IBL classroom, you get to see that happening in class, over and over. You get to listen as your students have genuine, deep conversations around mathematical ideas.

You gain a better understanding of what your students know. When lecturing, it's easy to believe that, since something was said and no one asked any questions, the students must have understood it. When you spend class listening to your students discuss math, you gain a much better sense of exactly what they've understood and what they're still working through.

Your students gain a better understanding of the practice of mathematics. Math lectures can give students mistaken beliefs about what it means to do math. Solutions immediately follow problems, with no sense of the time and effort it took people to first come up with the necessary ideas. Theorem statements and proofs arrive fully formed, without errors or dead ends.

In an IBL course, instructors still provide structure to guide students in the right direction, but you have the opportunity to show them that solving interesting mathematical problems takes time, that mistakes can provide valuable insight, that promising beginnings can lead to dead ends that can themselves lead to new insights. By emphasizing the notion of *productive struggle*, you can teach your students to persevere and become better problem-solvers alongside teaching them the course content.

Some history

Someone whose view of IBL was formed in the early 2000s might describe it very differently from the way I have. Here is some context; a more detailed exploration of this story can be found in [3].

There is a particular segment of the IBL community whose origins are intertwined with the Educational Advancement Foundation (EAF). The EAF has provided extensive support for IBL workshops, centers, and other initiatives that played an important role in training many mathematicians, including myself, in IBL methods. But the EAF was created in particular to spread awareness of

[4]*Hanna Bennett is a lecturer and co-director of the Mathematics Introductory Program at the University of Michigan. Her email address is* hbennett@umich.edu.

DOI: https://dx.doi.org/10.1090/noti1916

the “Moore Method,” which is a very specific approach to teaching developed by R. L. Moore, and this means Moore Method played a large role in discussions of IBL when the EAF first began funding IBL programs. Ultimately, though, the larger IBL community has chosen to pull away from this association with Moore, for a number of reasons: his approach forbade collaboration; instructors didn’t want the limitations that come with focusing on a single method developed by a single person; and Moore’s documented history of racism, sexism, and anti-Semitism was an impediment to the goals of inclusion and equity that the IBL community holds.

This community as it currently exists includes people who came to IBL from many origins, and is interested in nuanced conversations that value a variety of experiences and perspectives, with an emphasis on thinking about what we can do to make sure all students have positive mathematical experiences. If this sounds like a group you’d like to be a part of, then I hope you’ll come join the conversation.

How do I learn to teach this way?

You don’t have to radically change everything at once. You can start by mixing think-pair-share problems into your lecture, or spending one day a week with students working on problems in groups and presenting ideas, and lecturing other days. Any way you can find to get your students actively engaging with the material in class will be good for them.

Whether you make large- or small-scale changes, here are some additional things you can do.

Find a community. Taking a student-centered approach to teaching means relinquishing some control over what happens in class, and that can lead to a lot more variation in what difficulties you encounter. You’ll benefit from having a network of experienced instructors to go to for support and advice. You can find such a network by attending a workshop, or by becoming involved in the organizations listed below.

Find course notes. One of the hardest parts of teaching an IBL course is developing the materials that will lead students in the direction you want, scaffolded in such a way that students are challenged but ultimately able to succeed. *Journal of Inquiry-Based Learning in Mathematics* (`jiblm.org`) has peer-reviewed notes for many courses, and if you ask around you will find that IBL instructors are usually happy to share what they’ve tried.

Watch someone teach an IBL course. If you know someone teaching a course with IBL elements, ask to observe their class. Seeing IBL in action can give you a much better understanding of how it works.

Attend a workshop. There are a number of workshops, both stand-alone and attached to various conferences, intended to help instructors prepare to teach a course in an IBL style. See the organizations below for some options. Participating in these helps you get started and gives you a community of people to continue talking to.

Some IBL organizations:

This is by no means an exhaustive list, but gives you some places to start.

Academy of Inquiry-Based Learning: `www.inquirybasedlearning.org`

IBL SIGMAA: `sigmaa.maa.org/ibl`

Mathematics Learning by Inquiry: `www.mathlearningbyinquiry.org`

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Hanna Bennett

Credits

Author photo is courtesy of Hanna Bennett.

Lessons Learned Mentoring Senior Theses[5]

Background and Context

Supervising the senior thesis for an undergraduate student is an exciting and daunting opportunity often available to mathematics faculty in their early career. At some schools, a senior thesis is a requirement for graduation, and every faculty member is expected to mentor students in accomplishing this goal. I will use The College of Wooster as an example and give some advice on mentoring undergraduate senior theses to get you on your way.

The College of Wooster is a small private four-year residential liberal arts college located in rural Ohio. It enrolls approximately 2,000 students from 47 states and 45 different countries; 20 percent domestic students of color, 13 percent international students, and 55 percent students identifying as female. Each and every student is required, for graduation, to complete a two-semester capstone senior thesis that Wooster affectionately dubs *Independent Study* (IS). Students are encouraged to devote themselves to a sustained topic that they are passionate about; this experience allows them intellectual independence not necessarily available in other liberal arts curricula. Wooster touts this flagship centerpiece experience effectively as a class of 'one,' guided by a faculty mentor; IS takes the place of one of their four classes each semester of their final undergraduate year. Hence, IS positions Wooster as "America's Premier College for Mentored Undergraduate Research."

Every student develops a research idea (in their major or majors) to complete two semesters worth of in-depth examination. Students get individual support and guidance from a faculty member in their major. A student with a double major completes an integrated comprehensive project fulfilling both majors' requirements, advised by a faculty member in each major discipline. In the Department of Mathematics and Computer Science, students typically write 70-100 pages as a final thesis product. The process culminates with an oral defense during the second semester.

A continuing faculty member's course load is 5.5 courses per year; five IS students per year constitute a course in this system. Over the course of my 12 years at Wooster, I have advised 27 students; averaging 2-3 students per academic year. IS topics have ranged from *The 6,670,903,752,021,072,936,960 Reasons Why Sudoku is a Puzzle Worth Solving* and *Economic Benefits of Foot-Voting: Game Theoretic Approach*, to *Iteration Digraphs* and *Tighten Up: a Preliminary Study of Knots*. Some projects are publishable in undergraduate mathematics research journals, while others approach a new area in an innovative way for the individual student.

Nuts and Bolts

The best advice I was given when I started as a faculty member at Wooster was, "it's called *Independent* Study, not *Codependent* study!" It is difficult to advise thesis topics that are sometimes only tangentially related to your area of expertise. In getting started as an IS faculty advisor, this is a really experiential organic process—you are learning the mentoring 'system' as you go along. Faculty members are expected to meet their IS advisees one hour per week during the semester. Students expect to treat IS as a course, demanding the time equivalent of any other semester-long classroom course.

From a purely nuts and bolts perspective, I have learned, once your student has narrowed down a topic, facilitate your IS student in deciding on multiple small projects within the larger research project. Set small deadlines punctuated across the course of each semester. I do this with their narrowed down topic and a preliminary Annotated Bibliography due in the third week of the semester. Students are encouraged to add to this annotated bibliography throughout the project. An official project proposal—a one-page description of what their project will be about—is due the sixth week of the semester. In the twelfth week of the semester, students need to submit a detailed outline of their thesis (or a tentative table of contents and plan for the remainder of the project) and a comprehensive narrative, that could be a background exposition, a history chapter describing the general area of research, a literature review, or a later chapter in the planned thesis. In any case, it needs to be a substantial written portion of the thesis, at least 20 pages long. Keep in mind that grading for the first semester of IS is on a pass-fail basis.

In the second semester, an updated table of contents and a second chapter of writing are due in the second week of classes. The table of contents should closely reflect the planned-for final product. IS is always due the first Monday after the College's two-week spring break. As such, I require a final complete draft of the IS thesis one week prior to the start of spring break. I advise students that changes beyond this copy should only be editorial and a minimum of new material should be written after I read this final draft.

The final component of the project, during the last month of classes, is an hour-long oral defense (a 20-minute student presentation followed by a question-and-answer period from the advisor and a second reader) and public poster presentation. Evaluation of this project is on an Honors-Good-Satisfactory-No Credit (H-G-S-NC) basis. It is helpful to assess student gains across the whole two-semester project. Our IS rubric for Mathematics evaluates extent of material covered (investigation or application);

[5]*Jennifer R. Bowen is associate professor of mathematics and chair of the department of mathematics and computer science at The College of Wooster. Her email address is* JBowen@wooster.edu.

DOI: https://dx.doi.org/10.1090/noti1917

appropriate use of resources (prior literature and/or computational tools); writing quality; oral presentation; overall independence of learning; and student understanding and mastery of the subject—weighting each of these equally in the final project assessment.

Relax, Students are Human

Each spring, our department meets junior-level students preparing for their IS proposal (i.e. identifying a feasible topic for study). I always invite near-finished seniors to comment to the junior group about their best advice for IS. Across my time at Wooster, the best advice was, "If you never have any easy days, you'll never have any hard days." An easy quip, but it stays with me year after year.

Weekly meetings with IS students can be complex. It is helpful to vary your meeting location—your office, the dining hall, or the library. Throughout their final year, students commonly put off active job-seeking and future plans because they are "working on IS." I use preliminary meetings in the first semester, also looping back after fall break and winter break, to assign students to create an up-to-date resume for their dream job out of college, write the first draft of their personal statement for the Peace Corps, or research a compelling and plausible list of possible graduate programs.

Practice deep and active listening with your students. What other classes, projects, activities, or students are causing them distress? Often IS can be the great equalizer for students (and faculty): if you are at a program where each and every student has to complete IS to graduate, what also comes along with this is a common community and common peer pressure. Beware of signs of mental health stress throughout your student interactions and be able to connect them with appropriate institutional resources. Inevitably, sometimes weaker students shine in this process while strong students falter without a clear textbook syllabus. Senior students will constantly compare themselves to one another (independent of their major disciplines!). Careful assessment of a student's academic and community presence can prove helpful in gauging their progress in a two-semester senior thesis project.

Lessons

I offer a quick checklist as you prepare to mentor senior theses:

- Invite more experienced students to meet beginning students to give advice;
- Plan projects close to your area of expertise;
- Guide students in breaking their larger thesis topic into smaller topics;
- Devise a timeline that includes due dates for an annotated bibliography, project proposal, outline, etc.; and
- Circle back to address students' emotional wellbeing during the thesis experience.

Jennifer R. Bowen

Credits

Author photo is courtesy of Jennifer R. Bowen.

On-the-Job Teacher Training[6]

The training of teaching assistants is something most of us have taken part in during our early career. As graduate students we did this first as participants, and then perhaps as mentors. Later, as faculty members, we may help run or even design such programs in our departments. Teacher training activities in universities are not static, but evolve over time, and are influenced by similar programs in other departments or schools.

In this article, Doris Gluck and Herman Gluck talk about teacher training programs from kindergarten through college, and compare their methods. Besides the fact that these types of programs have common goals, some of the students we train for university-level teaching may later become involved in teacher education and training at the K–12 level.

Doris Gluck has been a math coordinator in several private schools in the Philadelphia area and has run workshops on elementary math teaching in local schools, as well as at state, regional, and national conferences for teachers. She is a winner of the Mathematical Association of America's Edyth May Sliffe Award for Distinguished Mathematics Teaching. Herman Gluck is a professor in the math department at the University of Pennsylvania and for the past 15 years has helped to organize and run the annual teacher training program there.

Who participates in your programs?

Herman: Our participants are mostly first- and second-year math grad students who will then serve as TAs. There are also advanced undergrads and grad students from other departments, and very recently, incoming math lecturers/postdocs, who are teaching for the first time since their graduate student days.

Besides myself, our trainers include another faculty member and five Master Teaching Assistants, experienced math graduate students selected for their outstanding performance as teachers and for personality traits which lead us to believe that they will be good mentors for our beginning teachers.

Doris: The teachers I work with, who have already been on the job for a number of years, usually teach all core subjects. They go into teaching for a variety of reasons, such as love of children or love of reading, but rarely for a love of math. Unlike Herman's group, I am usually the only trainer and organizer.

How are your training programs structured?

Herman: There are two parts to the teacher training program at Penn: a two-day total immersion just before classes begin and follow-up throughout the fall semester, involving classroom visits, filming of our teachers in action, and discussion afterwards.

The intensive two-day training period begins with demonstrations by our Master TAs of various ways to run a calculus recitation, and then continues with practice teaching in small groups. The training ends with a large group meeting where the Master TAs discuss many practical aspects of teaching, such as coordinating with the professor in charge of the course, keeping records, holding office hours, handling student concerns, and helping students in distress. For difficult issues, new teaching assistants are advised to "dump it on the professor."

The follow-up portion begins a couple of weeks later with visits by the Master TAs to the first-time teachers in their classrooms, and with discussions later that same day. Shortly thereafter, these teachers are filmed while teaching, and each then watches the film with one of the Master TAs. Seeing oneself in action is a powerful learning tool, especially after pausing many times for discussion.

Later in the semester, the math department holds a luncheon for all the program participants and many faculty, with wide-ranging unstructured discussion of various issues in the classroom. The faculty often jump-start this discussion by saying aloud what problems they are facing in their own teaching, and, with a little encouragement, the new teachers follow suit.

We often suggest to new teachers that they visit some of our most successful experienced teachers in their classrooms, with an eye to seeing what they do, and what may be worth trying to incorporate into their own teaching style.

Doris: The professional development I do depends on the needs of the school district or school at that particular time. It could focus on specific mathematics topics for all teachers, and is sometimes triggered by the adoption of a new curriculum.

As follow-up, I schedule individual or small group meetings with teachers monthly or more often. This is a time when we can discuss implementation of new programs or ideas, as well as problems that the teachers may be having in the classroom. We also discuss what's going to be happening in their math instruction the following few weeks. This can be as explicit as working through individual mathematics problems, or as general as discussing the overall goals of the next unit.

I aim to have teachers feel that they are in a risk-free environment when they are working with me, so that they are comfortable sharing mistakes and questions about

[6]*Doris Gluck is a former classroom teacher and math coordinator for elementary grades in the Radnor Township School District. Her email address is* dgluck@math.upenn.edu.

[6]*Herman Gluck, a professor at the University of Pennsylvania, works on geometry and topology with applications to molecular biology, hydrodynamics, and plasma physics. His email address is* gluck@math.upenn.edu.

DOI: https://dx.doi.org/10.1090/noti1918

content and delivery. At the same time, I am modeling the kind of environment I hope the teachers will carry over to their own classrooms.

How do you address teacher attitude and classroom atmosphere?

Herman: At the beginning of our training program, we say to the participants, "Have fun teaching and let it show! Genuinely care for your students, and let it show! If you don't yet have fun teaching, figure out how to. If you can't figure out how to, then fake it. The attitude you bring to the classroom and the atmosphere you create there is half the job of teaching."

And we continue by saying, "Be respectful of your students and their insecurities, and work very hard to create a risk-free environment for them. Do this by encouraging questions, and follow up by saying something positive about every question and every answer."

Doris: The math requirements for an Elementary Education Certificate are minimal in most universities. Many teachers feel unprepared and lack confidence in their ability to teach mathematics as a meaningful and thought-provoking subject.

I agree with everything Herman said, but rather than stating these things to my participants, I model them. I ask these teachers to do the activities and games as if they were a first grader or a fourth grader, and later we discuss how they felt, what they learned, and how they could adapt these activities for use in their own classrooms.

What are the anxieties and pressures felt by students and teachers?

Doris: The term "math anxiety" is used so often by math educators that it is easy to lose sight of just how awful it feels to those who have it: a kind of mental congestion which impedes thought and expression, and which involves tension, apprehension, nervousness, and worry.

Math anxiety can be thought of as a diffuse fear of math-related problems, impacting both children and adults, teachers, and parents.

When a person has math anxiety, working memory comes to a halt. Adults with math anxiety often transmit this fear and aversion to the children they work with. Children can develop math anxiety after having a bad experience, such as freezing when asked a question, or while taking a timed test in a math class. These experiences can lead to a more general fear of all mathematics.

No other subject is taught where every child is expected to learn the exact same thing in the exact same time. Reading is taught in small groups and children can often choose their own books; in writing, students write their own papers and no two students are expected to write in the exact same way. However, we often teach math to the entire class and expect every student to do the same problems and to memorize the same facts in the same time frame. Why wouldn't students who are struggling become anxious?

Herman: I can see two kinds of math anxiety among the students at Penn.

The first seems to me a continuation of what Doris described.

The second kind of anxiety hits strong math students when, for the first time, there is a creative, discovery-oriented component to the course. The students have to figure out why things are true, and back this up with well-reasoned arguments. Deeper thought is required. But there is a randomness to getting good ideas, and students who were best in their high school classes, who were the first to answer almost any question, may find themselves mute and trying to think, while other students are already answering the question. This is a humbling experience, which can lead to self-doubt.

I think we do not pay nearly enough attention to math anxiety at the college level. At Penn, we could do a better job in our teacher training program by candidly discussing math anxiety and ways to respond to it.

I am also aware of some of the pressures that our teachers, especially the beginning ones, feel. There is the pressure to do a good job in the classroom because you will be observed and judged by older faculty, who will later write a teaching recommendation for you when you go on the job market. There is the pressure to keep up with the pace of the curriculum, so that the students in your section are well equipped to deal with a common final exam. There is also the pressure to deal well with difficult students, but without the background experience to help you do this. With all these pressures, it's not so easy to relax and let your best self shine through, and create the kind of classroom atmosphere that puts students at ease and helps them to perform at their best.

Doris: Although some of the pressures felt by elementary school teachers, such as pacing of the curriculum or dealing with difficult students, may be similar to those mentioned by Herman, there are a number of other pressures we encounter.

Classroom teachers are answering to many other people: administrators, teachers at the next grade level or next school, and also parents. The expectations from each group are often very different, and can be mutually contradictory.

These are topics that should be brought up in training sessions, but often are not.

How do teachers and students share the responsibility for learning?

Doris: I sometimes think of the elementary school environment as a petri dish, and it is the responsibility of the classroom teacher to provide the nutrients for student growth. Students come to school curious about everything.

It is up to us, as teachers, to provide the activities and the problems that help them explore, discover, and construct their own understanding and learning.

When we present concepts and skills too slowly or too quickly, students lose that curiosity. If we go too slowly, they become bored and uninterested. If we go too quickly, they become confused and begin to lose confidence. Neither atmosphere provides for optimal intellectual growth.

It is the responsibility of teachers of young students to present material "just right": not too slowly and not too quickly. When this is done well, students become more confident and more independent. They are able to make appropriate choices to help themselves continue to learn.

This is made possible by teachers who understand the varying academic and developmental levels of their students, and have the flexibility to adapt the curriculum to meet their needs. Questions can be asked in such a way that students can answer correctly at many different levels of sophistication. This is sometimes called "differentiated teaching," and one way it can be achieved is by adapting lessons so that the students are given a variety of assignments with a range of degrees of difficulty.

Given that elementary math teachers, just as their students, are at many different academic and developmental levels, I must create an atmosphere in my workshops much as I would hope to create in a classroom of my own: risk-free, differentiated, and with choices.

Herman: As students get older, they naturally take on more and more responsibility for their own learning.

For math students at Penn, we get one view of how the division of responsibility evolves by simply counting hours that students spend on courses outside the classroom: perhaps 6 to 10 hours per week in freshman calculus, going up to 10 to 15 hours per week in more advanced classes.

As the mathematical level of the courses goes up, there is an increased emphasis on helping the students grow in mathematical power: the ability to successfully tackle problems of increasing complexity for which there may be no prescribed algorithm or strategy to lead to a solution. This mathematical power means the ability to purposefully hunt for and discover new mathematics, the ability to communicate this new understanding both in writing and in speech, and the ability to persevere through confusion and frustration to reach a solution. There is no possibility of going on in mathematics without these strengths.

For those math students who then do go on to graduate school, the division of responsibility for their learning changes once again. After a prescribed number of courses, they are on the hunt for a thesis advisor and a thesis problem, and while working on a thesis, even with the guidance of a caring and supportive advisor, they are on their own more than ever before. In some cases, this does not work out well, and so it becomes a time to reassess priorities and goals. But in the best of cases, this is the most exhilarating time of their lives, as they blossom forth to the attention of the world.

Conclusions

- For our youngest students, more time and concentration needs to be focused on problem-solving and critical thinking, less time on algorithms and computation.
- Attention to independent thinking in the early stages, including the development and nurturing of patience and perseverance, paves the way to creativity.
- Math anxiety and insecurity can continue from elementary school through college, and we must alert our beginning teachers to this, and encourage them to tell their students that these feelings are quite common, and can lessen with experience.
- Good teacher support should include regular contact between trainer and teacher.

Doris and Herman Gluck

Credits

Photo of authors is courtesy of Doris and Herman Gluck.

MEMORIAL TRIBUTE

Peter Swinnerton-Dyer (1927–2018)

Bryan Birch, John Coates, Jean-Louis Colliot-Thélène, and Alexei Skorobogatov

Peter Swinnerton-Dyer, whose work has greatly influenced the study of diophantine geometry in the twentieth century, died at his home near Cambridge on December 26, 2018, at the age of ninety-one. He attended Eton College, where the photograph at right was taken. There he became interested in diophantine equations from reading Heath's translation of *Diophantus of Alexandria*, and wrote his first paper [21] on the equation $x^4 + y^4 = z^4 + t^4$ while still at Eton.

Immediately after Eton, he went to Trinity College, Cambridge, and then, apart from a few years as a civil servant in London, he spent the rest of his life in Cambridge. In 1973 he became master of St. Catharine's College, Cambridge, and was vice-chancellor of the University of Cambridge from 1979 to 1981. In 1983, he took on the onerous task of distributing government funding to universities, first as chair of the University Grants Committee, and subsequently as chief executive of the Funding Council, during the years of the Thatcher government. He then happily returned to full-time mathematical work in Cambridge for the rest of his life. In the short article which follows, we have tried to remember both the mathematics and the personal qualities of this remarkable man, who has influenced profoundly all of our own mathematical

Figure 1. Peter at Eton around 1945.

Bryan Birch is an emeritus professor at Oxford University. His email address is birch433@btinternet.com.

John Coates is an emeritus professor at Cambridge University. His email address is jhc13@dpmms.cam.ac.uk.

Jean-Louis Colliot-Thélène is an emeritus director of research (CNRS) at the University of Paris-Sud, Orsay. His email address is jlct@math.u-psud.fr.

Alexei Skorobogatov is a professor at Imperial College London. His email address is a.skorobogatov@imperial.ac.uk.

Communicated by Notices *Associate Editor William McCallum.*

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DOI: https://doi.org/10.1090/noti1920

work and ideas, and from whose generous friendship we have all benefited so much. The reader will find other accounts of both his mathematics and his charismatic personality in a volume [16] dedicated to him on his seventy-fifth birthday.

Memories of Peter Swinnerton-Dyer, by Bryan Birch

Peter's first paper [21], written when he was a schoolboy at Eton, was about the arithmetic of diagonal quartic surfaces, as were several subsequent papers, for instance [18, 24, 30]; despite his being better known for the Birch–Swinnerton-Dyer Conjecture, it is fair to say that the arithmetic of algebraic surfaces was his lifetime mathematical interest. At first, he was a voice crying in the wilderness, but in the late 1960s Manin became interested in the subject, and for his last thirty years or so Peter was the senior member of a vigorous school on the subject, as Colliot-Thélène and Skorobogatov will describe later in this article.

Here, I will describe some of Peter's earlier work. In 1945 he went up to Trinity College Cambridge; after taking his BA he carried out research under the supervision of J. E. Littlewood. In 1950 he was awarded a Prize Fellowship at Trinity on the strength of his thesis on van der Pol's equation. Littlewood had worked on this equation during the war in collaboration with Mary Cartwright; Peter's thesis was not published at the time, but much later he published a group of papers in collaboration with Cartwright. During his Prize Fellowship Peter worked on various problems of number theory, which included a massive collaboration with E. S. Barnes [4] on the inhomogeneous minima of binary quadratic forms; in particular they determined which real quadratic fields are norm Euclidean. The most interesting paper from this period is [9], a joint paper with Ian Cassels in which they tried unsuccessfully to extend the inhomogeneous minimum results to indefinite ternary quadratic forms, and to products of three real linear forms. The mathematics is beautiful, but they were too far ahead of their time, and hit rock! The "natural" result they would have wished to prove for ternary quadratics was the Oppenheim conjecture, which was proved later by Margulis. As for products of three linear forms, their paper is quoted with approval by Lindenstrauss in his Fields Medal lecture, in the context of the Littlewood conjecture.

At the end of his Prize Fellowship, Peter spent the academic year 1954–5 with a Commonwealth Fund fellowship in Chicago; he went there intending to learn analysis with Zygmund, but was kidnapped by Weil, who converted him to algebraic geometry, particularly over the rationals. Weil's influence on Peter's mathematics was paramount; from that time on, Peter remained an arithmetic geometer, albeit with an unexpected affection for second order differential equations.

I first met Peter in the autumn of 1953, when he examined a second year prize essay I had written on the Theory of Games; he was very nice about it, though it was clear he would have preferred it to be somewhat shorter! While he was away in Chicago, I began research in the Geometry of Numbers under Ian Cassels, and when Peter returned as a teaching fellow I got to know him well; he taught me to love opera (I have happy memories of sitting on the floor listening to his recording of Callas singing Casta Diva), and we wrote a couple of (respectable but unremarkable) papers together. However, at the time, I was most interested by Davenport's analytic number theory. In turn, I went to Princeton with a Commonwealth Fund fellowship for the 1956–7 academic year, and while I was there I both wrote joint papers with Davenport by transatlantic mail, and also learnt a great deal of new mathematics. In particular I learnt of the beautiful reformulation of Siegel's work on quadratic forms in terms of a natural "Tamagawa measure" for linear algebraic groups. I seem to remember that Tamagawa gave a lecture, and Weil's comments made it exciting.

I returned to Cambridge, where Peter was now working in the computer laboratory, designing the operating system for TITAN, the machine planned to succeed EDSAC II. We wondered whether there was a similar phenomenon to the work of Siegel and Tamagawa for elliptic curves E over $\mathbb{Q}$. Specifically, was there a correlation between their local behaviour as described by their L-function $L(E,s)$, and their global behaviour, meaning their group $E(\mathbb{Q})$ of rational points? Our application for machine time was approved, with low priority—on certain nights we could use EDSAC from midnight until it broke down, which was typically after a couple of hours. The first task was to compute the rank of $E(\mathbb{Q})$ for a large number of elliptic curves. Cassels had provided the mathematics to do this via 2-descent, but at that time the computing problem was an awkward one because we needed to deal with several curves in parallel, and the amount of fast memory available on EDSAC was so very small. Peter was one of the very few people who could have managed it. The local behaviour presented more fundamental problems, because about the only thing we knew about $L(E,s)$ was that it converged when the real part of s was large enough. As John Coates describes later on, we were reduced to the naive expedient of computing the products $\prod(N_p/p)$. There was indeed a correlation, good enough to convince us but not enough for us to expect to convince anyone else! We needed to learn more mathematics! I think it was Davenport who told us that Hecke had dealt with $L(E,s)$ when E had complex multiplication. For the curves $y^2 = x^3 - Dx$, the critical value $L(E,1)$ is a finite sum of values of elliptic functions, easily machine computable if D is not too large, and computable by hand using some algebraic number theory when D is really small. Initially, Peter calculated about sixty critical values approximately, replacing the Weierstrass function by $1/u^2$ if I remember correctly. I plotted their logarithms against $\log D$, and found that the points lay on parallel lines of slope -1/4 about $\log D$ apart; thus the values $L(E_D, 1)$ were a constant multiple of $D^{-1/4}$, and a somewhat mysterious power of 2, which we managed to identify in terms of local factors and the Tate–Shafarevich group. Eureka! The critical value of the L-function really meant something!

From that point on, the investigation was a delight. As I have said, at the start we knew practically nothing about the analytic theory of $L(E, s)$, so we had to find everything out. There turned out to be an incredibly beautiful theory, with modular functions being just part of it. We had the joy of working in a fresh, partly new and partly forgotten, area of deep mathematics, which was so beautiful that it was certain to be important. No one else, except some close friends, had any idea of what we were finding, so we had no need to publish prematurely. Cassels described our conjectures in his lecture at the 1962 Stockholm ICM [8], and the papers [6] were published a little later.

The computer laboratory was a wonderful place to work: very informal, a trifle ramshackle (fire precautions were paramount, but the equipment was built without unnecessary frills), and enormously exciting. Everything seemed possible! I have particularly happy memories of those nighttime sessions. Gina Christ was the first computer assistant attached to the engineering laboratory. She shared an office with Peter and was qualified to turn the machine off, a complicated process, as one had to avoid electric surges. Thus she normally kept us company for the nighttime sessions, and these were of course an ideal environment for getting to know one another. We got married in the summer of 1961, and of course Peter was my best man.

In the autumn of 1962 I moved to a job in Manchester, and then in 1966 to Oxford, where I have remained since. I continued to see Peter often, but we no longer lived in the same city and it was long before the internet. Thus we read each other's papers, but collaborated much less.

Leaving our conjecture aside, Peter's most important achievement in the 1960s was the operating system for TITAN. He also wrote a dozen or more papers on a variety of other subjects, including the first counterexample to the local-global principle for cubic surfaces [22] and his paper with Atkin [3] where they published their conjecture concerning modular forms on non-congruence subgroups. I could say much more, but refer the reader to "In Lieu of Birthday Greetings" at the beginning of [16], where there is also a more comprehensive list of Peter's papers.

The Discovery of the Conjecture of Birch and Swinnerton-Dyer, by John Coates

The conjecture discovered jointly by Peter Swinnerton-Dyer and Bryan Birch in the early 1960s both surprised the mathematical world, and also forcefully reminded mathematicians that computations remained as important as ever in uncovering new mysteries in the ancient discipline of number theory. Although there has been some progress on their conjecture, it remains today largely unproven, and is unquestionably one of the central open problems of number theory. It also has a different flavour from most other number-theoretic conjectures in that it involves exact formulae, rather than inequalities or asymptotic questions.

As we will explain in a little more detail below, their conjecture grew out of a series of brilliant numerical experiments on the early EDSAC computers in Cambridge, whose aim was to uncover numerical evidence for the existence of some kind of analogue for elliptic curves of the beautiful exact formulae proven by Dirichlet for the class number of binary quadratic forms, and powerfully extended to all quadratic forms by Siegel. The work of Dirichlet and Siegel had been extended to linear algebraic groups

Figure 2. Bryan and Peter outside the Parish church in Solihull, July 1, 1961.

in general in the 1950s by Kneser, Tamagawa, Weil, and others. However, it was Birch and Swinnerton-Dyer alone who made the daring step of trying to find some analogue for elliptic curves. We now briefly recall their path-breaking computations, which were first published in [8] and [6]. An elliptic curve defined over the rational field $\mathbb{Q}$ is a non-singular projective curve of genus 1 defined over $\mathbb{Q}$ endowed with a given rational point $\mathcal{O}$. By the Riemann–Roch theorem, any such elliptic curve E will have a generalized cubic equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \qquad (1)$$

all of whose coefficients are rational integers, with $\mathcal{O}$ being the unique point at infinity on (1). Of course, such an equation is not unique, but we simply take any one having the property that the absolute value of its discriminant Δ is as small as possible amongst all such equations for E. Then the set $E(\mathbb{Q})$ of rational points on E has a natural structure of an abelian group, and the celebrated theorem of Mordell asserts that this abelian group is always finitely generated. For each prime number p, define N_p to be the integer such that $N_p - 1$ is the number of solutions of the equation (1) viewed as a congruence modulo p. In the autumn of 1958, Birch and Swinnerton-Dyer began computing the finite products

$$f_E(P) = \prod_{p \le P} N_p/p, \qquad (2)$$

where p runs over all primes $\le P$. They observed that the rate of increase of $f_E(P)$ as $P \to \infty$ seemed fairly closely related to the rank of $E(\mathbb{Q})$ as an abelian group, and were led to conjecture that $f_E(P)$ should be asymptotic as $P \to \infty$ to an expression of the form $C_E(\log P)^{g_E}$ for some strictly positive constant C_E; here, and in what follows, g_E will denote the rank of $E(\mathbb{Q})$. However, as they explain in [6], the value of $f_E(P)$ oscillates vigorously as P increases, and there seemed no hope of being able to guess a formula for the constant C_E from their numerical data. To overcome this difficulty, they quickly realized that they should instead work with the complex L-function $L(E,s)$ of E, which is defined by the Euler product

$$L(E,s) = \prod_{p|\Delta}(1 - t_p p^{-s})^{-1} \prod_{(p,\Delta)=1}(1 - t_p p^{-s} + p^{1-2s})^{-1},$$
$$\text{where } t_p = p + 1 - N_p. \qquad (3)$$

This Euler product converges in the half plane $\mathcal{R}(s) > 3/2$. Leaving aside all questions of convergence, one might expect that $L(E,1)$ should then be related formally to $f_E(\infty)^{-1}$. This led them to their first revolutionary conjecture, which assumes the analytic continuation of $L(E,s)$ to $s = 1$.

Conjecture 1 (Weak Birch–Swinnerton-Dyer conjecture). *$L(E,s)$ has a zero at $s = 1$ of exact order g_E.*

In these early computations described in [6], Birch and Swinnerton-Dyer worked with the family of curves

$$E_D : y^2 = x^3 - Dx \qquad (4)$$

where D is a non-zero integer that is not divisible by either 4 or the 4th power of an odd prime (more precisely, they considered all such D, for which the product of the odd primes dividing them is less than 108). Thus E_D is an elliptic curve with complex multiplication by the Gaussian integers $\mathbb{Z}[i]$, and Birch and Swinnerton-Dyer were using in [6] some nineteenth-century work, due originally to Eisenstein and Kronecker, proving the analytic continuation of $L(E_D,s)$ to the whole complex plane, and giving a closed formula for $L(E_D,1)$ in terms of values at points of finite order on E_D of Eisenstein series of weight 1. However, it must be stressed that this nineteenth-century work had in no way given the slightest hint of some possible connexion between $L(E_D,1)$ and the existence of non-trivial rational points on E_D. The first outcome of the EDSAC computations of Birch and Swinnerton-Dyer was to establish the apparent numerical validity of Conjecture 1 for the curve E_D for roughly 10^3 values of D, and this alone immediately convinced the world that they had uncovered something remarkable. However, at the same time, they took up the equally mysterious question of finding an exact arithmetic formula for $L(E,1)$ when this value is non-zero, probably thinking that it should somehow be in the spirit of Dirichlet's celebrated exact formula for the class number of an imaginary quadratic field K in terms of $L(\chi,1)$, with χ the non-trivial character of $\mathrm{Gal}(K/\mathbb{Q})$. Let $\omega_E = dx/(2y + a_1x + a_3)$ be the canonical holomorphic differential attached to the curve (1), and write Ω_E for the least positive real period of ω_E. When $L(E,1) \neq 0$, it seems they guessed that $L(E,1)/\Omega_E$ should be a rational number (an assertion that they proved for the curves E_D), which is closely related to the order of the mysterious and conjecturally finite Tate–Shafarevich group $\text{Ш}(E)$ of E. We recall that $\text{Ш}(E)$ is defined in terms of Galois cohomology by

$$\text{Ш}(E) = \mathrm{Ker}(H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), E(\overline{\mathbb{Q}}))$$
$$\to \prod_v H^1(\mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v), E(\overline{\mathbb{Q}_v})), \qquad (5)$$

where the bar denotes algebraic closure, v runs over all places of $\mathbb{Q}$, and $\mathbb{Q}_v$ is the completion of $\mathbb{Q}$ at v. They quickly found that the naive hope that, when $L(E,1) \neq 0$, one might have an arithmetic formula of the form $L(E,1)/\Omega_E = \#(\text{Ш}(E))/\#(E(\mathbb{Q}))^2$ failed for many of the curves E_D for the following reason. The one general result known about $\text{Ш}(E)$ at the time of their work, and indeed it is still the only general result known today, was the theorem of Cassels asserting that if Ш(E) is finite, then its order must be a perfect square. Their computations showed that while the exponent of each odd prime in the factorization of $L(E_D, 1)/\Omega_{E_D}$ was even for all D in the range considered, this failed to be true for the prime $p = 2$ and certain values of D. Prodded by Cassels, they then realized that their naive conjecture should be replaced by the following modified form arising from considering an analogue of the Tamagawa number of E:

Conjecture 2. *If* $L(E, 1) \neq 0$, *then*

$$L(E,1)/\Omega_E = \frac{\#(\text{Ш}(E))}{\#(E(\mathbb{Q}))^2} c_\infty(E) \prod_{p|\Delta} c_p(E).$$

Here $c_\infty(E)$ is the number of connected components of $E(\mathbb{R})$, and for p dividing the minimal discriminant Δ of E, $c_p(E)$ is the index $[E(\mathbb{Q}_p) : E_0(\mathbb{Q}_p)]$, where $E_0(\mathbb{Q}_p)$ is the subgroup of points in $E(\mathbb{Q}_p)$ with non-singular reduction modulo p. For the curves E_D, they explicitly determined the factors $c_p(E)$ at the primes dividing $2D$ in [6], showing that they were all powers of 2. Then, seemingly miraculously they found that, in the range of values of D they were considering, with $L(E_D, 1) \neq 0$, Conjecture 2 did indeed predict that the order of Ш(E_D) should be one of the values 1, 4, 9, 16, 25, 36, 49, or 81, and so always a square! In [6], they only explicitly discussed Conjecture 2 when $L(E, 1) \neq 0$, but it must have been known to them by this time that there was a fairly straightforward generalization of it to all elliptic curves E, involving the g_Eth derivative of $L(E, s)$ at $s = 1$, but with additionally the determinant of the canonical Néron-Tate height pairing on $E(\mathbb{Q})$ appearing on the numerator of the right-hand side. The conjunction of Conjecture 1, and this general version of Conjecture 2, is what is known today as the strong Birch–Swinnerton-Dyer conjecture. Today, this strong Birch–Swinnerton-Dyer conjecture has been tested numerically more extensively than any other conjecture in the history of number theory, with the possible exception of the Riemann Hypothesis. For the most systematic account of these computations, see the website www.lmfdb.org/EllipticCurve/Q, which gives numerical data on the conjecture for the 2,247,187 elliptic curves E with conductor $< 360,000$. The numerical results obtained have always been in perfect accord with the strong Birch–Swinnerton-Dyer conjecture, assuming that the mysterious square of an integer that arises in the calculations is indeed the order of the Tate–Shafarevich group.

The international echoes of their work after it became public in 1965 were enormous, starting with the celebrated Bourbaki lecture in Paris in 1966 by John Tate [37], discussing their conjecture for abelian varieties of arbitrary dimension over all global fields, and going a remarkably long way towards proving the geometric analogue of it. They themselves quickly realized that their conjecture also explained one of the ancient mysteries of number theory as to why every positive integer $N \equiv 5, 6, 7 \bmod 8$ should be the area of a right-angled triangle, all of whose sides have rational length (it is a simple classical exercise to prove that a positive integer N is the area of a right-angled triangle, all of whose sides have rational length if and only if the curve E_{N^2} has infinitely many rational points). Indeed, when $D = N^2$, Hecke's functional equation relating $L(E_D, s)$ and $L(E_D, 2 - s)$ shows that $L(E_D, s)$ has a zero at $s = 1$ of odd order precisely when $N \equiv 5, 6, 7 \bmod 8$. Unfortunately, it is still unknown how to prove the part of Conjecture 1 asserting that $L(E, 1) = 0$ implies that $g_E > 0$, and so the ancient question remains open at present. In yet another direction, Peter carried out the first systematic computations on whether or not all elliptic curves E over $\mathbb{Q}$ are modular in the following sense. On multiplying out the Euler product (3), we obtain a Dirichlet series $L(E, s) = \sum_{n=1}^{\infty} c_n n^{-s}$, whose coefficients c_n are rational integers. Let C_E be the conductor of E. We say that E is modular if, on writing $q = e^{2i\pi\tau}$, the function $f_E(\tau) = \sum_{n=1}^{\infty} c_n q^n$ is a classical primitive cusp form of weight 2 for the subgroup $\Gamma_0(C_E)$ of $SL_2(\mathbb{Z})$ consisting of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \equiv 0 \bmod C_E$. The question of whether E is modular was clearly very important for the Birch–Swinnerton-Dyer conjecture, since the work of Hecke shows, in particular, that $L(E, s)$ is an entire function when E is modular. In the late 1960s, Peter instigated the first systematic computations, described at the end of the volume [5], which listed all the modular elliptic curves $E/\mathbb{Q}$ with $C_E \leq 200$. It is now history that several great pieces of number theory emerged from these computations. Firstly, Birch realized the importance of the neglected idea of K. Heegner for constructing rational points on modular elliptic curves. Then beautiful theoretic work by Gross and Zagier on the one hand, and Kolyvagin on the other hand, led to the following best known theoretical result in the direction of the conjecture of Birch and Swinnerton-Dyer.

Theorem 3 (Kolyvagin–Gross-Zagier). *Assume that* $L(E, s)$ *has a zero at* $s = 1$ *of order at most 1. Then the order of this zero is equal to* g_E, *and* Ш(E) *is finite.*

Almost nothing is known about the conjecture of Birch and Swinnerton-Dyer when $L(E, s)$ has a zero at $s = 1$ of order strictly greater than 1. Unfortunately, we also still seem to be quite a long way from proving the exact formula for the order of Ш(E) predicted by the strong Birch–Swinnerton-Dyer conjecture when $L(E, s)$ has a zero of order at most 1, although the methods of Iwasawa theory have proven the p-part of this formula for many primes p. Note that there is now no assumption in the above theorem that E should be modular, because it is now history that Andrew Wiles [38], spurred along by mathematical ideas emerging from work on the Birch–Swinnerton-Dyer conjecture, found a marvellous proof that all E with square free conductor are indeed modular (this was then generalized in [7] to a proof that all elliptic curves over $\mathbb{Q}$ are modular). Moreover, it was shown by Ribet [17], prior to Wiles's work, that a proof that all elliptic curves over $\mathbb{Q}$ with square free conductor are modular would imply Fermat's celebrated conjecture asserting that, for any integer $n \geq 3$, the equation $x^n + y^n = z^n$ has no solution in integers x, y, z with $xyz \neq 0$. Curiously, Fermat had been led to this conjecture when he noted in the seventeenth century that his argument of infinite descent on the curve E_D for $D = 1$ implies his conjecture for $n = 4$, hinting at an almost spiritual connexion with the discoveries, made centuries later by Birch and Swinnerton-Dyer on the same family of curves!

Throughout all of the mathematical developments arising from his conjecture with Birch, Peter remained extremely level-headed, and carried on quietly his own mathematical work in both arithmetic and algebraic geometry, often using his great computational skills. He was also willing to take up heavy burdens for what he felt would be the general good of the University community. To graduate students in pure mathematics, Peter was always extremely generous and kind. On most evenings in the 1960s, he provided pre-dinner drinks for all comers in his beautiful rooms in Trinity College, and it was there that many of us first met distinguished international mathematical visitors, who were drawn to Cambridge by the work of Peter, and Ian Cassels. He continued to regularly attend the Departmental Number Theory Seminar until the end of his life.

Rational Points on Higher Dimensional Varieties, by Jean-Louis Colliot-Thélène

Peter Swinnerton-Dyer's mathematical work is many-sided. He would not follow fashion, but would take up a classical subject and introduce new ideas, which he often shared with his co-authors, leaving it to others to develop them into systematic theories. When working on a proof he would not refrain from applying brute force and would often embark on lengthy computations [23, 32].

The general mathematical public is well aware of the Birch and Swinnerton-Dyer conjecture. Peter made many other contributions to arithmetic geometry. His two most cited papers are his joint paper with Atkin [2], prompted by computations of Dyson, and [25], related to work of Deligne and Serre. Both are concerned with congruences for coefficients of modular forms, suggested by work of Ramanujan. Among his other papers that continue to generate research today, let me mention his work with Ian Cassels [9] on the geometry of numbers, his work with M. Artin [1] on the Tate conjecture for a class of $K3$ surfaces over a finite field, his work with B. Mazur [15] on the arithmetic of Weil curves and p-adic L-functions, his work on lattice points on a convex curve [26], and his recent work on the effect of twisting on the 2-Selmer groups [34]. Peter also made a number of contributions, in particular [20] and [33], to Manin's conjecture (1990) on counting points of bounded height. Here is a typical quote from [33]:

"This paper describes the mixture of ideas and computation which has led me to formulate more precise conjectures related to this problem. The process of refining (the initial guess) is iterative. One first formulates a more detailed conjecture. This then suggests computations which will provide evidence about the plausibility or otherwise of that more detailed conjecture; and if the evidence is confirmatory, it may suggest a further refinement of the conjecture. This process is of course only available to those who think that a conjecture should be supported by evidence."

Peter had a lifelong interest in rational points on some higher dimensional projective varieties over number fields: cubic surfaces and hypersurfaces, intersections of two quadrics of dimension at least 2, and also quartic surfaces. From the geometric point of view, the first two types of varieties are rationally connected varieties, whereas quartic surfaces are $K3$ surfaces.

Assuming that such a projective variety X over a field k is smooth, the first question is whether the set $X(k)$ of

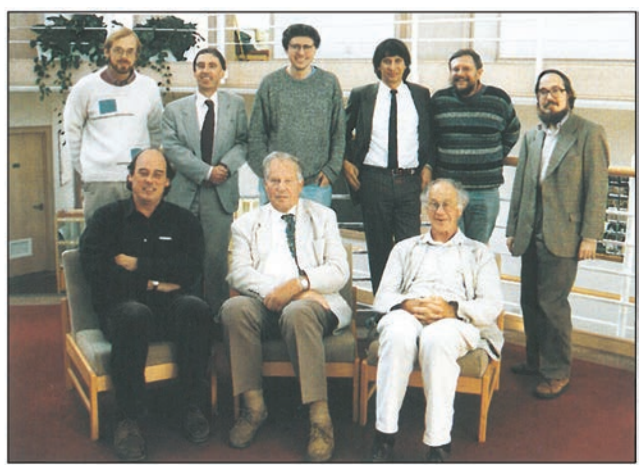


Figure 3. Diophantine Geometry and Differential Equations, a meeting in honour of Prof. Sir Peter Swinnerton-Dyer's seventieth birthday, Newton Institute, Cambridge (UK), 22nd–23rd September 1997. Standing, left to right: Richard Taylor, Noel Lloyd, Jan Nekovář, Jean-Louis Colliot-Thélène, Miles Reid, Don Zagier. Sitting, left to right: Colin Sparrow, Peter Swinnerton-Dyer, Bryan Birch.

rational points is dense in X for the Zariski topology. The next question is whether a class of varieties to which X belongs satisfies the Hasse principle: if the set $X(\mathbb{A}_k) = \prod_v X(k_v)$ of adèles of X is not empty, is there a rational point on X? A stronger question is if $X(k)$ is dense in the topological set $X(\mathbb{A}_k)$. In this case one says that weak approximation holds. In 1962 Peter found the first counterexamples to these properties for cubic surfaces.

Starting with results of Minkowski and Hasse, these questions were thoroughly investigated for a very special class of rationally connected varieties, namely (compactifications of) homogeneous spaces of connected linear algebraic groups. Counterexamples to both the Hasse principle and weak approximation were constructed. These questions also come up in the study of curves of genus one, where Tate–Shafarevich groups, which are conjecturally finite, measure the failure of the Hasse principle.

In 1970 Manin suggested a general framework to explain many known failures of the Hasse principle, including the examples produced by Peter in 1962. Calling in the Brauer–Grothendieck group, he noticed that the closure of the set $X(k)$ of rational points is included in the Brauer–Manin set $X(\mathbb{A}_k)^{\mathrm{Br}}$ consisting of adèles orthogonal to the Brauer group of X. When the closure $X(k)^{cl}$ coincides with $X(\mathbb{A}_k)^{\mathrm{Br}}$, we say that the Brauer–Manin property holds.

At the same time, both Swinnerton-Dyer and Manin drew attention to work of F. Châtelet (1958) on some special cubic surfaces, where one can apply a factorization process somewhat analogous to descent on elliptic curves. Peter asked how general this process was, and whether it could be iterated. I had the good fortune of spending the year 1969–1970 in Cambridge, with Peter Swinnerton-Dyer, then dean of Trinity College, as a mentor. He had just written his survey "Applications of algebraic geometry to number theory"—I still have the manuscript, in his beautiful handwriting. He suggested that I work on Châtelet's surfaces.

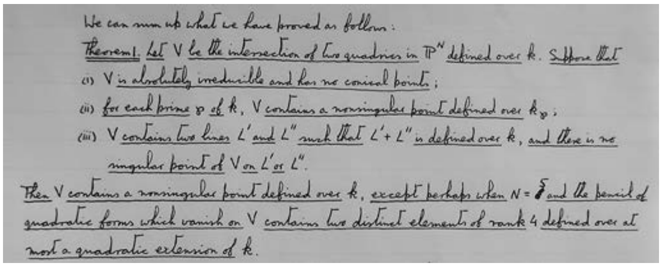
This would ultimately lead to at times intense exchanges of letters (in particular during Peter's time at the University Grants Committee and at the University Funding Council) and to a series of joint works (also with others)—the first one in 1984, the last one in 2012.

As a first answer to Peter's questions from 1970, a formal framework for this descent process (based on torsors under tori) was developed by Jean-Jacques Sansuc and me in the 70s. Our approach also clarified the connection with the Brauer–Manin set.

Starting around 1982, work of Peter, myself, Sansuc, Skorobogatov, and many younger authors, by now too many to be listed here, has resulted in a series of precise conjectures on rational points on rationally connected varieties over a number field. There are some unconditional theorems and some conditional theorems that tell us what to expect. There is also a further series of unconditional theorems in a different direction, where one asks for existence and density (in a suitable sense) of zero-cycles of degree one. In this direction the initial breakthrough is due to P. Salberger (1988). I shall restrict myself to a description of some results Peter was involved in.

- Weak approximation holds for smooth intersections of two quadrics with a rational point in projective space $\mathbb{P}^n$ for $n \geq 5$ [10].
- The Hasse principle holds for smooth intersections of two quadrics in $\mathbb{P}^n$ for $n \geq 8$ [10].
- The Brauer–Manin property holds for generalized Châtelet surfaces, which are given by an affine equation $y^2 - az^2 = P(x)$, where P is a polynomial of degree 3 or 4. In the special case when $P(x)$ is irreducible, the Hasse principle and weak approximation hold [10].
- The Brauer–Manin property holds for the total space of a pencil of Severi–Brauer varieties over $\mathbb{P}^1$ conditionally on Schinzel's hypothesis. (This is a common generalization of Dirichlet's theorem on primes in an arithmetic progression and of the twin prime conjecture; various versions of this hypothesis were discussed by Bouniakowsky, Dickson, Hardy and Littlewood, Schinzel, Bateman and Horn.) The unconditional proof of an analogous statement for zero-cycles instead of rational points [11, 13].
- Density of rational points on certain surfaces with a pencil of curves of genus one, including some diagonal quartic surfaces, conditionally on the combination of Schinzel's hypothesis and the conjectured finiteness of Tate–Shafarevich groups [14].
- The Hasse principle holds for diagonal cubic hypersurfaces in projective space $\mathbb{P}^n$ over the field of rational numbers for $n \geq 4$, conditionally only on the finiteness of the Tate–Shafarevich groups [19, 31].
- Various (unconditional) counterexamples: to the Hasse principle and weak approximation for cubic surfaces [22]; to an early conjecture on a geometric characterization of varieties on which rational points are potentially dense [12]; to an early conjecture on the structure of the closure of the set of rational points in the set of real points in a variety over $\mathbb{Q}$ where the rational points are dense for the Zariski topology [12].

Let us say a few words about the techniques involved. The long paper [10] builds upon a combination of the


We can sum up what we have proved as follows:

Theorem 1. Let V be the intersection of two quadrics in $\mathbb{P}^N$ defined over k. Suppose that

(i) V is absolutely irreducible and has no conical points;

(ii) for each prime $\mathfrak{p}$ of k, V contains a nonsingular point defined over $k_{\mathfrak{p}}$;

(iii) V contains two lines L' and L'' such that $L'+L''$ is defined over k, and there is no singular point of V on L' or L''.

Then V contains a nonsingular point defined over k, except perhaps when $N = 5$ and the pencil of quadratic forms which vanish on V contains two distinct elements of rank 4 defined over at most a quadratic extension of k.

Figure 4. Excerpt from preliminary version of [10] in Peter's beautiful handwriting, around 1984–85.

descent method and the fibration method. The paper solves the questions on Châtelet surfaces raised in 1970. It came out of a combination of the descent formalism mentioned above and a fibration method initiated by Peter in 1982. To make a long story short, if one starts with a smooth projective surface X over a number field k with affine equation $y^2 - az^2 = P(x)$ with $P(x)$ of degree 4, and with the property $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, the descent process on X produces a variety Y that projects onto X, is a (singular) intersection of two quadrics in a higher dimensional projective space, contains a pair of skew conjugate linear spaces, and satisfies $Y(\mathbb{A}_k) \neq \emptyset$. The rough idea now is to intersect Y with a suitable linear space so that the intersection Z is an intersection of two quadrics in $\mathbb{P}^4$ with a pair of skew conjugate lines, still satisfying $Z(\mathbb{A}_k) \neq \emptyset$. Such surfaces are known to satisfy the Hasse principle. One concludes $Y(k) \neq \emptyset$ hence $X(k) \neq \emptyset$. The details are delicate.

Hasse's proof of his principle for zeros of quadratic forms in four variables has four ingredients: Hensel's lemma, Dirichlet's theorem on primes in an arithmetic progression, the law of quadratic reciprocity, and the Hasse principle for quadratic forms in three variables. Replacing Dirichlet's theorem by Schinzel's hypothesis gives the Hasse principle for surfaces with equation $y^2 - az^2 = P(x)$, where $P(x)$ is irreducible of arbitrary degree (1979). The argument was extended to pencils of conics, quadrics, and Severi–Brauer varieties [11]. In [13], we pushed the idea further and gave precise arithmetic and geometric conditions on a fibration $X \to \mathbb{P}^1$ for the method to work. In these papers we also extracted the essence of Salberger's device (1988), which enabled us to transform a conditional proof for rational points as described above into an unconditional proof of a similar statement for zero-cycles of degree one.

These results concern families $X \to \mathbb{P}^1$ whose fibres satisfy the Hasse principle. A spectacular idea of Peter's [28], developed in [14], was a sophisticated version of these arguments applicable to some surfaces with a pencil of curves of genus one, which hitherto were not thought to be natural candidates for the Hasse principle. This will be detailed in Alexei Skorobogatov's contribution.

Sir Peter Swinnerton-Dyer, Mathematician and Friend, by Alexei Skorobogatov

I met Peter Swinnerton-Dyer in February 1989 on my first visit to the West. The Soviet Union had barely opened to the world, and nobody was sure how long this openness would last. To seize the opportunity, I managed to arrange for a private invitation, obtained a visa, and bought a train ticket from Moscow to London via East and West Berlin. I came to meet Peter in his grand office near Regent's Park. His status among young Russian mathematicians was that of a celebrity, not in the least due to his nobility, which added another twist to his fame. Peter was positively charming with his pleasant and benevolent manners, which—for a Russian—were striking if one bore in mind his elevated position in the British government. I knew that at the time he was the head of the University Funding Council (formerly the University Grants Committee), but I could not imagine the scale of the controversy related to his role in reforming British universities. We discussed mathematics of course. It was clear where Peter's heart was, so I was not too surprised that once his job in the government was over, he resumed his mathematical work at full blast.

The next time I saw him and his wife Harriet was at a soirée chez Jean-Louis Colliot-Thélène in the Parisian suburb of Massy, when Yuri and Xenia Manin were also present. The conversation was flowing a little less easily than the wine. Later, in a deliberate snub to the French and Russian schools of arithmetic geometry, largely centred on the legacy of Grothendieck, Peter insisted that he did not know what cohomology was, and was familiar with only the pre-1950 mathematics. The timing is important: 1954 was the year of his discipleship with André Weil in Chicago, and it is exciting to speculate if it led Peter and Bryan Birch to their famous conjecture. Peter's attitude to conjectures was also old-fashioned: he insisted that a conjecture should be made only when there was solid computational or theoretical evidence for it. He was not too sure if there was enough evidence for the influential Batyrev–Manin conjecture on rational points of bounded height. I think that on this occasion he said something like "Russian has no word for evidence." He himself also worked on this conjecture. Paper [27] contains this sentence: "$Z(s)$ can be analytically continued to the entire s-plane, but only as a meromorphic function with poles in somewhat unexpected places; and though it does satisfy a functional equation of a kind, it is not one which a respectable number theorist would wish to have anything to do with."

Not at all at ease with cohomology or the Brauer–Grothendieck group, Peter tried hard to make the Brauer–Manin obstruction to rational points explicit and amenable to calculation. In [28], one of the first papers written after he left the University Funding Council, Peter came up with a new method to prove the Hasse principle for rational points on a surface represented as a pencil of curves of genus one parameterised by the projective line. His technique was involved; some of the calculations passed through an explicit proof of the Tate duality for an elliptic curve over a local field, which he rediscovered. There were incredibly involved numerical computations with explicit choices of bases of vector spaces and lots of exotic conditions. Jean-Louis and I spent a couple of years trying to make sense out of this. One night at IHES, in a moment of illumination, I understood how this convoluted number theory could be reduced to linear algebra. This led the three of us to formulate a method which, under appropriate assumptions and conditionally on Schinzel's Hypothesis (H) and the Tate–Shafarevich conjecture on the finiteness of the Tate–Shafarevich groups of elliptic curves, proved that an everywhere locally solvable surface with a pencil of curves of genus one has a rational point [14]. The key idea, entirely due to Peter, is simple: find a rational point on the base such that the fibre is an everywhere locally solvable curve of genus one, and such that a suitable Selmer group of its Jacobian is so small that, unless the fibre has a rational point, it is incompatible with the fact that the order of the (conjecturally finite) Tate–Shafarevich group should be a square. It is indeed a theorem of Cassels that the Cassels–Tate pairing on the quotient of the Tate–Shafarevich group of an elliptic curve by its divisible subgroup is non-degenerate and alternating. Our paper was a first hint that it may be reasonable to expect that (at least some) K3 surfaces satisfy the Hasse principle when it is not obstructed by the Brauer group.

Peter used this method to prove that large families of diagonal cubic surfaces satisfy the Hasse principle [31]. The assumptions of his elegant theorem are easy to state; the result is conditional only on the finiteness of the Tate–Shafarevich group of elliptic curves, but not on Hypothesis (H). A similar result was later proved in our joint paper on Kummer surfaces, a particular kind of a K3 surface [19]. Much later, Yonatan Harpaz realised that this method can be simplified if one borrows some ideas from papers of Mazur and Rubin; as a result, the heavy linear algebra machinery of [14] was replaced by more natural arguments. This happened often with Peter's innovations: extremely complicated computations were either dramatically simplified or completely eliminated, but the main idea continued to shine. In fact, to this day this remains the only known approach to the local-to-global principle for rational points on families of curves of genus one!

One of the most influential papers of Peter written around his eightieth birthday is his work [34]. Like many of his papers, it uses the aforementioned linear algebra machinery of descent. A striking feature of this paper is that in it Peter used the main theorem of Markov chains to obtain an asymptotic distribution of the 2-Selmer rank in a universal family of quadratic twists of an elliptic curve. This was never done before, but turned out to be a very useful tool. Peter was quite happy with this unorthodox invention. "I am a computer scientist at heart," he commented.

Peter was fearless in his choice of problems but was never ashamed to produce an extremely convoluted solution if necessary. He did not care about the level of the journal. Once he had a new idea how one could prove the local-to-global principle for rational points on conic bundles with any number of degenerate fibres. It did not quite work and he was only able to do the case of six degenerate fibres (still unsurpassed). The result was the paper [29]; when I asked him about it, he remarked with melancholy that "this paper was not supposed to be read."

In line with the classical tradition of number theory, Peter was interested in rational points on cubic and quartic hypersurfaces, on intersections of two quadrics, on pencils of quadrics, and so on. In particular, the arithmetic of diagonal quartic surfaces is a recurrent theme in his work, from his first paper [21] (published under the pseudonym of P. S. Dyer while still at Eton), revisited in [18, 24, 30, 35], until his last paper [36]—published seventy-three years later! "My first paper was on diagonal quartics and my last paper will be on diagonal quartics," said Peter. The prophecy has been fulfilled.

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Credits

Figure 1 is courtesy of Harriet Crawford.
Figure 2 is courtesy of Bryan Birch.
Figure 3 is courtesy of Isaac Newton Institute for Mathematical Sciences.
Figure 4 is courtesy of Jean-Louis Colliot-Thélène.

Excerpt from *The Grothendieck I Knew: Telling, Not Hiding, Not Judging*

By Paulo Ribenboim

ABSTRACT. In this paper, I write about my lifelong friendship with Grothendieck.

Le tout est composé de ses parties, même des plus petites.

Introduction

I had the privilege of being a lifelong friend—in fact the only one—of Grothendieck, whom I refer to by his nickname Schurik. As a close friend, I had the opportunity of knowing him as friends do, exchanging confidences, and leaving mathematics largely aside. This paper is a recollection of what came spontaneously to my mind.

Every episode is in truth what I witnessed—unless I indicate the contrary. No judgment is made.

The variety of Schurik's attitudes in different situations will leave readers perplexed—as it did me. I do not hide what I observed. But what I write would be described in another way by another person. In the Japanese movie *Rashomon*, different witnesses of a crime describe the event in as many different ways.

The reader will probably find incongruences with respect to the chronology and a variety of other incorrect references. I will be grateful to anyone who wishes to point out my errors. Despite it all, I hope to have achieved a truthful text that illustrates the unclassifiable person Grothendieck was. A genius, yes, and more?

Paulo Ribenboim is a Professor Emeritus of Mathematics at Queen's University. His email address is pauloribenboim13@gmail.com.

DOI: https://dx.doi.org/10.1090/noti1909

Before Day 1

To understand my relationship with Grothendieck (called Schurik by his close friends) it is appropriate to tell a little bit about myself. I was born on March 13, 1928—Schurik was born fifteen days later. I lived in Rio de Janeiro, received my BSc in mathematics in 1948 at age twenty, and became immediately an assistant researcher at the new Brazilian Center for Research in Physics (CBPF).

Early in 1950, my former physics professor José Leite Lopes stopped me in a hallway of the CBPF and, without preamble, said, "Paulo, I have a fellowship from the French government for you to study in France." Such fellowships were created to reestablish the traditional French cultural and scientific exchanges with Brazil. Startled by the surprising offer, I answered, "I will ask my mother if she would allow me to travel so far away." Old times!

She readily accepted since my first cousin Leopoldo Nachbin, six years older, who was becoming a recognized mathematician, was already in the USA. Mom obviously had a certain rivalry with her sister.

Already in 1949, I was studying the notes of the lectures by Jean Dieudonné at the Universidade de São Paulo (USP), translated into Portuguese by the young mathematician Luiz Henrique Jacy Monteiro. They concerned the theory of commutative fields, including Galois theory and a large part on valuation theory. I was taken by the subject, appreciating the clarity, rigor, and ample breadth of the lectures. This study awakened in me a great appetite to study, since it made me aware of my great ignorance. I had chosen to work with Dieudonné, who readily agreed to be my mentor and had always shown a favourable posture towards Brazilian mathematicians.

In Nancy, where Dieudonné lived, I found a mansarde room (these are the cheap attic rooms, usually rented to students) in a hotel that, unbeknown to me, was on the same street as Dieudonné's apartment and just in front of the one of Laurent Schwartz. With his natural kindness, Schwartz invited me with regularity to have afternoon tea, prepared by his wife Marie Hélène. They definitely both liked me—and this will come up soon.

My arrival in Nancy was in early April, which would be in Brazil the start of a new academic year. But in France, I had to wait until October. What I will tell now was also important in my friendship with Schurik. My mother came from the old Austria and thus gave much importance to education in classical music. For this reason, already as a child of five, I began studying piano with a Polish lady who, of course, taught me the easier pieces. After her I had a Brazilian teacher, Dona Lubelia de Souza Brandão, who introduced me to Bach and made me play Beethoven, Mozart, Debussy, etc. All this would come up in conversations with Schurik. And the Bach Festival in Strasbourg in June 1950 was one more topic for us.

As I gradually became more fluent in French, I was able to read the classics in literature, psychology, history, and philosophy. Please do not infer that I am trying to say good things about myself. I was just a young man, age twenty-two, with a variety of interests, who could communicate with a new friend. A friend of unexpected and surprising noble genius who showed a great receptivity to what I said. Soon we would become the greatest friends.

Enough from Paulo, and now we come to...

Day 1

As was becoming customary, Marie Hélène invited me for afternoon tea. That day, Schwartz came to me and said, "Ribenboim, I have observed that you are a well-rounded young man with a pleasant manner. Today you'll meet one of my students. He has such an intense interest in mathematics that in the long run it might overcome him. Maybe you will become friends and together you'll enjoy the other pleasures of life."

Soon after, the heavy metal door was opened and a young man entered the front garden and put his bicycle against a wall. He was dressed in Bermuda shorts and had dark abundant hair. After entering the residence, Schwartz said, "Here is my student Grothendieck, and here is Ribenboim, who comes from Rio de Janeiro."

From the first moment it was clear that we were pleased to have made each other's acquaintance and we anticipated the development of a friendship. I was as curious to meet a French student as he was to meet one from Brazil.

As we left Schwartz's home, I invited him to visit my room, a normal custom in Brazil. As it happened, my hotel was just across from Schwartz's residence. My room was in the mansarde, on the top floor of the building. Formerly it was the maid's room; then it became a room for students to rent. It had minimal furniture, in particular no bookshelves. My books were stacked in a rather high pile on the floor. Schurik asked about all these books, and I explained, "After the war ended, Brazil started importing books. I would buy any book which I thought might be interesting for my further studies." Schurik heard this, looked at the high pile, and said, prophetically, "Paulo, you will never read these books." Much later, I learned that he rarely read math books. Over the years, it became clear that Schurik did not learn mathematics, but instead created his own.

The Days and Weeks after Day 1

We met very often, walking around the city and talking about a variety of subjects, but never about mathematics. It was the beginning of summer, and there were no classes.

Literature and cinema

Schurik told me of his love for the new school of Italian movies. He was very impressed by the movie *The Bicycle Thief* by Vittorio De Sica. I told him that I was taking private lessons in French from a doctoral student in French literature. I wanted to read classical authors and was directed to read Kafka. The plots of *The Trial, The Castle,* and *Amerika* predicted what would be the ordinary human condition confronted by the power of dictatorship, bureaucracy, and money. The short story "The Metamorphosis" leaves an indescribable feeling of the impotence of man, which must leave the reader forever deeply impressed. In one of his books there is the satirical piece "A Report to an Academy," which I could not forget when this happened to me (a Brazilian monkey).[1] Schurik and I discussed Kafka, but I don't know if he ever read any of those books.

Music

Another topic of our conversations was classical music. I had attended the Bach Festival in Strasbourg (1950). On my return, I told Schurik much about the concerts and artists who performed there. As a result he would go with me to the Salle Poirel to listen to recitals and orchestras. Schurik was enthusiastic, often the last person to stop applauding. All this was very new for him, and I was happy to see his reaction.

Having learned that I played the piano, he decided to learn it too. I said, "You need to have a piano and to find a teacher." To which he answered, "I will not need a teacher, but I need a piano which I will not buy, but just rent." He stopped and then said, "Nobody will rent a piano to me. Look how I am dressed. You will be able to rent the piano for me."

I did it. This was the origin of many changes of address. Schurik studied math the whole day and when he was tired, he started the piano, often at midnight. How many times did he have to move? It was not important, since I had never visited him in any of his rooms.

[1] *Editor's note: In "A Report to an Academy," an ape named Red Peter presents to an academy the story of how he learned to behave like a human.*

The mother

I am sure to be the only friend of Schurik who had the privilege of visiting his mother. She was resting in bed but quite alert, since her tuberculosis was in remission. Obviously she knew that I was a close friend of her son. Amongst the facts I retained from our conversation was that she was writing, in German, a novel inspired by her own experiences. This is what I learned.

Schurik was born in Berlin in 1928, being just fifteen days younger than I. His father was Jewish, but his mother was not. I don't know her religion or if she was an atheist. In those years in Germany, the communists, nationalists, of the extreme left, or right, fought each other. And there were also the anarchists—it was a time of many ideologies.

With the advent of the Nazis and the difficulties that followed, Schurik's mother and the little boy went, eventually, to France and settled in the region of Montpellier. There, Schurick had all his education, receiving his university undergraduate degree at the Université de Montpellier.

Schurik had a tender love for his mother, as he fully understood the difficulties she must have had to escape to France. At the time I met Schurik he was an "apatride," which means that he had no nationality.

This is a good moment to tell why Schurik went to Nancy to work with Schwartz. While still an undergraduate, Schurik created a theory. For this reason he was advised to go to Paris and to show his work to Henri Cartan. It turned out to be a variant of the Lebesgue integration theory.

Seeing the potential of the young man, Cartan allowed him to attend a seminar. The material was too sophisticated for Schurik, so Cartan arranged for his friend Schwartz to guide the young Schurik.

The bicycle ride

It was a warm summer day, one when nobody would like to be studying mathematics. So, it was not a surprise when Schurik said, "Paulo, let us have a bicycle ride to Pont-à-Mousson. It is only 25 km from Nancy." I replied, "I would like to go, but I don't have a bicycle." "No problem," said Schurik. And he continued, "Monsieur Rudlof, who is the 'appariteur' of the building where the courses and seminars take place, will let you use his bike." And so it was. *L'appariteur* had his lodging in the building and his job was a compensation for the heavy injuries he sustained in the war. For this reason, his bicycle was old, heavy, and in need of lubrication.

Having secured a bicycle, the excursion was set for the next Saturday. It was apparent that Schurik was far stronger than I, who never rode long distances. And his bicycle was incomparably better than mine. The road had many hills to climb. It was easy for Schurik, who always waited for me on the top of the hills with a glorious smile. Eventually, we reached our aim, and our prize was to drink an ice-cold beer.

With my Agfa camera, the old black-and-white way, I took the picture below.

Anyone who met Schurik as he grew old will be struck by his appearance at age twenty-two: a vigorous young man, with hair that would start falling out prematurely.

Alexander "Schurik" Grothendieck at age twenty-two, June 1950

A mathematical lesson

One day Schurik gave me a mathematical lesson, something that had never happened between us and would never happen again! I can only speculate about the reason for this lesson. To mitigate my obvious ignorance? To show me a beautiful theorem? To practice his pedagogical skills on me? I guess that the reason was a mixture of these.

Schurik, now disguised as Professor Grothendieck, stated and proved the Stone theorem on representations of Banach algebras. I state here unequivocally that Professor Grothendieck, when he wanted, could motivate the statement and the proof, rendering the lesson a model of clarity. Bravo!! But why was this not the case with most of his writings, carefully made opaque?

Many years later, exactly in 1958, I presented my theorem on the representations of lattice ordered abelian groups, using P. Jaffard's theory of "filets." This theorem was related to what I learned from Schurik about representations of Banach algebras.

A New Semester Begins

As classes began (finally!) I made new friends, like Georges Glaeser, and attended various classes.

Schwartz had just received the Fields Medal and was lecturing on the theory of distributions, which he had invented, an alternative to a theory by Sobolev. Of course I attended his lectures. Schwartz was, as expected, unassuming. His lectures were as transparent as required by the explanation of the new concepts. Schurik was always present, and at the end of the presentation he would invariably discuss with Schwartz—on equal terms, as I observed. Not only could Schurik anticipate what was to come, but he had probably a more penetrating view of the subject.

There were also the lectures by Jean Delsarte on Lie groups and Lie algebras, following the recent book by Pontryagin. For me it was a time to learn, and the summer walks with Schurik were temporarily interrupted. As classes stopped in December, I went on vacation to Combloux in the Alps to fulfill the Brazilian dream: to see, to marvel, to drink the white powder. Just in front of Mont Blanc. At that time I did not fare well in my efforts to learn how to ski and ended up with a twisted ankle.

Having much later moved to Canada, I could not avoid hating how much snow there was.

The Year 1951

Huguette

Another person became the number three in my friendship with Schurik. Let me tell the story, just stressing the relevant facts.

My mother's birthday was in January. For the first time I would not be present, but I wanted to give her a gift, so I chose to offer a French quality perfume. While I was shopping, another customer arrived. I was not shy and immediately was struck by her beauty. Not only the naturally blond hair, the blue-green perfect eyes, the delicate facial features, the body proportions and posture, but no less important, the frank, honest regard—and, best of all—directed to me. You know, a shampoo can be bought more quickly than a perfume. She left before I did, and I wondered where I would ever meet her again. Surprise, there she was, waiting for me! This lucky encounter led to our wedding, and at the moment I'm writing it has already lasted for sixty-seven years, with greetings from Queen Elizabeth II on the occasion of our sixtieth anniversary.[2]

Now I explain the determinant role of Schurik.

When he was over, Schurik joined me in my walks with Huguette. I was very pleased that they liked each other very much. One day, when it was just Schurik and I, Schurik asked point blank, "Paulo, what are you doing with Huguette?" And my answer was immediate: "I will marry Huguette." This was something I had never expressed. It was very irresponsible. Indeed, age twenty-two, on a fellowship that would not be renewed, no career position waiting for me, etc. But, you'll see that I am a person who makes quick decisions. The provoking question of Schurik had a crucial importance in my life. He was not a matchmaker. But like Cupid he awakened me, and I understood also that I had to fight to develop my career.

Terry Mirkil

The close relationship between the universities of Nancy and Chicago became known as Nancago and was attracting numerous very gifted young mathematicians, who came with post-doctoral fellowships. Terry and Presocia Mirkil came from Dartmouth College in New Hampshire. Terry's goal was to work with Schurik on functional analysis. Schurik, and I too, liked Terry and his wife very much. Quite often I met Terry without Schurik, who was busy with his own research. Terry confided to me, "When I see Schurik, I tell my ideas to him, which results I want to reach. After some thought he tells me much stronger facts than I would anticipate, even stating the theorems and sketching their proofs." After a pause, Terry would continue, "It becomes depressing to realize that all that I wanted to do erupts from his head so easily. I never met anyone like him. Schurik is incredible."

I have to say that knowing Schurik so well, I avoided doing the same as Terry did.

The fourteen questions

Classes resumed, as interesting as they were challenging. From Dieudonné I learned about algebraic numbers, p-adic numbers—so many topics that today have accompanied and fostered my research. Besides Schurik, among the doctoral students there were Jean-Louis Lions, Bernard Malgrange, and Paul Malliavin. All three became in due time members of the Académie des Sciences de Paris. We could also see, quite often, Jean Braconnier. I became a friend of Georges Glaeser, who was a doctoral student of Marie Helène, daughter of Paul Levy—what a lineage! Eventually, Glaeser became a leader in the pedagogy of mathematics.

In the midst of this effervescent mathematical atmosphere much attention was given to a new paper by Dieudonné and Schwartz on new kinds of topological vector spaces. At the request of my cousin Leopoldo, I reported on the paper. It finished with a list of fourteen open problems that the creators of the new theory were unable to solve. Twenty-two-year-old Schurik solved them all!

[2] *Editor's note: Citizens of UK Commonwealth realms and Overseas Territories who are celebrating their 60th, 65th, or 70th wedding anniversaries are eligible to receive, if a friend or relative files an application in advance, a congratulatory, personalized card from the Queen.*

Schurik in Brazil

Schurik in São Paulo

I returned to Rio de Janeiro in June 1952 and assumed my assistant position in the Department of Mathematics at the CBPF. A smaller department could not exist. For this reason I could not secure the funds to invite my friend, who, on top of that, was not a physicist, just a mere mathematician.

But, with the help of Candido L. da Silva Dias, a professor of the Department of Mathematics at the Universidade de São Paulo (USP), Schurik was invited. He stayed two years at the USP, and there he produced work of central interest in analysis involving tensor products and much more. This research made him internationally famous. It is not out of place to commend the math department of the USP for the quality of their invitees viz. Fantappier, Zariski, Weil, Dieudonné, Schwartz, Delsarte, and now, Grothendieck.

For me the only regret was that I could not participate in the activity. At that time, a railway trip from Rio de Janeiro to São Paulo would have taken about seven hours.

The visit to Rio

Schurik was invited to give a talk in Rio, where he chose to stay at the Icaraí beach in Niteroi, just across the bay of Guanabara. He had come with his mother, whose health had improved substantially.

At that time we were living in my parents' apartment. My mother prepared a nice afternoon party for Schurik, and many of my friends were present. Schurik and his mother expressed their joy during the party in Brazilian style.

The next day I met Schurik in Icaraí. While walking along the beach I told him about my impending trip to Bonn to study ideal theory with Krull. I also told him that I was going to submit a long monograph about the theory of commutative rings, where I used the new ideas about injective properties that had been developed by Schurik.

When I explained the results in my work, Schurik said, "Paulo, do not compete with this paper. You can do far better work."

My inner feeling was that he was right despite the fact that I was using inductive limits, a very new concept at that time. Once more I proved that I am a man of quick decisions. In the first instance, Schurik was the catalyst in my marriage. In Icaraí he was preventing me from having a mediocre paper for a "livre-docência." I said, "Schurik, I will follow your advice."

Few people would do as I did. My monograph was later published in the Brazilian series *Notas de Matemática*.

I tell now a third instance of a quick decision even though it has no bearing to Schurik. It happened in 1962, when I moved to Canada. To buy a house in Kingston, I arranged with the realtor to see houses the same day of my arrival. Mel Blachfond showed me four houses, but none pleased me. Then he said that he would be able to show another home, but he did not have the key with him. To which I said he should show me the house anyway. Without entering the house, I applied an idea that is basic in the theory of representations of groups, namely, enough representation often suffices to know the structure of the group. In this case that there were two floors and a basement with windows. From the windows I could accurately surmise the number and location of the bathrooms and bedrooms and where the living room, dining room, and kitchen were. The quality of the large windows indicated the quality of the whole. I couldn't see inside, but Mel said, "The floors are hardwood." Knowing the likely price range, I said, "I'll buy this house."

And Mel said that in his entire career this had never happened. The lesson of this episode and my previous quick decisions is to follow good advice and educated intuition. To this day I live in the same house!

I have drifted away from Schurik, but only apparently. For him, as for me, the mathematical creator needs: 1) appetite, which means the compulsion to discover; 2) ignorance, because knowing the path of one's predecessors who did not succeed should be avoided; and 3) imagination, this being essential for the solution of complicated problems. Coincidentally or not, Schurik's work had these characteristics. I am convinced that he never studied a book from page one to the end. If you remember his comment on Day 1 about my pile of books, it was not satirical or scornful, it was his attitude on the matter. To some extent I do it, too, but any comparison between Schurik and me is pure fantasy.

From 1953 to 1968

The flying career of Schurik

The achievements of Schurik were of the greatest importance, not only in the theory of topological vector spaces and homological algebra, but also in algebraic geometry, which he was revisiting on much more sophisticated bases. Some people were convinced that he was preparing the ground for a proof of Fermat's Last Theorem. But one cannot find one single reference to theorems in his voluminous writings. Was he, or was he not?

In this respect, I tell a conversation I had with him. "Schurik," I said, "I have been looking at some of your writings, in particular the theorems you obtained. Often they are corollaries. I searched the proof in the developments and comments that precede the theorems. I see no calculations that could be the essence of the proof."

Schurik interrupted me and said, "My theories are like enormous trees from which the theorems fall like ripe fruits."

Well said. In this respect, I like to tell a story, which you have to believe because it is true. In Petropolis, in the property of my brother, I was sitting under an avocado tree. These trees are very tall and their fruits are like big, hard soccer balls. Like one of Grothendieck's theorems, a ripe fruit fell: since I am telling the story, it did not hit me.

During the years from 1953 to 1968, Schurik worked nonstop in the development of his long-range, monumental construction of the new—I better say *his*—algebraic geometry, waiting for applications of those general concepts. I heard from a disgruntled reader of EGA-0 the unjust comment, "It is the globalization of the platitude." But Schurik was also achieving other very sensational results, as I will tell now.

Recognition of the leading and innovative quality of Schurik's results was coming. I mention some of these by memory. There must have been more. Some are mentioned below, but I didn't try to find when they were offered.

Schurik became a member of the Bourbaki brotherhood. He entered into conflict concerning the program. Not succeeding in imposing his point of view, he resigned.

Schurik was nominated to the Institut des Hautes Études Scientifiques (IHES) in Bures-sur-Yvette, subsequently he was for a short period in Collège de France, and finally he took up a professor position in Université de Montpellier.

In 1966, Schurik, at age thirty-eight, was awarded the Fields Medal. But as a protest to the treatment of the Russian dissidents Yuli Daniel and Andrei Sinyavski, he refused the medal and did not attend the IMU meeting in Moscow. He made his view widely known. Soon I will mention his Carnival muzhik fantasy. It was for the young man a genuine elegance, but it did not make him a communist sympathiser.

My view about what was for Schurik an acceptable posture for his research had these characteristics: 1) It should not have any obvious military applications, nor even any remote possibility thereof. 2) It should essentially be without any practical use, definitely not polluting.

But then, was there any justification and glorification for this endeavor?

Jacobi gave the answer: Mathematics is for "L'Honneur de l'Esprit Humain," which is the title of a remarkable book by Dieudonné.

My whereabouts from 1953 up to 1968

From 1953 to 1956 I was in Bonn working with Krull. There I produced the first counterexample to a conjecture of Krull. And it was there that I prepared an entirely new thesis, this time on valuation theory. It was the confirmation that Schurik understood that I could do a better work than the one I described to him in Icaraí. My thesis was approved in the USP in 1957. Without entering into more details, I was an associate professor at the University of Illinois in Urbana–Champaign from 1959 to 1962 as a Fulbright Fellow. Being unable to extend my USA visa, I went to Kingston. I have already related how I bought my house. In all these years I maintained a rarefied correspondence by handwritten letters with Schurik. I regret that these letters have been lost, but they only had topics of personal interest.

In 1964, I went to France to participate in a meeting in Clermont-Ferrand. While in Paris, I visited Schurik, who was living in the suburb of Bois-Colombes. It was my first meeting with Mireille, the companion of Schurik. He was thirty-six years old and had adopted a "genre" dressing in black, like a Russian "muzhik," with shaven hair.

Mathematically he had just completed his famous work on the Riemann–Roch theorem, enlarging the validity of the theorem proved earlier by Hirzebruch. He was deservedly proud of this result.

I would not meet Schurik again until 1969.

The Academic Year 1969–1970

My stay in Bourg-la-Reine

After the tumultuous year 1968, the Sorbonne was broken into thirteen universities. I was invited to give a graduate course at the Université Pierre et Marie Curie, also called Paris VI. During the year I rented the apartment of Pierre Samuel, who was spending the year at Harvard.

While there, Samuel was writing the book entitled *Amazones, Gaillardes et Guerrières*, a glorification of strong women from legend and history. A peculiar subject, treated with respect and scholarly methods. Of course, my point was not to stress the high category of Samuel as a mathematician. Later in this article, Samuel will reappear.

Schurik in 1969–1970

Schurik was living in Massy-Verrières, a suburb close to Bourg-la-Reine. I could visit him quite often. I describe my observations about various aspects of Schurik at age forty-one.

Political views

Despite all his years in France, Schurik had not acquired the French nationality. He was an "apatride," which means a person without any nationality. His title of "séjour" (like the USA green card) could be revoked. But this possibility was not sufficient to deter Schurik. In his backyard there were quite often tents with various types of dissidents. I remember some Buddhist monks and Spanish dissidents.

Denunciation from neighbours brought him a threat of expulsion. But his international reputation was his safeguard. We have already seen how he quit the Collège de France. And even though this did not happen in 1969, but some years later, Schurik went to Antwerp to disrupt the important meeting on algebraic groups. The first day was a mess. You can guess the reason: NATO support.

Schurik's mathematics

You already know that I am incompetent to judge Schurik's mathematical achievements, but I am giving in a disorderly way the signs of their importance. Out of curiosity, I went once to the IHES to listen to one of his seminar lectures. The auditorium was totally full, many people standing. I could recognize some of the best-known French mathematicians and, as I was told, up-and-coming mathematicians from all over the world. Grothendieck had become an icon. What could be more revealing than Dieudonné's decision to abandon his own research to become the secretary of Schurik? One could see that at the end of each seminar lecture, Dieudonné received from Schurik the handwritten notes that were to be put in shape and typed. This was a difficult task not the least because Schurik's handwriting was even worse than mine! (Hard to imagine, so I take this opportunity to thank the patient person who is typing my text.)

Dieudonné's work prevented the important lectures of Schurik from becoming just words that would disappear forever. Mathematics must be grateful to Dieudonné, whose work was for "l'honneur de l'esprit humain."

The family

In the intervening years, Schurik's mother had died. As was rather common, her mortuary mask was prepared and, as I was told, was in his bedroom as a symbol watching her son's sleep and occasionally more.

Schurik was living with Mireille. Some twelve years earlier, if I am correct, Schurik had been invited to give a series of lectures at Harvard. Mireille could only have a USA visa if they were married. So, they were. Three children were born: Johanna, the oldest, and two boys.

Very often I would go with Huguette and my eight-year-old son Eric to visit Schurik. I could hardly speak with him, due to the uninterrupted flow of mathematicians arriving to see him. For example, I remember when, one day, Brieskorn, one of the most high-profile German geometers, spent hours discussing math in the pleasant backyard of Schurik's home.

With Huguette we sat around a table, talking with Mireille, who, we could say in short, was a good soul.

The Visit to Kingston

I invited Schurik to visit Queen's University. He arrived by the end of 1971 and stayed for two months. I offered to share my office with Schurik. He chose to give lectures on algebraic groups.

I was very disappointed with his presentation. No doubt his categorical view of the subject was the more embracing approach, but the total absence of illustrations, with special cases, made the lectures ineffective. Schurik could not step down from his podium.

Schurik's presence in Kingston was for me a great pleasure. On January 7 he was invited to have dinner in our home. For the evening there was an alert for a strong snowstorm. I recommended that he wear his heavier shoes, not just the usual sandals.

He didn't follow my advice. As soon as I picked him up in his room with my car, the big storm began. In no time the streets were covered by snow. In this situation the best practice is to follow the car in front. At a turn of the road the car in front was stuck in the snow, and so was mine. It is well known what to do, just to rock the car with small movements forward and backward and little pushes outside. I asked Schurik to sit at the driver's seat, rocking the car, while I was outside giving the little pushes. He said, "I cannot do it. I don't know how to drive." I had no alternative but to tell him, "Then I will stay in the car. You go out and give the pushes."

Schurik had to step outside, just wearing his sandals, which left his feet exposed to the snow, for the fifteen minutes we needed to get the car back in the road. We still needed about fifteen more minutes to reach my home. Schurik entered our warm home and his feet became very painful, but he endured the acute pain, which was diminishing little by little.

Huguette received as a gift a reproduction of a very tender drawing by the German artist Käthe Kollwitz. It depicted a mother holding her baby in an attitude that expressed the universality of a mother's love. A choice that reflected the exceptional attachment of Schurik to his mother.

Huguette had prepared food that we knew would be appreciated by Schurik. Despite its rough beginning, the evening was pleasant. When it was the time to leave, I said, "With so much snow on the ground, I will call a taxi to take you to your home."

But Schurik did not want it. Nothing would change his mind, as was always the case. About midnight, in his tight and wet sandals, Schurik began walking the three miles to reach his room.

And for the following days, he continued walking around wearing only his sandals. Just like in the picture that appeared on the first page of the local newspaper. From the top to the bottom of the page, sandals visible and the caption, "Cold feet, warm heart."

The activity of Schurik was by no means limited to his lectures. This was the period when Schurik was putting on paper his thoughts about the survival of mankind.

He expressed in his voluminous writings his warnings about nuclear plants to produce energy, the danger of leaks and explosions, stressing also the extreme importance of the treatment of nuclear waste. The unrestricted use of fossil fuels, the CO_2 pollution of the atmosphere, and the predictable consequences of climate change: the melting of the polar ice, the rise of ocean level, the impoverished people who lived in low-level artificial islands in Florida, the probable disappearance of islands, etc. The cooling of the Gulf Stream would bring an important climate change, violent storms would become more common.

The melting of the ice would open a seaway between the islands north of Canada and the continent. The region is supposedly very rich in all sorts of minerals, requiring high military protection against other countries. More military… And unexpectedly, it would be a death sentence for one of the most perfect creatures in existence, the majestic white polar bears.

The analysis of these questions by Schurik was pioneering, well documented. Often Schurik would tell me what he was typing with the help of his typewriter.

Still to come was the group founded by Schurik in France, called something like "Survivre." It did not last long before he was expelled from it, due to his intransigence in compromising. But the seed he sowed was at the origin of the green movement, like "Les Amis de la Terre," headed by Pierre Samuel.

When it became known that Schurik was in Kingston, the phone calls were numerous, with invitations from universities. If he found one worthwhile, he would answer, "Yes, I accept, but besides my mathematical lecture I will give another lecture on the survival of mankind."

When the arrangements were agreed upon, he would perform as expected.

I will tell the episode when Coxeter, the world-famous geometer from the University of Toronto, invited Schurik. My friend enlarged important results of Coxeter, stressing the value of Coxeter's work. Naturally there was a mutual attraction and admiration. In Toronto, Coxeter was treating Schurik with his inherent class, and among other things, he invited Schurik for a dinner "à trois." Mrs. Coxeter was a classy lady, the daughter of the famous Dutch mathematician Brouwer, who had proved a celebrated fixed point theorem and was also the creator of the doctrine of intuitionism in mathematical logic. The usual logic states that the negation of the negation is the affirmation and it is related with Boolean lattices. The intuitionistic logic is related to the so-called Brouwerians. This is what happens when I want to mention the excellent dinner prepared by Mrs. Coxeter: it makes me write about Coxeter connections which included the famous painter Escher, often inspired by mathematics. And, hoping that you are not tired of my verbiage, I'll add that my very first research paper, in 1949, contained the proof that Brouwerian lattices are equationally definable, this being a desirable property.

On his return from Toronto I asked Schurik about his visit. He was fully pleased, and answering my question, he said, "Mrs. Coxeter prepared a beautiful dinner, but I said, 'Today is Tuesday, the weekly day when I fast in protest to the war waged by the Americans in Vietnam.'"

Once he would not wear heavy shoes and now he would not eat good food. It was time for him to return to France.

The Third Phase

When analysing the lives of creators, it is customary to divide into three phases the evolution of their careers. The first period is the connection with the past; in the second phase we witness the coming of maturity. Typically, this is done for Beethoven.

In many instances, the creator repeats basically what they were doing. Digging deeper and deeper with their techniques, there comes a point when the creator is in a hole, and he sees no more. This was not what happened with Schurik. As I have already described, he had a constant interest for many subjects outside mathematics, like cinema; literature; music; anti-war attitudes; "incorrect" political views; a well-hidden, sort of mystical posture, which, for lack of a name, I would call an "ultra-religion." I am not judging Schurik, but for those who wish to do it, the great mathematics he invented is only a part of him.

At this point, it is appropriate to devote some space to these special features and to the new directions. A chronology for these transformations is meaningless.

Leaving (Part 1)

Leaving the survival movement

I have already told about the frenetic writing about the problems for planet Earth, which were a main activity while Schurik was in Kingston for his USA and Canadian lectures. Schurik preached so his prophecies would be heard. Back in France he founded the survival movement.

It did not last long: he had to leave his own movement because of disagreements with his companions. Without knowing, the seeds that he and others had planted gave rise to important political movements.

Leaving Paris

It came as a surprise when Schurik left Paris and went to his alma mater in Montpellier. On that occasion Schurik stopped giving seminar lectures and, as far as I know, did not publish any new papers. But in Montpellier he still advised PhD students (how many I don't know).

His extensive correspondence with Quillen, about one thousand pages long, has been edited and is now available.

I still maintained a sparse correspondence by letters with Schurik. Once I proposed him to make me the guardian of his unpublished manuscripts including his letters. I was afraid that they could be lost or destroyed by Schurik. No reply from him, but I hope that they were not destroyed.

There has been much speculation about Schurik abruptly ceasing to produce more papers. It has been said that when Deligne had proved Weil's Conjecture, which was one of the main aims of Schurik's research, he felt it unnecessary to continue his research. This is more a calumny than a fact. My idea is that he had already spent all his years in that same endeavor and was irresistibly attracted by his own nihilism. There cannot be an explanation expressible in simple sentences.

Récoltes et Sémailles

This is the work by Schurik produced in mimeographed form. It consists of five densely typed softcover books.

Normally the seeds come first and later is the time to harvest the grain. But in this work all is backwards. While writing this text, I began grasping the reason for this inversion. It is a metaphor to state that what he became was already in the womb of his mother. This statement is deep in meaning and supported by genetics.

I read only parts of this long text and was mystified by the analysis of mathematical creation. An anthology text as only Kafka could have written.

The quite unusual style of writing—with sentences, subsentences, and the excessive use of cross references—intrigued me. One of my friends, a professional psychoanalyst, declared that the author was paranoiac.

This was in agreement with the fact that the book contains allegations against most of Schurik's pupils, followers, admirers, and friends. Basically, Schurik accused them of using his ideas to write their papers.

Leaving (Part 2) and our Last Encounter

I am totally confused about the chronological order of what I will tell now. Probably between age sixty and sixty-five Schurik left Montpellier and went to an undisclosed little village in the Ariège. Until much later, I could not find out his address. He lived by himself in a monastic way.

Planting his vegetables, probably not eating meat, maybe drinking milk. According to his credo of recycling, his own excrements would serve, maybe mixed with horses', to enrich the soil where his vegetables were growing.

One day, as a total surprise, he called me on the phone and asked whether he could be in my apartment in Paris for a short visit. I had to agree that nobody would be told of this visit. Both Huguette and I were excited about the intended visit. Schurik came and explained that he came to Paris to propose to an editor the publication of his "Récoltes et Sémailles."

It was so good to talk with Schurik, who appeared to be in excellent health and, could I say, possess the serenity of a Buddhist monk. In our conversation he confirmed his way of growing his vegetables. When I asked if he was writing anything, like a book, I was astonished by his positive answer. He said with enthusiasm that he was writing a book of his dreams. Every time he had a dream he would wake up and take note of the dream. His book was becoming voluminous. At some point, he said that he had a dream that the world would end in the year 2000, still a few years away. When asked, he narrated the dream. Instead of the apocalypse, there was someone playing a guitar and other strange happenings. This was a sign of mental derangement.

The next day, we three went to see the editor. We waited in a small garden, those that we see in Paris. Schurik went alone and about one hour later he said, "The editor would accept if I make substantial cuts." Of course, Schurik could not accept it. He took leave from us to return to his village. I never saw or talked to him again.

Leila Schneps, a mathematics professor at the Université Pierre et Marie Curie, visited Schurik a few times and provided his address to me. My letter to him was returned unopened with a notification that the addressee refused to receive it. The same happened to a second letter.

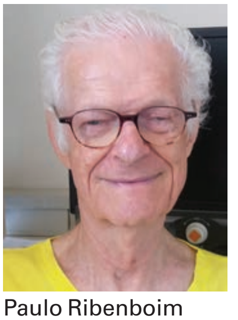
Postfacio

My text is sincere in telling, not hiding, not judging. If I would have to find a short and striking epithet for Schurik, I would say: genius or more.

Clearly, he was a genius in mathematics. "More" refers to the unclassifiable characteristics of his complex personality.

EDITOR'S NOTE. Ribenboim's original piece contains some additional facts that are not included in this excerpt. Readers interested in the full text should contact the author.

ACKNOWLEDGMENT. The author wishes to thank Sophia Merow for her careful study of the text and suggestions for improving the style of the presentation.



Paulo Ribenboim

Credits

Author photo and photo of Grothendieck are courtesy of Paulo Ribenboim.

Mental Health in the Mathematics Community

Mikael Vejdemo-Johansson, Justin Curry, and Julie Corrigan

Note: The opinions expressed here are not necessarily those of Notices.

On March 7th, 2019, Kelly Catlin—a three-time cycling world champion, winner of a silver medal in the 2016 Summer Olympics, and Stanford graduate student pursuing a masters in computational and mathematical engineering—died by suicide.

Her death is an absolute tragedy. The world has lost an amazing individual who pushed the boundaries of what humans are capable of in realms both physical and intellectual. She also reminds us of the mental health crisis that is quietly raging in our classrooms, our students' dormitories, and our own offices. To hear of a graduate student's death by suicide leaves us shaken, but not surprised.

Mental illness is a widespread problem, but it has a uniquely devastating presence in the university [1–4]. In a recent international study of 2,279 masters and PhD students, 39 percent were evaluated as having moderate to severe depression, compared with 6 percent of the general population [5]. Multiple studies [5,6,7] have also shown that rates of depression and anxiety in graduate students who are women, people of color, or LGBTQ are higher than among those students who are not. This makes individuals in these groups, who are already underrepresented in STEM programs and consequently STEM careers, at increased risk. Failure to address mental health problems in the university makes addressing inequality in society at large harder to do.

Even within the university, rates of mental illness vary by department. A 2014 Berkeley survey of graduate students indicated that 64 percent of students in the arts and humanities met a threshold for depression, compared with 43–46 percent of students in science and engineering and 28 percent of business students [6]. The Berkeley report also measured the top 10 predictors for graduate student well-being, which put career prospects as the number one predictor. Unfortunately, we find it discouraging that another study found that 43 percent of interviewed faculty had at least a mild mental disorder, even when 92 percent of those interviewed held permanent positions [8].

To our knowledge no study has focused on mental health in the mathematics community specifically, and thus we are left with anecdotal evidence and subjective analysis. Below we provide some of our own perspectives on mental health in mathematics—a narrative of our own struggles with mental illness and recommendations for how we as a community might do better. Our goal is that perhaps, in another universe, Kelly Catlin—and others lost to their conditions—may have instead finished her studies and gone on to pursue further intellectual achievements.

An Analysis of Our Community

If we examine the constellation of cultural images of mathematicians, we need not go too far to encounter examples of mental illness. John Nash, who was portrayed by Russell Crowe in Ron Howard's film *A Beautiful Mind*, had his career

Mikael Vejdemo-Johansson is an Assistant Professor of Data Science in the Department of Mathematics at CUNY College of Staten Island. His email address is mvj@math.csi.cuny.edu.

Justin Curry is an Assistant Professor in the Department of Mathematics and Statistics at the University at Albany SUNY. His email address is jmcurry@albany.edu.

Julie Corrigan is a design consultant in Charleston, SC. Her email address is jcorrigan20@gmail.com.

DOI: https://dx.doi.org/10.1090/noti1910

interrupted for many years by a schizophrenic breakdown. Kurt Gödel starved himself to death out of fear of being poisoned. Alexander Grothendieck went into isolation for over two decades. However, in these cultural images, we make mental illness into a sort of superpower that imparts a competitive edge in creative work [9]. This glorification is deeply problematic and creates a culture where intense, exhaustive work—often done in isolation—is praised at the sacrifice of an individual's well-being. Moreover, a "survival of the fittest" mentality tends to prevail in our departments with the idea that people who aren't cut out to be mathematicians will eventually leave.

Although the above individuals are exceptional mathematicians, their illnesses did not empower them to do great things. Yes, anecdotally, the mathematical community tends to have a higher percentage of neurodiverse individuals, but co-occurring depression, anxiety, trauma, and mood disorders create a community with a disturbingly high attrition. Often stress is the primary factor that precipitates or exacerbates mental illness, of which there is no shortage in our careers—from undergraduate school through tenured professor.

Beginning graduate school in mathematics often means stepping into a world without clear structure in place for success. The familiar metrics for evaluation used as an undergraduate with graded homework is replaced with more open-ended study and formulation of potential research projects. The role of the advisor can be variable and the quality of one's results is often marked by subjective social signaling, which can make us hyper-aware of criticism or outright dismissal as we progress through postdoctoral work and faculty life.

Collaboration in mathematics is also a thorny subject. Generally speaking, mathematicians tend to only have a few collaborators, which can lead to feelings of isolation. This can be contrasted with the applied sciences where a lab can help provide social support. Of course, collaborative work in the sciences can also lead to abusive power structures where multiple professors appear as co-authors on a paper written mostly by a single graduate student.

Many of the above sources of stress are endemic to an academic career, with no easy alleviation, but there is at least one thing that can be easily dismantled: the stigma of struggling with a mental illness. Stigmatizing not only increases the stress, but also raises the threshold to seek and get help. Our hope is that by starting a dialogue about mental health in the mathematics profession, we can help people feel okay with asking for help and seeking what they need to lead fuller, happier, and more productive lives.

Our Own Experiences

At the Joint Mathematics Meetings 2019, we organized a panel discussion on mental health in the mathematics profession. In addition to the authors—Mikael Vejdemo-Johansson and Justin Curry, both tenure-track faculty with diagnosed mental illnesses under treatment, and Julie Corrigan, who was unable to continue her graduate studies due to insufficient support for managing her chronic PTSD—the panel also hosted Kate Farinholt, the Executive Director of NAMI Maryland (the National Alliance on Mental Illness) [10]. On the panel, we talked about our lived experiences with mental illness, and thoughts and ideas on how we would like our colleagues to support us. Between us, the authors represent three different types of mental health problems: chronic mood disorder, maintained enough to survive in academia; situational depression, with an identifiable trigger; and anxiety that did not meet sufficient accommodations. Below we have included each of our stories.

[Mikael] I have always suffered from mood swings. When I first started dating my wife, she suggested I seek help, and helped me through the process. It is difficult for me to articulate, or even remember, my moods, so the first time I called in to a psychiatric clinic, I couldn't say much more than "sometimes I get sad." The day after, she sat down with me and we wrote out a cheat sheet together: how sad, how often, what impact does it have on me, on my family. I called back to the clinic, and using the cheat sheet I was immediately started on outpatient medication and therapy.

Through therapy, and through sheer experience, I have now built up an awareness of my moods and mood swings. I can recognize many of the warning signs, and I know some of the ways that a meltdown can be triggered. As with very many mental health care routines, regular food and good sleep are very important to keep me in balance. I have found it difficult to think of how my workplace could possibly provide accommodations for me—and the ones I have thought of are connected to this: I need my schedule to allow for a meal break, and I need my schedule to allow me to keep my sleep routines in check.

I first started deliberately talking openly about my mental health when I saw my hackerspace/makerspace community step up and build support systems and boost visibility for depression and mental illness in the wake of a sequence of high profile suicides. I joined the visibility boosting activities—and posted a text on my personal website [11] about my mood disorder and how it feels to live with it. I posted the link on Twitter, and it was read by Ian Gent, a Professor of Computer Science at the University of St Andrews, where I had just finished a postdoc year. Ian emailed me and told me about his experiences with depression—and seeing how we had worked side by side for a year, sharing coffee breaks and lunch breaks, without either of us having any idea of what the other one was going through sparked us to start work on visibility in academia and in mathematics. We founded a group blog [12], and I started looking for ways to build in mathematics what I saw them build in the hackerspace/makerspace community.

Looking for things that have helped me, some concrete ones come to mind. Getting help recalling the structure and extent of my mood swings was critical in getting help to begin with. Through many years and with help from therapists and family I have learned to recognize my moods and my triggers so that I am able to stave off problems or remove myself from difficult situations. Medication has worked to some extent, though with a lot of fine tuning and experimentation. My department and my chair accommodating a schedule that helps me to maintain food and sleep routines has been immensely helpful.

Since that conversation with Ian Gent, I have been deliberately radically open with my mental health. Nearly all my colleagues across several years of postdoc-ing and these past years on the tenure track know of my diagnosis. I have not noticed any bias or hostile reactions. Admittedly, this could be because I am oblivious, or because I have been lucky in whom I interact with. One effect I have noticed is that people open up to me. I bring up my diagnosis in class—and my students talk to me about their issues and how to best handle their course work and stress. I talk about my diagnosis and mental health with colleagues or at conferences—and I am approached by others carrying loads and never having had someone to talk to who would understand.

[Justin] Although I have probably struggled with episodes of depression my entire life, I didn't really seek expert help until I was a graduate student. My six years as a PhD student were filled with some extreme highs and some deep lows. I was extremely close to my dad, who had recurrent bladder cancer that went through multiple courses of treatment throughout my graduate career, starting with radical surgery to remove his bladder in 2010, chemotherapy and radiation to treat metastatic cancer from 2011 to 2012, and concluding with his death in October 2013. As an only child with divorced parents, making trips home to care for my dad, in addition to all the challenges of earning a PhD, was a major source of stress; this was only barely made manageable with the incredible support of my advisor, Robert Ghrist, as well as my close friends.

One of the problems with mental illness is that you often don't know what's going on with you.

In 2011, I was away at a conference when I learned that my father's cancer had metastasized to a nearby lymph node. For a few days, I heard a constant whispered thought, "metastasis," which precipitated a deep shift inside of me. This was a shift that marked the transition from a relatively upbeat, happy existence to one filled with struggle and suffering. I would at random times be affected by a bodily sensation, like someone was pinning me down in a chair from which I could not get up. Over the next year or two I began to experience extremes in emotion: oscillating between despair and almost a divine sense of hope and transcendent optimism when my research started to take off. I started to see a therapist in 2012 and began taking medication (Zoloft) in 2013, as prescribed by my primary care physician.

At first, Zoloft imbued me with an almost hyperactive level of energy: I couldn't sleep without taking Benadryl before bed, I would have nervous ticks with my hands and heard a rushing in my ears, but side effects aside, it was worth it because I was able to finally write up my thesis. Medication enabled me to speed through my last year of grad school when I was applying for jobs and when my dad died. There was definitely a conscious sense of keeping myself together so that I could graduate, which I did in May of 2014.

I fell apart shortly after starting my postdoc at Duke. Many of my prior social support systems disintegrated and, for the first time, I was no longer a student. For my first year, I was left with the uninspiring task of writing up journal versions of my thesis results. This was nearly impossible as I felt chronic fatigue—even doing one hour of work a day felt exhausting. I kept a couch in my office so that I could nap frequently. Life felt like such a struggle that I began to conclude that not existing would be easier than existing, which is a form of suicide ideation. I didn't have active plans to end my life, but if I were to not wake up one day, then that seemed like a preferable outcome. Remarkably, I didn't think I was depressed, so it took me a while before I started to see a therapist again. Luckily I connected with a good cognitive behavioral therapist and started to wean myself off of Zoloft. Off of Zoloft, I started to feel inspired to do math again, but would also experience extreme ranges in emotion depending on circumstances in my personal life.

I think a lot of my depression was precipitated from the sheer exhaustion of writing my thesis and caring for my father. I started to shirk my responsibility as a postdoc and took up exercise as a passion. I fell in love with movement: weight lifting, gymnastics, and yoga especially. In 2017 I enrolled in a yoga teacher training program led by Nina Be and Bart Westdorp at Global Breath Studio in MindBody Centering Yoga. By discussing my traumas in a supportive group setting, I benefited from more in those three months than I did in several years of therapy. Reading foundational texts in yoga philosophy provided me with a new framework for living. I also learned skills for managing my energy more effectively.

Since starting my tenure-track job at Albany, I feel that some of my old energy levels are returning as well as a new sense of purpose and responsibility to mathematics. Teaching can be draining, but directing my own research program and having students to discuss research with brings me joy. I still feel fatigued at times, and I used part of my start-up funds to buy a Thai massage mat so that I can rest whenever I feel run down. My friends, many of whom I met through yoga, provide me with a valuable support system that I can lean on when I'm feeling down. I contemplate going back on medication sometimes, but I feel good about the self-care techniques that I'm currently using to manage my depression.

[Julie] PTSD makes it difficult to separate reality both from inventions of the mind and from echoes of the past. One time I stopped attending a class without withdrawing because the octave of a gentleman's voice who sat two seats behind me reminded me vividly of the aggressive and threatening arguments my parents had before their divorce. The sensation of an impending screaming match, of which I had the misfortune of being physically in the middle of, was so overwhelming that I couldn't approach the classroom door without hyperventilating. It took two semesters until I was even able to get close to that general set of classrooms without feeling an overwhelming sense of dread. I didn't withdraw because I was past the deadline to do so without the professor's signature. That meant, in the middle of my panic episodes, finding the strength to go to my professor, explain why I was dropping, and ask for their permission. That is not only dealing with the anxiety from my past traumas, but also from the spiralling thoughts of: "How exactly do I explain this without sounding like I'm lying? How do I ask my professor to sign a withdrawal form so I don't just fail because an impulse in my head won't let me go near the classroom? Who do I ask for help in telling my professor? How do I get help in resolving this problem so that it doesn't affect my academic career?"

It's hard to say when I had my first PTSD episode. I remember having issues "controlling my anger" that started when I was 11, but my father recounts times from as young as 5 or 6 where I would become inconsolable when we went to leave the house, but calmed instantly the second I was in the car. What I was experiencing wasn't even identified as panic attacks until I was 19, in 2006. I started seeing doctors and therapists whose diagnoses ranged from being emotional, to having anxiety, to even suggesting epilepsy or OCD. I have had my bloodwork done more times than I can remember to check for thyroid issues or any other physically evident misnomers that could cause my apparent irrationality. I've tried all the Benzos and SSRIs.

Eventually, I landed on a mixture of beta-blockers (propranolol) and sedatives along with healthy living and exercising to help me manage my anxiety during the day and cure my insomnia. It helped, but it was always just cheap duct tape over a foundation crack. The cocktail, along with therapy and exercise, definitely helped curb the day-to-day issues over time, but the severe lack of self-worth, insecurity, fear of being abandoned, judged, abused, et cetera, would, and still do, surge in full force from time to time.

When I was able to work up the courage to approach my professors about my constant "sick" days or peculiar needs, I was often met with eye rolls, scoffs, or a general lack of compassion. I often heard things like: the syllabus is the syllabus, every student needs to be held to the same standards, maybe this isn't for me if I can't handle the workload. At the time I wasn't properly diagnosed. My various therapists and doctors all attributed my symptoms to Generalized Anxiety Disorder, which many people do suffer from, but that diagnosis didn't exactly come with a pamphlet on my legal rights and a list of possible reasonable accommodations. My university's Disability Student Services department told me that all they could do for me was give me extra test time or a copy of lecture notes. How does extra test time help me work out what to do when I can't go to class because I woke up unable to leave my house?

When I could make it to class, it was incredibly difficult to sit still for the whole lecture time. Thanks to technology, having my textbooks on a tablet eased the strain of having to be focused in one place for 50+ minutes by making the internet available as a distraction. My professors did note my "lack of attention," though, and it became a hurdle for me to ask for support when I needed extensions after days of rolling panic attacks prevented me from completing my homework. I dreaded having to discuss my poor test scores after freezing up whenever my professor walked behind me during an exam; the severe anxiety that I was being silently judged made me unable to recall information or sometimes even finish the test.

PTSD affects your ability to separate reality from invention. I highly doubt, and even doubted at the time, that my professor was looking at my work so far thinking, "Hrm, she sucks. That's completely wrong. Her attempts are laughable." That's just what my PTSD was telling me was happening because of the trauma from the abuse I suffered growing up and that continued into my late 20s. It sounds reasonable and rational in the moment, even though, in retrospect, I may see it as obviously absurd.

The stress of the studies exacerbated my condition, increasing the difficulty of me handling it on my own. I did have friends and peers who were understanding and supportive, but it wasn't enough. I needed support from my university, from the administration, from the field of mathematics. I always felt competent, but my disability didn't allow me to show my competency in the way that I was expected to by the community. I felt like I was being judged by my professors and peers for being lazy, unattentive, unfocused, and simply not that good at mathematics. The overwhelming insecurity being fed by my past and ongoing abuse made it impossible for me to bolster my own ego enough to continue on. Thus, I left the academic world behind with only my bachelor's degree completed.

The next several years were still a struggle for stability even outside academia. Mathematics didn't cause my disorder, it was just a field unwilling to accommodate the way I needed to learn and demonstrate my knowledge. Leaving academia, however, meant physically returning to the abuse from my family.

After a particularly traumatic series of events in 2016, I made a call to check myself into a treatment facility outside of Santa Fe, NM called the Life Healing Center. I came to the realization that I was spiralling severely and knew if I did not seek serious help, I would likely kill myself. During my time there, I was forced to explore the past, acknowledge the abuse from my family, and realize that I had c-PTSD[1] from years of trauma that stemmed from extreme, systemic family dysfunction. I was set on a course for recovery. I started seeking proper treatment in terms of trauma therapy and have found myself improving year after year. I now use CBD[2] for the day-to-day anxiety and keep up with yoga, healthy eating, and therapy to help me learn to love myself and feel valuable. I do still have issues with self-worth, insecurity, and feeling accomplished, but as I build a more stable life and continue in my treatment, it gets better every year.

So What Can We Do?

Mental illness causes us to lose mathematicians at all stages of their career. A few die from their mental illnesses like Kelly Catlin. Some are forced to leave the field, taking all their potential with them. This happens at the beginning of careers, like with Julie Corrigan, but also to prominent mathematicians who have made a significant mark. For example, Andreas Floer, the namesake of Floer Homology, died from his depression at the age of 34 [13]. In this section we provide some of our own thoughts on how to address mental health issues in the mathematics community.

The first step towards any actionable change is knowledge. Before we can lobby institutions or organizations for support, we must promote awareness of mental health issues as individuals. Many faculty are aware of reasonable accommodations that are made for students with physical and sometimes learning disabilities, but we often forget that those who struggle with mental illness should have analogous reasonable accommodations, as promised by the Americans with Disabilities Act. Due to the unseen nature of most psychiatric disabilities, accommodations often feel abstract or are seen as a ploy to get out of the heavy workload that all students and faculty feel. To be clear, reasonable accommodations are not about changing the net amount of work; they are about providing support so that each person can achieve the same goal in their own time and in their own way. Existing resources such as [14] provide detailed recommendations for promoting inclusivity and access for faculty with mental illnesses. If you find that you are in need of an accommodation, you can try talking to your mentor, supervisor, HR, or even a union representative to advocate on your behalf.

However, many members in our community might not even be aware that their struggles are of legitimate concern and require professional help. Not being able to perform in a way that other people appear to be performing can create further anxiety and depression, which fuels withdrawal and makes it difficult to go out and seek help. This is why we feel that it is important to normalize the discussion around mental illness. We need to normalize seeing therapists and psychiatrists, mentioning your depression, mentioning that a situation is making you anxious or impacting your disability. Just as you might get a yearly physical, seeing a therapist at least once a year to have a conversation about your mood or outlook on life should be standard practice. Moreover, taking psychiatric medication shouldn't be regarded as any different than managing your blood pressure with medication. If you are currently seeing a therapist or taking medication to manage a mental illness, consider opening up to colleagues and students about your experiences—doing so can mark you as a "safe person" to talk to when times are tough and help normalize the conversation inside our community.

In addition to encouraging one another to seek professional help, we as a community can take further steps towards promoting wellness and life satisfaction in our community. Too often we valorize over-working at the expense of our own physical and mental health. Maintaining personal hygiene, getting enough sleep, and cooking yourself regular meals are fundamental to life and should be treated as such. Spending time outdoors, exercising, meditating, or socializing can all provide much needed resets to your mood or mental state. Having hobbies or a notion of self separate from work can help insulate ourselves from depression when work isn't going well. Make self-study and journaling a regular part of your life and read up on any issues you might be having. There are many books written on how to manage most mental disorders, with techniques and coping strategies that you can learn and adopt. Mental illness doesn't have to be a life sentence, many people recover and go on to live content, meaningful lives.

Even with a perfect community, supportive and promoting balance, there will be more tragedies. Mental illnesses change the way we perceive the world: inhibiting our ability to recognize and internalize affection from others, our ability to see value in ourselves; causing us to withdraw, and to believe ourselves not worthy of help. Suicide, when it occurs, is not the selfish act many take it for, but a desperate attempt to end suffering—for oneself, but also for friends and family. By raising visibility and removing stigmatization, we can at least help pave the way for when our community members start reaching out.

Our hope in writing this article is to start a community-wide conversation, so that instead of suffering in silence and isolation, our community members will reach out, ask for, and get support from the rest of us. As such, we are in the process of creating a support network,[3] so that our community members can reach out to peers who not only combat mental illnesses, but do so knowing the ins-and-outs of academia and of mathematics in particular.

[1] *Complex Post-Traumatic Syndrome*

[2] *Cannabidiol*

[3] *We have started trying to gather enough people to start a mailing list. You can sign up here:* https://mailman-mail5.webfaction.com/listinfo/mda-info

We will organize another panel discussion at the Joint Mathematics Meetings in Denver 2020.

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Mikael Vejdemo-Johansson

Justin Curry

Julie Corrigan

Credits

Author photos are courtesy of the authors.

MathJobs at the University of California–R.I.P.

Abigail Thompson

Note: The opinions expressed here are not necessarily those of Notices.

MathJobs for faculty hiring is dead at the University of California (UC), killed by the UC administration. This is a brief history of what happened, some conjectures as to why, and how optimists might try to avoid its demise at other institutions.

What happened

The department chairs in each UC math department were notified in April of 2018 that as of July 1, 2018 all mathematics department faculty recruitments would be run through UC Recruit, the University of California's in-house job application system. Within a week the chairs of mathematics of eight UC campuses wrote a joint letter in protest to the University of California Council of Executive Vice-Chancellors, the system-wide decision-making body behind this process. This letter made the obvious points that the decision would put UC at a competitive disadvantage with respect to other institutions in mathematics hiring, enormously increase the workload both for applicants and reference letter writers, force UC faculty to use a cumbersome and inefficient system, and have a serious negative impact on UC's ability to attract a diverse applicant pool.

By early May the chairs had received no response to the concerns raised in their letter; they were simply informed that UC was going ahead with the move to the new system. In late May, the department chairs together with more than 270 UC mathematics faculty from eight campuses signed a letter of protest and sent it to UC President Janet Napolitano. Various individual UC mathematics faculty and faculty committees also sent letters of protest to President Napolitano. One suggestion for compromise that was made at this time (following consultation with the MathJobs team at the AMS) was that it should be possible to coordinate the two systems so that applications and reference letters would automatically be fed from MathJobs to UC Recruit. After considerable further correspondence, consultation with some Executive Vice-Chancellors, and a demonstration of MathJobs to UC Vice Provost Susan Carlson and the UC Recruit team and Governing Board, in July the chairs received a reply offering a glimmer of hope. They heard from UC Provost Michael T. Brown that, while the decision to move to UC Recruit was final, the process of coordinating the "necessary analysis of a technical connection between the two systems" would begin. In the next few months, some consultation on ways to improve UC Recruit was conducted, but coordination and consultation on data transfer between MathJobs and UC Recruit was postponed. In December 2018, the chairs were told, "UC has decided that our best option in future years is to use the UC Recruit as the basis of our recruitments in mathematics…UC will not be pursuing a technical feed between the two systems."

Abigail Thompson is a professor in and the chair of the Department of Mathematics at the University of California, Davis. She is also a vice president of the AMS. Her email address is thompson@math.ucdavis.edu.

DOI: https://dx.doi.org/10.1090/noti1911

Here are some important notes: Several other universities and programs already automatically download data from MathJobs to their own institutional systems (see https://academicjobsonline.org/ajo/intro for a quote from the Institute for Advanced Study). There are various levels of data transfer possible from MathJobs, ranging from the maximal: (a) transfer (and updating) of entire applications, including reference letters, to the minimal: (b) automatic transfer of letters of recommendation. For (a), while data transfer is easy according to the technical team at MathJobs, the main difficulty is in treating incoming data in a way that is consistent with local institutional practices designed to meet legal requirements, particularly if the applicant revises their file. This does require cooperation on the part of the institution to which information is being transferred. However, MathJobs can be tailored to create a version (via settings) that clones an institution's practices; then at appropriate times data can automatically transfer. On the other hand, implementing (b) is very easy on all counts, as data transfer and institutional rules for handling letters of recommendation are straightforward to incorporate. While this would not save applicants from having to apply in a different system, it would save an enormous overhead for reference letter writers.

The UC mathematics departments engaged in extensive efforts to facilitate some form of automatic data transfer. In this, the MathJobs team at the AMS and the AMS leadership (especially then-President Ken Ribet, who met with Carlson and Brown in June 2018) were extremely helpful, and by all accounts were eager, willing, and able to help facilitate a technical solution. In contrast, the University of California administration ultimately declined even to commit the minimal effort required for (b), and their idea of "consultation" appeared to be along the lines of asking "how can we make UC Recruit work better for you?"

Another point, and the reason for the history recital, is that I can't think of anything more we could have tried. UC mathematicians were united, vocal, active, open to compromise, and anxious for a solution. None of that seemed to matter. Effectively, once the edict was handed down, the decision had been made.

Why this happened

So why was this decision made, and apparently immediately carved in stone? Mathematicians had developed and run a terrific tool for job applications for years. Why couldn't they leave us alone?

The ostensible reasons have to do with compliance with state and federal law. In a letter from May 2018, UC Vice Provost Susan Carlson, who spearheaded the move to UC Recruit, said, "…the benefits of the move to UC Recruit, particularly bringing UC into compliance with California and Federal recruitment standards, outweigh the negative consequences…." This compliance issue makes no sense, and this assertion was never supported; MathJobs can be set to have almost any kind of desired data collection and any needed privacy and compliance settings.

I believe that one reason for the extraordinary obstinacy is related to the alarming trend of increasing administrative control over faculty hiring at the University of California. More and more faculty positions are in some "special" category controlled largely by administrators; departments starved for faculty can select from a pool of pre-screened applicants or else get no one at all. In short, UC Recruit ensures that the administration now will have first crack at filtering the files.

Another possible factor is UC Vice Provost Susan Carlson's special interest in the potential uses of the UC Recruit system. From the abstract of her $1.18 million 2015–19 NSF grant (#1535509), which aims to address the questions "are there gender and racial/ethnic disparities in STEM faculty hiring? If so, what conditions, processes and social contexts generate/mitigate these disparities?", we see that the PIs "will construct and use a unique dataset from an online administrative system that compiles information from all faculty recruitments at all University of California campuses. This rich data source includes detailed information on applicant pools, applicant credentials and achievements, hiring processes and committees, and candidates' progression from application through the short list, interview, and offer steps in the process." While this is perhaps an interesting research project, it is hard to see why UC mathematics faculty should be compelled to participate.

It is especially ironic that the implementation of UC Recruit will override the most efficient mechanism for actually promoting diversity in mathematics faculty hiring, namely using MathJobs for faculty recruitment to ensure the most inclusive possible pool.

How to fight against this

I see no reason for optimism. The only effective way I've heard of to counter this in the long term has unacceptable consequences in the short term; one could simply refuse to post letters of recommendation anywhere except on MathJobs. Several mathematicians have in fact adopted this policy. The difficulty is that the short-term damage would be to job applicants, the people whom we most wish to support. So I can't advocate for this position.

If, or more likely when, an attack on MathJobs arises at your own institution, here are some things that may be helpful to know:

- It is possible for files/letters to be imported from MathJobs to an in-house system, and the AMS and MathJobs are interested in and experienced at making this work.
- The claim that the use of an in-house system is essential to ensure compliance with state and federal laws appears to be totally specious. There was no supporting documentation ever offered for this claim, and no credible reason that using MathJobs should be problematic.

- Most faculty outside of mathematics have no idea what we are talking about. Faculty searches in other disciplines are often highly focused and have relatively small applicant pools. Faculty members in other disciplines typically spend endless hours posting letters of recommendation to tediously contrived individual university portals, and job applicants spend even more time doing the same. While we are all experienced with the nightmare of dealing with letters of recommendation for our undergraduate students applying to multiple graduate programs ("please rate the candidate on the following seventeen largely irrelevant characteristics…"), in many disciplines this incredibly inefficient and annoying process has long been the norm for job applicants as well. People in other fields don't really see why we shouldn't suffer, too.

Conclusion

At the beginning of this century, mathematicians developed and implemented an effective system for online job applications that was way ahead of its time. It has taken university administrators quite a few years to get around to trying to destroy this system, but they are really making an effort. I am afraid we soon will all be back to the endless uploading of reference letters and the equally frustrating task of submitting individual job applications. I can't see a way out of this quagmire; perhaps others will find a path.

Abigail Thompson

Credits

Author photo is courtesy of University of California, Davis.

WHAT IS...

a Markov Basis?

Sonja Petrović

The June 25th, 2014 *New York Times* article "Shinzo Abe's Bid to Shake Up Corporate Japan" by Hiroko Tabuchi discussed share ownership of seventeen Japanese corporations and cited Prime Minister Abe's claim that they are connected in interlocking ways, owning shares in one another "to create relationships that can protect them from outside interference." A graphical representation of this relationship is depicted in Figure 1. Can this claim be verified? How confident are we that it is anything beyond basic intuition?

Questions like this are at the heart of statistical reasoning: given an observed data set, we wish to find out how surprising it is given some assumption about the world. The fact is, we face such questions on a regular basis: do male faculty members have higher salaries than their female counterparts? We answer them by looking at the data on salaries and breaking it down by gender, discipline, etc. Our hope that gender is independent of salary translates in a very intuitive way into an expectation: we expect to see a certain distribution of numbers in the salary data; when we do not, we suspect that our assumption of independence could be wrong. We shall put this everyday intuition into a formal framework: the assumption of independence is captured by a statistical model, a family of probability distributions for the salary data that take a specific form; the observed data is evidence for or against such a model. The evidence carries with it a weight, a confidence level that measures its strength.

Sonja Petrović is an associate professor of applied mathematics at Illinois Institute of Technology. Her e-mail address is `sonja.petrovic@iit.edu`.
Communicated by former Notices *Associate Editor Cesar E. Silva.*

DOI: https://doi.org/10.1090/noti1904

The Algebra Behind the Intuition

While the idea of testing a model based on observed data is a simple one in statistics, an exciting development in the 1990s has brought to bear tools from commutative algebra to help solve the problem for a special class of statistical models that we can define using an integer matrix. A Markov basis is a set of vectors in the null space of that matrix that allows us to generate synthetic data, starting from the observed, and use the resulting sample to gather evidence against or for the proposed model.

Mathematically, given a statistical model defined by

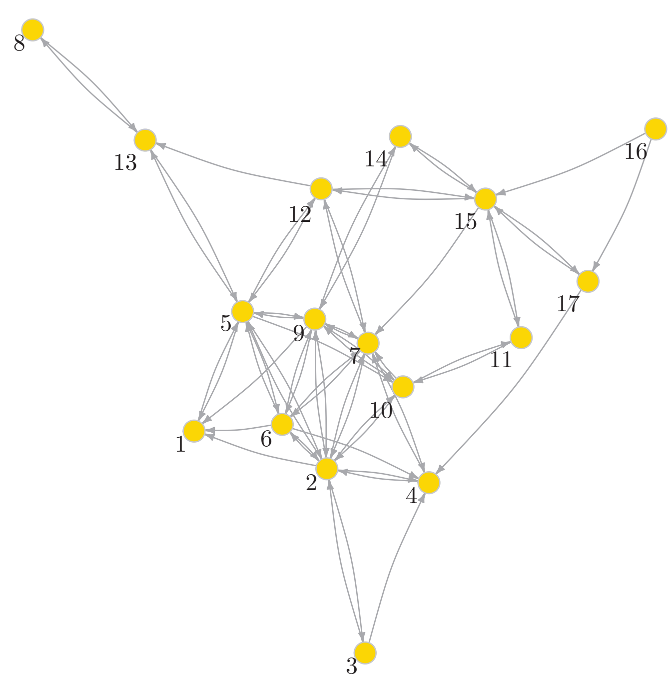

Figure 1. A graphical summary of the shareholding relationships of those corporations who disclosed them. Nodes in the graph represent the corporations. A directed edge from i to j means company i owns shares in company j.

$A \in \mathbb{Z}^{m\times r}$, a *Markov basis for the model* is a set of vectors $\{b_1, \dots, b_n\} \subset \ker_{\mathbb{Z}} A$ such that for every pair of vectors u, v for which $Au = Av$, there exists a choice of basis vectors satisfying

$$u + b_{i_1} + \dots + b_{i_N} = v,$$

where each partial sum results in a non-negative vector, that is, $u + \sum_{j=0}^{N} b_{i_j} \geq 0$, componentwise for any $j = 1 \dots N$. Set $b_i^+ = \min(0, b_i)$ and $b_i^- = \max(0, b_i)$, so that each vector in the basis can be written as a difference of nonnegative vectors $b_i = b_i^+ - b_i^-$.

Theorem ([5]). *A set of vectors is a Markov basis* if and only if *the corresponding set of binomials* $\{x^{b_i^+} - x^{b_i^+}\}$ *generates the toric ideal* $I_A := (x^u - x^v : u - v \in \ker_{\mathbb{Z}} A)$.

One of the remarkable consequences of this theorem is that the existence of a finite Markov basis for *any* model that can be defined by such a matrix is now guaranteed by the Hilbert basis theorem. Besides this being a lovely mathematical result connecting commutative algebra with statistics, it turns out that Markov bases are a necessary tool for reasoning with certain types of data such as the Japanese corporate example.

Formal Reasoning with Data

In order to restate our opening questions more formally, let us think like statisticians: buying shares is a random event that occurs with some probability. The *New York Times* article suggests that this probability is governed in large part by reciprocity: If you own shares in my company, I am likely to buy shares in yours. If we can construct a statistical model that produces relational data where reciprocity matters, the question then becomes whether such a model fits the observed set of relationships. In other words, we seek to find out whether such a model can adequately explain how the share ownership data was generated.

The statistics literature provides us with a model that was designed to capture precisely this type of a reciprocal relationship [3, 4]. The model comes equipped with an integer matrix: it is *log-linear* in form, an example of a discrete exponential family, defined in the next section. We will test its goodness of fit to the observed data using an *exact conditional test*, defined in the following section, which requires an understanding of a certain conditional probability distribution. Markov bases are a key ingredient to this step.

Statistical reasoning then proceeds as follows: The *New York Times* article's claim can be restated from the perspective of statistical models as follows: *Under the assumption* that the random event of buying shares is governed by reciprocity, the relationship between Japanese corporations is *not* unusual. If this is correct, then the model on dyadic relationships with non-zero reciprocation effect fits the corporate network data, while the one in which reciprocation effect is set to zero does not. That is, the observed data is extreme/an outlier in the latter case, and not so in the former. As there are too many possible share-buying scenarios within 17 companies to which we should compare the observed data, we instead sample from a carefully chosen reference set, one that makes statistical sense. Markov bases, defined for any log-linear model, are used to design an MCMC sampling scheme of this reference set.

The A Matrix: Log-Linear Models

Log-linear models are a class of statistical models for discrete data for which logarithms of joint probabilities are captured by a linear map as follows. Let $X_1, \dots, X_k$ be discrete random variables with X_i taking values in $[d_i]$. A k-way *contingency table* $u \in \mathbb{Z}_{\geq 0}^{d_1\times\cdots\times d_k}$ is a nonnegative integer table whose $(i_1, \dots, i_k)$-entry counts the number of times the event $\{X_1 = i_1, \dots, X_k = i_k\}$ occurred; we think of u as a realization of a random table U. A typical example of the use of such tables is in a cross-classification of items into k categories (e.g., salary levels by gender). Fix an integer matrix $A \in \mathbb{Z}^{m\times(d_1\cdots d_k)}$ such that $(1, \dots, 1)$ is in its row span.[1] Flatten the data table u into a $d_1 \cdots d_k$ column vector. We may interpret the vector Au as a summary of the data table u.

Definition. With the above setup, the statistical model for which the summary Au suffices to capture the probability of u is called *the log-linear model* $\mathcal{M}_A$ *for* k*-way tables associated to the matrix* A. It is the family of probability distributions of the following form:

$$P_\theta(U = u) = \exp\{\langle Au, \theta\rangle - \psi(\theta)\},$$

where $\theta \in \mathbb{R}^m$ is the vector of model parameters and $\psi(\theta)$ is the normalizing constant.[2] The i-th entry of the vector Au is called the *minimal sufficient statistic* for the parameter θ_i. The matrix A that computes the sufficient statistics is called the *design matrix* of $\mathcal{M}_A$.

Consider the simple example of independence of two discrete random variables, X and Y, taking values in $[d_1]$ and $[d_2]$, respectively. Let $\alpha_i = P(X = i)$ and $\beta_i = P(Y = i)$ denote the marginal probabilities of X and Y. The model $\mathcal{M}$ of independence postulates that the joint probabilities factor as $P(X = i, Y = j) = \alpha_i\beta_j$. In the language of [2, Definition 1.1.9], $\mathcal{M}$ is the toric model associated to A, because the monomial joint probabilities $\alpha_i\beta_j$ parameterize a toric variety. Data on X, Y can be organized in a 2-way table, where the ij-entry counts the number of occurrences of the event $\{X = i, Y = j\}$. Under the model $\mathcal{M}$, to know the probability of observing a

[1] *A normalizing assumption so that all of the details make sense for probability vectors that must sum to* 1.

[2] *This is simply to ensure the probabilities are nonnegative and sum to* 1.

given data table u it suffices to know the marginal probabilities α_is and β_js. The corresponding sufficient statistics are marginal counts—row and column sums—of the data table u. As computing these marginals is a linear operation, it can be presented as a linear map $u \mapsto Au$, where $A \in \mathbb{Z}^{(d_1+d_2)\times d_1 d_2}$ and u is flattened to a $d_1 d_2 \times 1$ vector.

The Weight of the Evidence: Exact Conditional p-Value

What, then, is the conditional test for whether a log-linear model $\mathcal{M}_A$ fits the observed data table u? As we seek to answer whether u is more-or-less expected under $\mathcal{M}_A$, the test approximates the *exact conditional p-value* of u: the probability of a data table being more extreme (less expected) than u, conditional on the observed values of the sufficient statistics. Since sufficient statistics offer a summary of u that fully captures its probability of occurring under $\mathcal{M}_A$, it is reasonable to condition on the value of Au and explore the resulting distribution and set of tables. The set

$$\mathcal{F}_A(u) := \{v \in \mathbb{Z}_{\geq 0}^{d_1\times\ldots\times d_k} : Au = Av\}$$

is called *the fiber of u under the model $\mathcal{M}$*, since it is a fiber of the linear map defined by A.

Definition. A *Markov basis* of the model $\mathcal{M}_A$ is any set of tables $\mathcal{B} := \{b_1, \ldots, b_n\} \subset \mathbb{Z}^{d_1\times\ldots\times d_k}$, called 'moves,' for which

$$Ab_i = 0$$

and such that for any data table $u \in \mathbb{Z}_{\geq 0}^{d_1\times\ldots\times d_k}$ and for any $v \in \mathcal{F}_A(u)$, there exist $b_{i_1}, \ldots, b_{i_N} \in \mathcal{B}$ that can be used to reach v from u:

$$u + b_{i_1} + \ldots + b_{i_N} = v$$

while walking through elements of the fiber:

$$u + \sum_{j=0}^{N} b_{i_j} \geq 0, \text{ componentwise}$$

for any $j = 1 \ldots N$.

Note that $Au = A(u + b_i)$ means that adding a move b_i to any data table does not change the values of the sufficient statistics, so to remain on the fiber, we only need to ensure that adding a move did not produce negative table entries.

What's in a Basis?

The notion of a Markov basis is different (stronger) than that of a basis in linear algebra. Fixing a model and an observed data point results in a fixed conditional distribution of interest. Think of the finitely many points in this distribution as lying on an integer lattice and Markov moves as vectors that can be added to a fixed starting point to create a random walk on the lattice. The set is a basis in the sense that such a random walk is guaranteed to connect all points on the fiber without "stepping outside."

Every Markov basis contains a linear-algebra basis of the null space of A. Although the latter can be used to reach all tables in the fiber, it will generally fail to satisfy the second condition that each intermediate step is a legal table, since adding one of the basis elements to some data table may inadvertently make some table entries negative, even while preserving the values of the sufficient statistics. To satisfy the non-negativity condition, a combination of several null space basis elements may have to be used as a single Markov move in order to reach or move away from a particular table in the fiber. Algebraically, this can be stated as the fact that a generating set of the toric ideal I_A can be obtained by saturation from the lattice basis ideal defined by A; for more details, see [2, §1.3].

The Algebraic Advantage Has Its Challenges

Markov bases are one of the two popular ways to sample from the conditional distribution on the fiber (the other is called sequential importance sampling). Fibers $\mathcal{F}_A(u)$ are generally far too large to enumerate for most reasonably sized matrices A in practice, and thus exploring them via random walks is a natural alternative. The basic idea of Markov bases is therefore quite straightforward, yet it has provided a multitude of open problems over the past two decades. The terminology was coined in [5] and it has become a cornerstone of one area of algebraic statistics.

Let us revisit the algebraic setup. Looking back to the independence model example, let us arrange the joint probabilities of X and Y in a table $p \in [0,1]^{d_1\times d_2}$. A probability table p is in the model if and only if it is of rank 1, or, equivalently, can be written as an outer product of the two marginal probability vectors ($p_{ij} = \alpha_i\beta_j$). Rank-one is of course a determinantal condition: $p_{ij}p_{kl} - p_{il}p_{kj} = 0$ for all i, j, k, l. This binomial corresponds to a Markov move that replaces the 1s in positions il and kj of the table with 1s in positions ij and kl. It is one of the defining polynomials of the toric ideal associated to the design matrix of the independence model. Hilbert's basis theorem guarantees that this ideal is finitely generated; in fact, *any set of generators of this ideal is a Markov basis of the model.* The correspondence between bases connecting the fibers and generating sets of toric ideals is often called the Fundamental Theorem of Markov Bases [2, §1.3]. Random walks on fibers constructed using Markov moves come with certain convergence guarantees—if they are used as proposal moves in a Metropolis-Hastings algorithm to sample the fiber, the stationary distribution of that Markov chain will—by design—be precisely the conditional distribution we are interested in sampling.

How complicated are Markov bases? In a wonderful theorem about decomposable graphical models, a special class of models that can be broken into components recursively, Dobra provided a divide-and-conquer strategy to compute Markov bases for all such models on tables of any dimension. Then came the fundamental bad news result of De Loera and Onn: the easiest of non-decomposable models on three-dimensional contingency tables is such that Markov bases can be "as complicated as you can imagine" if two of the dimensions are allowed to grow. (If two dimensions are fixed, then the moves are of bounded complexity. All of these results are summarized in [2, §1.2].) So a natural question arises: if the model is not decomposable, how does one compute a Markov basis or verify that a proposed set of moves constitutes one? One general strategy is to use the so-called distance-reducing method [6]: consider two arbitrary points in the fiber and show that the distance between them can be reduced by applying some of the proposed moves.

This hints at an obvious limitation of Markov bases: *most* of the moves are unnecessary in many scenarios! Specifically, the bases are *data-independent* by definition: for a fixed A, they connect the fiber $\mathcal{F}_A(u)$ for *any* data table u, so that for a specific observed data table u, many of the moves are inapplicable. For example, the following move (right) from the independence model is not applicable to the table on the left:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Even though it preserves the row and column sums, it produces a negative entry, which is not a 'legal' data table that counts the number of occurrences of the event $\{X = 3, Y = 1\}$. Other restrictions—such as sampling constraints, maximum on table entries—make the issue worse.

The literature offers a myriad of results on Markov bases that address these problems: structural results of and complexity bounds for moves for many classes of models, dynamic algorithms that construct only applicable moves and not an entire basis, larger bases that guarantee connectedness of restricted fibers such as those consisting only of 0/1 tables, etc.

The p-Value of Shareownership

Let us finally address Prime Minister Shinzo Abe's aim to diversify the interlocking Japanese corporations. The model of interest for this data is also log-linear in form as shown in [4]: its sufficient statistics are, for each company i, the number of companies in which i owns shares, the number of companies that own shares in i, and the number of times i reciprocated a shareholding relationship. Computing these counts is a linear operation on the set of companies since it amounts to counting neighbors in the graph above.

We sample the fiber of the observed Japanese corporation data using the dynamic Markov bases implementation for log-linear network models from [7]. For each of the $100,000$ sampled data points, we compute a goodness-of-fit statistic—in this case the chi-square statistic—which measures the distance of the data point from what is expected under the model. We do this for two model variants: first when there is a positive reciprocation effect, and second when the reciprocation effect is zero. We take a look at the histogram of this statistic: *the number of times a 'more extreme' data point is encountered* is the volume of the histogram to the right of the vertical red line marking the observed value. Since these data are *farther* from the expected value than the observed data, the size of the histogram to the right of the line gives the p-value of the data. The p-values of the data under the models with non-zero and zero reciprocations are 0.319 and 0.002, respectively.

So, is there statistical evidence to support Prime Minister Shinzo Abe's claim about strong reciprocation effect in interlocking corporate directories? Indeed, you may decide that there is, by looking at the histograms in Figure 2.

Under the model in which reciprocation effect is present, as many as 31.9% of data points in the sample of

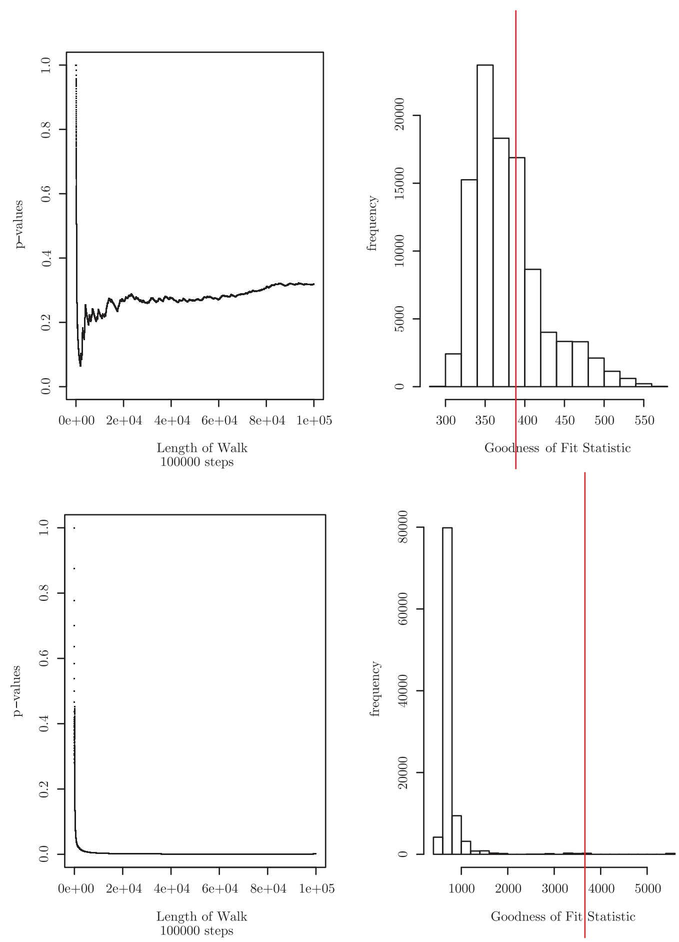

Figure 2. Histogram of the sampling distribution of the goodness-of-fit statistic. Length of each random walk: 100,000 steps. Top: model with nonzero reciprocation effect. Bottom: model with no reciprocation effect. p-values: $p = 0.319$ and $p = 0.002$, respectively.

100,000 are less expected than the observed data, whereas under the model with zero reciprocation, that number is merely 0.2%. Therefore, the model that sets the reciprocation effect to zero does not fit the data. Perhaps the Prime Minister knew about Markov bases.

ACKNOWLEDGMENT. The author is grateful to two PhD students, Lily Silverstein at University of California at Davis and Dane Wilburne at Illinois Institute of Technology, for valuable feedback on a draft of this article in 2018.

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Sonja Petrović

Credits

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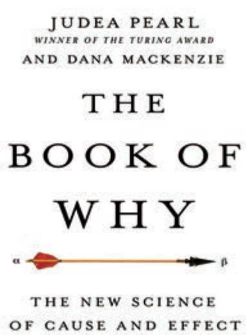
The Book of Why

A review by Lisa R. Goldberg



The Book of Why
The New Science of Cause and Effect
Judea Pearl and Dana Mackenzie
Basic Books, 2018
432 pages
ISBN-13: 978-0465097609

Judea Pearl is on a mission to change the way we interpret data. An eminent professor of computer science, Pearl has documented his research and opinions in scholarly books and papers. Now, he has made his ideas accessible to a broad audience in *The Book of Why: The New Science of Cause and Effect*, co-authored with science writer Dana Mackenzie. With the release of this historically grounded and thought-provoking book, Pearl leaps from the ivory tower into the real world.

The Book of Why takes aim at perceived limitations of *observational studies*, whose underlying data are found in nature and not controlled by researchers. Many believe that an observational study can elucidate association but not cause and effect. It cannot tell you *why*.

Perhaps the most famous example concerns the impact of smoking on health. By the mid 1950s, researchers had established a strong association between smoking and lung cancer. Only in 1984, however, did the US government mandate the phrase "smoking causes lung cancer." The holdup was the specter of a latent factor, perhaps something genetic, that might cause both lung cancer and a craving for tobacco. If the latent factor were responsible for lung cancer, limiting cigarette smoking would not prevent the disease. Naturally, tobacco companies were fond of this explanation, but it was also advocated by the prominent statistician Ronald A. Fisher, co-inventor of the so-called gold standard of experimentation, the Randomized Controlled Trial (RCT).

Subjects in an RCT on smoking and lung cancer would have been assigned to smoke or not on the flip of a coin. The study had the potential to disqualify a latent factor as the primary cause of lung cancer and elevate cigarettes to the leading suspect. Since a smoking RCT would have been unethical, however, researchers made do with observational studies showing association, and demurred on the question of cause and effect for decades.

Was the problem simply that the tools available in the 1950s and 1960s were too limited in scope? Pearl addresses that question in his three-step Ladder of Causation, which organizes inferential methods in terms of the problems they can solve. The bottom rung is for model-free statistical methods that rely strictly on association or correlation. The middle rung is for interventions that allow for the measurement of cause and effect. The top rung is for counterfactual analysis, the exploration of alternative realities.

Early scientific inquiries about the relationship between smoking and lung cancer relied on the bottom rung, model-free statistical methods whose modern analogs dominate the analysis of observational studies today. In one of *The Book of Why*'s many wonderful historical anecdotes, the predominance of these methods is traced to the work of Francis Galton, who discovered the principle of regression to the mean in an attempt to understand the process that drives heredity of human characteristics. Regression to the mean involves association, and this led Galton

Lisa Goldberg is a co-director of the Consortium for Data Analytics in Risk and an adjunct professor of Economics and Statistics at University of California, Berkeley. She is a director of research at Aperio Group, LLC. Her email address is lrg@berkeley.edu.

Communicated by Notices *Book Review Editor Stephan Ramon Garcia.*

DOI: https://doi.org/10.1090/noti1912

and his disciple, Karl Pearson, to conclude that association was more central to science than causation.

Pearl places deep learning and other modern data mining tools on the bottom rung of the Ladder of Causation. Bottom rung methods include AlphaGo, the deep learning program that defeated the world's best human Go players in 2015 and 2016 [1]. For the benefit of those who remember the ancient times before data mining changed everything, he explains,

> The successes of deep learning have been truly remarkable and have caught many of us by surprise. Nevertheless, deep learning has succeeded primarily by showing that certain questions or tasks we thought were difficult are in fact not.

The issue is that algorithms, unlike three-year-olds, do as they are told, but in order to create an algorithm capable of causal reasoning,

> ...we have to teach the computer how to selectively break the rules of logic. Computers are not good at breaking rules, a skill at which children excel.

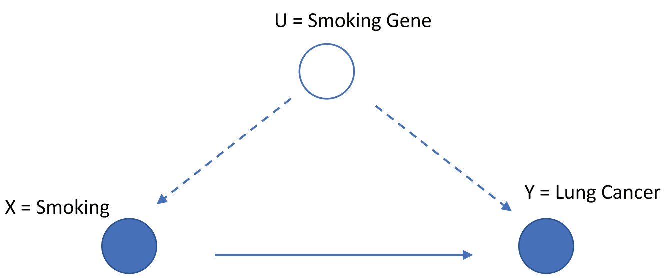


Figure 1. Causal model of assumed relationships among smoking, lung cancer, and a smoking gene.

Methods for extracting causal conclusions from observational studies are on the middle rung of Pearl's Ladder of Causation, and they can be expressed in a mathematical language that extends classical statistics and emphasizes graphical models.

> Various options exist for causal models: causal diagrams, structural equations, logical statements, and so forth. I am strongly sold on causal diagrams for nearly all applications, primarily due to their transparency but also due to the explicit answers they provide to many of the questions we wish to ask.

The use of graphical models to determine cause and effect in observational studies was pioneered by Sewall Wright, whose work on the effects of birth weight, litter size, length of gestation period, and other variables on the weight of a 33-day-old guinea pig is in [2]. Pearl relates Wright's persistence in response to the cold reception his work received from the scientific community.

> My admiration for Wright's precision is second only to my admiration for his courage and determination. Imagine the situation in 1921. A self-taught mathematician faces the hegemony of the statistical establishment alone. They tell him "Your method is based on a complete misapprehension of the nature of causality in the scientific sense." And he retorts, "Not so! My method is important and goes beyond anything you can generate."

Pearl defines a *causal model* to be a directed acyclic graph that can be paired with data to produce quantitative causal estimates. The graph embodies the structural relationships that a researcher assumes are driving empirical results. The structure of the graphical model, including the identification of vertices as mediators, confounders, or colliders, can guide experimental design through the identification of minimal sets of control variables. Modern expositions on graphical cause and effect models are [3] and [4].

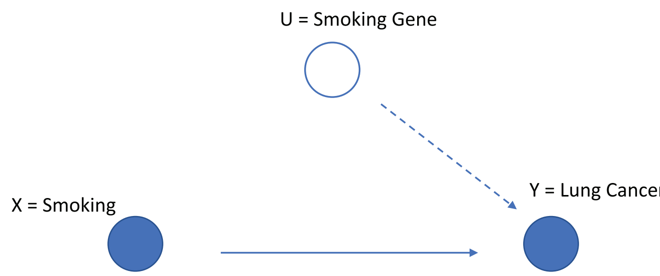


Figure 2. Mutated causal model facilitating the calculation of the effect of smoking on lung cancer. The arrow from the confounding smoking gene to the act of smoking is deleted.

Within this framework, Pearl defines the *do operator*, which isolates the impact of a single variable from other effects. The probability of Y do X, $P[Y|\mathrm{do}(X)]$, is not the same thing as the conditional probability of Y given X. Rather $P[Y|\mathrm{do}(X)]$ is estimated in a mutated causal model, from which arrows pointing into the assumed cause are removed. *Confounding* is the difference between $P[Y|\mathrm{do}(X)]$ and $P[Y|X]$. In the 1950s, researchers were after the former but could estimate only the latter in observational studies. That was Ronald A. Fisher's point.

Figure 1 depicts a simplified relationship between smoking and lung cancer. Directed edges represent assumed causal relationships, and the smoking gene is represented by an empty circle, indicating that the variable was not observable when the connection between smoking and cancer was in question. Filled circles represent quantities that could be measured, like rates of smoking and lung cancer in a population. Figure 2 shows the mutated causal model that isolates the impact of smoking on lung cancer.

The conclusion that smoking causes lung cancer was eventually reached without appealing to a causal model. A crush of evidence, including the powerful sensitivity analysis developed in [5], ultimately swayed opinion. Pearl argues that his methods, had they been available, might have resolved the issue sooner. Pearl illustrates his point in a hypothetical setting where smoking causes cancer only by depositing tar in lungs. The corresponding causal diagram is shown in Figure 3. His *front door formula* corrects for the confounding of the unobservable smoking gene without ever mentioning it. The bias-corrected impact of smoking, X, on lung cancer, Y, can be expressed

$$P[Y|\text{do}(X)] = \sum_{Z} P[Z|X] \sum_{X'} P[Y|X', Z] P[X'].$$

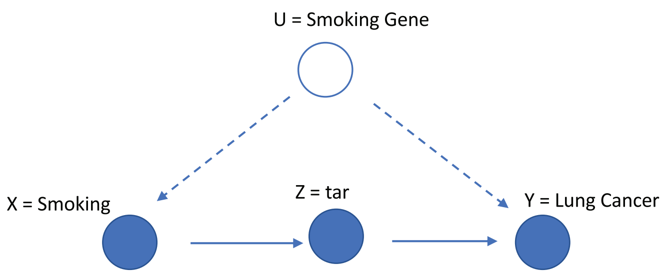


Figure 3. Pearl's front door formula corrects for bias due to latent variables in certain examples.

The Book of Why draws from a substantial body of academic literature, which I explored in order to get a more complete picture of Pearl's work. From a mathematical perspective, an important application is Nicholas Christakis and James Fowler's 2007 study described in [6] arguing that obesity is contagious. The attention-grabbing claim was controversial because the mechanism of social contagion is hard to pin down, and because the study was observational. In their paper, Christakis and Fowler upgraded an observed association, clusters of obese individuals in a social network, to the assertion that obese individuals cause their friends, and friends of their friends, to become obese. It is difficult to comprehend the complex web of assumptions, arguments, and data that comprise this study. It is also difficult to comprehend its nuanced refutations by Russell Lyons [7] and by Cosma Shalizi and Andrew Thomas [8], which appeared in 2011. There is a moment of clarity, however, in the commentary by Shalizi and Thomas, when they cite Pearl's theorem about *non-identifiability* in particular graphical models. Using Pearl's results, Shalizi and Thomas show that in the social network that Christakis and Fowler studied, it is impossible to disentangle contagion, the propagation of obesity via friendship, from the shared inclinations that led the friendship to be formed in the first place.

The top rung of the Ladder of Causation concerns counterfactuals, which Michael Lewis brought to the attention of the world with his best selling book, *The Undoing Project* [9]. Lewis tells the story of Israeli psychologists Daniel Kahneman and Amos Tversky, experts in human error, who fundamentally changed our understanding of how we make decisons. Pearl draws on the work of Kahneman and Tversky in *The Book of Why*, and Pearl's approach to analyzing counterfactuals might be best explained in terms of a question that Kahneman and Tversky posed in their study [10] of how we explore alternative realities.

> How close did Hitler's scientists come to developing the atom bomb in World War II? If they had developed it in February 1945, would the outcome of the war have been different?
>
> —The Simulation Heuristic

Pearl's response to this question includes the *probability of necessity* for Germany and its allies to have won World II had they developed the atom bomb in 1945, given our historical knowledge that they did not have an atomic bomb in February 1945 and lost the war. If Y denotes Germany winning or losing the war (0 or 1) and X denotes Germany having the bomb in 1945 or not having it (0 or 1), the probability of necessity can be expressed in the language of potential outcomes,

$$P\left[Y_{X=0} = 0 \mid X = 1, Y = 1\right].$$

Dual to the *probability of sufficiency*, the probability of necessity mirrors the legal notion of "but-for" causation as in: but for its failure to build an atomic bomb by February 1945, Germany would probably have won the war. Pearl applies the same type of reasoning to generate transparent statements regarding climate change. Was anthropogenic global warming responsible for the 2003 heat wave in Europe? We've all heard that while global warming due to human activity tends to raise the probability of extreme heat waves, it is not possible to attribute any particular event to this activity. According to Pearl and a team of climate scientists, the response can be framed differently: There is a 90% chance that the 2003 heat wave in Europe would not have occurred in the absence of anthropogenic global warming [11].

This formulation of the impact of anthropogenic global warming on the earth is strong and clear, but is it correct? The principle of garbage-in-garbage-out tells us that results based on a causal model are no better than its underlying assumptions. These assumptions can represent a researcher's knowledge and experience. However, many scholars are concerned that model assumptions represent researcher bias, or are simply unexamined. David Freedman emphasizes this in [12], and as he wrote more recently in [13],

> Assumptions behind models are rarely articulated, let alone defended. The problem is exacerbated because journals tend to favor a mild degree of novelty in statistical procedures. Modeling, the search for significance, the preference for novelty, and the lack of interest in assumptions—these norms are

> likely to generate a flood of non-reproducible results.
>
> —Oasis or Mirage?

Causal models can be used to work backwards from conclusions we favor to supporting assumptions. Our tendency to reason in the service of our prior beliefs is a favorite topic of moral psychologist Jonathan Haidt, author of *The Righteous Mind* [14], who wrote about "the emotional dog and its rational tail." Or as Udny Yule explained in [15],

> Now I suppose it is possible, given a little ingenuity and good will, to rationalize very nearly anything.
>
> —1926 presidential address to the Royal Statistical Society

Concern about the impact of biases and preconceptions on empirical studies is growing, and it comes from sources as diverse as Professor of Medicine John Ioannides, who explained why most published research findings are false [16]; comedian John Oliver, who warned us to be skeptical when we hear the phrase "studies show" [17]; and former *New Yorker* writer Jonah Lehrer, who wrote about the problems with empirical science in [18] but was later discredited for representing stuff he made up as fact.

The graphical approach to causal inference that Pearl favors has been influential, but it is not the only approach. Many researchers rely on the Neyman (or Neyman–Rubin) potential outcomes model, which is discussed in [19], [20], [21] and [22]. In the language of medical randomized control trials, a researcher using this model tries to quantify the difference in impact between treatment and no treatment on subjects in an observational study. Propensity scores are matched in an attempt to balance inequities between treated and untreated subjects. Since no subject can be both treated and untreated, however, the required estimate of impact is sometimes formulated as a missing value problem, a perspective that Pearl strongly contests. In another direction, the concept of fixing, developed by Heckman in [23] and Heckman and Pinto in [24], resembles, superficially at least, the do operator that Pearl uses. Those who enjoy scholarly disputes may look to Andrew Gelman's blog, [25] and [26], for back-and-forth between Pearl and Rubin disciples (Rubin himself does not seem to participate—in that forum, at least) or to the tributes written by Pearl [27] and Heckman and Pinto [24] to the reclusive Nobel Laureate, Trygve Haavelmo, who pioneered causal inference in economics in the 1940s in [28] and [29]. These dialogs have been contentious at times, and they bring to mind Sayre's law, which says that academic politics is the most vicious and bitter form of politics because the stakes are so low. It is this reviewer's opinion that the differences between these approaches to causal inference are far less important than their similarities. Support for this includes constructions by Pearl in [3] and by Thomas Richardson and James Robins in [30] incorporating counterfactuals into graphical cause-and-effect models, thereby unifying various threads of the causal inference literature.

Figure 4. National Transportation Safety Board inspectors examining the self-driving Uber that killed a pedestrian in Tempe, Arizona on March 18, 2018.

Late one afternoon in July 2018, Pearl's co-author Dana Mackenzie spoke on causal inference at UC Berkeley's Simons Institute. His presentation was in the first person singular from Pearl's perspective, the same voice used in *The Book of Why*, and it concluded with an image of the first self-driving car to kill a pedestrian. According to a report [31] by the National Transportation Safety Board (NTSB), the car recognized an object in its path six seconds prior to the fatal collision. With a lead time of a second and a half, the car identified the object as a pedestrian. When the car attempted to engage its emergency braking system, nothing happened. The NTSB report states that engineers had disabled the system in response to a preponderance of false positives in test runs.

The engineers were right, of course, that frequent, abrupt stops render a self-driving car useless. Mackenzie gently and optimistically suggested that endowing the car with a causal model that can make nuanced judgments about pedestrian intent might help. If this were to lead to safer and smarter self-driving cars, it would not be the first time that Pearl's ideas led to better technology. His foundational work on Bayesian networks has been incorporated into cell phone technology, spam filters, bio-monitoring, and many other applications of practical importance.

Professor Judea Pearl has given us an elegant, powerful, controversial theory of causality. How can he give his theory the best shot at changing the way we interpret data? There is no recipe for doing this, but teaming up with science writer and teacher Dana Mackenzie, a scholar in his own right, was a pretty good idea.

ACKNOWLEDGMENT. This review has benefitted from dialogs with David Aldous, Bob Anderson, Wachi Bandera, Jeff Bohn, Brad DeLong, Michael Dempster, Peng Ding, Tingyue Gan, Nate Jensen, Barry Mazur, Liz Michaels, LaDene Otsuki, Caroline Ribet, Ken Ribet, Stephanie Ribet, Cosma Shalizi, Alex Shkolnik, Philip Stark, Lee Wilkinson, and the attendees of the University of California, Berkeley Statistics Department social lunch group. Thanks to Nick Jewell for informing me about scientific studies on the relationship between exercise and cholesterol, which enhanced my appreciation of *The Book of Why*.

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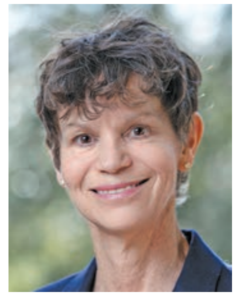

Lisa R. Goldberg

Credits

Figures 1–3 are by the author.

Figure 4 is courtesy of the National Transportation Safety Board (NTSB).

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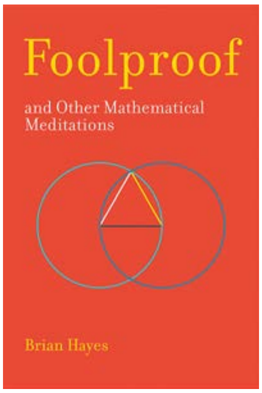


MIT Press, 2017, 248 pages.

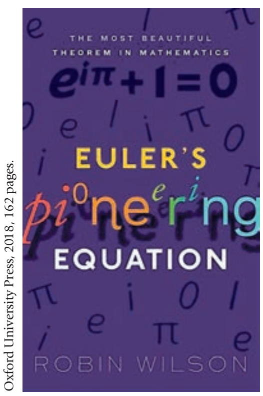
Foolproof and Other Mathematical Meditations
by Brian Hayes

This book is a collection of thirteen essays, each about fifteen pages or so, on a range of topics that includes enumerating Sudoku puzzle grids, self-avoiding random walks, the nature of mathematical proof, space-filling curves, and the numerical computation of π. The essays are composed with relatively few equations or formulas, and the book includes plenty of gray-scale illustrations. It is accessible to anyone with an undergraduate mathematics degree.

Although the typical mathematician may have a passing familiarity with many of the subjects covered in the book, I suspect that most readers will find plenty of new material or, at least, new takes on old favorites. Moreover, it is not just the mathematics that is surveyed, but also the history and evolution of the topics. The first essay, "Young Gauss Sums It Up," furnishes a perfect example. Most mathematicians are familiar with the anecdote about the harsh schoolmaster who assigned the young Gauss and his classmates the sum 1+2+…+100. However, many of us will be surprised to learn that there are several distinct variants of this tale and that the modern version, familiar to us all, was the result of years of evolution. In this opening essay, Hayes goes back to the original sources, traces different renditions of the Gauss story throughout time, and includes several fascinating figures that indicate the frequency of various occurrences over the years.

The remaining topics are varied, although analysis, probability, and combinatorics feature most prominently. The book is self-contained but augmented with occasional bits of pseudocode with which the interested reader might verify Hayes's calculations or explore on their own. Most of the essays contain a healthy dose of historical context and background. For example, Hayes's discussion of Markov chains, a fairly standard topic in linear algebra and probability courses, spends a good deal of time recounting Markov's use of the technique to study Russian poetry.

Overall, this book covers a wide range of material in a thought-provoking and inviting manner. It should satisfy a wide audience, and most mathematicians will have something to gain from it.



Oxford University Press, 2018, 162 pages.

Euler's Pioneering Equation: The Most Beautiful Theorem in Mathematics
by Robin Wilson

Euler's beautiful formula $e^{i\pi}+1=0$ unites in one elegant equation the five most important constants in mathematics. Each of the five numbers involved in this famous identity has its own unique history and peculiar properties. Wilson studies each constant in turn, paying careful attention to the historical development of the mathematics and the notation behind it.

Although this book might be regarded as a "popular science" title, it understandably involves more mathematics than one usually sees at that level. For example, infinite series, integration, and complex numbers need to be reckoned with. There are lots of equations and diagrams, along with a fair number of pictures.

Most mathematicians will find this a quick and enjoyable read, while perhaps picking up a historical anecdote or cute result along the way. On the other hand, analysts, number theorists, and historians of mathematics might find little in this book that they do not already know. The ideal reader might be a curious student of calculus, who would find this book a fascinating invitation to pure mathematics.

Note: The author Robin Wilson is an emeritus faculty member at the Open University in the UK and should not be confused with the Robin Wilson who is on the faculty at California Polytechnic University and is on the editorial board of the Notices.

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Joining a Mathematical Research Community

Leslie Hogben and T. Christine Stevens

Mathematics has always had research communities, although that terminology was not used and they were usually not as open as they are today. A research community can serve as a professional neighborhood, providing friendship (networking) and access to local knowledge (professional development). Research communities play an important role in keeping people engaged in research, helping them succeed in their careers, and shaping the discipline of mathematics. Traditional research communities have frequently been unintentionally exclusive, often being based on informal power networks and sometimes requiring substantial resources for travel.

In many cases those underrepresented in research mathematics, such as women, minorities, persons with disabilities, and faculty in departments without doctoral programs, have been even more underrepresented in research communities. To ensure that all mathematicians have access to the benefits of research communities, there has been an intentional effort over the past ten to twenty years to build strong research communities with well-publicized open application processes. All four programs profiled here welcome applications and are clear about who is eligible. Each is supported by grants from the National Science Foundation and other organizations, and such support has played an important role in the development of research communities with open application processes.

By a research community, we mean a group of mathematicians with some common interests who collaborate on research with some other members of the community and who support each other professionally, but which is larger than a group of co-authors even if they work together on an ongoing basis. This definition is intentionally rather loose and broad, and is perhaps best illustrated by the four quite different research community programs we have chosen to profile. Three of them—the Mathematics Research Communities (MRC), the Research Collaboration Conferences for Women (RCCW), and the Park City Math Institute (PCMI)—build a research community around a single research theme. The fourth, the Research Experiences for Undergraduate Faculty (REUF), builds a community of researchers who support research with undergraduate students, with subgroups focused on different research topics. MRC, RCCW, and REUF have limited target audiences, serving, respectively, junior researchers, women, and faculty at undergraduate institutions. PCMI is vertically integrated, bringing together groups of researchers, graduate students, undergraduate faculty, undergraduate students, and secondary school teachers. MRC, RCCW, and REUF are each one week long and include continuation activities, whereas PCMI is three weeks long. PCMI is the oldest, beginning in 1991 as a Regional Geometry Institute and as PCMI in 1995, with the others operating first in 2008 (in at least some preliminary form). Each has a major research emphasis (and produces published research), but also includes professional development and networking opportunities. Each includes multiple groups of collaborators. Each MRC has either twenty or forty participants, and an RCCW usually has thirty to fifty people, with several workshops each year. REUF has about twenty participants per annual workshop,

Leslie Hogben is Dio Lewis Holl Chair in Applied Mathematics, professor of mathematics, and associate dean for graduate studies and faculty development at the College of Liberal Arts and Sciences at Iowa State University, and associate director for diversity at the American Institute of Mathematics. Her email address is hogben@aimath.org.

T. Christine Stevens is associate executive director for meetings and professional services at the AMS. Her email address is tcs@ams.org.

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DOI: https://dx.doi.org/10.1090/noti1913

and PCMI has about 300 people in several subprograms that meet in parallel at the same site each year. PCMI, REUF, and MRC all meet in the summer, as do many but not all RCCW workshops.

Here we have profiled four research community programs, but there are many others. For example, the NSF Mathematical Sciences Institutes offer many research opportunities for which there are open applications [8], and some of these create research communities. Additional information about opportunities to join a mathematical research community (and about other professional development opportunities and communities, including those that support teaching) can be found in the Mathematics Opportunities section of each month's *Notices* [9] and on the listing of Awards, Fellowships, and Other Opportunities on the AMS website [2].

Mathematics Research Communities (MRC)

The AMS developed the Mathematics Research Communities (MRC) as a national program to support and guide young mathematicians as they begin their research careers. Currently serving 120 early career mathematicians each year, the program fosters the creation of collaborative research groups and provides the participants with mentoring from leading scholars in their fields. Each year, three to five mathematical research areas are chosen as topics for MRCs, each of which includes twenty or forty participants.

An MRC is a year-long experience that begins with an intensive one-week conference in the summer. A typical MRC conference includes little or no lecturing, with the participants instead spending most of their time working in groups on open problems related to the topic of the conference. To make this format possible, the participants may interact electronically with the organizers before the conference and do some background preparation. The week also includes professional development sessions that address the concerns of graduate students and recent PhDs, such as applying for jobs, choosing appropriate venues for publications, and writing grant proposals. The conferences are currently held at Whispering Pines Conference Center in West Greenwich, Rhode Island.

MRC participants spend most of their time working collaboratively on research problems, as this group did in 2018.

The MRC experience continues at the Joint Mathematics Meetings (JMM) the following January and in research collaborations throughout the year. Three of the participants in each summer conference are invited to organize an AMS Special Session at the JMM on the topic of the MRC. Although some of the talks in this Special Session are given by the MRC participants themselves, this event also serves as an opportunity to expand the research community to include people at all career stages who did not attend the summer conference. Both before and after the JMM, participants are eligible for travel funding for follow-up collaborations that enable them to finish the projects that they launched at the summer conference and to initiate new ones. After their year of formal MRC participation is over, MRC alumni can request funding for "microconferences," which are gatherings of ten to fifteen mathematicians that advance the mathematical work emanating from their original MRC summer conference.

Through 2018, there have been forty different MRCs on topics that range throughout the mathematical sciences [3]. Pending funding, there will be five MRCs in 2020, on dynamical models of the ecology of infectious diseases, using combinatorics and linear algebra to attack inverse problems, combinatorial applications of computational topology and algebraic geometry, analysis in metric spaces, and new problems in several complex variables. The application portal for 2020 will open in the summer of 2019.

MRC funding is open to early career individuals who are affiliated with US institutions or are US citizens or permanent residents. A few international participants may be accepted. Applicants from academic institutions of all types are welcome, as well as from private industry and government laboratories and agencies. The goal is to create a collaborative research community that is vibrant, productive, and diverse. All participants are expected to be active in the full array of MRC activities—the summer conference, Special Sessions at the Joint Mathematics Meetings, and follow-up collaborations.

Since its inception in 2008, the MRC program has helped some 1,300 young mathematicians to launch their research careers. Their collaborations are productive, with many reporting publications that emerged from their work at the MRCs. Here are some comments that members of the 2013 cohort made in 2018 when they were asked to reflect upon the impact of their MRC experience:

- My MRC mentor was my mentor for my first postdoc, we wrote a paper that was published in *Geometry and Topology*, and we continue to talk to this day, even after

I have moved to another institution.... MRC had a positive impact on the social aspect of my career too, as I continue to see many of my fellow participants at conferences and meetings.

- The MRC has played an important role in my career. The one week of intense mathematical activity shaped my research plans for the coming years. The program led to two papers and equally importantly, it gave me the opportunity to interact with several active postdoctoral researchers and graduate students working in tropical geometry. The mentoring provided by the two experts was very valuable.

The organizers also praise the format of the MRCs, often commenting on the satisfaction they derive from "seeing the participants from such different backgrounds gel and start making real serious progress on the problems." As a 2018 organizer put it:

> I am really glad that we went "all in" with the philosophy of the MRC program and devoted as much of the scheduled time to group work as possible. This was a very different approach than most of us were used to, but we got a lot of strong feedback during the week about how participants preferred the approach to other summer schools, etc. they had done.

More information about organizing an MRC can be found in [10]. Pending funding, proposals for the 2021 MRC conferences will be due in August 2019.

Who: Graduate students, postdocs, and early career faculty—people with PhD age in the interval [-2, 5]. In each given week, there is either one MRC with forty participants or two with twenty each, for a total of 120 participants per year.

What: One-week summer conference, Special Session at the Joint Mathematics Meetings (JMM), and follow-up collaborations.

When: Summer conference is in June; JMM is in January.

Where: For 2018 and 2019, the conference site was the Whispering Pines Conference Center in West Greenwich, RI.

Website: www.ams.org/programs/research-communities/mrc

Apply: Online via MathPrograms.org. The application portal for each summer conference opens in the previous summer; applications are due in February.

Park City Mathematics Institute (PCMI)

The Park City Mathematics Institute (PCMI) builds a vertically integrated community by annually running several parallel programs around a chosen research theme. Its constituent programs include the Research Program (RP), Graduate Summer School (GSS), Undergraduate Faculty Program (UFP), Undergraduate Summer School (USS), and Teacher Leadership Program (TLP), each lasting for three weeks, and the Workshop on Equity and Mathematics Education (WEME), which takes place during the first of the three weeks. Participants engage in program-specific activities but also interact through social and cross-program academic activities. Promoting such interaction between participants from across the entire spectrum of the mathematics profession is one of the key goals of PCMI, which is an outreach effort of the Institute for Advanced Study (IAS) and is now in its twenty-ninth year.

Participants at PCMI work collaboratively, as this group did in 2018.

The research theme for PCMI 2020 is "Number Theory Informed by Computation," organized by Jennifer Balakrishnan (Boston University), Kristin Lauter (Microsoft Research), Bjorn Poonen (MIT), and Akshay Venkatesh (IAS and Stanford). Research themes from the recent past include "Quantum Field Theory and Manifold Invariants" (2019), "Harmonic Analysis" (2018), "Random Matrices" (2017), and "The Mathematics of Data" (2016).

The GSS consists of several mini-courses delivered by lecturers selected by the organizers of that year's research theme in consultation with the PCMI Steering Committee. The courses range from relatively introductory to advanced, and TA sessions accompanying each lecture help students keep up with the rapid development of material. The RP is a research workshop with seminar talks and plenty of time for participants to work together or form new collaborations. The organizers invite a core of senior researchers to the RP, but applications to this program are also welcomed from researchers at all levels post-PhD. Many RP participants attend the graduate mini-courses, and conversely, many graduate students attend RP seminars and interact closely with the researchers in attendance. One job of the Steering Committee and organizers is to make sure that RP attendees do not regard this as "just another conference" where they talk to only a handful of people, but instead talk with participants from across PCMI.

The UFP provides an opportunity for faculty at undergraduate institutions to spark their research, as well as for any university faculty member to engage with themes related to undergraduate education. For example, the UFP participants in 2016 spent the session under the direction of Richard De Veaux (Williams College) designing an undergraduate data science curriculum, which was later published [7]. By contrast, the 2019 UFP session provided an intensive course on the topology of 4-manifolds led by John Etnyre (Georgia Tech) and Paul Melvin (Bryn Mawr), to enable participants to do computational research in this area (with or without undergraduate students). The core of the USS is a set of two parallel lecture courses for undergraduates related to that year's research theme. This is accompanied by opportunities for these students to engage in small-group open-ended research projects, with one another or with other PCMI participants. USS participants are also able to interact with graduate students and faculty from many universities, which leads to more informed decisions about graduate programs to which they might apply.

The TLP is a high-level professional development opportunity for K–12 teachers from around the country. The theme of the WEME changes each year, but focuses broadly on equity and social justice issues as they relate to mathematics education and the mathematics profession.

By building a vertically integrated research community, PCMI creates meaningful interactions between members from different parts of the profession, in particular the mentoring of students and younger researchers by more established researchers. Some hint of participant experience is conveyed by the following quotes:

- Extended period of time with many of the same people allows you to get beyond the "getting to know someone" stage and have "substantive conversations," not only about actual math but also in terms of making connections with people you would not otherwise have heard about. This is a very good thing. (2018 RP)
- I got overwhelming support from professors, fellow undergraduates, and teachers I met. I got so much out of PCMI that it would be impossible to imagine it was only 3 weeks long. (2018 USS)

Who: RP: Researchers (60), GSS: Graduate students (80), USS: Undergraduate students (45), UFP: Faculty at undergraduate colleges and universities (15), WEME: Faculty (10), TLP: K–12 teachers (60).

What: Three-week summer session (WEME takes place only during the first week of PCMI).

When: Summer (the first three weeks of July).

Where: Park City, UT.

Website: `https://www.ias.edu/pcmi`

Apply: Online through the PCMI website. The application page is open from early November; applications for the RP, GSS, UFP, and TLP are due in mid-January; applications for the USS and WEME are typically due a few weeks later.

Research Collaboration Conferences for Women (RCCW)

The Association for Women in Mathematics (AWM) nurtures the building of Research Networks (RNs), which are research communities of women in particular mathematical areas. An RN usually begins with a Research Collaboration Conference for Women (RCCW), which is itself a research community, and an RN may later welcome more new members through additional RCCWs. Here we focus on the RCCW and its subsequent activities because an RN usually grows out of an RCCW, and participating in an RCCW is the way participants join an RN.

Each RCCW is a one-week workshop held at a mathematics institute where junior and senior women collaborate on specific research projects in one area. Women who are junior faculty, postdocs, and advanced graduate students can apply to participate in an RCCW through the host institute. Each participant is placed into a research group led by one or more senior women who plan the research projects in advance. The week of the RCCW is spent doing research collaboratively in teams. A few of the RCCWs have included some male participants, but the majority of

RCCW participants spend most of their time in working groups, such as this group at the Institute for Mathematics and its Applications in 2016.

participants and the leaders are female. To date there are nineteen RNs [6], with five of them scheduled to have their first RCCWs during the next few months.

In most cases the research is only started at the RCCW, and the group continues working together after the conference via electronic communication and follow-up meetings, which include special sessions at the AWM workshops at the Joint Mathematics Meetings and the SIAM Annual Conference and at the AWM Research Symposium (held every two years). In addition to completing a research paper, this continued collaboration helps to form lasting bonds and build a community. Each RCCW is encouraged to publish an associated *Proceedings* volume of the AWM–Springer Series [4], and most do so.

RCCWs and RNs were designed to address the serious underrepresentation of women in mathematical research. The first RCCW was in number theory and was held at the Banff International Research Station (BIRS) in 2008 and led to the creation of the Women In Numbers (WIN) network [12], which has now run six RCCWs and published five volumes of research papers [13]. The success of the WIN network inspired the RCCW and RN model.

RCCWs have been hosted at institutes all over the world including BIRS, Institute for Computational and Experimental Research in Mathematics (ICERM), Institute for Pure and Applied Mathematics (IPAM), Institute for Mathematics and its Applications (IMA), American Institute of Mathematics (AIM), Mathematical Biosciences Institute (MBI), Centre International de Rencontres Mathématiques (France), Lorentz Center (Netherlands), and Nesin Mathematical Village (Turkey).

RCCW participants have been very enthusiastic about the experience, with one commenting:

> The workshop was fantastic. The collaborative and supportive atmosphere was totally different than any other workshop I've attended, in the most wonderful way. I think we all felt comfortable, respected and valued as mathematicians.

Thus far we have described opportunities to participate in an RCCW that has already been organized. There are also opportunities to initiate a new RCCW and develop an RN. The AWM ADVANCE Project provides mentorship and support to organizers wishing to create an RN by organizing its first RCCW, including help finding a conference venue, help developing and submitting a conference proposal, and help soliciting travel funding for participants [5].

Who: Women (faculty, postdocs, often advanced graduate students, usually thirty to fifty per workshop).

What: One-week workshop, subsequent special session at the AWM workshop at the Joint Mathematics Meetings or the SIAM Annual Conference and at the AWM Research Symposium.

When: Dates vary.

Where: A mathematical sciences institute.

Website: `https://awmadvance.org/rccws/`

Apply: Online through the host institute (upcoming RCCWs are listed on the RCCW website under Year 5).

Research Experiences for Undergraduate Faculty (REUF)

The Research Experiences for Undergraduate Faculty (REUF) program builds a community of researchers who support undergraduate research by engaging faculty in departments without doctoral programs in small research groups as part of an annual one-week workshop held at the American Institute of Mathematics (AIM) or at the Institute for Computational and Experimental Research in Mathematics (ICERM). Since it began in 2008, more than 250 undergraduate researchers have been supervised by REUF alumni.

REUF models an undergraduate research experience by throwing participants into research problems for which they have limited background knowledge. They work in research teams that are led by senior faculty with expertise engaging undergraduates in research, and the topics are accessible to undergraduates. As undergraduate researchers often do, research groups frequently begin by experimenting and making conjectures, initially rediscovering known results, acquiring background along the way, and eventually discovering new results. Most of the time at the workshop is spent doing research in teams, but there are also whole-group discussions related to undergraduate research. One participant commented:

> I have greatly benefited from the variety of problems described, the discussion on the logistics of running an undergraduate research program, and the camaraderie with other liberal arts faculty. This has easily been the most worthwhile workshop or conference I've ever attended.

REUF was designed to enhance the ability of faculty at undergraduate colleges and universities to engage their students in research, but has become much more, involving faculty participants in long-term research collaborations, developing expertise in new research areas that have problems accessible to undergraduates, and helping them rediscover the joy of mathematics. Faculty come to REUF in

REUF participants spend most of their time doing research in groups with guidance from the project leader, as this group did at the American Institute of Mathematics in 2018.

a variety of career stages, and some use REUF to rejuvenate their research.

Some groups have continued their collaborations for many years. Members of the generalized symmetric spaces group from the 2013 REUF workshop continue their work, having met for a week in the summer multiple times, in addition to regular electronic meetings during the academic year. They have two papers published as a group and another by a group member with her undergraduate student [11]; more results are in preparation. About their participation in REUF, Jennifer Schaefer (Dickinson College) said:

> I was looking for a research community to develop meaningful collaborations and hoping to expand my research into an area that was more accessible to undergraduate students. Thanks to the REUF program, I found both!

and Vicky Klima (Appalachian State University) commented:

> Our monthly electronic meetings have kept me motivated to stay engaged with our projects throughout the academic year. Setting small goals that take into account our relatively high teaching and service expectations allows us to work throughout the year in a supportive environment in which no one fears letting others down. Having thought regularly about smaller questions throughout the academic year, we are prepared to jump directly into the bigger problems during our summer meetings.

Some faculty use REUF as a springboard to join an existing research community: After learning about the graph parameters zero forcing and power domination in REUF 2015, Mary Flagg (University of St. Thomas, TX) participated in the AIM workshop "Zero forcing and its applications" in January 2017 [1], where she co-authored a paper that has been published and another that is under review, in addition to two papers published with her REUF group [11]. According to Flagg:

> Thanks to the 2015 REUF workshop and my group's continuation meeting in 2016, I was able to go from knowing nothing about the power domination and zero forcing problems to being totally comfortable and able to contribute in such a high level research workshop. I feel like I am now part of a new research community. This was a game-changer for me.

REUF also supports the community of those who supervise undergraduate researchers through regular communication via a listserv and two annual lunch meetings of REUF alumni at the Joint Mathematics Meetings and MAA MathFest.

Who: Faculty at undergraduate colleges and universities (about twenty per workshop).

What: One-week workshop with possible continuation meeting(s).

When: Summer (date varies).

Where: AIM (San Jose, CA) or ICERM (Providence, RI).

Website: `reuf.aimath.org/`

Apply: Online through the host institute (the next workshop is listed on REUF website). The application is usually due in late February or early March and becomes available in December or early January.

ACKNOWLEDGMENT. The authors thank PCMI Director Rafe Mazzeo for providing information about PCMI, and AWM ADVANCE PI Kristin Lauter and Project Director Magnhild Lien for providing information about RCCWs and Research Networks.

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Leslie Hogben


T. Christine Stevens

Credits

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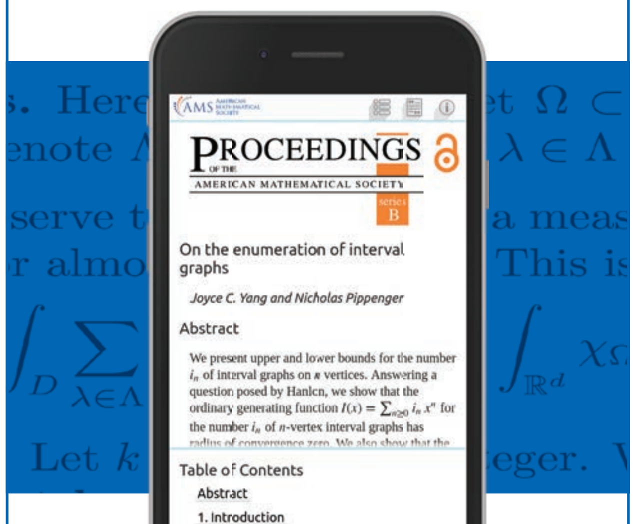

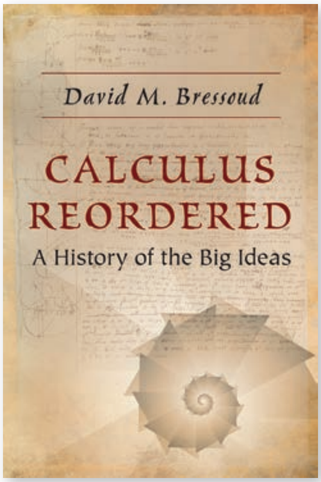

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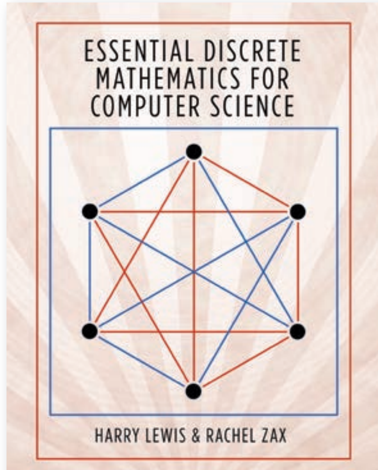

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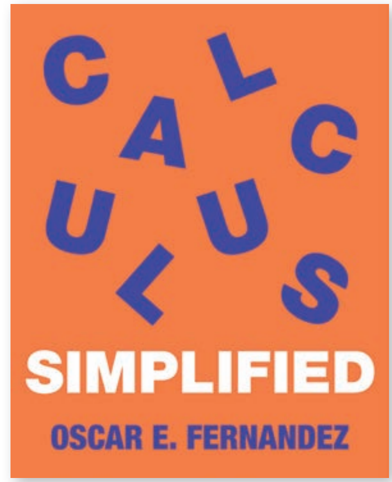

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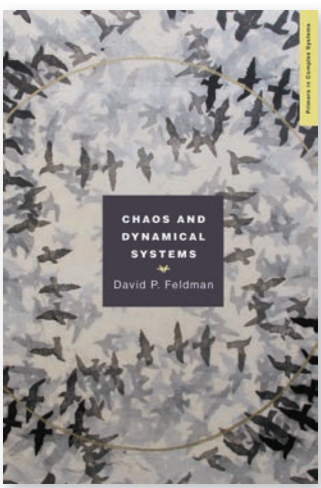

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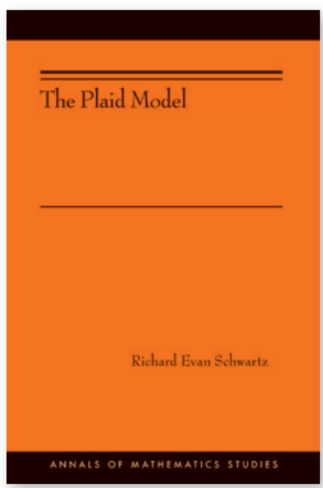

Outer billiards provides a toy model for planetary motion and exhibits intricate and mysterious behavior even for seemingly simple examples. *The Plaid Model*, which is a self-contained sequel to Richard Schwartz's *Outer Billiards on Kites*, provides a combinatorial model for orbits of outer billiards on kites.

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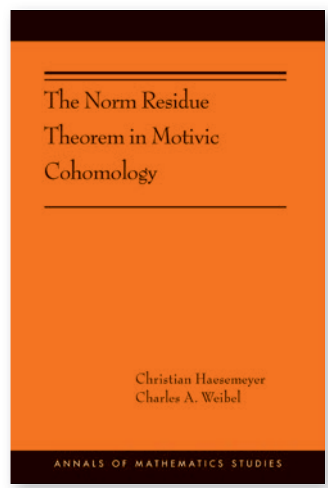

This book presents the complete proof of the Bloch-Kato conjecture and several related conjectures of Beilinson and Lichtenbaum in algebraic geometry. Brought together here for the first time, these conjectures describe the structure of étale cohomology and its relation to motivic cohomology and Chow groups.

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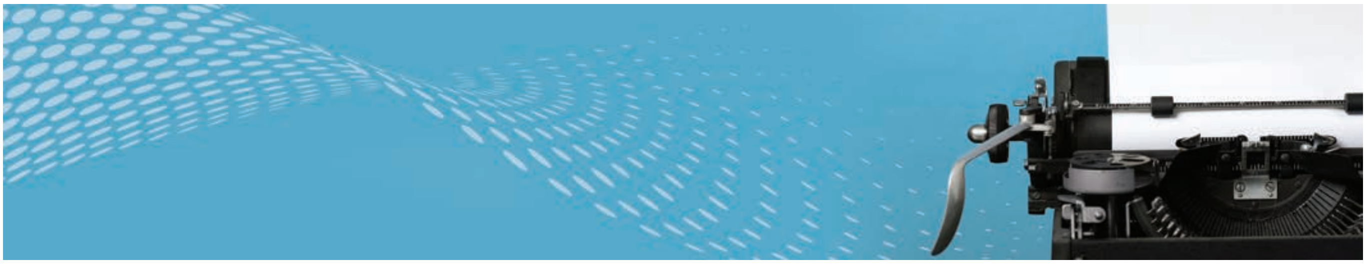


Combinatorial Inequalities

Igor Pak

Introduction

Combinatorics has always been a battleground of tools and ideas. That's why it's so hard to do, or even define. The inequalities are a particularly interesting case study as they seem to be both the most challenging and the least explored in enumerative and algebraic combinatorics. Here are a few of my favorites, with some backstories.[1]

We start with *unimodality of binomial coefficients:*

$$\binom{n}{k-1} \le \binom{n}{k}, \quad \text{for all } 1 \le k \le n/2. \qquad (1)$$

This is both elementary and well known—the proof is an easy calculation. But ask yourself the following natural question: Does the difference $B(n,k) := \binom{n}{k} - \binom{n}{k-1}$ count anything interesting? It should, of course, right? Imagine there is a natural injection

$$\psi : \binom{[n]}{k-1} \to \binom{[n]}{k}$$

from $(k-1)$-subsets to k-subsets of $[n]$, where $[n] := \{1, \ldots, n\}$. Then $B(n,k)$ can be described as the number of k-subsets of $[n]$ that are not in the image of ψ, as good an answer as any. But how do you construct the injection ψ?

Let us sketch the construction based on the classical *reflection principle* for the *ballot problem*, which goes back to the works of Bertrand and André in 1887. Start with a $(k-1)$-subset X of $[n]$, and let ℓ be the smallest integer s.t. $|X \cap [2\ell+1]| = \ell$. Such ℓ exists since $k \le n/2$. Define

$$\psi(X) := (X \setminus [2\ell+1]) \cup ([2\ell+1] \setminus X).$$

Observe that $|\psi(X)| = k$ and check that ψ is the desired injection. This gives an answer to the original question: $B(n,k)$ is the number of k-subsets $Y \subset [n]$, s.t. $|Y \cap [m]| \le m/2$ for all m.

At this point you might be in disbelief at me dwelling on the easy inequality (1). Well, it only gets harder from here. Consider, e.g., the following question: Does there exist an injection ψ as above, s.t. $X \subset \psi(X)$ for all $X \in \binom{[n]}{k-1}$? We leave it to the reader as a challenge.[2]

There is also a curious connection to algebraic combinatorics: $B(n,k) = f^{(n-k,k)}$, the dimension of the irreducible S_n-module corresponding to the partition $(n-k,k)$. To understand how this could happen, think of both sides of (1) as dimensions of permutation representations of S_n. Turn both sides into vector spaces and modify ψ accordingly, to make it an S_n-invariant linear map. This would make it more natural and uniquely determined. As a consequence, we obtain a combinatorial interpretation $B(n,k) = |\mathrm{SYT}(n-k,k)|$, the number of standard Young

Igor Pak is a professor in the Department of Mathematics at UCLA. His email address is pak@math.ucla.edu.

[1] *To save space and streamline the exposition we omit all of the many references. The interested reader can find them in the expanded version of this article at the author's website:* https://tinyurl.com/y26efyoj.

DOI: https://doi.org/10.1090/noti1919

[2] *We give the answer in the expanded version, see the previous footnote.*

tableaux of shape $(n-k,k)$, a happy outcome in every way.

Consider now *unimodality of Gaussian coefficients:*

$$p(n,k,\ell-1) \le p(n,k,\ell), \quad \text{for all } 1 \le \ell \le k(n-k)/2, \text{ where} \tag{2}$$

$p(n,k,\ell)$ is the number of integer partitions $\lambda \vdash \ell$ that fit into a $k \times (n-k)$ rectangle; i.e., λ has parts of size at most $(n-k)$, and has at most k parts. To understand the context of this inequality, recall:

$$\sum_{\ell=0}^{k(n-k)} p(n,k,\ell)\, q^\ell = \binom{n}{k}_q := \frac{(n!)_q}{(k!)_q \cdot ((n-k)!)_q},$$

$$\text{where} \quad (n!)_q := \prod_{i=1}^{n} \frac{q^i-1}{q-1}.$$

To connect this to (1), note that $\binom{n}{k}_1 = \binom{n}{k}$, and that $\binom{n}{k}_q$ is the number of k-subspaces of $\mathbb{F}_q^n$. In (2), we view $\binom{n}{k}_q$ as a polynomial in q and compare its coefficients. Now, the Schubert cell decomposition of the Grassmannian over $\mathbb{F}_q$ or a simple induction can be used to give the partition interpretation.

The inequality (2) is no longer easy to prove. Conjectured by Cayley in 1856, it was established by Sylvester in 1878; the original paper is worth reading even if just to see how pleased Sylvester was with his proof. In modern language, Sylvester defined the $\mathfrak{sl}_2(\mathbb{C})$ action on certain homogeneous polynomials, and the result follows from the highest weight theory (in its simplest form for $\mathfrak{sl}_2$).

Let's continue with the questions as we did above. Consider the difference $C(n,k,\ell) := p(n,k,\ell) - p(n,k,\ell-1)$. Does $C(n,k,\ell)$ count anything interesting? Following the pattern above, wouldn't it be natural to define some kind of nice injection from partitions of size $(\ell-1)$ to partitions of size ℓ, by simply adding a corner square according to some rule? That would be an explicit combinatorial (as opposed to algebraic) version of Sylvester's approach.

Unfortunately we don't know how to construct such a nice injection. It's just the first of the many frustrations one encounters with algebraic proofs. Most of them are simply too rigid to be "combinatorialized." It doesn't mean that there is no combinatorial interpretation for $C(n,k,\ell)$ at all. There is one very uninteresting interpretation due to Panova and myself, based on a very interesting (but cumbersome) identity by O'Hara. Also, from the computer science point of view, it is easy to show that $C(n,k,\ell)$ as a function is in #P. We leave it to the reader to figure out why (or what that even means).

To finish this story, we should mention Stanley's 1989 approach to (2) using finite group actions. More recently, Panova and I introduced a different technique based on properties of the *Kronecker coefficients* of S_n, via the equality $C(n,k,\ell) = g((n-k)^k, (n-k)^k, (k(n-k)-\ell,\ell))$. Here the Kronecker coefficients $g(\lambda,\mu,\nu)$ can be defined as structure constants for products of S_n characters: $\chi^\mu \chi^\nu = \sum_\lambda g(\lambda,\mu,\nu)\, \chi^\lambda$. Both approaches imply stronger inequalities than (2), but neither gets us closer to a simple injective proof.

We turn now to *log-concavity of independent sets:*

$$a_{k-1}(M) \cdot a_{k+1}(M) \le a_k(M)^2, \text{ where} \tag{3}$$

$a_k(M)$ is the number of independent k-subsets of a matroid M. Note that the log-concavity implies unimodality, and in the special case of a *free matroid* (all elements are independent) this gives (1).

The inequality (3) is a celebrated recent result by Adiprasito, Huh, and Katz (2018), which showed that a certain "cohomology ring" associated with M satisfies the hard Lefschetz theorem and the Hodge–Riemann relations. This resolved conjectures by Welsh and Mason (1970s).

It would be naïve for us to ask for a direct combinatorial proof via an injection, or by some other elementary means. For example, Stanley in 1981 used the Aleksandrov–Fenchel inequalities in convex geometry to prove that the log-concavity is preserved under taking truncated sums with a free matroid, already an interesting but difficult special case proved by inherently non-combinatorial means.

There is also a computational complexity version of the problem that might be of interest. Let $A(k,M) := a_k(M)^2 - a_{k-1}(M) \cdot a_{k+1}(M)$. Does $A(k,M)$ count any set of combinatorial objects?

For the sake of clarity, let $G=(V,E)$ be a simple connected graph and M the corresponding matroid; i.e. bases in M are spanning trees in G. Then $a_k(M)$ is the number of spanning forests in G with k edges. Note that computing $a_k(M)$ is #P-complete in full generality. Therefore, computing $A(k,M)$ is #P-hard.

Now, $A(k,M)$ is in GapP; i.e. equal to the difference of two #P-functions. Does $A(k,M)$ lie in #P? This seems unlikely, but the current state of the art of computational complexity doesn't seem to provide us with tools to even approach a negative solution.

To fully appreciate the last example, consider the *log-concavity of matching numbers:*

$$m_{k-1}(G) \cdot m_{k+1}(G) \le m_k(G)^2, \text{ where} \tag{4}$$

$m_k(G)$ is the number of k-matchings in a simple graph $G=(V,E)$; i.e., k-subsets of edges that are pairwise disjoint. For example, $m_n(K_{2n}) = (2n-1)\cdots 3\cdot 1$. While perfect matchings don't necessarily define a matroid, they do have a similar flavor from a combinatorial optimization point of view. The inequality (4) goes back to Heilmann and Lieb (1972) and is a rare case when the injection strategy works well. The following argument is due to Krattenthaler (1996).

Take a $(k-1)$-matching β whose edges we color *blue* and a $(k+1)$-matching γ whose edges we color *green*. The union $\beta \cup \gamma$ of these two sets of edges splits into connected components, which are either paths or cycles, all alternately colored. Ignore for the time being all cycles and paths of even lengths. Denote by $(r-1)$ the number of odd-length paths that have extra color blue. There are then $(r+1)$ odd-length paths that have extra color green.

Now, allow switching colors in any of the $2r$ odd-length paths. After recoloring, we want to have r odd-length paths extra color blue and the same with green. This amounts to a constructive injection from $(r-1)$-subsets of $[2r]$ to r-subsets of $[2r]$, which we already know how to do as a special case of proving (1).

We leave to the reader the problem of finding an explicit combinatorial interpretation for $M(k,G) := m_k(G)^2 - m_{k-1}(G) \cdot m_{k+1}(G)$, proving that this function is in #P. Note that computing $m_k(G)$ is famously #P-complete, which implies that so is $M(k,G)$. This makes the whole connection to computational complexity even more confusing. What exactly makes matchings special enough for this argument to work?

If there is any pattern to the previous examples, it can be summarized as follows: the deeper one goes in an algebraic direction, the more involved are the inequalities and the less of a chance of a combinatorial proof. To underscore this point, consider the following three *Young tableaux inequalities*:

$$(f^\lambda)^2 \le n!, \quad (c^\lambda_{\mu\nu})^2 \le \binom{n}{k}, \quad c^\lambda_{\mu\nu} \le c^\lambda_{\mu\vee\nu,\mu\wedge\nu}, \quad \text{for all } \lambda \vdash n,\ \mu \vdash k,\ \nu \vdash n-k. \tag{5}$$

Here $f^\lambda = |\mathrm{SYT}(\lambda)|$ is the number of standard Young tableaux of shape λ, equal to the dimension of the corresponding irreducible S_n-module as above. Similarly, $c^\lambda_{\mu\nu} = |\mathrm{LR}(\lambda/\mu,\nu)|$ is the *Littlewood–Richardson coefficient*, equal to the number of Littlewood–Richardson tableaux of shape λ/μ and weight ν. It can be defined as a structure constant for products of Schur functions: $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$. Finally, $\mu \vee \nu$ and $\mu \wedge \nu$ denote the union and intersection, respectively, of the corresponding Young diagrams.

Now, the first inequality in (5) is trivial algebraically, but its combinatorial proof is highly nontrivial—it is a restriction of the RSK correspondence. The second inequality is quite recent and follows easily from the definition and the Frobenius reciprocity. We believe it is unlikely that there is a combinatorial injection, even though there is a nice double counting argument.

Finally, the third inequality in (5) is a corollary of the powerful inequality by Lam, Postnikov, and Pylyavskyy (2007) using the curious Temperley–Lieb immanant machinery. The key ingredient in the proof is Haiman's theorem, which in turn uses the Kazhdan–Lusztig conjecture proven by Beilinson–Bernstein and Brylinski–Kashiwara. While stranger things have happened, we would be very surprised if this inequality had a simple combinatorial proof.

We conclude on a positive note, with a combinatorial inequality where everything works as well as it possibly could. Consider the following *majorization property of contingency tables*:

$$\mathrm{T}(\mathbf{a},\mathbf{b}) \le \mathrm{T}(\mathbf{a}',\mathbf{b}') \quad \text{for all } \mathbf{a}' \trianglelefteq \mathbf{a},\ \mathbf{b}' \trianglelefteq \mathbf{b}. \tag{6}$$

Here $\mathbf{a} = (a_1,\ldots,a_m)$, $a_1 \ge \ldots \ge a_m > 0$ and $\mathbf{b} = (b_1,\ldots,b_n)$, $b_1 \ge \ldots \ge b_n > 0$ are two integer sequences with equal sum:

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j = N.$$

A *contingency table* with margins $(\mathbf{a},\mathbf{b})$ is an $m \times n$ matrix of non-negative integers whose i-th row sums to a_i and whose j-th column sums to b_j, for all $i \in [m]$ and $j \in [n]$. $\mathrm{T}(\mathbf{a},\mathbf{b})$ denotes the number of all such matrices. Finally, for sequences $\mathbf{a}$ and $\mathbf{a}'$ with the same sum, we write $\mathbf{a} \trianglelefteq \mathbf{a}'$ if $a_1 \le a'_1$, $a_1 + a_2 \le a'_1 + a'_2$, $a_1 + a_2 + a_3 \le a'_1 + a'_2 + a'_3$, … In other words, the inequality (6) says that there are more contingency tables when the margins are more evenly distributed.

Contingency tables can be viewed as adjacency matrices of bipartite multi-graphs with given degree distribution. They play an important role in statistics and network theory. We learned the inequality (6) from a paper by Barvinok (2007), but it feels like something that should have been known for decades.

Now, I know two fundamentally different proofs of (6). The first is an algebraic proof using Schur functions that amounts to proving the following standard inequality for Kostka numbers: $K_{\lambda\mu} \le K_{\lambda\nu}$ for all $\mu \trianglerighteq \nu$, where $K_{\lambda\mu}$ is the number of semistandard Young tableaux of shape λ and weight μ. This inequality can also be proved directly, so combined with the RSK we obtain an injective proof of (6).

Alternatively, one can prove (6) directly for $2 \times n$ rectangles and $(+1,-1)$ changes in row (column) sums. Combining these injections together gives a cumbersome, yet explicit injection. In principle, either of the two approaches can then be used to give a combinatorial interpretation for $\mathrm{T}(\mathbf{a}',\mathbf{b}') - \mathrm{T}(\mathbf{a},\mathbf{b})$.

In conclusion, let us note that we came full circle. Let $m = 2$, $a_1 = n-k+1$, $a_2 = k-1$, $a'_1 = n-k$, $a'_2 = k$, and $b_1 = \ldots = b_n = b'_1 = \ldots = b'_n = 1$. Observe that $\mathrm{T}(\mathbf{a},\mathbf{b}) = \binom{n}{k-1}$ and $\mathrm{T}(\mathbf{a}',\mathbf{b}') = \binom{n}{k}$. The inequality (1) is a special case of (6) then.

Igor Pak

Credits

Author photo is courtesy of Igor Pak.

Report of the Executive Director: State of the AMS, 2018

Courtesy of Maré Studios.

Catherine A. Roberts, Executive Director

Our American Mathematical Society is the professional home for thousands of mathematicians around the world. Since our founding in 1888, the two major activities of the AMS continue to be publications and meetings. We advance mathematics in other ways, too, such as through prizes and awards. We support our profession through many programs, as well as through government relations, membership benefits, your philanthropy, and public outreach. The articulation of our mission continuously evolves over time to reflect the priorities of the communities we serve and represent. This report contains information for our members about AMS activities in 2018; it is based on a report that was presented and discussed at the spring 2019 meetings of the Council and Board of Trustees.

The AMS is in Excellent Financial Health

Our financial statements are available in the annual "Report of the Treasurer" that appears in the *Notices of the AMS* (see `https://www.ams.org/about-us/annual-reports/annual-reports`). With close oversight by the Board of Trustees and input from volunteer committees such as Investment, Audit and Risk, and Long Range Planning, employees work hard to conduct the work of our Society within a balanced budget and with a view to the future. We make many choices each year in support of these goals. Moreover, we are midway through an ambitious five-year Strategic Plan designed to focus our efforts and further secure our long-term future.

Did you know that our largest sources of revenue come from library subscriptions to MathSciNet® and AMS journals, as well as from sales of the books we publish? Our 2017 purchase of the Mathematical Association of America's MAA Press book program is now incorporated into our own acquisitions, production, sales, printing, and distribution systems. Consequently, we expect to publish around one hundred new books annually. Other sources of revenue include membership dues, external grants, meeting registration fees, and income from Mathjobs.org. Our carefully managed investments provide additional operating income and also endow many AMS awards and prizes. Although our investments lost 5.1% in 2018, we had already more than recovered during the first quarter of 2019. We are also fortunate to receive generous philanthropic donations from members and friends, which come to us in many forms such as cash gifts and bequests. In 2018, the quiet phase of fundraising for our new Fund for the Next Generation took place. Generous donors pledged about 80% toward a $1.5 million goal to meet a challenge from a benefactor who has pledged up to $1.5 million in matching funds.

What about our expenses? The AMS employs about 200 people, which is our largest expenditure. Our talented staff join forces with dedicated volunteers to ensure the work of our Society is executed responsibly. Our income-generating activities support some of our important work in government relations, diversity, education, and outreach. Another major expense is meetings. Since our meetings registration fees are intentionally kept low to promote accessibility, we use other income to help run meetings. A third major expense is our member journals, *Notices of the AMS* and *Bulletin of the AMS*. We seek outside funding for some projects; for example, we receive NSF funding for our Mathematics Research Communities, which covers most of the costs. Other expenses include a growing number of membership activities, such as our Graduate Student Chapters, as well as outreach activities to increase awareness of mathematics and the work of our profession.

Where do we Conduct the Work of the AMS?

AMS headquarters are in Rhode Island. The AMS has satellite offices in Michigan, Washington DC, and North Carolina.

Our Society has a small Office of Government Relations in Washington DC, which advocates for research funding for mathematics and for national policies supporting mathematics and graduate education. Staff in Washington support two policy committees of the Council (Committee on Education and Committee on Science Policy). They also help place mathematicians on key policy panels, host Congressional briefings, and cooperate effectively with other scientific organizations.

In Ann Arbor, Michigan we create and maintain MathSciNet® and its database Mathematical Reviews. This work is conducted with editorial oversight from the volunteer Mathematical Reviews Editorial Committee and the tremendous help of around 22,000 reviewers. Eighteen PhD mathematicians serve as editors for Mathematical Reviews, while other staff support the editing, bibliographic, and technical aspects of keeping this authoritative gateway to the scholarly literature accurate, up-to-date, and relevant. An IT department in Ann Arbor helps keep everything running smoothly so MathSciNet® is available to researchers across the globe.

At our Providence, Rhode Island headquarters, we produce, market, and sell books and journals. We run a wide variety of AMS programs (outreach, travel grants, employment services, membership, meetings, development, data collection, and the AMS website) and support other policy committees of the Council (Committee on the Profession, Committee on Meetings and Conferences, Committee on Publications). The finance, development, human resources, and computer services departments reside in Providence. The computer services team maintains our computing systems, creates in-house tools to run our programs and activities, creates and maintains `www.ams.org`, and develops innovative products such as MathViewer and Open Math Notes. Nearby Pawtucket, RI is home to our Printing and Distribution Center, which handles nearly all of our Society's needs in this regard, including journals, books, posters, and promotional material.

The Office of the Secretary currently resides in the Department of Computer Science at North Carolina State University in Raleigh. There, our AMS Secretary works with a small staff to coordinate and support the work of the Council, the Secretariat, and the entire committee structure.

How was 2018 for the AMS?

It was, in many ways, a year of changes. Some had to do with people. Searches are underway for a new AMS Secretary and a new AMS Treasurer, who will both start in 2021. The early edge of an anticipated large wave of staff retirements also washed ashore. For example, longtime employee Barbara Beeton retired after fifty-six years of dedicated service. Searches initiated last year led to recent hires of four new editors at Mathematical Reviews in Ann Arbor, a new Director of Education in DC, and several other staff. While the loss of experienced staff is always challenging, we aim to leverage this employee turnover to progress towards more modern workflows.

Other 2018 changes related to projects. Many of these are associated with our five-year Strategic Plan, which is now halfway through its implementation. The Plan focuses on diversity and inclusion; advocacy, awareness, and visibility; membership; portfolio coherence; and research and publishing. The Strategic Plan is further detailed in the 2017 State of the AMS report and other communications. In early 2018, you were introduced to the new AMS logo. We are working to increase awareness about all that we do at the AMS to advance research and create connections. The very active Campaign for the Next Generation volunteer committee continues to raise the remaining dollars needed to meet our goal of $1.5 million, which is being matched dollar-for-dollar by a generous benefactor. These funds will provide many small grants to early-career mathematicians to address the particular needs of this cohort. And then, there were a few unanticipated projects, such as a burst pipe that led us to renovate the kitchen and cafeteria in Providence.

I would like to mention some additional highlights from 2018.

- Our Society ended the year with a 3% increase over budget for our operating revenue. Our sales of AMS electronic book backlists and MathSciNet® subscriptions exceeded expectations, although sales of MAA Press books did not meet budget (we were a bit too optimistic about how quickly we would be ready to begin selling these books). Our operating expenses were lower than budgeted, mainly due to some staff positions left unfilled and our shift to a high-deductible insurance plan for employees.
- Our investments lost 5.1% in 2018. Fortunately, we quickly recovered from those losses in the first quarter of the new year.
- We completed the first year of providing management services to the Association for Women in Mathematics. We are pleased to continue working together.
- We began looking for new office space in both Ann Arbor, MI and Washington DC. Our current arrangements will not be able to serve our needs going forward.
- We replaced a roof and a HVAC unit in Rhode Island buildings. Such facilities projects over the years have cut our electricity use in half since 2007.
- We made multiple privacy and security enhancements. You may have noticed the *Combined Membership List* is gone, and the *AMS Member Directory* is now behind a secure login.
- You can now make recurring donations to the AMS with a credit card.
- We continue to see an increase in donations to the AMS, particularly in light of the Campaign for the Next Generation. Bequests continue to play an important role in our financial security.
- Our US and EU sales of books increased in 2018, although sales in Asia declined. Sales of individual books declined, yet our sales of AMS e-book backlist collections grew. The AMS book series *Contemporary Mathematics* proved particularly successful, as did *Graduate Studies in Mathematics* and *Mathematical Surveys and Monographs*.
- Two AMS books won 2018 Prose Awards: *Harmonic Maass Forms and Mock Modular Forms: Theory and Applications* by Kathrin Bringmann, Amanda Folsom, Ken Ono, and Larry Rolen won the Mathematics Category and *An Illustrated Theory of Numbers* by Martin H. Weissman won honorable mention in the same category.
- We developed a plan to publish more books in applied mathematics.
- We integrated MAA Press books into the AMS Book Program, with the goal of creating a comprehensive and unified book program to serve potential readers at all levels. We plan to ultimately introduce an electronic backlist of the MAA Press collection.
- AMS MathViewer—an online, interactive, dual-panel reading experience for AMS journals—was expanded to include *Mathematics of Computation*.
- The AMS Board of Trustees approved a plan to eliminate the backlog of *Transactions of the AMS* by publishing a double-volume of this journal in 2019.
- We progressed the redesign of our website ams.org. New versions of the homepage and *Government Relations* and *Giving to the AMS* sections were released.
- The *Notices of the AMS* website was enhanced to support fresh types of content introduced by the new Editor in Chief, Erica Flapan. *Notices* content can now be syndicated to appear on other areas of our website.
- Mathematical Reviews added over 129,000 new items to the database behind MathSciNet®, plus more than 40,000 new author profiles.
- MathSciNet® itself continued to evolve, with enhancements such as auto-suggest for author and journal searches and faceted search.
- We saw the number of graduate student AMS members increase for the fourth year in a row, and in 2018 we saw an increase in the number of regular members for the first time in at least eight years. Although our total membership numbers continued to decline (as is being experienced across all professional societies, so this is not unique to the AMS), numbers decreased at a slower rate than in previous years.
- Our *Headlines & Deadlines* electronic newsletter was upgraded and is now sent to AMS members twice monthly.
- The Mathematics Research Communities moved to Rhode Island. We involved more AMS staff in this important professional development activity for early-career mathematicians.
- We served the needs of more early-career mathematicians through our Mathematics Research Communities, blogs (including our newest "Math Mamas"), AMS Graduate Student Travel Grants, and the AMS Simons Travel Grants. Our annual Congressional Fellow and Mass Media Fellow programs continued. We recently began participating in the Science and Technology Policy Fellowships for PhD data scientists and the Catalyzing Advocacy in Science and Engineering Workshop for graduate students.

- We are actively developing plans to enhance awareness of and opportunities for careers in business, industry, government, and nonprofits.
- Our Congressional lunch briefings on Capitol Hill are now joint with the Mathematical Sciences Research Institute, and they occurred twice last year.
- We increased our involvement with the National Academies, the American Association for the Advancement of Science, and the White House's Office of Science and Technology Policy.
- We redesigned the annual meeting of the Committee on Education to include a one-day mini-conference. The 2018 conference focused on "Next steps in the evolution of mathematics education: moving beyond pilots."
- The Department Chairs' Workshop at the Joint Mathematics Meetings attracted seventy people (a 43% increase over the previous two years).
- Many conversations took place about diversity, inclusion, and equity issues in our community. These discussions will help inform our next best steps forward in this arena.

Two important things happened in early 2019 that I'd like to mention here, rather than wait until next year's annual report. You may be familiar with MathJax, a cross-browser open-source library that displays mathematical notation in web browsers. This innovative project has been co-managed by the AMS and SIAM for over a decade. This now-mature initiative recently graduated to join NumFOCUS, an independent and professional home for projects in the open-source scientific data stack. I think this is a terrific example of how professional societies can support and initiate new products to benefit the entire scientific community.

Next, the AMS is an inaugural member of a new coalition of 100+ scientific societies called the Societies Consortium on Sexual Harassment in STEMM (Science, Technology, Engineering, Mathematics, and Medicine). This consortium will advance professional and ethical conduct, climate, and culture across our respective fields. Professional societies have a unique responsibility as standard-setters for STEMM fields in addressing the pervasive problem of sexual and gender harassment. Initial work will focus on model policies and procedures for society honors and awards, which will be overseen by the new Prize Oversight Committee.

Future of the Joint Mathematics Meetings

As many of you know, under a 1998 agreement the AMS and the MAA co-manage the Joint Mathematics Meetings (JMM). Starting with the January 2022 meetings, the AMS will become solely responsible for all management (selecting and working with the site location, gathering and evaluating proposals and abstracts, assembling the program, running the meeting). The MAA also announced a plan to shift their energies toward their summer MathFest meeting, and thus they intend to reduce the content they currently organize at the JMM. As you can imagine, there are many discussions going on as we begin to plan and re-imagine what the JMM will look like three years from now. Several mathematical groups and organizations are stepping forward with offers to help.

We remain in the input-gathering and idea-generation stage and invite you to submit your thoughts at `https://www.ams.org/jmm-reimagined`. To help communicate our intention to offer a large and valuable mathematics meeting each January, both the Council and the Board of Trustees recently endorsed the following statement:

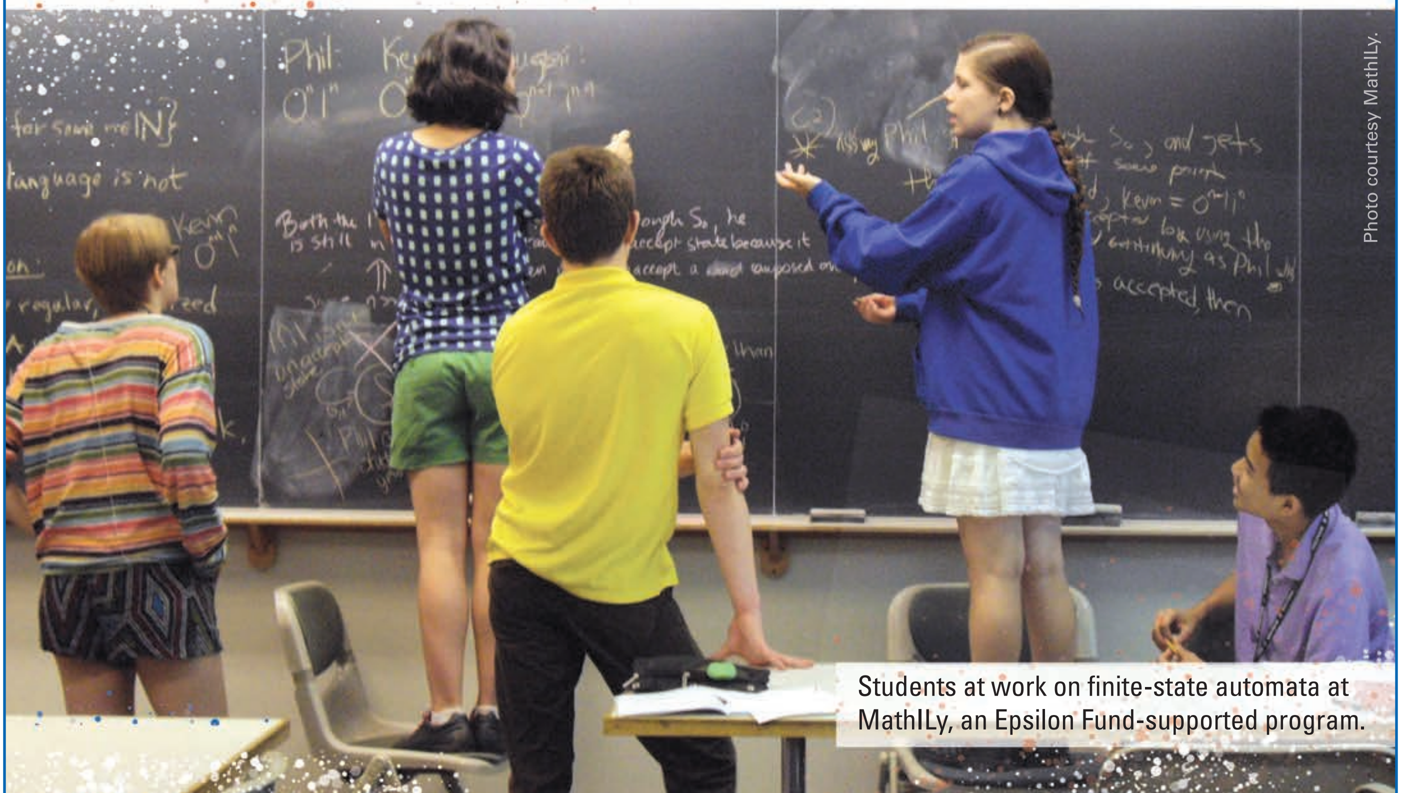
> *The Joint Mathematics Meetings will strive to represent the full spectrum of interests of the mathematical community.*

A new Joint Meetings Planning Committee has been formed to help us move forward. Information will be available on our meetings website. A new page at `https://www.ams.org/meetings` provides a mechanism where you can propose sessions, panels, meetings, and other events for future JMMs. Working together, we can ensure that this vibrant and important gathering of mathematicians continues to represent all of our interests.

In Closing

When I reflect back on 2018 and my three years with the AMS, the impact of all we do to advance research and create connections in our community resonates deeply. Serving you as AMS Executive Director is something I truly value. I am fortunate to see all this great work of our Society up close, and it is impressive. I recall years ago at a JMM, after a brief and friendly exchange with then AMS Executive Director John Ewing, thinking to myself that he had an interesting, challenging, and impactful job. I am so grateful to all of you for your contributions to this important work we are doing together. The success of the AMS is the result of thousands of dedicated people—staff, volunteers, members, and friends. We could not do it without you!

Catherine A. Roberts
Executive Director
May 2019



Students at work on finite-state automata at MathILy, an Epsilon Fund-supported program.

Photo courtesy MathILy.

Doctoral Degrees Conferred

2017–2018

ALABAMA

Auburn University (7)

DEPARTMENT OF MATHEMATICS AND STATISTICS

Dempsey, Emily, Automorphisms of subspace designs

Denu, Dawit, Analysis of stochastic vector host epidemic model with direct transmission

Ford, Jeffrey, Measure-preserving dynamical systems on $\mathbb{R}^3$ with all trajectories bounded

Liu, Jianzhen, Toepliz matrices are unitarily similar to symmetric matrices

McQuaig, Bradley, Morita-equivalence between strongly non-singular rings and the strutcher of the maximal ring of quotients

Ngoma Koumba, Bertran Sedar, Inverse source problem and inverse diffusion coefficient problem for parabolic equations with applications in geology

Sun, Wei, Rank-based methods for single-index varying coefficient models

University of Alabama (4)

DEPARTMENT OF MATHEMATICS

Dinh, Khanh, Inexact methods for the chemical master equation with constant or time-varying properties and applications to parameter inference

He, Xuan, Variational models with elastica energies: A comparison, a new model, and new algorithms

Liu, Sijie, Develop interval and non-interval methods for solving multi-objective optimization problems

Luo, Xin, Development of model interval algorithm for solving continuous minimax problems

University of Alabama at Birmingham (5)

DEPARTMENT OF BIOSTATISTICS

Hillegass, William, Comparative performance and clinical utility of indirect treatment comparison estimators

Li, Yan, Sample size re-estimation for confirmatory two-stage multi-arm trials with normal and binary outcomes

Lirette, Seth, A statistical approach to computed tomography perfusion

Liu, Yuliang, Univariate frailty model for competing risks data analysis

Turley, Falynn, Statistical tests of confounding

University of Alabama–Huntsville (1)

DEPARTMENT OF MATHEMATICAL SCIENCES

Sewell, Jonathan, Vortex based distinguishing collections

ARIZONA

Arizona State University (22)

MATHEMATICS, COMPUTATIONAL AND MODELING SCIENCES CENTER

Burkow, Daniel, Intramyocellular lipids and the progression of muscular insulin resistance

Espinoza, Baltazar, Consequences of short term mobility across heterogeneous risk environments: The 2014 West African Ebola outbreak

Manning, Miles, Patterns in knowledge production

Messan, Komi, Prey-predator “host-parasite” models with adaptive dispersal: application to social animals

Moreno, Victor, Understanding the impact of social factors on the transmission dynamics of infectious diseases across highly heterogeneous risk environments

Nazari, Fereshteh, Mathematical model for IL-6-mediated tumor growth, invasion and targeted treatment

SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES

Baez, Javier, Mathematical models of androgen resistance in prostate cancer patients under intermintent androgen suppression therapy

Dassanayake, Mudiyanselage Maduranga, A study of components of Pearson’s chi-square based on marginal distributions of cross-classified tables for binary variables

Frank, Kristin, Examining the development of students’ covariational reasoning in the context of graphing

Gilg, Brady, Critical coupling and synchronized clusters in arbitrary networks of Kuramoto oscillators

Irimata, Katherine, Essays on the identification and modeling of variance

Irimata, Kyle, Three essays on correlated binary outcomes: Detection and appropriate model

Kayser, Kirk, The economics of need-based transfers

Kim, Soohyun, Optimal experimental designs for mixed categorical and continuous responses

Kim, Younghwan, On the uncrossing partial order on matchings

Kuper, Emily, Sparky the saguaro: Teaching experiments examining students’ development of the idea of logarithm

Pampel, Krysten, Perturbing practices: A case study of the effects of virtual manipulatives as novel didactic objects on rational function instruction

Scarnati, Theresa, Recent techniques for regularization in partial differential equations and imaging

Walker, Philip, Effective-diffusion for general non-autonomous systems

Wang, Bei, Three essays on comparative simulation in three-level hierarchical data structure

Wang, Zhongshen, Locally D-optimal designs for generalized linear models

Zhou, Lin, Optimum experimental design issues in functional neuroimaging studies

The above list contains the names and thesis titles of recipients of doctoral degrees in the mathematical sciences (July 1, 2017, to June 30, 2018) reported in the 2019 Annual Survey of the Mathematical Sciences by 263 departments in 186 universities in the United States. Each entry contains the name of the recipient and the thesis title. The number in parentheses following the name of the university is the number of degrees listed for that university.

University of Arizona (13)

Department of Mathematics

Gunton, Cody, Crystalline representations and Néron component groups in the semi-stable case

Pounder, Kyle, Nearly singular Jacobi matrices and applications to the finite Toda lattice

Rossi, Daniel, Fields of values in finite groups: Characters and conjugacy classes

Shahar, Doron, Hydrodynamic limits for long range asymmetric processes and probabilistic opinion dynamics

Stone, Megan, Eigenvalue densities for the Hermitian two-matrix model and connections to monotone Hurwitz numbers

Program in Applied Mathematics GIDP

Henscheid, Nicholas, Quantifying uncertainties in imaging-based precision medicine

Kappler, Nicholas, Effects of parasites on the structure and dynamics of food webs

Kilen, Isak, Non-equilibrium many body influence on mode locked vertical external-cavity surface-emitting lasers

Lim, Soon Hoe, Effective dynamics of open systems in non-equilibrium statistical mechanics

McEvoy, Erica, A numerical method for the simulation of Skew Brownian motion and its application to diffusive shock acceleration of charged par

Meissen, Emily, Invading a structured population: A bifurcation approach

Nabelek, Patrik, Applications of complex variables to spectral theory and completely integrable partial differential equations

Shearman, Toby, Geometry and mechanics of leaves and the role of weakly-irregular isometric immersions

ARKANSAS

University of Arkansas at Fayetteville (2)

Department of Mathematical Sciences

Cheng, Wanqing, Π-Operators in Clifford analysis and its applications

Mahdi, Ghadeer, Hierarchical Bayesian regression with application in spatial modeling and outlier detection

CALIFORNIA

California Institute of Technology (1)

Department of Mathematics

He, Siqi, The Kapustin–Witten equations with singular boundary conditions

Claremont Graduate University (10)

Institute of Mathematical Sciences

Allan, Collins, Probabilistic microsimulation modeling of heterogenous traffic flow

Calhoun, Peter, Novel random forest and variable importance methods for clustered data

Cuevas, Daniel, Bridging the genomic gaps: genomescale metabolic network tools for bioinformatics analyses

Kang, Di, Modeling and analysis of thin viscous liquid films in spherical geometry

Koblick, Darin, Re-purposing the advanced solar photon thruster as a constellation of solar reflectors to track debris in geosynchronous earth orbit

Ma, Anna, Stochastic iterative algorithms for large-scale data

Pierret, Julien, Climate data computing: optimal interpolation, averaging, visualization and delivery

Spinella, William, A systematic investigation of exotic matter in neutron stars

Waynelovich, John, Synthesis of nervous systems in hybrid roots utilizing hierarchical Q-learning and temporal shifting

Wilcox, Bruce, A time series data mining and unobserved component modeling approach to credit risk correlation

Naval Postgraduate School (2)

Department of Applied Mathematics

Martinsen, Thor, Correlation immunity, avalanche features, and other cryptographic properties of generalized boolean functions

Roginski, Jonathan, The distance centrality: Measuring structural disruption in a network

Stanford University (21)

Department of Mathematics

Alvarez-Gavela, Daniel, The flexibility of caustics

Cheng, Da Rong, Geometric variational problems: Regular and singular behavior

Dozier, Benjamin, Saddle connections on translation surfaces

Klang, Inbar, Factorization theory of Thom spectra, twists, and duality

Lazaer, Oleg, Flexible Weinstein structures and applications to symplectic and contact topology

Li, Chao, Singularity and comparison theorems for metrics with positive scalar curvature

Madnick, Jesse, Nearly-Kähler 6-manifolds of cohomogeneity two: Local theory

Pan, Donghai, Pencils of Fermat hypersurfaces and Galois cyclic covers of the projective line

Rosengarten, Zev, Tate duality in positive dimension and applications

Savvas, Michail, Generalized Donaldson–Thomas invariants via Kirwan blowups

Szucs, Gergely, The equivariant cobordism category

Warner, Evan, Adic moduli spaces

Zhai, Lin, Asymptotics of Gaussian processes and Markov chains

Zhou, Yang, Higher-genus wall-crossing in Landau–Ginzburg theory

Department of Statistics

Arthur, Joseph, Detection and validation of genomic structual variation from DNA sequencing data

DiCiccio, Cyrus, Hypothesis testing using multiple data splitting

Fan, Zhou, Eigenvalues in multivariate random effects models

Guu, Kelvin, Learning to generate text programs (and beyond) from weak supervision

Le, Ya (Elaine), Topics in statistical learning with a focus on large-scale data

Panigrahi, Snigdha, An approximation-based framework for post-selective inference

Sankaran, Kris, Discovery and visualization of Latent structure with applications to the microbiome

University of California, Berkeley (35)

Department of Mathematics

Agrawal, Shishir, Deformations of overconvergent isocrystals on the projective line

Au, Benson, Rigid structures in traffic probability: With a view toward random matrices

Bechor, Elan, Two models of coagulation with instantaneous gelation

Brereton, Justin, A method of constructing invariant measures at fixed mass

Chen, Justin, On Betti tables, monomial ideals, and unit groups

Fang, Kuan-Ying, Geometric constructions of mapping cones in the Fukaya category

Gavrus, Cristian Dan, Global well-posedness and parametrices for critical Maxwell–Dirac and massive Maxwell–Klein–Gordon equations with small Sobolev data

Gerig, Christopher, Seiberg–Witten and Gromov invariants for self-dual harmonic 2-forms

Karp, Steven, Total positivity for Grassmannians and amplituhedra

Kerber, Alvin, Quasi-Fuchsian surface subgroups of infinite covolume Kleinian groups

Ladd, Watson, Algebraic modular forms on $SO_5(\mathbb{Q})$ and the computation of paramodular forms

Lin, Bo, Combinatorics and computations in tropical mathematics

Lowengrub, Daniel, Applications of the intersection theory of singular varieties

Manber, Shelly, Asymptotics of the Tate–Shafarevich group

Melvin, George, Crystals and mirror constructions for quotients

Neuman, Anh Martina, Functions of nearly maximal Gowers–Host–Kra norms on Euclidean spaces

Qadeer, Saad, Simulating nonlinear Faraday waves on a cylinder

Ramsey, Samuel Nicholas, Independence, amalgamation, and trees

Schmaltz, Wolfgang, Gromov–Witten axioms for symplectic manifolds via polyfold theory

Spivak, Amelia, Multisymplectic geometry in general relativity and other classical field theories on manifolds with boundaries: A deobfuscating role

Van Andel, Ethan, Eulerian simulation of elastic membranes and shells

Vargas Pallete, Franco, On renormalized volume

Williams, Brandon, Computing modular forms for the Weil representation

Wilson, Patrick, Asymptotically conical metrics and expanding Ricci solitons

Zhou, Qiao, Applications of toric geometry to geometric representation theory

Zorn, Alexander, A combinatorial model of Lagrangian skeleta

Department of Statistics

Boyd, Nicholas, Sets as measures: Optimization and machine learning

Choi, Hye Soo, The Doob–Martin compactification of Markov chains of growing words

Kuang, Christine, Predictive and interpretable text machine learning models with applications in political science

Mukherjee, Soumendu, On some inference problems for networks

Saha, Sujayam, Information theory, dimension reduction and density estimation

Wei, Yuting, A geometric perspective on some topics in statistical learning

Wilson, Ashia, Lyapunov arguments in optimization

Group in Biostatistics

Ju, Cheng, Variable and model selection for propensity score estimators in causal inference

Perraudeau, Fanny Gabrielle Solange Marie, Statistical and computational methods for single-cell transcriptome sequencing and metagenomics

University of California, Davis (19)

Department of Mathematics

Bassett, Robert, Stochastic and convex optimization in statistical estimation

Berrian, Alexander, The chirped quilted synchrosqueezing transform and its application to bioacoustic signal analysis

Goh, Gabriel, Optimization with costly subgradients

Haddock, Jamie, Projection algorithms for convex and combinatorial optimization

Paramonov, Kirill, Essays in combinatorics: Crystals on shifted primed tableaux, bigraded Fibonacci numbers and data mining on social graphs

Samperton, Eric, Computational complexity of enumerative 3-manifold invariants

Smothers, Evan, Self-similar solutions and local wavefront analysis of a degenerate Schrödinger equation arising from nonlinear acoustics

Tam, Patrick, Nearly finitary matroids

Xu, Yuanyuan, On several problesm in random matrix theory and statistical mechanics

Department of Statistics

Bandyopadhyay, Rohosen, Benchmarking the observed best predictor

Dai, Xiongtao, Principal component analysis for Riemannian functional data and Bayes classification

Dao, Cecilia Uyen, Goodness-of-fit tests for generalized linear mixed models

Lee, Olivia Yuh Ru, Data-driven computation for pattern information

Li, Shuyang, Joint models for partially observed longitudinal data

Namdari, Jamshid, Estimation of spectral distributions of a class of high-dimensional linear processes

Roy, Tania, Discovery and visualization of the information content in data through histograms and phylogenetic trees

Sonmez, Ozan, Structural breaks in functional time series data

Wang, Nana, Analysing dependence in stochastic networks via Gaussian graphical models

Zhang, Chunzhe, Uncertainty quantification and sensitivity analysis in statistical machine learning

University of California, Irvine (10)

Department of Mathematics

Chen, Taiji, Deformation quantization of vector bundles on Lagrangian subvarieties

Fang, Jun, Ray-based finite element method for high-frequency Helmholtz equations

Fider, Nicole, Color categorization: Quantitative methods and applications

Kassir, Ali, Absorbing Markov chains with random transition matrices and applications

Kelleher, Casey, On existence and regularity theory of Yang–Mills fields

Ma, Timmy, A nonlinear approach to learning from an inconsistent source (with some applications)

Porter, Michael, Graphs based on polynomials over finite fields

Rackauckas, Christopher, Simulation and control of biological stochasticity

Simonyan, Aghavni, Non-receptors, feedback, and robust signaling gradients in biological tissue patterning

Wood, Karen, Mathematical modeling of cooperation based diversification and speciation

University of California, Los Angeles (26)

Department of Biostatistics, Fielding Sch of Publ Hlth

Conn, Daniel, Utilization of low dimensional structure to improve the performance in high dimensions

Li, Qian, Hierarchical integration of heterogenous highly structure data: The case of functional brain imaging

Tolkoff, Max, Phylogenetic factor analysis and natural extensions

Wang, Lu, Bayesian curve registration and warped functional regression

Department of Mathematics

Azzam, Alexander, Doubly critical semilinear Schrödinger equations

Bellis, Ben, Resolvent estimates and semigroup expansions for non-self-adjoint Schrödinger operators

Cadegan-Schlieper, William, On the geometry and topology of hyperplane complements associated to complex and quaternionic reflection groups

Cheng, Peter, Spline deferred correction

Clyde, David, Numerical subdivision surfaces for simulation and data-driven modeling of woven cloth

Fu, Chuyuan, The material point method for simulating eastoplastic materials

Gim, Geunho, Stabilization of a tower of universal deformation rings

Hughes, Joseph, Modular forms associated to real cubic field

Laackman, Donald, Degree three cohomological invariants of reductive groups

Lagkas Nikolos, Ioannis, Localization and modules in derivators

Luo, Xiyang, Analysis and application of graph semi-supervised learning methods

Meng, Zhaoyi, High performance computing and real time software for high dimensional data classification

Menke, Michael, Some results on fillings in contact geometry

Nguyen Luu, Danh, The computational complexity of Presburger arithmetic

Norwood, Zach, Combinatorics and absoluteness of definable sets of reals

Ntalampekos, Dimitrios, Potential theory on Sierpinski carpets with applications to uniformization

Rooney, Jacob, On cobordism maps in embedded contact homology

Royston, Michael, A Hopf–Lax formulation of the eikonal equation for parallel redistancing and oblique projection

Siegel, Jonathan, Accelerated first-order optimization with orthogonality constraints

Yang, Yilong, Shapes of finite groups through cove properties and Cayley graphs

Zhang, Fangbo, A blob method for advection-diffusion-reaction systems with application to robotic swarm

Zhang, Yunfeng, Strichartz estimates for the Schrödinger flow on compact symmetric spaces

University of California, Riverside (9)

Department of Mathematics

Arauza, Andrea, Spectral triples and fractal geometry

Chun, Daniel, Asymptotic syzygies of normal crossing varieties

Coya, Brandon, Circuits, bond graphs, and signal-flow diagrams: A catergorical perspective

Murray, Kayla, Graded representations of current algebras

Ogaga, James, Function theory on open Kähler manifolds

Ruth, Lauren, Two new settings for examples of von Neumann dimension

Sherbetjian, Alex, Rigidification of algebras over algebraic theories in diagram categories

Simanyi, John, The Poisson cohomology of k-step Nilmanifolds

Tousignant, Jordan, Koszulity of directed graded k-linear categories and their quadratic dual

University of California, San Diego (9)

Department of Mathematics

Bodnar, Michelle, Rational Catalan combinatorics

Chen, Jie, Prediction in time series models and model-free inference with a specialization in financial return data

Drimbe, Daniel, Some rigidity results for coinduced actions and structural results for group von Neumann algebras

Liu, Yuchao, Detection and localization of a submatrix: Theory, methods and algorithms

Pornopparath, Donlapark, Well-posedness and modified scattering for derivative nonlinear Schrödinger equations

Prinyasart, Thanakorn, An effective equidistribution of diagonal translates of certain orbits in $\mathrm{ASL}(3, Z)\backslash \mathrm{ASL}(3, R)$

Sangha, Luvreet, Generating functions for descents and levels over words which avoid a consecutive pattern

Tang, Xiudi, Symplectic stability and new symplectic invariants of integrable systems

Zhu, Tingyi, Kernel methods in nonparametric functional time series

University of California, Santa Barbara (19)

Department of Mathematics

Blacker, Casey, The moduli space of flat connections over higher dimensional manifolds

Curtis, Amanda, On projectors for the sl3 spider

Dougherty, Michael, The geometry and topology of the dual braid complex

Hake, Kathleen, The geometry of the space of knotted polygons

Jin, Zhongmin, Homeomorphism finiteness theorem with integral curvature bound

Kaminsky, John, On the stochastic closure theory of homogeneous turbulence

Merkx, Peter, Global symmetries of six dimensional super conformal field theories

Pankau, Joshua, On stretch factors of pseudo-Anosov maps

Ricci, Joseph, Congruence subgroup and quantum representations of mapping class groups

Wen, David, On minimal models and canonical models of elliptical fourfold with section

Department of Statistics and Applied Probability

He, Jingyi, Fixing mixed effects models with big data

Hu, Ruimeng, Asymptotic methods for portfolio optimization in random environments

Mousavi, Seyyed Mostafa, Financial markets with delay

Ning, Patricia, Topics in financial math (uncertain volatility, ross recovery and mean field games on random graph)

Risk, James Kenneth, Three applications of Gaussian process modeling in evaluation of longevity risk management

Rodriguez Hernandez, Sergio, Generalized probabilistic bisection for stochastic root-finding

Shi, Jian, Some contributions to smoothing spline density estimation and inference

Xu, Danqing, Fitting smoothing splines to massive data

Yang, Xuwei, Games in energy markets

University of California, Santa Cruz (6)

Department of Mathematics

Barsotti, Jamison, The unit group of the Burnside ring for some solvable groups

Ferrara, Joseph, Stark's conjectures for p-adic L-functions

Gottesman, Richard, The algebra and arithmetic of vector-valued modular forms on $\Gamma_0(2)$

Jackman, Connor, Free homotopy classes in some n-body problems

Martins, Gabriel, The Hamiltonian dynamics of magnetic confinement and instances of quantum tunneling

Nguyen, Danquynh, Fusion rules for the lattice vortex operator algebra V_L

University of Southern California (9)

Department of Mathematics

Bhattacharjee, Chinmoy, Stein's methods and its application in strong embeddings and Dickman approximations

Gerhardt, Spencer, Topological generation of classical algebraic groups

Hankin, Michael, Control of false discovery rate and related metrics for sequential testing of multiple hypotheses under arbitrary dependence conditions

Kim, Gene B., Distribution of descents in matchings

Nguyen, Dinh Trung, Random walks on finite groups and their irreducible representations

Ozel, Enes, Cycle structures of permutations with restricted positions

Sun, Rentao, Conditional mean-fields SDEs and application

Wang, Fei, On regularity and stability in fluid dynamics

Wu, Cong, Controlled McKean–Vlasov equations and related topics

COLORADO

Colorado State University (8)

Department of Mathematics

Blumstein, Mark, A geometric formula for degree of equivariant cohomology rings

Davis, Brent, The numerical algebraic geometry approach to polynomial optimization

Emerson, Tegan, A geometric data analysis approach to dimension reduction in machine learning and data mining in medical and biological sensing

Hashmi, Bahaudin, Pattern formation in reaction diffusion systems and ion bombardment of surfaces

Maglione, Josh, On automorphism groups of p-groups

Neville, Rachel, Topological techniques for characterization of patterns in differential equations

Department of Statistics

Fu, Ran, Improving survey estimators through weight smoothing

Scharf, Henry, Statistical models for dependent trajectories with application to animal movement

University of Colorado, Boulder (13)

Department of Applied Mathematics

Fairbanks, Hillary, Low-rank, multi-fidelity methods for uncertainty quantification of high dimensional systems

Kalchev, Delyan, Dual least squares finite element methods for hyperbolic problems

Nardini, John, Partial differential equation models of collective migration during wound healing

Southworth, Benjamin, Seeking space aliens and the strong approximation property: A disjoint study in dust plumes on planetary satelites and nonsymmetric algebraic multigrid

Wills, Peter, Studies in the analysis of stochastic processes

Department of Mathematics

Burkett, Shawn, Subnormality and normal series in supercharacter theory

Coston, Natalie, Spectral properties of products of independent random matrices

Frinak, Joshua, Degeneration of Prym varieties: A computational approach to the indeterminacy of the Prym period map and degenerations of cubic threefolds

Lamar, Jonathan, Lattices of supercharacter theories

Ledbetter, Sion, Heisenberg codes and channels

Long, Ian, Spectral Hutchinson-3 measures and their associated operator fractals

Rosenbaum, Ryan, On the poles of Mellin transforms of principal series Whittaker functions

Shriner, Jeffrey, Hardness results for the subpower membership problem

University of Colorado, Denver (5)

Department of Biostatistics and Informatics

Dalwani, Manish, Machine learning in neuroimaging based modalities using support vector machines with wavelet kernals

Vestal, Brian, A computationally efficient spatial point process framework for characterizing lung computed tomography scans

Department of Mathematical and Statistical Sciences

Sigler, Devon, Multi-objective optimization under uncertainty

Walsh, Scott, Simulation-based optimal experimental design: Theories, algorithms and practical considerations

Yorgov, Daniel, Combined admixture and association mapping for complex traits

University of Northern Colorado (2)

School of Mathematical Sciences

Hancock, Brent, Undergraduates' collective argumentation regarding integration of complex functions within three worlds of mathematics

Hancock, Emilie, The sociocultural mediation of metacognition in undergraduate mathematics classroom communities of practice

CONNECTICUT

University of Connecticut, Storrs (18)

Department of Mathematics

Dong, Mengxia, Best constants, extremal functions and stability for geometric and functional inequalities

Evans, Kyle, Investigating the relationship between mathematics education and global citizenship education through K-12 mathematics teacher perspectives

Feng, Qi, Topics in stochastic analysis and Riemannian foliations

Lemay, Steven, Teachers' navigation of mathematical representations of argumentation

Martin, Daniel, Mass in general relativity

Moran, Rebecca, Traveling waves in a suspension bridge

Nicholson, Marie, Applications of computability theory to partial and linear orders

Rabideau, Michelle, Continued fractions in cluster algebras, lattice paths, and Markov numbers

Tang, Huili, National retirement sustainability index

Department of Statistics

Bapat, Sudeep, Multistage sampling strategies and inference in health studies under appropriate Linex loss functions

Bishoyi, Abhishek, Application of Gaussian process priors on Bayesian regression

Deshpande, Ved, Statistical methods for analyzing bivariate mixed outcomes

Jiang, Yujing, Marginal score equations for spatial extremes modeling with latent signals and applications in fingerprinting changes in climate extremes

Luo, Chongliang, On integrative reduced-rank models and applications

Mishra, Aditya, On sequential estimation of multivariate associates

Shi, Daoyuan, Statistical methods for information assessment and data compatability with applications

Vaughan, Gregory, Stagewise estimating equations

Zhang, Yaohua, Structure learning and break detection in high-frequency data

Yale University (16)

Biostatistics Department

Cameron, Briana, Extensions to the two-stage randomized trial for testing treatment, self-selection, and treatment preference effects

DeVeaux, Michelle, Innovative statistical methods for early phase clinical trials

Hu, Yiming, Integrative analysis of multi-omics data improves genetic risk prediction and transcriptome-wide association analysis

Jung, Taehyun, A joint model for recurrent events and semi-competing risk in the presence of multi-level clustering: An application to HIV-infected US veterans from OPTIMA trial

Liu, Yiyi, Statistical methods for cell heterogeneity and cell drug-response study

Department of Mathematics

Abrikosov, Efim, Potentials for moduli spaces of A_m-local systems on surfaces

Corey, Daniel, Initial degenerations of Grassmannians

Montealegre, Daniel, Probabilistic methods in combinatorics

Pan, Wenyu, Dynamics of Kleinian groups

Shen, Jifeng, Break divisors and compactified Jacobians

Sheydvasser, Arseniy, Classifying integral crystallographic packings

Weng, Daping, Cluster Donaldson–Thomas transformations of Grassmannians and double Bruhat cells

Department of Statistics and Data Science

Brinda, William D., Adaptive estimation with Gaussian radial basis mixtures

Klusowski, Jason, Density, function, and parameter estimation with high-dimensional data

Lu, Yu, Statistical and computational guarantees for learning latent variable models

Zhang, Anderson Ye, Community detection: Fundamental limits, methodology, and variational inference

DELAWARE

Delaware State University (3)

Department of Mathematical Sciences

Sutton, Brielle, A qualitative simulation of blood flow through an elastic cerebral saccular aneurysm using an immersed boundary method

Xu, Penglong, Iteration based temporal subcycling finite-difference time-domain algorithm and through-the-wall radar detection analysis

Xu, Yanan, Optical soliton propagation in metamaterials; evolutionary pattern formation for competing populations under seasonal forcing

University of Delaware (7)

Department of Mathematical Science

Bailey, Zachary, Some inverse problems for hyperbolic partial differential equations

DeTeresa Trueba, Irene, A symptotic methods in inverse scattering for inhomogeous media

McGinnis, Matthew, Combinatorial and spectral properties of graphs and association schemes

Qirko, Klajdi, A saddle point least squares method for systems of linear PDEs

Rezac, Jake, Direct methods for inverse scattering with time dependent and reduced data

Yuan, Tao, Radon transform spherical means and an inverse problem for the wave equation

Zhou, Yingxiang, Estimation and inference in problems from imaging and biophysics

DISTRICT OF COLUMBIA

George Washington University (2)

Department of Mathematics

Bedi, Harpreet, Cohomology of line bundles of rational degree over perfectoid space

Lu, Jiajun, Pattern formation in binary systems with inhibitory long-range interaction

FLORIDA

Florida Atlantic University (6)

Department of Mathematical Sciences

Gallolu Kankanamalage, Hasala Senpathy, Output stability analysis for nonlinear systems with time delays

Joseph, Jean, A constructive theory of ordered sets and their completions

Kepley, Shane, The circular restricted four body problem is non-integrable: A computer assisted proof

Langenberg, Brandon, Quantum circuits for symmetric cryptanalysis

Robinson, Angela, Quantum-resistant key agreement and key encapsulation

Thomack, Andrew, Random harmonic polynomials

Florida Institute of Technology (5)

Department of Mathematical Sciences

Aal Rkhais, Habeeb, On the qualitative theory of the nonlinear degenerate second order parabolic equations modeling reaction-diffusion-convection processes

Alharbi, Majed, On logconcavity of multivariate discrete distributions

Aljaber, Noha, Boundary value problems in a multidimensional box for higher order linear and quasi-linear hyperbolic equations

McDougall, Jeffrey, The capacitated transfer point covering problem (TPCP): Expanding delivery network coverage with minimal resources

Onyejekwe, Osita, Parametric and nonparametric regression models with applications to climate change

Florida State University (42)

Department of Mathematics

Almalki, Yahya, Sorvali dilatation and spin divisors on Riemann and Klein surfaces

Cellat, Serdar, Metric learning for shape classification: A fact and efficient approach with Monte Carlo methods

Dobreva, Atanaska, Using mathematical tools to investigate the autoimmune hair loss disease alopecia areata

Ebadi, Sepideh, Evolutionary dynamics of bacterial persistance under nutrient/antibiotic actions

Flores Diaz, Diana, An electrophysiological and mathematical modeling study of developmental and sex effects on neurons of the zebra finch song system

Galvis, Daniel, Distributed neural network models for birdsong production

Hancock, Matthew, Algorithmic lung module analysis in chest tomography images: Lung nodule malignancy likelihood prediction and a statistical extension of the level set image segmentation method

Imamoglu, Erdal, Algorithms for solving linear differential equations with rational function coefficients

Khanmohamadi, Omid, High-order, efficient, numerical algorithms for integration in manifolds

Li, Jian, Modeling of biofilms with implementations

Lin, Hua-Yi, Optimal portfolio execution under time-varying liquidity constraints

Marchand, Melissa Sue, Low-rank Riemannian optimization approach to the role extraction problem

Pei, Chaoxu, Space-time spectral element methods in fluid dynamics and materials science

Shen, Yunyi, Landscapes in non-commutative geometry

Sparaco, Leona, Character varieties of knots and links with symmetries

Tsai, Wan-Yu, Monte Carlo scheme for a singular control problem: Investment consumption under proportional transaction costs

Tzeng, Yu-Ying, Quasi-Monte Carlo and Markov chain quasi-Monte Carlo methods in estimation and prediction of time series models

Valdes, Yaineli, The 1-type of k-theory for Waldhausen category as a miltifunctor

Wang, Jian, Ensemble methods for capturing dynamics of limit order books

Wesolowski, Sergiusz Jan, Developing SRSF shape analysis techniques for applications in neuroscience and genomics

Williams, Ethan, Affine dimensions of smooth curves and surfaces

Xu, Wen, Third order A-hypergeometric functions

Zhang, Xiping, Characteristic classes and local invariants on determinantal varieties (temperate)

Zhou, Chenchen, On the multidimensional default threshold model for credit risk model

Department of Statistics

Ahn, Kyungmin, Elastic functional regression model

Baker, Danisha, A Bayesian wavelet based analysis of longitudinally observed skewed heteroscedastic responses

Bhingare, Apurva, Semiparametric Bayesian regression models for skewed responses

Chen, Qiusheng, Tests and classifications with application in adaptive designs

Duncan, Adam, Statistical shape analysis of nueronal tree structures

Duncan, Megan, Elastic functional principal component analysis for modeling and testing of functional data

Geneus, Vladimir, Nonparametric change point detection methods for profile variability

Guo, Ruite, Testing for the equality of two distributions on high dimensional object space

Hu, Guanyu, Spatial statistics and its applications in biostatistics and environment statistics

Kucukemiroglu, Saryet, Examining the effect of treatment on the distribution of blood pressure in the population using observational studies

Li, Hanning, Bayesian modeling and variable selection for complex data

Liu, Shuyi, Generalized Mahalanobis depth in point process and its application in neural decoding and semi-supervised learning in bioinformatics

Mukherjee, Anwesha, Building a model performance measure for examing clinical relevance using net benefit curves

Sabnis, Gautum, Scalable and structured high dimensional covariance matrix estimation

Tran, Hoang, Non-parametric and semi-parametric estimation and inference with application to finance and bioinformatics

Varbanov, Roumen, Wavelet-based Bayesian approaches to sequential profile monitoring

Wang, Libo, Regression methods for skewed and heteroscedastic response with high-dimensional covariates

Yang, Liu, Semi-parametric generalized estimating equations with kernel smoother: A longitudinal study in financial data analysis

University of Central Florida (2)

Department of Mathematics

Cheng, Cheng, Sampling and reconstruction of spatial signals

Gomez, Tyler, Filtering problems in stochastic tomography

University of Florida (18)

Department of Mathematics

Amarasinghe, Ashwini, On acyclicity properties of complements of subsets in the Hilbert cube

Dey, Agnish, Collapsing of non-homogeneous Markov chains

Hiller, Joshua, On some variations of the multistage model of carcinogenesis

Hocutt, Jeramiah, Twisted duality in various Clifford structures

Li, Xianqi, Fast first-order optimization methods with applications to inverse problems

Saucier, Chase, Strict contactomorphisms of hyperquadrics

Uncu, Ali, Some topics in q-series and partitions

Vallejo, Celeste, Some techniques for analyzing stochastic models with application to epidemiology

Wijesooriya, Udeni, Finite rank isopairs

Zhang, Wei, Accelerated bundle level type methods for large scale convex optimization

Department of Statistics

Bhattacharjee, Abhishek, Identifying active factors in multi factor trials and empirical Bayes intervals for the selected mean

Myung, Jiyoun, Small area estimation, nonparametric median estimation and measurement error models: A Bayesian approach

Rahman, Syed, Cholesky-based model selection and estimation in graphical models

Shi, Runmin, Statistical computing methods for big data problems

Skripnikov, Andrey, Topics in joint estimation of vector autoregressive models

Tang, Xueying, Bayesian data analysis under shrinkage priors

Wang, Pei-Li, Identifying active factors in multi factor trials and empirical Bayes intervals for the selected mean

Zhang, Liyuan, Trace class Markov chains for Bayesian shrinkage models

University of Florida College of Public Health (4)

Department of Biostatistics

Liu, Jing, Improving power for testing genetic effects on binary and survival outcomes using auxiliary information

Sharker, M. A. Yushuf, Pairwise accelerated failure time models for infectious disease transmission data

Sikdar, Sinjini, Statistical methods for anlayzing genomics data

Xu, Suwa, Learning high-dimensional Bayesian networks for general types of random variables

University of Miami (3)

Department of Mathematics

Bajo Caraballo, Carlos, Fluid limit and stochastic stability for a genetic model

Ellzey, Brittney, On chromatic quasisymmetric functions of directed graphs

McKeown, James, On the combinatorics of the Waldspurger decomposition

University of South Florida (12)

Department of Mathematics and Statistics

Churchill, Gregory, On extending Hansel's theorem to hypergraphs

Churchill, Indu Rasika, Contributions to quandle theory: A study of f-quandles, extensions and cohomology

Devamitta Perera, Muditha, Statistical analysis and modeling of ovarian and breast cancer

Gao, Chao, Statistical analysis and modeling of stomach cancer data

Garapati, Kumar Vijay, Structural analysis of poloidal and toroidal plasmons and fields of multilayer nanorings

Hitigala Kaluarachchilage, Pubudu Kalpani, Cybersecurity: Stochastic analysis and modelling of vulnerabilities to determine the network security and attackers behavior

McAnally, Morgan, Generalized D-Kaup-Newell integrable systems and their integrable couplings and Darboux transformations

Na, Shuang, Time series online Bayesian kernel segmentation: Applications in real time activity recognition using smartphone accelerometer

Rajasooriya, Sasith, Cybersecurity: Probabilistic behavior of vulnerability and life cycle

Rodrigo, Pulahinge Hansapani, Bayesian artificial neural networks in health and cyber-security

Saghafi, Abolfazl, Real-time classification of biomedical signals, Parkinson's analytical model

Zhou, Yuan, Lump, complexition and algebro-geometric solutions to solition equations

GEORGIA

Augusta University (3)

Department of Population Health Sciences

Daniel, Jeannie, A modified bump hunting approach with correlation-adjusting kernel weight for detecting differentially methylated regions on the 450K array

Lee, Jaeeun, A modified information criterion in the 1d fused lasso for DNA copy number variant detection using next generation sequencing data

Lee, Taejin, Mathematical and stochastic modeling of HIV immunology and epidemiology

Emory University (18)

Department of Biostatistics and Bioinformatics

Ainslie, Kylie Ellen, Estimation of the effectiveness of influenza vaccination from observational studies

Deng, Yi, Statistical methods for incomplete big data

He, Qing, Machine learning methods in large scale neuroimaging study

Jeffers, Caprichia, Statistical methods for correlated count data

Jin, Zhuxuan, Statistical methods for omics data integration

Li, Ben, Novel model-based methods for high-throughput genomics data analysis

Li, Ziyi, Statistical learning methods for big biometical data

Liao, Peizhou, Novel statistical methods for analyzing next generation sequencing data

Noreen, Samantha, Quantifying the impact of local SUTVA violations in spatiotemporal causal models

Zhu, Wanzhe, Statistical methods for handling missing data in functional data analysis

Department of Mathematics and Computer Science

Haase, Bastian, Patching and local-global principles for gerbes with an application to homogeneous spaces

Keller, Ariel, On cycles, chorded cycles, and degree conditions

Lupo Pasini, Massimiliano, Deterministic and stochastic acceleration techniques for Richardson-type iterations

Morrissey, Charles, Topics in tropical analytic geometry

Schneider, Robert, Eulerian series, zeta functions and the arithmetic of partitions

Shull, Warren, On spanning trees with few branch vertices

Trebat-Leder, Sarah, Connections between classical and umbral moonshine

Viguerie, Alexander, Efficient, stable, and reliable solvers for the steady incompressible Navier–Stokes equations in computational hemodynamics

Georgia Institute of Technology (14)

School of Mathematics

Bolding, Mark, Topics in dynamics: First passage probabilities and chaotic properties of the physical wind-tree model

Dang, Thanh, Minors of graphs of large path-width

de Viana, Mikel, Results on invariant whiskered tori for fibered holomorphic maps and on compensated domains

Du, Rundong, Nonnegative matrix factorization for text, graph, and hybrid data analytics

Hou, Yanxi, Statistical inference for some risk measures

Kunwar, Ishwari, Multilinear dyadic operators and their commutators

Mena Arias, Dario, Characterization of matrix valued BMO by commutators and sparse domination of operators

Ralli, Peter, Curvature and isoperimetry in graphs

Sampson, Donald, Inflation of a planar domain

Spencer, Timothy Scott, Weighted inequalities via dyadic operators and a learning theory approach to compressive sensing

Tossounian, Hagop, Mathematical problems concerning the Kac model

Walsh, Joseph Donald, The boundary method and general auction for optimal mass transportation and Wasserstein distance computation

Wang, Yan, Subdivisions of complete graphs

Zhang, Lei, Analysis and numerical methods in solid state physics and chemistry

Georgia State University (12)

Department of Mathematics and Statistics

Alacam, Deniz, Modeling rhythm generation in swim central pattern generator of Melibe leonina

Baidya, Robin, Capacities and cancellation

Chamberlain, Darryl, Investigating the development of proof comprehension: The case of proof by contraction

DeCamp, Linda, Regularized numerical algorithms for stable parameter estimation in epidemiology and implications for forecasting

He, Xiuxiu, Integrated mathematical and experimental study of cell migration and shape

Jeter, Russell, The stability and control of stochastically switching dynamical systems

Liu, Bing, Bayesian methods in brain connectivity change point detection in EEG data and genetic algorithm

McCammon, Jeffrey, Preservice teachers' understanding of geometric definitions and their use in the concept of special quadrilaterals

Ng, Shuenn Siang, Characterizing F-rationality of Cohen–Macaulay rings via canonical modules

Reimbayev, Reimbay, Synchronization in neuronal networks with electrical and chemical coupling

Sirb, Benjamin, Decentralized parameter estimation

Yankey, David, An estimation of county-level vaccination coverage for human papillomavirus vaccine among adolescents aged 13–17 years in Southeastern United States of America using Bayesian and spatial effects models

University of Georgia (13)

Department of Mathematics

Hobson, Natalie, Vector bundles of conformal blocks in types A and C from a combinatorial approach

Lanterman, James, A generalization of bivariate splines over polygonal partitions and applications

Parshall, Hans, Point configurations over finite fields

Schaffler, Luca, The KSBA compactification of a 4-dimensional family of polarized Enriques surfaces

Stevenson, Richard "Bret", Barcodes and quasi-isometric embeddings into Hamiltonian diffeomorphism groups

Varghese, Abraham, An application of von Neumann algorithm to matrix completion and other data problems

Department of Statistics

Bedoui, Adel, Bayesian empirical likelihood for linear regression and penalized regression

Cao, Sha, Towards understanding the interplays between cellular stresses and cancer development

Hasan, Mohamad, Optimal p-value weighting with independent information

Liu, Yiwen, Dimension reduction and multisource fusion for big data with applications in bioinformatics

Sun, Xiaoxiao, Nonparametric methods for big and complex datasets under a reproducing kernel Hilbert space framework

Xing, Xin, Statistical methods with applications in epigenomics, metagenomics and nueroimaging

Zhang, Wei, Optimal designs for the panel mixed logit model

HAWAII

University of Hawaii at Manoa (2)

Department of Mathematics

Guillen, Alejandro, On the generalized word problem for finitely presented lattices

Wong, Ka Lun, Sums of quadratic functions with two discriminants and Farkas' identitites with quartic characters

ILLINOIS

Illinois Institute of Technology (7)

Applied Mathematics Department

Chang, Yu-Sin, Markov chain structures with applications to systemic risk

Cheng, Zhuan, Non-Gaussian assimilation

Hernandez, Francisco, A boundary integral method for computing the forces of moving beads in a 3-dimensional linear viscoelastic flow

Huang, Yicong, Wiener–Hopf factorization for time-inhomogeneous Markov chains

Kabre, Julienne, An energy-preserving scheme for the Poisson–Nernst–Planck equations

Mudrock, Jeffrey, On the list coloring problem and its equitable variants

Wilburne, Dane, Quantifying uncertainty in random algebraic objects using discrete methods

Northern Illinois University (3)

Department of Mathematical Sciences

Dhar, Sougata, Lyapanov-type inequalities and applications to boundary value problems

Dmitrieva, Tatiana, Improved approximate Bayesian computation methods for censored and uncensored data

Wang, Yiqing, A dependent competing risks model

Northwestern University (14)

Department of Mathematics

Elmanto, Elden, Motivic contractibility of the space of rational maps

Garcia, Xavier, Asymptotics on the number of critical points for two high-dimensional analytical models

Heleodoro, Aron, On the geometry of higher Tate spaces

Konter, Johan, The Goodwillie tower of infinite loops as derived functors

Looper, Nicole, Uniformity in polynomial dynamics: Canonical heights, primitive prime divisors, and Galois representations

Reiser, Andrew "Bif", Pushforwards of measures on real varieties under maps with rational singularities

Shi, Nan, Almost Gelfand property of symmetric pairs

Vankoughnett, Paul, Localizations of E-theory and transchromatic phenomena in stable homotopy theory

Williams, Brian, The holomorphic sigma model and its symmetries

Department of Statistics

Ktsanes, Rachel, Design and analysis of trials for developing adaptive treatment strategies in complex clustered settings

Yoon, Grace, Topics of variable selection in biomedical data mining

Engineering Science and Applied Mathematics Dept

Autry, Eric, Traveling waves in models of population dynamics with nonlocal interactions

Berahas, Albert, Methods for large scale nonlinear and stochastic optimization

Hill, Kaitlin, Bifurcation analysis of a piecewise-smooth arctic energy balance model

Southern Illinois University (6)

Department of Mathematics

Alhuraiji, Abdulkarem, Coupling of quadratic lattices

Althubiti, Saeed, Stochastic functional differential equations with infinite memory

Gamachchige, Nirosh, Double-change covering designs with block size $k = 4$

Gumus, Mehmet, On the Lyapunov-type diagonal stability

Pelawa Watagoda, Lasanthi, Inference after variable selection

Rupasinghe, Hasthika, Bootstrapping analogs of the one way Manova test

University of Chicago (24)

Department of Mathematics

Apisa, Paul, Dynamics in the moduli space of translation surfaces

Butler, Clark, Characterizing symmetric spaces by their Lyapunov spectra

Casto, Kevin, Representation theory and arithmetic statistics of generalized configuration spaces

Chan, Henry Yi-Wei, Topological Hochschild homology of maximal orders of simple $\mathbb{Q}$-algebras

Chen, Lei, From point-picking to sections of surface bundles

Chi, Jingren, Geometry of generalized affine Springer fibers

Chonoles, Zev, The $RO(G)$-graded cohomology of the equivariant classifying space $B_G SU(2)$

Frankel, Ian, On Euclidean and Teichmüller distances

He, Yan Mary, Some theorems in Kleinian groups and complex dynamics

Leal, Isabel, Topics in ramification theory

Rubin, Jonathan, Equivariant categorical coherence theory

Sagatov, Sergei, Logarithmic differential operators on the wonderful compactification

Scher, Henry, The Bernstein–Sato b-function for the complement of the open SL_n-orbit on a triple flag variety

Vishnepolsky, Rachel, Random walks on Cartesian products of certain nonamenable groups and integer lattices

Department of Statistics

Goswami, Subhajit, Some metric properties of planar Gaussian free field

Li, Ang, Multiple testing with prior structural information

Li, Li, Two problems in percolation theory

Tang, Si, High-dimensional first passage percolation and occupational densities of branching random walks

Tang, Yunfan, Models and inference for microbiome data

Wang, Wei, Applications of adaptive shrinkage in multiple statistical problems

Wong, Sze Wai, Geometric methods in statistics and optimization

Xu, Wanting, On the optimal estimation, control, and modeling of dynamical systems

Zhang, Danna, Estimation and inference for high dimensional time series

Zhu, Xiang, A Bayesian large-scale multiple regression model for genome-wide association summary statistics

University of Illinois at Chicago (18)

Mathematics, Statistics, and Computer Science Department

Bering, Edgar, Compatible trees and outer automorphisms of a free group

Bliss, Nathan, Computing series expansions of algebraic space curves

Cheng, Qianshun, Novel algorithm for constrained optimal design and information-based subdata selection for logistic models

Cole, Samuel, An iterative spectral approach to recovering planted partitions

Dannenberg, Ellie, Circle packings on surfaces with complex projective structures

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Wang, Xin, Optimal design for nonlinear models with random effects and information-based subdata selection for LASSO

University of Illinois, Urbana–Champaign (46)

Department of Mathematics

Addabbo, Darlayne, Q-systems and generalizations in representation theory

Albar, Wafaa, Non commutative version of arithmetic geometric mean inequality and crossed product of ternary ring of operators

Camacho, Santiago, Truncation in differential Hahn fields

Chaubey, Sneha, Correlations of sequences modulo one and statistics of geometrical objects associated to visible points

Du, Xiumin, A sharp Schrödinger maximal estimate in $\mathbb{R}^2$

Etedadi Aliabadi, Mahmood, Generic behaviour of a measure transformation

Ferguson, Timothy, Dynamical systems on networks

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Karr, William, Convexity and curvature in Lorentzian geometry

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Kydonakis, Georgios, Gluing constructions for Higgs bundles over a complex connected sum

Liu, Shiya, Asymptotically optimal shapes for counting lattice points and eigenvalues

Loeb, Sarah, Coloring and covering problems on graphs

Malik, Amita, Partition asymptotics and zeros of zeta functions

Meng, Xianchang, The distribution of k-free numbers and integers with fixed number of prime factors

Michiels, Daan, Symplectic foliations, currents, and local Lie groupoid

Obiero Oyengo, Michael, Chebyshev-like polynomials, conic distribution of roots, and continued fractions

Panagiotopoulos, Aristotelis, Structures and dynamics

Pandey, Ashish, Modulational instability in some shallow water wave models

Qin, Wei, Information, insider trading and takeover announcements

Romney, Matthew, Metric geometry of the Grushin plane and generalizations

Skabelund, Dane, Covers and invariants of Deligne–Lusztig curves

Smith, Mychael, Equivariant E-infinity algebras

Sun, Hao, W-operators and generating functions of Hurwitz numbers

Tebbe, Amelia, Computing the Goodwillie–Taylor tower for discrete modules

Tian, Hongfei, On the center of the ring of invariant differential operators on semisimple groups over fields of positive characteristics

Tokcan, Neriman, Relative waring ranks of binary forms

Turmunkh, Bolor, Nakajima (q, t)-characters as quantum cluster variables

Uyanik, Caglar, Dynamics of free group automorphisms and a subgroup alternative for $\mathrm{Out}(F_N)$

Villeta-Garcia, Juan, Stabilizing spectral functors of exact categories

Wagner, Zsolt Adam, On some problems in extremal, probabilistic, and enumerative combinatorics

Wolbert, Seth, Symplectic toric stratified spaces with isolated singularities

Yi, Bingji, On intrinsic ultracontractivity of perturbed Lévy process and applications of Lévy process in actuarial mathematics

Department of Statistics

Chen, Yinghan, Sampling for network motif detection and estimation of Q-matrix and learning trajectories in DINA model

Huang, Weihong, Statistical algorithms using multisets and statistical inference of heterogeneous networks

Huang, Xichen, Fast algorithms for Bayesian variable selection

Kinson, Christopher, Longitudinal principal components analysis for binary and continuous data

Lee, Chung Eun, Statistical inference of multivariate time series and functional data using new dependence metrics

Ouyang, Yunbo, Scalable sparsity structure learning using Bayesian methods

Park, Yeon Joo, Effect size estimation and robust classification for irregularly sampled functional data

Paul, Subhadeep, Consistent community detection in uni-layer and multi-layer networks

Tang, Xiwei, Individualized learning and integration for multi-modality data

Yang, Fan, Statistical inference based on characteristic functions for intractable likelihood problems

Yao, Shun, Dependence testing in high dimension

Zhu, Xiaolu, Heterogeneity modeling and longitudinal clustering

INDIANA

Indiana University-Purdue University Indianapolis (4)

Department of Mathematical Sciences

Behrouzvaziri, Abolhassan, Thermoregulatory effects of psychostimulants and exercise: Data-driven modeling and analysis

Cosper, David, Periodic orbits of piecewise monotone maps

Petrovic, Drazen, Exact solution of the dimer model on the square and triangular lattice

Xu, Heng, Universally optimal designs for the two-dimensional interference model

Indiana University, Bloomington (15)

Department of Mathematics

Allen, Samantha, Relationships between the nonorientable genus and the normal Euler number of nonorientable surfaces whose boundry is a knot

Coleman, Neal, Laplace subspectrality

Cyr, Justin, Stationary determinantal processes on Z^d and some existence results for SPDEs with Lévy noise

Horton, Henry, A symplectic instanton homology via traceless character varieties

Hwang, Won Tae, On a classification of the automorphism group of a polarized abelian surfaces over finite fields

Lam, Wai Kit, Topics in critical and first passage percolation

Nie, Hongmin, Iteration at the boundry of Newton maps

Park, Eunhee, Some problems in boundary layer theory and stochastic partial differential equations

Shi, Fangye, Sharp decouplings for three dimensional manifolds in $\mathbb{R}^5$

Sprunger, David, Logics for coalgebras of finitary net functors

Wang, Xiaoyan, Numerical approximation of a variational inequality related to the humid atmosphere

Xu, Chen, Wavelet analysis of financial time series in asset pricing theory and power law exponent estimation

Yang, Ruiyu, Differential methods for phylogenetic reconstruction and their properties

Zuñiga, Andres, Geometric problems in the calculus of variations

Department of Statistics

Shields, Jacob, Intrinsic random functions on spheres: Theory, methods, and application

Purdue University (24)

Department of Mathematics

Barrios, Alexander, Minimal models of rational elliptic curves with non-trivial torsion

Bond, Jacob, On the computation and composition of Belyi maps and dessins d'enfants

He, Cuiyu, Robust a posteriori error estimation for various finite element methods on interface problems

Kepley, Paul, Techniques for the reconstruction of a Riemannian metric from boundary data via the boundary control method

Li, Qinfeng, Geometric measure theory with applications to shape optimization problems

Miller, Nicholas, Some geodesic theorems for arithmetic orbifolds

NG, Ngai Fung, Properties of caratheodory measure hyperbolic universal covers of compact Kähler manifolds

Sammartano, Alessio, Infinite free resolutions and blowup algebras

Shapiro, Jacob, Semiclassical resolvent bounds and wave decay in low regularity

Solapurkar, Partha, Some new surfaces of general type with maximal Picard number

Wang, Yingwei, Fast structured spectral methods

Department of Statistics

Chen, Chen, Parallel construction of large-scale gene regulatory networks

Cheng, Lin-Yang, Pragmatic statistical methods for effective protein biomarker discovery

Dixon Hamil, Kelly-Ann, Analysis of data of different spatial support: A multivariate process approach

Feldman, Guy, Sampling laws for multi-objective simulation optimization on finite sets

Harry, April, Design and statistical analysis of mass spectrometry imaging experiments

Hass, Zachary, Division of credit modeling for team sports with an emphasis on NCAA women's volleyball

Henry, Courtney, The design and analysis of calcium-41 clinical trials to study treatments for bone loss

Huang, Whitney, Statistics of extremes with climate applications

Li, Cheng, Optimal interative threshold kernel estimation of jump-diffusion processes

Liu, Yaowu, Advanced statistical tests for large-scale genomic data analysis

Medina, Patrick, Wave TDA: Bayesian statistics, wavelets, and topological data analysis to assess statistical shape

Oh, Ji Hwan, Graphical models for non-Gaussian continuous data with applications to genomics datasets

Schloerke, Barret, Generalized plot mattrices, automatic cognostics, and efficient data exploration

University of Notre Dame (10)

Applied and Computational Mathematics and Statistics

Bowen, Claire, Data privacy via integration of differential privacy and data synthesis

Liddell, Alan, Applications of Newton homotopies

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Pancaldi, Francesco, Mathematical models of bacteria polarity and fibrin networks mechanics

Department of Mathematics

Culver, Dominic, On the cooperations algebra for the second Brown–Peterson spectrum at the prime 2

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Lewis, Benjamin, Quantization commutes with reduction on compact Lie groups under the adjoint action

Lichtenfelz, Leandro, The geometry of the Euler and the Navier–Stokes equations

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IOWA

Iowa State University (36)

Department of Mathematics

Bozeman, Chassidy, Connections to zero forcing: Tree covers and power

Dairyko, Michael, On exponential domination of graphs

Goertz, Jeremiah, Reflection group diagrams for a sequence of Gaussian Lorentzian lattices

Guthrey, Pierson, Regionally implicit discontinuous Galerkin methods for solving the relativistic Vlasov–Maxwell system

Hotchkiss, Calvin, Fourier bases on the skewed Sierpinski gasket

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Huang, Keguo, Topics in queueing theory

Lazar, Robert, Association schemes and designs in symplectic vector spaces over finite fields

Munsi, Monalisa, Magneto-hydrodynamic flow in closed channels

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Zhou, Sen, Estimation of the risk-neutral density from option prices

Department of Statistics

Alvarez-Castro, Ignacio, Bayesian analysis of high-dimensional count data

Cheng, Hao, Contributions to improve the accuracy and computational efficiency of genomic prediction

Clark, Nicholas, Self-exciting spatio-temporal statistical models for count data with applications to modeling the spread of violence

Curley, Brenna J., Nonlinear models with measurement error: Application to vitamin D

DaSilva, Natalia, Bagged protection methods for supervised classification in big data

Downey, Jillian, Accounting for structure in education assessment data using hierarchical models

Hellams, Luvenia, Leveraging genetic time series data to improve detection of natural selection

Johnson, Margaret, Methods for analysis and uncertainty quantification for processes recorded through sequences of images

Kaplan, Andrea, On advancing MCMC-based methods for Markovian data structures with applications to deep learning, simulation, and resampling

Li, Yihua, Investment decision-making in clean energy under uncertainties: A real options research

Lithio, Andrew, Statistical methods for estimation, testing, and clustering with gene expression data

Liu, Chenhui, Three essays on crash frequency analysis

Martin-Schwarze, Adam, Extending removal and distance-removal models for abundance estimation by modeling detections in continuous time

Mei, Yong, Modeling and control to improve blood glucose concentration for people with diabetes

Mittman, Eric, Application of Bayesian hierarchical models in gene expression and product reliability

Nguyen, Yet Tien, Multiple hypothesis testing and RNA-seq differential expression analysis accounting for dependence and relevant covarities

Pan, Lanfeng, Mixture model and subgroup analysis in nationwide kidney transplant center evaluation

Peng, Liuhua, Topics in statistical inference for massive data and high-dimensional data

Price, Michael, Penalized B-splines and their application with an in depth look at the bivariate tensor product penalized B-spline

Ries, Daniel C., Measurement error modeling of physical activity data

Sage, Andrew, Random forest robustness, variable importance, and tree aggregation

Sun, Yaxuan, Statistical methods in modeling disease surveillance data with misclassification

Tyner, Samantha C., Graphical discovery in stochastic actor-oriented models for social network analysis

Wakeland, Kenneth, Exploring dependence in binary Markov random field models

University of Iowa (23)

Applied Mathematical and Computational Sciences

Aiello, Gordon, The classical moment problem and scattering asymptotic in Euclidean relativistic quantum theory

Bao, Minli, A moving-window penalization method and its applications

Cai, HanQin, Accelerating truncated singular-value decomposition: A fast and provable method for robust principal component analysis

Caples, Christine, A tabulation of tangle families

Czuprynski, Kenneth, Numerical analysis in energy dependent radiative transfer

Rhomberg, Patrick, On the parallelization of network diffusion models

Stiegler, Cole, Efficient local optimization for low-rank large-scale instances of the quadratic assignment problem

Wang, Tianming, Non-convex methods for spectrally sparse signal reconstruction via low-rank Hankel matrix completion

Department of Biostatistics

Barnes, Janel, Regression and boosting methods to inform precisionized treament rule using data from crossover studies

Ghattas, Andrew, Medical imaging segmentation assessment via Bayesian approaches to fusion, accuracy and variability estimation with application to head and neck cancer

Langenfeld, Natalie, A novel sequential ABC algorithm with applications to the opioid epidemic using compartmental models

Lou, Yiyue, Principal stratification: Applications and extensions in clinical trials with intermediate variables

Riedle, Benjamin N., Probabilistic pairwise model comparisons based on discrepancy measures and a reconceptualization of the p-value

Seedorff, Michael, Methods for testing for group differences in highly correlated, nonlinear eyetracking data

Tamegnon, Monelle, Avoiding the redundant effect on regression analysis of including an outcome in the imputation model

Zahrieh, David, Bayesian point process modeling to quantify excess risk in spatial epidemiology: An analysis of stillbirths with a maternal contextual effect

Zeng, Yaohui, Scalable sparse machine learning methods for big data

Department of Mathematics

Ardila, Rene, Morita equivalence of W*-correspondences and their Hardy algebras

De Santiago, Rolando, Structural results for von Neumann algebras of polyhyperbolic groups

Kindred, Thomas, Checkerboard plumbings

Liu, Suhui, Projected Wirtinger gradient descent for spectral compressed sensing

Pant, Sujan, Structural results in group von Neumann algebra

Department of Statistics and Actuarial Science

Wang, Xiao, On pricing barrier options and exotic variations

KANSAS

Kansas State University (13)

Department of Mathematics

Alrayes, Norah, Approximation of p-modulus in the plane with discrete grids

Aryal, Ashok, Geometry of mean value for uniformly elliptic divergence from operators

Chen, Hui, Counting representations of deformed preprojective algebras

Flesher, Paul, Proof validation in Euclidean geometry: A comparison of novices and experts using eye tracking

Ren, Jie, Cohomological Hall algebras and 2 Calabi–Yau

Van, Cong Tuan Son, Numerical solution to some inverse problems

Department of Statistics

Jia, Weijia, Goodness-of-fit tests in measurement error models with replications

Li, Xintong, Modeling for spatial and spatio-temporal data sets with applications

Li, Xiongya, Robust miltivariate mixture regression models

Mahmood, Sharif, Finding common support and assessing matching methods for causal inference

Mawella, Nadeesha, A robust test of homogeneity

Natarajan, Balasubramaniam, Asymptotic properties of non-parametric regression with beta kernels

Opoku Nsiah, Richard, A computationally efficient bootstrap-equivalent test for ANOVA in skewed populations with a large number of factor levels

University of Kansas (6)

Department of Mathematics

Chen, Wei-Da, Reimannian geometry on some noncommutative spaces

Claassen, Kyle, Stability of periodic waves in nonlocal dispersive equations

Jaramillo Gil, Arturo, Limit distributions for functionals of Gaussian processes

Lewis, Peter, Regularity of stochastic Burgers'-type equations

Wang, Bin, Balancing domain decomposition by constraints algorithms for incompressible Stokes equations with nonconforming finite element discretizations

Zhou, Hongjuan, Parameter estimation for stochastic differential equations driven by fractional Brownian motion

University of Kansas Medical Center (2)

Department of Biostatistics

Karanevich, Alex, Novel statistical methodology development and applications in ALS research

Zhong, Yi, Feature selection and classification for high-dimensional biological data under cross-validation framework

Wichita State University (4)

Department of Mathematics, Statistics, and Physics

Chen, Juan, Pluripolar hulls and convergence sets

Sahraei, Saman, Computation of plane potential flow past multi-element airfoils using conformal mapping revisited

Thompson, Nathan, Inverse tomography for the weighted transport equation

Wei, Xin, On strict Whitney arcs and t-quasi self-similar arcs

KENTUCKY

University of Kentucky (10)

Department of Biostatistcs

Liu, Meng, A predictive probability interim design for Phase II clinical trials with continuous endpoints

Department of Mathematics

Deterding, Stephen, Bounded point derivations on certain function spaces

Happ, Alexander, A combinatorial miscellany: Antipodes, parking cars and descent set powers

Liu, Jiaqi, Gloval well-posedness for the derivative nonlinear Schrödinger equation through inverse scattering

Meyer, Marie, Polytopes associated to graph Laplacians

Olsen, McCabe, Hilbert bases, descent statistics, and combinatorial semigroup algebras

Russell, Brandon Chase, Homogenization in perforated domains and with soft inclusions

Department of Statistics

Ellis, Amanda, Accounting for matching uncertainty in photographic identification studies of wild animals

Fan, Liangdong, Estimation in partially linear models with correlated observations and change-point models

Janse, Sarah, Inference using Bhattacharyya distance to model interaction effects when the number of predictors far exceeds the sample size

University of Louisville (10)

Department of Bioinformatics and Biostatistics

Kendrick, Sarah, Functional data analysis methods for predicting disease status

Li, Xiaohong, Sample size calculations and normalization methods for RNA-seq data

Liu, Ruiqi, Estimation of three key parameters and the lead time distribution in lung cancer screening

You, Wu, Bayesian approach on short-time-course data of protein phosphorylation, casual inference or ordinal outcome and casual analysis of dietary and physical activity in T2DM using NHANES data

Department of Mathematics

Bandara, Udika, Likelihood-based methods for analysis of copy number variation using next generation sequencing data

Hapuarachchi, Sujeewa, Regularized solutions for terminal problems of parabolic equations

King, Sarah, Extended difference of votes rules on three voting methods

Li, Heng, Some problems arising from mathematical model of ductal carcinoma in SITU

Luke, Michael, Residuated maps, the way-below relation, and contractions on probabilitstic metric spaces

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LOUISIANA

Louisiana State University, Baton Rouge (9)

Department of Mathematics

Ceniceros, Jose, On braids, branched covers and transverse invariants

Fontaine, Victoria, Characterizations of some classes of graphs that are nearly series-parallel

Livesay, Neal, Moduli spaces of flat GSp-bundles

McHenry, Emily, Electromagnetic resonant scattering in layered media with fabrication errors

Sinz, Paul, Extraction of displacement fields in heterogeneous media using optimal local basis functions

Thakur, Lokendra, Reduced order models for beam-wave interaction in high power microwave sources

Wang, Shilin, On structures of large rooted graphs

Wetzler, Kristen, The graphs and matroids whose only odd circuits are small

Zhai, Jiayu, General stochastic integral and Itô formula with applications to stochastic differential equations and mathematical finance

Louisiana Tech University (2)

Program of Mathematics and Statistics

Joseph, Grace, Cost-effective use of follow-up colonoscopies and fecal immunochemical tests in colorectal cancer screening

Liu, Rui, Mutual information in the frequency domain for application in biological systems

Tulane University (8)

Department of Global Biostatistics and Data Science

Xu, Chao, Hypothesis testing for high-dimensional regression under extreme phenotype sampling continuous traits

Department of Mathematics

Boroojeni, Asma Azizi, Mathematical models for predicting and mitigating the spread of chlamydia sexually transmitted infection

Fenske, Ellis, Anonymity and linkability

Gossman, Roseanna, The elastohydrodynamics of a simplified model of human birth

Kesarwani, Aashita, Theory of the generalized modified Bessel function $K_{z,w}(x)$ and 2-acid valuations of integer sequences

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Ugurlu, Ozlem, Counting Borel orbits in classical symmetric varieties

University of Louisiana at Lafayette (3)

Department of Mathematics

Heider, Blaise John, Annihilators and extensions of idempotent generated ideals

Petrovic, Vojislav, The $K(n)$-local E_n-Adams spectral sequence and a cohomological approximation of its E_2-term

Zhao, Lihong, Modeling, estimation and approximation in structured models

MARYLAND

Johns Hopkins University (8)

Department of Biostatistics

Deng, Detian, Scalable semi-parametric methods in biostatistics

Du, Yu, Statistical methods in clincial trial design

Fu, Jack, Statistical and computational methods for investigating human genetic variation via sequencing data

Huang, Emily, Statistical methods for the fraction who benefit using a randomized trial

Jiang, Dan, Adjustment procedure to permutation tests in epigenomic differential analysis

Lenis, David, Estimating causal effects with non-experimental data

Myint, Leslie, Evidence-based methods in studies of biology and data anlaysis

Prior, Thomas, Novel statistical methods in conjoint analysis and Pade approximation of the profile likelihood

Johns Hopkins University, Baltimore (7)

Department of Applied Mathematics and Statistics

Drivas, Theodore, Anomalous dissipation, spontaneous stochasiticity and Onsager's conjecture

Tang, Runze, Robust estimation from multiple graphs

Department of Mathematics

Diamond, Benjamin, Smooth surfaces in smooth fourfolds with vanishing first chern class

Kauffman, Christopher, Global stability for charged scalar fields in space times close to Minkowski

Luo, Chenyun, On the motion of the free surface of a compressible liquid

Wyman, Emmett, Explicit bounds on integrals of eigenfunctions over curves in surfaces of nonpositive curvature

Xi, Yakun, Kakeya–Nikondym problems and geodesic restriction estimates for eigenfunctions

University of Maryland, Baltimore County (9)

Department of Mathematics and Statistics

Boukouvalas, Zois, Development of ICA and IVA algorithms with application to medical image analysis

Donovan, Preston, Homogenization theory for solute motion accounting for obstructions, interactions, and path length effects

Gauran, Iris Ivy, Multiple testing procedures controlling false discovery rate with applications to genomic data

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Hosseini, Mina, Statistical modeling of self and proxy responses with gerontological applications

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Popuri, Sai Kumar, Prediction methods for semi-continuous data with applications in climate science

Zylstra, John, Generation and analysis of synthetic data for privacy protection under the multivariate linear regression model

University of Maryland, College Park (25)

Department of Mathematics

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Becker, Matthew Harrington, Modeling the transfer of drug resistance in solid tumors

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Bui, Quan Minh, Fast solvers and preconditioners for multiphase flow in porous media

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Fan, Yimei, Statistical learning with applications in high dimensional data and health care analytics

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Hunter, Ryan, The Monge–Ampere iteration

Jie, Cheng, Decision making under uncertainty: New models and controls

Law, Judith Neill, Estimation of a function of a large covariance matrix using classical and Bayesian methods

Li, Weilin, Topics in harmonic analysis, sparse representations and data analysis

Li, Xia, Misspecified weights in weight-smoothing method

Li, Yiran, Feature extraction in image processing and deep learning

Lu, Chunting, Hydrodynamic limits of the Boltzmann equation

Lum, Oliver, Solving, generating, and modeling arc routing problems

Maggino, Mark, Constant amplitude zero autocorrelation sequences and single pixel camera imaging

Maschal, Robert, Parabolic higgs bundles and Deligne–Simpson problem for loxodromic conjugacy classes in $PU(n,1)$

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Punshon-Smith, Samuel David, Capturing the large scale behavior of many particle systems through coarse-graining

Sun, Hechao, Describing urgent event diffusion on Twitter using network statistics

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Whiteway, Matthew, A latent variable modeling framework for analyzing neural population activity

Yao, Xuan, Extended estimating equations and empirical likelihood

MASSACHUSETTS

Boston College (5)

Department of Mathematics

Banfield, Ian, Baskets, staircases and sutured Khovanov homology

Cai, Yuanqing, Theta representations on covering groups

Crawford, Thomas, A stronger Gordon conjecture and analysis of free bicuspid manifolds with small cusps

Hayden, Kyle, Knot theory of holomorphic curves in Stein surfaces

Leslie, Winston, Theta liftings on higher covers of symplectic groups

Boston University (7)

Department of Mathematics and Statistics

Chang, Eric, A Sierpinski Mandelbrot spiral for rational maps of the form $z^n + \lambda/z^d$

Kang, Xinyu, Statistical methods for topology inference, denoising, and bootstrapping in networks

Lavitas, Liliya, On some new advances in self-normalization approaches for inference on time series

Li, Jun, Statistical methods for certain large, complex data challenges

Mottet, Clementine, Optimization-based approaches to non-parametric extreme event estimation

Tao, Long, Contributions to statistical analysis methods for neural spiking activity

Weber, Karoline, Advances in point process modeling: Feature selection, generalized goodness-of-fit, and novel applications

Boston University School of Public Health (7)

Department of Biostatistics

Ananthakrishnan, Revathi, On the designs of early phase oncology studies

Chaudhuri, Anamika, Profiling medical sites based on adverse events data for multi-center clinical trials

Chen, Yuning, Evaluation of mark density for population stratification adjustment and of a family-informed phenotype imputation method for single variant and variant-set tests of association

Deng, Xuan, Multiple testing problems in classical clinical trial and adaptive designs

Gerlovin, Hanna, Effective exposure lag-parametered exponential models for exposure risk

Ghosh, Pranab, Design of adaptive multi-arm multistage clinical trials

Yao, Baiyun, Bayesian approaches to address the issue of high placebo response in clinical trials using sequential parallel comparison design

Brandeis University (6)

Department of Mathematics

Kim, Moses, Weighted quivers, pro-picture groups, and periodic red-green trees

Lemus Vidales, Cristobal, Lattice path enumerization and factorization

Li, Yiting, Rigidity of eigenvalues for beta ensemble in multi-cut regime

Mirzadeh, Shahrias, Upper estimates for Hausdorff dimensions of exceptional sets in homogenous spaces

Murray, Devin, The Morse boundary and CAT(0) geometry

Zhuang, Yan, Non-commutative symmetric functions and permutation enumeration

Harvard University School of Public Health (17)

Department of Biostatistics

Cai, Tianyi, Statistical methods for the analysis of observational data with multiple correlated outcomes

Cheng, David, Estimating causal effects in pragmatic settings with imperfect information

Correia, Katharine, Statistical methods for assessing complex multi-exposure data in HIV and genetic epidemiology

Cummiskey, Kevin, Methods for estimating the health effects of exposure to point sources of emissions using large-scale and diverse data sources

Devick, Katrina, Nonparametric regression methods for causal mediation analysis

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Jazic, Ina, Design and analysis of nested case-control studies in the presence of a terminal event

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Li, Fei, Statistical methods for evaluating the relationships among social, spatial, and genetic networks

Liu, Lin, Contributions to evolutionary dynamics and causal inference

Papadogeorgou, Georgia, Casual inference methods in air pollution research

Peskoe, Sarah, Correcting for biases arising in epidemiologic research

Pita Juarez, Yered, Three aspects of gene expression: Pathway coexpression, cross-species analysis of RNA-seq data, and bias in gene coexpression

Schwager, Emma, Statistical methods for sequence-based microbial community assays

Slade, Emily, Statistical methods for multi-omics data

Wang, Yan, Toward a causal health effect of fine particulate air pollution

Wong, Hong Wen Benedict, Misclassification in the partial population attributable risk

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Department of Mathematics

Campbell, Christopher Justin, Nearby cycles of Whittaker sheaves

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Zhu, Huanhuan, Statistical methods for analyzing multivariate phenotypes and detecting causal rare variants

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Duo, Siwei, Mathematics and computation on nonlocal Schrödinger equation

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Hammuda, Muna, Semiparametric estimation under a Cox-type model for recurrent event data and model validation

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Bayat Mokhtari, Elham, Effect of neuromodulation of short-term plasticity on information processing in hippocampal interneuron synapses

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Drenska, Nadejda, A PDE approach to a prediction problem involving randomized strategies

Fennell, James, Two topics in the theory of nonlinear Schrödinger equations

Han, Jihun, Spontaneous oscillation and fluid-structure interaction of cilia

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Wang, Jun, Integral equation methods for the heat equation in moving geometry

Wang, Zhe, A driven tagged particle in one-dimensional exclusion processes

Rensselaer Polytechnic Institute (3)

Department of Mathematical Sciences

Crodelle, Jennifer, The role of electotonic junctions between pyramidal cells in the cortex

DeCourcy, Brendan, Parameter sensitivity of acoustic propagation in models of curved fronts over uniform slopes

Leisman, Katelyn, On solutions to integrable and non-integrable wave equations

Stony Brook University (35)

Department of Applied Mathematics and Statistics

Chen, Xiaolei, Collision response and rigid body dynamics in FronTier++ with application to parachute system simulations

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Fang, Han, Graphical and machine learning algorithms for high-throughput genomics data

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Mahadeo, Vinay, A new scaling law for K62 with applications to particle clustering in turbulent flow

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Wang, Xiaoying, Longitudinal learning dynamics and the conceptual restructuring of evolutionary understanding

Wang, Xingyu, Adaptive Lagrangian particle algorithms for partial differential equations and applications

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Yang, Jianzhao, Modeling financial durations in ultra-high-frequency data: A Markov-switching multifractal approach

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Zhang, Dingxin, Degeneration of slopes

Syracuse University (5)

Department of Mathematics

Cuneo, Daniel, Mappings between annuli of minimal p-harmonic energy

Edmond, Jennifer, The structure of 4-clusters in fullerenes

Heffers, James, Lelong numbers and geometric properties of upper level sets of currents on projective space

Huang, Qian, Stable sparse orthogonal factorization of ill-conditioned banded matrices for parallel computing

Roy, Robert, Auslander–Reiten sequences over Gorenstein rings of dimension one

University at Albany, SUNY (2)

Department of Mathematics and Statistics

Briggs, Carly, On combinatorial models for Kirillov–Reshetkihim crystals of type B

Cahill, Patrick, Bounded algebra over coarse spaces

University at Buffalo, SUNY (15)

Department of Biostatistics

Chen, Yang, Contributions to the theory and applications of statistical distances

Liu, Qian, Mining for the genetic gemstones in the post-GWAS era

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Department of Mathematics

Dougan, Kelly, Permutation equivalence of quartic 2-rotation symmetric Boolean functions

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University of Rochester (7)

Department of Biostatistics and Computational Biology

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Department of Mathematics

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Shi, Shuhui, Multiple zeta functions over $\mathbb{F}_q[t]$

NORTH CAROLINA

Duke University (10)

Department of Mathematics

Ghadyali, Hamza, Applications of sliding window embeddings and topological data analysis for learning on novel features of time-varying dynamical systems

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Huang, Jingxian, Sigma models with repulsive potentials

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Xu, Chao, The continuum limit of the Thomas–Fermi–Dirac–von Weizsacker model

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Department of Statistical Science

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North Carolina State University (16)

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Skau, Erik, Relaxations to sparse optimization problems and applications

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Strait, Melissa, Phase-field models of two-fluid flow in a capillary tube and Hele–Shaw cell

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University of North Carolina at Chapel Hill (36)

Department of Biostatistics

Barkley, Brian, Causal inference from observational studies in the presence of partial interference

Berchuck, Samuel, Statistical methods for modeling the spatial structure on the visual field in glaucoma research

Chaudhari, Monica, Study design and methods for evaluating sustained unresponsiveness to peanut sublingual immunotherapy

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Choi, Wansuk, Semiparametric approaches for auxiliary and incomplete covariate under outcome dependent sampling designs

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Zhou, Xuan, New statistical learning methods for personalized medical decision making

Department of Mathematics

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Fang, Fang, Improving semiparametric estimation of longitudinal data with covariance function

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Liu, Pengyu, The braid indices of alternating links

Sohrabi, Elham, Option valuation under a regime-switching model using the fast Fourier transform

Zhang, Meijiao, Robust generalized likelihood ratio test based on penalization

University of North Carolina at Greensboro (6)

Department of Mathematics and Statistics

Chen, Wei, Spectral estimation for random processes with stationary increments

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Milstead, Jonathan, Computing Galois groups of Eisenstein polynomials over p-adic fields

Morris, Quinn, Analysis of classes of superlinear semipositone problems with nonlinear boundary conditions

Payne, Catherine, Approximation of neutral delay-differential equations

Son, Byungjae, Analysis of classes of singular steady state reaction diffusion equations

NORTH DAKOTA

North Dakota State University, Fargo (2)

Department of Mathematics

Brownlee, Erin, Maximally edge-colored directed graph algebras

Issaka, Aziz, Analysis of variance based financial instruments and transition probability densities: Swaps, price indices, and asymptotic expansions

OHIO

Bowling Green State University (7)

Department of Mathematics and Statistics

Basalamah, Doaa, Statistical inference for a new class of skew t distribution and its related properties

Bhusal, Raju, Error analysis of the cubic front tracking and RKDG method for one dimensional conservation laws

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Schrader, Paul, Hom-tensor categories

Xiong, Jingwei, A penalized approach to mixed model selection via cross validation

Case Western Reserve University (1)

Department of Mathematics, Applied Mathematics and Statistics

Zhang, Yue, Sparse modeling in image processing and machine learning: Models and theories

Kent State University, Kent (8)

Department of Mathematical Sciences

Alexander, Matthew, Combinatorial and discrete problems in convex geometry

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Ashu, Tom, Non-smooth SDE's and hyperbolic lattice SPDE's expansions via the quadratic covariation differentiation theory and applications

Bissler, Mark, Character degree graphs of solvable groups

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Ohio State University, Columbus (34)

Department of Mathematics

Adali, Ali, Applications of quasi solution method to interfacial and other nonlinear problems

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Dinitz, Thomas, Cox processes and their use in mathematical models of sleep-wake regulation

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Khan, Gabriel, On the Hermitian geometry of k-Gauduchon orthogonal complex structures

Kim, Tae Eun, Quasi-solution approach to nonlinear integro-differential equations: Applications to 2-D vortex patch problems

Lin, Yongxiao, Subconvex bounds for twists of GL(3) L-functions

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Moore, Daniel, An intrinsic theory of smooth automorphic representations

Nash, Evan, Extended tropicalization of spherical varieties

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Senay Aras, Betul, Investigation of some cell morphology using phase field method

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You, Fenglong, A mirror theorem for toric stack bundles

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Department of Statistics

Bean, Andrew, Transformations and Bayesian estimation of skewed and heavy-tailed densities

Chen, Ziyue, Generalizing results from randomized trials to target population via weighting methods using propensity score

Gory, Jeffrey, Marginally interpretable generalized linear mixed models

Jiang, Jieyi, Realistic predictive risk: The role of penalty and covariate diffusion in model selection

Kunkel, Deborah, Anchored Bayesian mixture models

Li, Liubo, Trend-filtered projections for smoothed principal component analysis

Li, Meng, Simultaneous inference procedures in the presence of heteroscedasticity

Lu, Yi, Function registration from a Bayesian perspective

Stark, Amy, Early clinical trial design recommendations in oncology based on overall success across phases I, II, and III

Xie, Weiyi, A geometric approach to visualization of variability in univariate and multivariate functional data

Yan, Zhifei, Semidefinite programming approaches to network clustering and smoothing

Zhang, Han, Detecting rare haplotype-environmental interaction and nonlinear effects of rare haplotypes using Bayesian LASSO on quantitative traits

Zheng, Jiayin, Calibrated Bayes factor and Bayesian model averaging

Ohio University, Athens (5)

Department of Mathematics

Akinyemi, Lanre, The interface dynamics in the Hele–Shaw cell

Bossaller, Daniel, Some topics in infinite dimensional algebra

McClatchey, Nathaniel, Tensors: An adaptive approximation algorithm, convergence in direction, and connectedness properties

Ronquillo-Rivera, Javier, Extremely amenable groups and Banach representations

Xin, Ying, Complex dynamical systems: Definitions of entropy, proliferation of epithelia and spread of infections and information

University of Cincinnati (5)

Department of Mathematical Sciences

Hanandeh, Ahmad Ali, Nonstationary nearest neighbors Gaussian process models

Liu, Jinzhong, Bayesian inference for treatment effect

Ramanathapuram Vancheeswaran, Saraswathy, Zero-knowledge proof for knowledge of RLWE (Ring-learning with errors) secret keys

Ren, Sheng, New methods of variable selection and inference on high dimensional data

Shi, Hongxiang, Hierarchical statistical models for large spatial data in uncertainty quantification and data fusion

OKLAHOMA

Oklahoma State University (1)

Department of Statistics

Ye, Shangyuan, Bayesian analysis for sparse functional data

University of Tulsa (1)

Department of Mathematics

Le, Trang, A differential privacy-based and dimensionality reduction framework to optimize the generalizability of machine learning techniques for high-dimensional biological data

OREGON

Oregon State University (14)

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Klangwang, Jetjaroen, Zeros of certain combinations of products of Eisenstein series

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Titus, Mathew, Mixing times for diffusive lattice-based Markov chains

Department of Statistics

Brintz, Benjamin J., A normal approximation to n-mixture models with applications in large abundance estimation and disease surveillance

Comiskey, Christopher W., Take me out to (analyze) the ballgame: Visualization and analysis techniques for big spatial data

Li, Kai, Regularization-based data integration and its application to genome-wide association studies

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Wang, Chunxiao, Matrix-free method in spatial-temporal modeling

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Portland State University (3)

Fariborz Maseeh Department of Mathematics and Statistics

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University of Oregon (9)

Department of Mathematics

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Merrill, Leanne, Periodic Margolis self maps at $p = 2$

Schmidt, Karl, Factorizable module algebras, canonical bases, and clusters

PENNSYLVANIA

Bryn Mawr College (1)

Department of Mathematics

Pezzimenti, Samantha, Immersed Lagrangian fillings of Legendrian submanifolds via generating families

Carnegie Mellon University (6)

Department of Mathematical Science

Eisenbeis, Jeff, Nonlocal aggregation dynamics with near-neighbor weighting and the associated geometry of measures

Iglesias, Jennifer, Approximation algorithm for faster communication and cheaper networks using linear programming

Kirov, Slav, Models and methods for one-dimensional approximations to point cloud data

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Drexel University (5)

Department of Mathematics

Akin, Myles, Multiplex network analysis of local topology in simulated and in vitro spiking neural networks

Faver, Timothy, Nanopteron-stegoton traveling waves in mass and spring dimer Fermi–Pasta–Ulam–Tsingou lattices

Onderdonk, Alexander, Activity patterns in lateral-inhibition type neural fields with asymmetric excitatory distal components

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Sulon, David, Analysis for periodic traveling interfacial hydroelastic waves

Lehigh University (2)

Department of Mathematics

Scheirer, Steven, Topological complexity of graph configuration spaces

Short, Robert, The relative topological complexity of a pair

Pennsylvania State University (15)

Department of Mathematics

Banerjee, Shilpak, Real-analytic ABC method on the torus

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Chen, Dong, On some problems in Lagrangian dynamics and Finsler geometry

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Temple University (10)

Department of Mathematics

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Department of Statistical Science

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University of Pennsylvania (15)

Department of Epidemiology, Biostatistics and Informatics

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Jia, Cheng, Statistical methods for whole transcriptome sequencing: From bulk tissue to single cells

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Department of Mathematics

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Wharton Statistics Department

Deshpande, Sameer, Bayesian model selection and estimation without MCMC

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University of Pittsburgh (24)

Department of Biostatistics

Abberbock, Judah, Usage of surrogate endpoints in the design and analysis of clinical trials

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Liu, Yimeng, Efficient estimation of the quantile residual lifetime using auxiliary covariates

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Sun, Zhaowen, Power and sample size determinations in dynamic risk prediction

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Department of Mathematics

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PUERTO RICO

University of Puerto Rico, Rio Piedras (1)

Department of Mathematics

Li, Yanyan, Fine regularity of solutions to the Dirichlet problem associated with the regional fractional Laplacian

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Brown University (21)

Department of Biostatistics

Zhang, Yi, Pathway analysis of genomic data

Department of Mathematics

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Department of Mathematics

Bertrand, Elliott, Global dynamics of some competitive and cooperative discrete dynamical systems

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Staniszewski, Christopher, On possible limit functions on a Fatou component in non-autonomous iteration

SOUTH CAROLINA

Clemson University (13)

Department of Mathematical Sciences

Dranichak, Garrett, Robust solutions to uncertain multiobjective programs

Grove, Ryan, Discretizations and efficiently linear solvers for problems related to fluid flow

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Ruiz Ramirez, Javier, Time-dependent Stokes–Darcy flow with deposition

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Medical University of South Carolina (6)

Department of Public Health Sciences

Bauza, Colleen, Determining the potential joint effect of obesity and diabetes on clinical outcomes following an ischemic stroke and issues related to the measures of obesity

Dai, Lin, Multilevel modeling of zero-inflated count data from complex surveys using pseudo-likelihood and Bayesian methods

Davis, Melanie, Addressing geographic confounding through spatial propensity score analysis for hierarchical data

Jiang, Yunyun, Bayesian response adaptive randomization in phase III confirmatory clinical trials with a binary endpoint

Small, James, Personal and family history of skin cancer: A possible marker for increased internal cancer risk and mortality

Speiser, Jaime, Decision tree and random forest methodology for clustered and longitudinal binary outcomes

University of South Carolina (7)

Department of Mathematics

Ahmed, Sameed, Theory, computation, and modeling of cancerous systems

Cochran, Garner, Quick trips: On the oriented diameter of graphs

Farman, Blake, Geometry of derived categories on noncommutative projective schemes

Musulin, Rade, Classical and quantum Kac's chaos

Department of Statistics

Huang, Songqiao, Functional data smoothing methods and their applications

Qiang, Beidi, Improved simultaneous estimation of location and system reliability via shrinkage ideas

Tang, Chuan-fa, Nonparametric inference for orderings and associations between two random variables

SOUTH DAKOTA

South Dakota State University (6)

Department of Mathematics and Statistics

Aburweis, Mohamed, Comparative study of the distribution of repetitive DNA in model organisms

Armstrong, Douglas, Development and properties of kernel-based methods for the interpretation and presentation of forensic evidence

Elmesmari, Nasir, Threshold models for genome-wide association mapping of familial breast cancer incidence in human

McDermaid, Adam, Statistical algorithms and bioinformatics tools development for computational analysis of high-throughput transcriptomic data

Ommen, Danica, Approximate statistical solutions to the forensic identification of source problem

Robertson, Joseph, The impact of data sovereignty on American Indian self-determination: A framework proof of concept using data science

TENNESSEE

Middle Tennessee State University (1)

Department of Mathematical Sciences

Alrowaili, Dalal, Mathematical modeling for ring systems in molecular networks

University of Memphis (2)

Department of Mathematical Sciences

Carraway, Latia, New procedures of estimation for binary data

Popielarz, Kamil, Problems in extremal graph theory

University of Tennessee, Knoxville (18)

Department of Mathematics

Cummings, John, Mathematical modeling of mixtures and numerical solution with applications to polymer physics

Feng, Wenqiang, Linearly preconditioned nonlinear solvers for phase field equations involving p-Laplacian terms

Heines, Elizabeth, Assessing the economic tradeoffs between prevention and suppression of forest fires

Higginbotham, Logan, Coarse constructions

Jantsch, Peter, Efficient methods for multidimensional global polynomial approximation with applications to random PDEs

Jensen, Ryan, Localization of large scale structures

Marchese, Andrew, Statistical and topological techniques for classification and clustering

McClurkin, Grace, Generalizations and variations of zero-divisor graphs

Mike, Joshua, Understanding data through computational topology and geometry

Rooker, Kelly, Modeling the evolution of visual sexual signaling, receptivity, and sexual signal reliability among female primates

Schnake, Stefan, Numerical methods for non-divergence form second order linear elliptic partial differential equations and discontinuous Ritz methods for problems from the calculus of variations

Sinclair, Kevin, Generalizations of coarse properties in large scale spaces

Sonnanburg, Kevin, Blow-ups of two-convex, type-I mean curvature flow

Starnes, Andrew, Investigating hulls from the Loewner equation

Tu, Eddie, Dependence structures in Levy-type Markov processes

Weber, Darrin, Various topics on graphical structures placed on commutative rings

Worley, Chase, Construction and classification results for commuting squares of finite dimentional *-algebras

Yilmaz, Faruk, Approximation of invariant subspaces

Vanderbilt University (8)

Department of Mathematics

Das, Sayan, Poisson boundaries of finite von Neumann algebras

Gaslowitz, Joshua Z., Characterizations of graphs without certain small minors

Klaus, Colin, Parabolic variational problems of 1-Laplacian type and finite element models of protein diffusion and homogenized core photoreceptor visual transduction

Lee, Chang-Hsin, Analysis of signal reconstruction algorithm based on consistency constraints

Michaels, Timothy, Node generation on surfaces and bounds on minimal Riesz energy

O'Connell, Kelly, Characterisation and Hamiltonicity of $K_{1,1,t}$-minor-free graphs: A fan-based approach

Petrosyan, Armenak, Dynamical sampling and systems of lectors from iterative actions of operators

Ren, Yunxiang, Skein theory of planar algebras and some applications

Vanderbilt University, School of Medicine (4)

Department of Biostatistics

D'Agostino McGowan, Lucy, Improving modern techniques of causal inference: Finite sample performance of ATM and ATO doubly robust estimators, variance estimation for ATO estimators, and contextualized tipping point sensitivity analyses for unmeasured confounding

Mercaldo, Nathaniel, Design and analysis consideration for complex longitudinal and survey sampling studies

Mercaldo, Sarah, On optimal prediction rules with prospective missingness and bagged empirical null inference in large-scale data

Tripp Saunders, Christina, On estimating causal mediation effects from a single regression model

TEXAS

Baylor University (14)

Department of Mathematics

Armour, David, Similarity of blocks in parabolic category O and a wonderful correspondence for modules of covariants

Courtermance, Jordan, Local automorphisms of finitary incidence algebras

Frymark, Dale, Boundary conditions associated with left-define theory and the spectral analysis of iterated rank-one perturbations field: Operator theory and functional analysis

Jones, Tiffany, Conventional and asympototic stabilities of decomposed compact methods for solving highly oscillatory wave problems

Osborn, John, Moment representations of exceptional orthogonal polynomials

Padgett, Joshua, Solving degenerated stochastic Kawarada partial differential equations

Pennington, Brian, Boundary data smoothness for solutions of nonlocal boundary value problems for nth order differential equations

Department of Statistical Sciences

Casement, Chris, Graphical methods in prior elicitation

Drevets, Madeline, Bayesian inference for bivariate Poisson data with zero-inflation

King, Clay, Applications of Bayesian quantile regression and sample size determination

Nelson, Tyler, Multilevel binomial and Poisson regression with misclassified outcomes and binary covariates: A Bayesian approach

Odom, Gabriel, Three applications of linear dimension reduction

Stanley, Sarah, Beta regression for modeling a covariate-adjusted ROC

Zhou, Kate (Qi), Bayesian propensity score analysis for clustered data with misclassified response and its sensitivity to unmeasured confounding

Rice University (17)

Department of Computational and Applied Mathematics

Brimkov, Boris, Graph coloring, zero forcing, and related problems

Hendryx, Emily, Subset selection and feature identification in the electrocardiogram

Takhtaganov, Timur, Efficient estimation of coherent risk measures for risk-average optimization problems governed by partial differential equations with random inputs

Terentyev, Igor, Nonlinear waveform inversion with surface-oriented extended modeling

Zhai, Jian, Analysis of inverse boundary value problems for elastic waves

Department of Mathematics

Bernazzani, Daniel, Interval exchange transformations

Bucaj, Valmir, Anderson localization for discrete one-dimensional random operators

Li, Chao, Global regularity for Euler vortex patch in bounded smooth domains

Nakahara, Masahiro, Cohomology classes responsible for Bauer–Manin obstructions, with applications to rational and K3 surfaces

VandenBoom, Thomas, Isospectral dynamics of reflectionless Jacobi operators

Department of Statistics

Baker, Yulia, Methods and applications for mixed graphical models

Campbell, Frederick, Statistical machine learning methodology and inference for structured variable selection

Chapple, Andrew, Bayesian models for clinical trials and survival analysis

Ginley, Matthew, An old dog learns new tricks: Novel applications of kernel density estimators on two financial datasets

Kosar, Robert, Methods and applications in multivariate analysis: Skewering, the Carnegie classification, and stochastic regularization

Melnikov, Oleg, Dynamic characterization of multivariate time series

Zhang, Liangcai, Statistical applications in cancer genomics and Bayesian statistical modeling in early phase clinical trials

Southern Methodist University (2)

Department of Mathematics

Juhnke, Fritz, High-order implementations of the double absorbing boundary

Swenson, Jennifer, Swelling as a stabilizing mechanism during ion bombardment of thin films: An analytical and numerical study

Texas A&M University (26)

Department of Mathematics

Barrera, Roberto, Local cohomology: Combinatorics and D-modules

Chung, Yeong Chyuan, Quantitative K-theory for Banach algebras and its applications

Dutta, Sourav, Mathematical models and numerical methods for porous media flows arising in chemical enhanced oil discovery

Fu, Shubin, Some applications of the generalized multiscale finite element method

Ghesmati, Arezou, Residual and goal-oriented-h and hp-AFEM; applications for elliptic and saddle point problems

Ji, Bingbing, A local minimax method using the generalized Nehari manifold for finding differential saddles

Kha, Minh, Spectral edge properties of periodic elliptic operators

Kogan, Roman, Measures induced by automata and their actions

Leung, Wing-Tat, Adaptivity and online basis construction for generalized multiscale finite element methods

Li, Meiqin, Finding multiple saddle points for defocused nonlinear problems and G-differentiable functionals

Noles, Joseph, On upper-triangular forms in tracial von Neumann algebras

Patty, Spencer, An energy formulation for surface tension or Willmore force for two phase flow

Samurkas, Suleyman, Bounds for the rank of the finite part of operator K-theory and polynomially full groups

Scholze, Sam, Signal construction from frame and sampling erasures

Terzioglu, Fatma, Compton camera imaging and the cone transformation

Williams, Robert, Restrictions on Galois groups of Schubert problems

Zeng, Guchao, Theta operators on v-adic modular forms and v-adic families of Goss polynomials and Einstein series

Zhang, Zhidong, Inverse problems for fractional diffusion equations

Department of Statistics

Cai, Quan, Topics in partially linear sing-index models for longitudinal data

Lee, Hyuneui, Goodness-of-fit test for large number of small data sets

Malloure, Matthew, Goodness-of-fit testing using cross-validation

Nikooienejad, Amir, Bayesian variable selection in high dimensional genomic studies using nonlocal priors

Payne, Richard, Two-stage metropolis hastings; Bayesian conditional density estimation and survival analysis via partition modeling, Laplace approximations, and efficient computation

Shin, Minsuk, Priors for Bayesian shrinkage and high-dimensional model selection

Shin, Yei-Eun, Statistical research on covariate matching, monotone functional data and binary spatio-temporal data modeling

Xue, Jingnan, Robust model-free variable screening double parallel Monte Carlo and average Bayesian information criterion

Texas Christian University (2)

Department of Mathematics

Aguirre, Luis, On linking multiple lines

Smith, Jeremy, Indices of algebraic integers in cubic fields

Texas State University (4)

Department of Mathematics

Akbuga, Enes, Motivation intervention through science and engineering integrated calculus tasks

Cheshire, Daniel, On the axiomatic formalization of mathematical understanding: Continuous functions in the transition to topology

Hurdle, Zachariah, Aspects that arise in the transition from the Montessori method to a traditional method: Fourth grade mathematics view

Lindsey, Joni, Evolving mathematical identity in post-secondary students

Texas Tech University (16)

Department of Mathematics and Statistics

Acharya, Gangadhar, A study of university mathematics outreach programs—findings and implications

Bandara, Dhanamalee, A neighborhood hypothesis test for high dimensional object data analysis

Calandrini, Sara, Fluid-structure interaction simulations for medical applications

Capodaglio, Giacomo, Multigrid methods for finite element applications with arbitrary-level hanging node configurations

De Silva, Mihiri, Continuous-time models of plankton interactions and a discrete system of Larch Budmoth population

Huff, Krystin, Modeling the early states of within-host viral infection and clinical progression of hantavirus pulmonary syndrome

Jayawardhana, Rangana, Iterative learning control for discrete-time MIMO systems and applications in cooperative learning control

Karunarathna, Sanjeewa, Customized contact lens design for regular and irregular vision defects

Mayer, Joshua, Modeling drug sensitivity: Variable selection, inference and prediction

Premarathna, Iresha, Classification of protein binding ligands using their structural dispersion

Rai, Shiva, Cardinality of pseudo-endpoints of chainable continua

Rush, Simon, Reconstruction of fluid flow using discrete data to determine wake location

Walty, Michelle, Integer representations of small groups and their cohomology

Wang, Hongwei, The action of Kauffman bracket skein algebra on the torus on skein module of 3-twist knot complement

Zhang, Wei, Analytic continuation of Laurent series to domains of minimal capacity

Zhang, Yang, An investigation of some public key exchange cryptosystems

University of Houston (11)

Department of Mathematics

Cheng, Wanli, A motion of freely oscillating droplet of a yield stress fluid: Analysis and numerical studies

Chiu, Shang Huan, Three dimensional DLM/FD methods for simulating the motions of spheres in bounded shear flows of oldroyd-B fluids

Kayasandik, Cihan, Geometric multiscale analysis and applications to neuroscience imaging

Lopez, Juan, Well-posedness for weak solutions of axisymmetric div-curl systems

Muravina, Viktoria, Statistical analysis of biomedical data

Ouegnin, Francois, Non-parametric estimation of stochastic differential equations

Platt, Eric, Active manipulation of acoustic fields

Sarkisov, Sergey, Parameters estimation for stochastic genetic evolution of asexual populations

Stolarczyk, Simon, Decision making in social networks

West, James, Two problems in graph algebras and dynamical systems

Zhang, Peixin, A comprehensive method for integrated volatility estimation

University of North Texas (7)

Department of Mathematics

Allen, Cristian, An analysis of the homogeneity of countable products of subsets of real numbers

Kuhns, Nehemiah, Uniserial representations of Vec($\mathbb{R}$)

O'Dell, Connor, Non-resonant uniserial representations of Vec($\mathbb{R}$)

Puente, Philip, Crystallographic complex reflection groups and the Braid conjecture

Reid, James, Numerical values of the Hausdorff and packing measures for limit sets of iterated function systems

Yan, Yujie, A general approach to Buhlmann credibility theory

Ziegler, Caleb, On factors of rank one subshifts

University of Texas at Arlington (15)

Department of Mathematics

Bolat, Emel, A study on the rotational B-family of equations

Campbell, Robert, Characterizing college algebra students' mathematical problem solving

Cavaness, Andrew, Simple weight modules of the Lie algebra of vector fields on $\mathbb{C}^2$

Choi, Sl Ghi, Image reconstruction from incomplete radon data and generalized principal component analysis

Dong, Yinlin, Mathematical methods for vortex identification with applications on shock wave vortex ring interaction

Ercan, Ramazan, Scattering and inverse scattering on the line for a first-order system with energy-dependent potentials

Karaduman, Gul, Numerical solution of saddle point problems by projection method

Luo, Ting, A study on traveling wave solutions in the shallow-water-type systems

Ojeda Ruiz, Ivan, Normalized cut problems with generalized linear constraints

Seo, Sat byul, Evoked and spontaneous neurotransmitter releases for independent synaptic currents: Mathematical modeling and analysis

Smith, Hillary, Discrete time risk models with random premiums

Sun, Junwei, A study on the nonlocal shallow-water model arising from the full water waves with the Coriolis effect

Tang, Jie, Stability study on shear flow and vortices in late boundary layer transition

Tomlin, Derek, Projective geometry associated to some quadratic, regular algebras of global dimension four

Yang, Yong, High order DNS for vortex structure in late flow transition

University of Texas at Austin (26)

Institute for Computational Engineering and Sciences

Crestel, Benjamin, Advanced techniques for multi-source, multi-parameter, and multi-physics inverse problems

Fuentes, Federico, Various applications of discontinuous Petrov–Galerkin (DPG) finite element methods

Gholaminejad, Amir, Fast algorithms for biophysically-constrained inverse problems in medical imaging

Hawkins, John, Investigations in integrative and molecular bioscience

Le, Ellen, Data-driven strategies for Bayesian inverse problems

Malhotra, Dhairya, Fast integral equation solver for variable coefficient elliptic PDEs in complex geometries

Nagaraj, Sriram, DPG methods for nonlinear fiber optics

Tao, Zhen, Numerical analysis of multiphase flows in porous media on non-rectangular geometry

Zhu, Hongyu, Inverse problems for basal properties in a thermomechanically coupled ice sheet model

Department of Mathematics

Carson, Timothy, Pinched manifolds becoming dull

Chu, Michelle, Quantifying virtual properties of Bianchi groups

de Azevedo Sodre, Antonio Carlos, Three essays on dynamics on point processes and unimodular networks

Derryberry, Richard, Towards a self-dual geometric Langlands program

Duque Alvarez, Luis Felipe, The double obstacle problem and the two membranes problems

Fayvisovich, Roman, Martingale-generated control structures and a framework for dynamic programming principle

Guadagni, Roberta, Lagrangian torus fibrations for symplectic torid degenerations

Hunt, Joseph, Academic math mindset interventions in first year college calculus

Li, Zheng, Optimal investment with high-watermark fee in a multi-dimensional jump diffusion model

Magee, Timothy, GHK mirror symmetry, the Knutson–Tao hive cone, and Littlewood–Richardson coefficients

Mihaila, Cornelia, Stability and behavior of critical points in the capillary droplet problem

Miller, Allison, Metabelian techniques in knot concordance

Neumayer, Robin, Stability and minimality in Sobolev and isoperimetric inequalities

Rezende de Macedo, Alessandro, Differential fppf descent obstructions

Yang, Yunan, Optimal transport for seismic inverse problems

Zakharevich, Valentin, K-theoretic computation of the Verlinde ring

Zhong, Yimin, Inverse problems in photoacoustic imaging: Analysis and computation

University of Texas at Dallas (12)

Department of Mathematical Sciences

Adjei, Francis Kwabena, Inversion of the covering map for indefinite spin groups

Arenas-Navarro, Isnardo, Numerical simulations for turbulent drag reduction using liquid infused surfaces

Bilson-Darku, Francis, Study on parameter estimation via multistage sampling with applications

Chowdhury, Marzana, Prediction of individualized risk of contralateral breast cancer

Huang, Xin, Robust analysis of nonparametric space-time clustering

Kaderli, Jordan, An analytic solution to a coupled system of equations for modeling photoacoustic trace gas sensors and a full waveform inversion approach to microseismic source estimation

Kwame, Eyram, Dynamics of macroeconomics models with hysteresis operators

Perez-Nagera, Pedro, Numerical solutions for a class of singular neutral functional differential equations

Tian, Yahui, Nonparametric and robust methods for community detection in complex networks

Wang, Cheng, Extensions of semiparametric single index models

Wu, Hao-pin, Applications of degree theory to dynamical systems with symmetry (with special focus on computational aspects and algebraic challenges)

Wu, Jiayi, Wavelet analysis of big data contaminated by large noise in an FMRI study of neuroplasticity

UTAH

Brigham Young University (1)

Department of Mathematics

Zhao, Junyilang, Dynamics for a random differential equation: Invariant manifolds, foliations, and smooth conjugacy between center manifolds

University of Utah (13)

Department of Mathematics

Brooks, Heather, Dynamics and structure: From microtubule networks to population networks

Bydlon, Andrew, Counterexamples of Bertini theorems for test ideals

Carapezza, Leonard, Minimal sequences in finite graphs and unique equilibrium states for some $\beta\alpha$ shifts

Carroll, Samuel, Spatiotemporal dynamics of orientation-selective neural populations in the visual cortex

Gupta, Radhika, Relative currents and loxodromic elements in the relative free factor complex

Lau, Chung Ching, On positive properties of algebraic subvarieties

Levien, Ethan, Noise propagation in biochemical reaction networks

Ma, Jie, Stochastic modeling of random drug taking processes and the use of singular perturbation methods in pharmacokinetics

Miles, Christopher, A hop, switch, and jump: Stochasticity in models of motor-mediated intracellular transport

Miller, Anna, Mathematical modeling of epithelial cell division: Evaluating the effects of human papillomavirus infection

Romanov, Anna, A Kazhdan–Lusztig algorithm for Whittaker modules

Sampson, Christian, Multiscale models of sea ice phenomena

Wigglesworth, Derrick, CT's and the geometry of Out(F_n)

Utah State University (7)

Department of Mathematics and Statistics

Dai, Xiaotian, Novel statistical models for quantitative shape-gene association selection

Furtak-Cole, Eden, Three environmental fluid dynamics papers

Jordan, Scott, Modeling the spread of alfalfa stem nematode: Insights into their dynamics and control

Li, Chunyang, Extracting and visualizing data from mobile and static eye trackers in R and Matlab

Quach, Anna, Extensions and improvements to random forests for classification

Schwartz, Sarah, Exact approaches for bias detection and avoidance with small, sparse, or correlated categorical data

Scott, Marcus, A series of papers on detecting examinees who used a flawed answer key

VERMONT

University of Vermont (1)

Department of Mathematics and Statistics

Morse, Ada, Networks, (K)nots, nucleotides, and nanostructures

VIRGINIA

George Mason University (6)

Department of Mathematical Sciences

Jensen, Alathea, Self-polar polytopes and sequential importance sampling algorithms

Revay, Shauna, Unions of Reisz bases of exponentials for bandlimited signals

Torrejon, Diego, Generalized master equations for continuous time random walks and their application to modeling coarsening

Department of Statistics

Du, Chengan, Networks analysis with nodal covariates

Marchese, Scott, Semiparametric regression models for mixed-type data analysis

Yin, Lixuan, Semiparametric transformation models with applications to diagnostic biomarker data and clinical trials

Old Dominion University (2)

Department of Mathematics and Statistics

Timalsina, Asim, A partitioned approach for computing fluid-structure interactions with application to tumor modeling and simulation

Working, Amanda, Methods for analyzing attribute-level best-worst discrete choice experiments

University of Virginia (12)

Department of Mathematics

Berman, John, Categorified algebra and equivariant homotopy theory

Courtney, Kristin, C^*-algebras and their finite-dimensional representations

Gates, Zachary, Finite presentability of groups acting on locally finite twin buildings

Mak, Kin Hei, Constraints on basic classes of Lefschetz fibrations

Osborne, Christina, Decomposing the classifying diagram in terms of classifying spaces of groups

Rebelo Grifo Pires, Eloisa, Symbolic powers and the containment problem

Reeks, Michael, The trace and center of the twisted Heisenberg category

Shalotenko, Veronica, In search of bounds on the dimension of ext between irreducible modules for finite groups of Lie type

Simone, Jonathan, Cut-and-paste operations and exotic 4-manifolds

Wan, Xiang, Global well-posedness and exponential stability for a nonlinear thermoelastic Kirchhoff–Love plate system

Weaver, Bradley, Non-abelian groups of order eight and the local lifting problem

Willis, Michael, Stable limits of the Khovanov homology and L-S-K spectra for infinite braids

Virginia Commonwealth University (5)

Department of Statistical Sciences and Operations Research

Mazicioglu, Dogucan, Adversarial decision making in counterterrorism applications

Piri, Sepehr, Parametric, nonparametric and semiparametric approaches in profile monitoring of Poisson data

Saleck Pay, Babek, Decomposition algorithms in stochastic integer programming: Applications and computations

Salmani-Jajaei, Ghasemali, Rotating supporting hyperplanes and snug circumscribing simplexes

Sorrell, Toni, Tuning optimization software parameters for mixed integer programming problems

Virginia Commonwealth University, Medical Center (2)

Department of Biostatistics

Kang, Jian, Estimating the respiratory lung motion model using tensor decomposition on displacement vector field

Lehman, Rebecca, The generalized monotone incremental forward stagewise method for modeling longitudinal, clustered, and overdispersed count data: Application predicting nuclear bud and micronuclei frequencies

Virginia Polytechnic Institute and State University (15)

Department of Mathematics

Grant, Holly, A viscoelastic constitutive model for thixotropic yield stress fluids: Asymptotic and numerical studies of extension

Grimm, Alexander, Parametric dynamical systems: Transient analysis and data driven modeling

Krometis, Justin, A Bayesian approach to estimating background flows from a passive scalar

Krueger, Justin, Parameter estimation methods for ordinary differential equation models with applications to microbiology

Marx, Gregory, Noncommutative kernels

Nikin-Beers, Ryan, Immunoepidemiological modeling of Dengue viral infection

Sariaydin, Selin, Randomization for efficient nonlinear parametric inversion

Swirydowicz, Katarzyna, Strategies for recycling Krylov subspace methods and bilinear form estimation

Withrow, Camron, The moment graph for Bott–Samelson varieties and applications to quantum cohomology

Department of Statistics

Chu, Shuyu, Change detection and analysis of data with heterogeneous structures

Crandell, Ian, Semi-supervised anomaly detection and heterogeneous covariance estimation for Gaussian processes

Gao, Zhenguo, Variance change point detection under a smoothly-changing mean trend with application to liver procurement

Guan, Ting, Novel statistical methods for multiple-variant genetic association studies with related individuals

Zhang, Lin, Semiparametric Bayesian kernel survival model for highly correlated high-dimensional data

Zhao, Meng, Analysis and evaluation of social network anomaly detection

WASHINGTON

University of Washington (22)

Department of Applied Mathematics

Buvoli, Tommaso, Polynomial-based methods for time-integration

Harris, Kameron, The brain is a mess: Inference, random graphs, and biophysics to disentangle neuronal networks

Price, Jacob, Multiscale techniques for nonlinear dynamical systems: Applications and theory

Wang, Yue, Some problems in stochastic dynamics and statistical analysis of single cell biology of cancer

Ye, Xiaofeng, Stochastic dynamics: Markov chains, random transformations and applications

Department of Biostatistics

Haris, Asad, Toward more flexible models in high dimension

Marsh, Tracey, Distribution-free approaches to assessing the potential clinical impact of biomarkers

Shi, Xu, Multivariate inference and surveillance using population scale data

Zhuang, Yingying, Evaluation of treatment effect modification by post-randomization biomarker-defined principal strata with application to vaccine efficacy trials

Department of Mathematics

Bragg, Daniel, Twistor spaces for supersingular K3 surfaces

Cameron, James, On the Duflot filtration in equivariant cohomology

DeVleming, Kristin, Compact moduli of surfaces in three-dimensional projective space

Iyer, Karthik, Inverse problems for linear and non-linear elliptic equations

Palacios, Benjamin, The inverse problem of thermoacoustic tomography in attenuating media

Roy, Scott, Algorithms for convex optimization with applications to data science

Scholl, Travis, Abelian varieties with small isogeny classes and applications to cryptography

Zheng, Hailun, On the g_2-number of various classes of spheres and manifolds

Department of Statistics

He, Yanjun, Coevolution regression and composite likelihood estimation for social networks

Lin, Lina, Methods for estimation and inference for high-dimensional models

Wan, Yali, Topics in graph clustering

Wang, Yu-hsuan, Linear structural equation models with non-Gaussian errors: Estimation and discovery

Weihs, Luca, Parameter identification and assessment of independence in multivariate statistical modeling

Washington State University (5)

Department of Mathematics and Statistics

Higgins, Abigail, Examining student agency in an active-learning business calculus class

Hu, Yunfeng, Median shapes

Kasigua, Michael, Eventual cone invariance

Noorazar, Hossein, An energy-based interaction model for population opinion dynamics with topic coupling

Torres, Patrick, Stability analysis, convex hulls of matrix powers and their relations to p-matrices

WEST VIRGINIA

West Virginia University (9)

Department of Mathematics

Cheng, Jian, Integer flows and circuit covers of graphs and signed graphs

Elsrrawi, Fariha, Existence of global solutions for nonlinear magenetohydrodynamics with finite Larmor radius correction

Ghaderi, Shadisadat, On the matroid intersection conjecture

Han, Miaomiao, Graph coloring problems and group connectivity

Li, Jiaao, Group connectivity and modulo orientations of graphs

Mohamed, Fatma, On some parabolic type problems from thin film theory and chemical reaction-diffusion networks

Short, Christopher, Reducing spatial stochastic models of membrane receptors to approximately equivalent chemical reaction networks through coarse graining

Solomon, Daniel, A new compactification for celestial mechanics

Xu, Murong, A study on graph coloring and digraph connectivity

WISCONSIN

Marquette University (4)

Department of Mathematics, Statistics and Computer Science

Ahsan, Golam Mushih Tanimul, Motivational and intervention systems and monitoring with mHealth tools

Rizzo, Benjamin, Compressed sensing for few-view multi-pinhole SPECT with applications to preclinical imaging

Rutarindwa, Regis, Computational strategies in uncertainty quantification for hazard mapping

Saxena, Piyush, Feature space augmentation: Improving prediction accuracy of classical problems in cognitive science and computer vision

Medical College of Wisconsin (1)

Division of Biostatistics

Sahr, Natasha, Variable screening and selection for survival and competing risks data with grouped covariates

University of Wisconsin, Madison (23)

Department of Mathematics

Biswas, Chandan, Sharp inequalities in harmonic analysis

Brunner, James, Polynomial dynamical systems and interaction networks

Charles, Zachary, Algebraic and geometric structure in machine learning and optimization algorithms

Cheng, Jingrui, Some study on semi-geostrophic equations and related models

Han, Jiyuan, Deformation theory of asymptotically locally Euclidean Kähler surfaces

Johnston, Reese, Computability in uncountable binary trees

Kim, Jongchon, Endpoint estimates for multiplier transformations

Lim, Tau Shean, Propagation of reactions in Lévy diffusion

Liu, Liu, Uncertainty quantification for multi-scale kinetic equations and quantum dynamics

Makuluni, Tamvana, Complexity classifications in model theory and computable structures

McCarthy, Ethan, Some results and applications of computability theory

Mitchell, Will, Analysis and computation of sedimentation and erosion in viscous flow

Ramos, Eric, Homological invariants of FI-modules and FI_G-modules

Shu, Ruiwen, Uncertainty quantification and sensitivity analysis for multiscale kinetic equations with random inputs

Xi, Haokai, Anisotropic local law and its application in random matrix theory

Yang, Yang, Some problems related to the equitable presentation for the quantum algebra

Ye, Dongxi, Modular forms, Borcherds lifts and Gross–Zagier type CM value formulas

Yu, Peng, CM values of Green functions associated to special cycles on Shimura varieties with applications to Siegel 3-fold case

Department of Statistics

Li, Xiaomao, Optimal recommendation of individual dose intervals

Nguyen, Duy, Statistical methods for differential analysis of Hi-C and ChIP-Seq Data

Song, Xinyu, Volatility analysis with unified discrete and continuous time models by combining low-frequency, high-frequency and option data

Xie, Bingying, Nonparametric estimation of conditional expectation with auxiliary information and dimension reduction

Zhang, Ying, Efficient treatment effect estimation with dimension reduction

University of Wisconsin, Milwaukee (6)

Department of Mathematical Sciences

Gould, Andrew, God's number in the simultaneously-possible turn metric

Iyiola, Olaniyi, Exponential integrator methods for nonlinear fractional reaction-diffusion models

Pietsch, Brian, Z-structures and semidirect products with an infinite cyclic group

Poudyal, Chudamani, Robust estimation of parametric models for insurance loss data

Sriwongsa, Songpon, Orthogonal abelian Cartan subalgebra decompositions of classical Lie algebras over finite commutative rings

Vieten, Martin Gerhard, Numerical solution of stochastic control problems using the finite element method

WYOMING

University of Wyoming (6)

Department of Mathematics

Huang, Bang, On solving large-scale Ill-conditioned linear systems

Johnson, Russell, Petrov–Galerkin FEM for solving second-order IVPs and its a posteriori analysis

King, Gavin, Uniformly balanced sets

McCaskill, Bradley, Numerical methods for porous media flow

Department of Statistics

Jackson, Eugenie, Exploration and analysis of longitudinal vaginal micro biome data

Yin, Tianzhixi "Tim", Data mining application to power grid PMU data

Photos by Tongtong Wang

Report on the 2016–2017 New Doctorate Recipients

Amanda L. Golbeck, Thomas H. Barr, and Colleen A. Rose

This report presents a statistical profile of recipients of doctoral degrees awarded by departments in the mathematical sciences at universities in the United States during the period July 1, 2016 through June 30, 2017. Information in this report was provided by 299 of the 321 doctoral-granting departments surveyed, with additional information provided by the individual new doctoral recipients.

The Report on the 2016–2017 Employment Experiences of New Doctoral Recipients immediately following this report provides an analysis of the fall 2017 employment plans of the 635 PhD recipients who responded to this survey, as well as a summary of their demographic characteristics.

Detailed information, including tables not appearing in this report, is available on the AMS website at www.ams.org/annual-survey.

Doctorates Awarded

In mathematical sciences 1,957 PhDs were awarded by 299 doctorate-granting departments. Of these, 16 departments awarded no doctorate.

The highest percentage, 31% (615), of the new PhDs had a dissertation in statistics/biostatistics, followed by algebra/number theory with 14% (280) and applied mathematics with 14% (271).

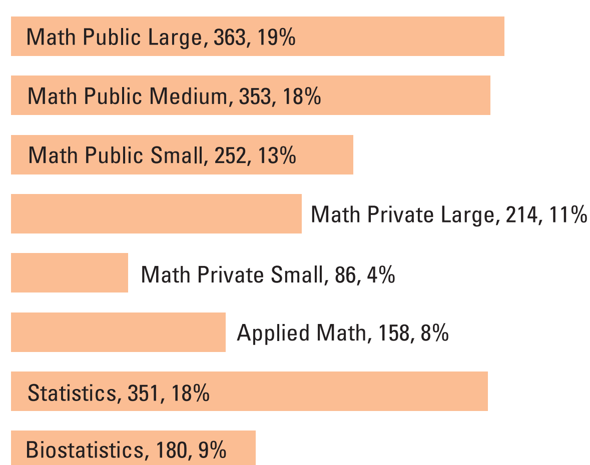
Comparing PhDs awarded in 2016–17 to 2015–16 the number of PhDs awarded:

- Increased about 2% from 1,921 to 1,957. In the 280 departments that responded both this year and last year the number of PhDs awarded decreased from 1,921 to 1,826.
- Increased in all groups except Math Public Large, Math Private Large, and Math Private Small.
- Increased 21% in Statistics, 14% in both Math Public Medium and Applied Math, 4% in Math Public Small, and 2% in Biostatistics.
- Decreased 15% in Math Public Large, and 11% in both Math Private Large and Math Private Small.

Comparing PhDs awarded in 2016–17 with those awarded in 2006–07:

- PhDs awarded increased by 47%.
- Degrees awarded by Doctoral Math and by Statistics/Biostatistics combined increased by 46% and 49%, respectively.

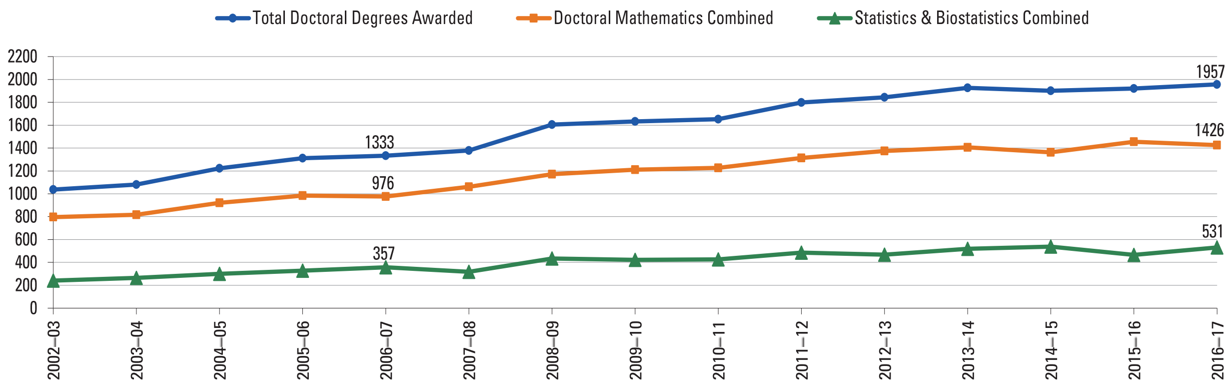
Figure A.1: Number and Percentage of Degrees Awarded by Department Grouping*
Total Degrees Awarded: 1,957



*See page 1157 for a description of the department groupings.

Amanda L. Golbeck is associate dean for academic affairs and professor of biostatistics in the Fay W. Boozman College of Public Health at University of Arkansas for Medical Sciences. Thomas H. Barr is AMS special projects officer. Colleen A. Rose is AMS survey analyst.

Figure A.2: New PhDs Awarded by Group



Employment

The employment status as of late 2017 was known for 1,671 of the 1,957 doctoral recipients. Figure E.1 gives a percentage breakdown by employment locale and seeking status. Figure E.2 shows the overall percentages of these PhDs reporting employment in various job sectors, and Figure E.3 provides a breakdown of the same information by citizenship. Most of these distributions are close to the ones for 2015–16.

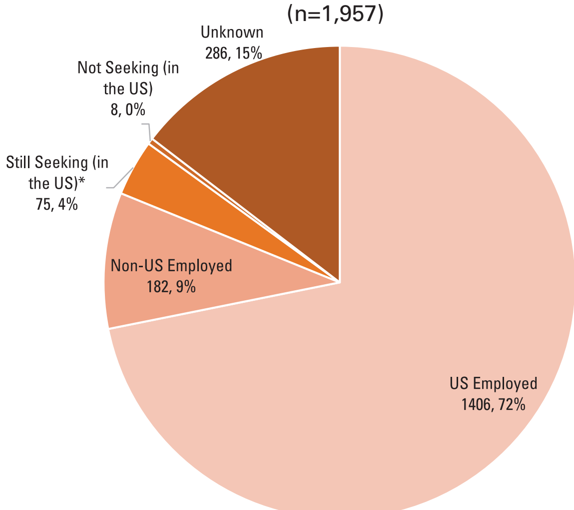
Of the US Citizens whose employment status is known, 89% (762) are employed in the US, and of these:

- 29% are employed in PhD-granting departments.
- 39% are employed in all other academic categories.
- 32% are employed in government, business and industry.

About 33% of the 2016–17 PhDs were in postdoc positions, which marks a decrease of about 8 percentage points from 2015–16. Most were in doctorate-granting departments, and their distribution is shown in Figure E.4. The counts of postdocs in various job sectors are shown in Figure E.5, broken down by citizenship. Of the PhDs in US academic jobs, 47% are postdocs.

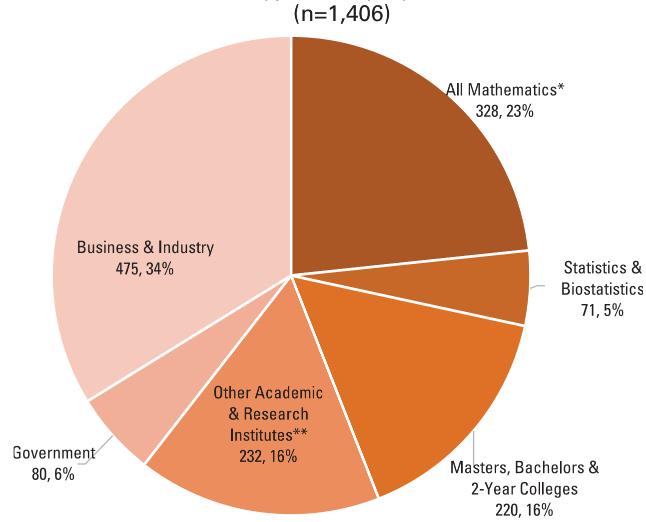
Figure E.6 tracks the overall and women's unemployment of new PhDs over a ten-year period. These rates have tended to parallel each other, though in all but two of these years, the unemployment rate has been slightly lower for women. The highest unemployment rate in 2016–17 was approximately 8% in the Math Public Medium group, and the lowest was about 1% in the Biostatistics group.

Figure E.1: Employment Status (n=1,957)



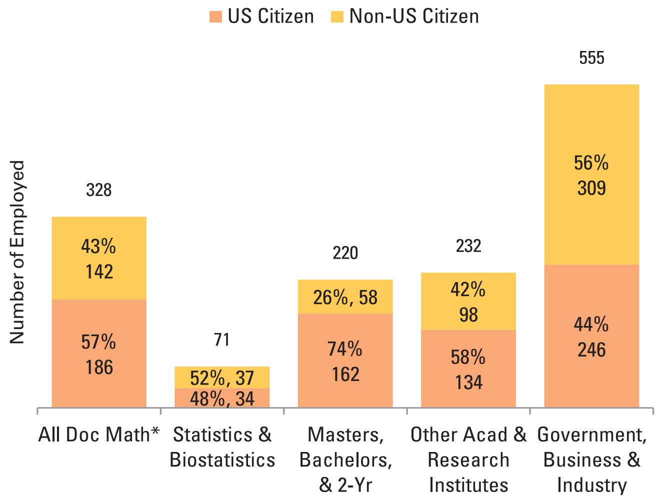
* If the unemployment percentage is adjusted by computing with a denominator that excludes those employed outside the US, then the value rounds to the same whole number percentage, 4%. Even if, in addition, those whose employment status is unknown are also removed from the denominator, the unemployment rate would be slightly more than 5%.

Figure E.2: US Employed by Type of Employer (n=1,406)



* Includes all Math Public, Math Private, and Applied Math departments.
** Other Academic consists of departments outside the mathematical sciences including numerous medical-related units.

Figure E.3: Employment in the US by Type of Employer and Citizenship (n=1,406)



*Includes all Math Public, Math Private, and Applied Math departments.

Figure E.4: PhDs Employed in Postdocs by Degree-Granting Department Group (n=1,957)

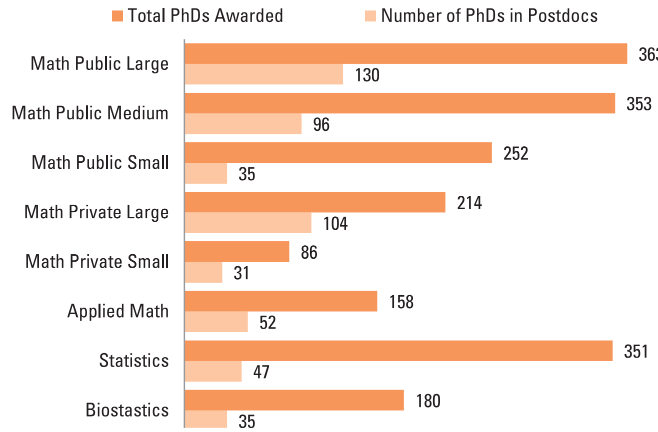
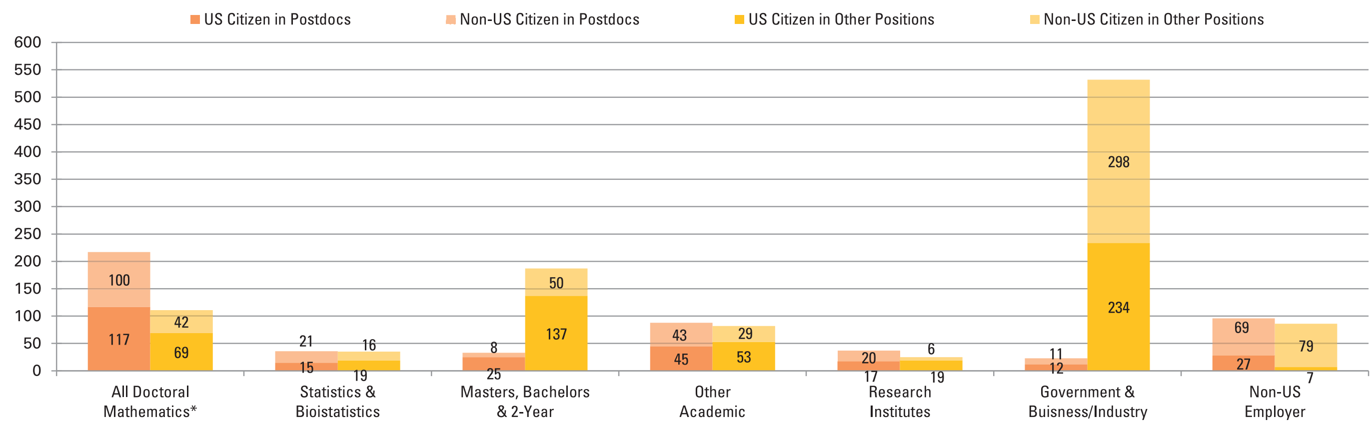
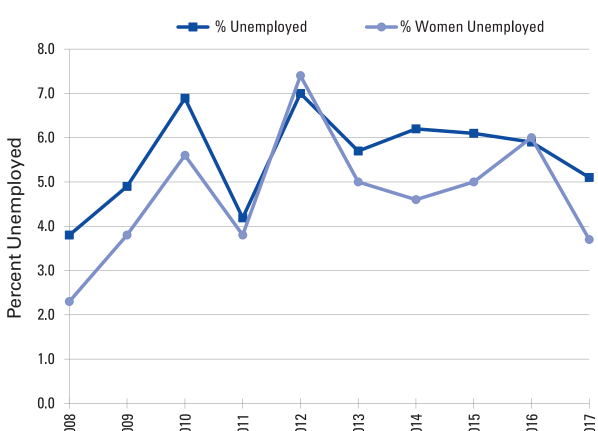


Figure E.5: New PhDs Employment by Citizenship, Type of Position, and Type of Employer (n=1,588)



* Includes all Math Public, Math Private, and Applied Math departments.

Figure E.6: Percentage of New Doctoral Recipients Unemployed 2008–17*



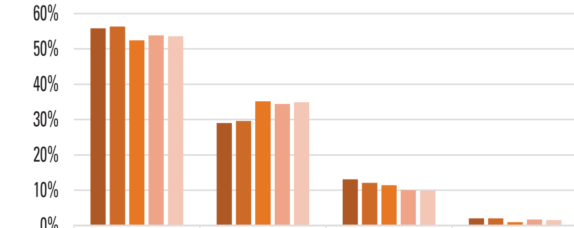
Demographics

Gender and citizenship were known for all 1,957 new PhDs reported for 2016–17. Figure D.1 gives a breakdown by departmental grouping of the recipients' gender, and Figure D.2 provides the same categorical breakdown by citizenship. Overall, 49% (957) of recipients were US citizens, 29% (577) were women, and 8% (81) were members of underrepresented minority groups. Figure D.3 shows the gender breakdown of the US citizens, and Figure D.4 shows the overall size of the PhD cohort and citizenship breakdown for 2016–17 and the preceding five years.

Here are a few other features of the 2016–17 data:

- 54% of the PhDs awarded by Math Public Large and Medium groups were to US citizens; 34% of the PhDs awarded by the Statistics group were to US citizens.

Figure E.7: Percentage of Employed New PhDs byType of Employer



	US Academic*	US Nonacademic	Non-US Academic	Non-US Nonacademic
Fall 2013	56%	29%	13%	2%
Fall 2014	56%	30%	12%	2%
Fall 2015	52%	35%	11%	1%
Fall 2016	54%	34%	10%	2%
Fall 2017	54%	35%	10%	2%

* Includes other academic departments and research institutes/other non-profits.

Figure D.1: Gender of Doctoral Recipients by Degree-Granting Grouping (n=1,957)

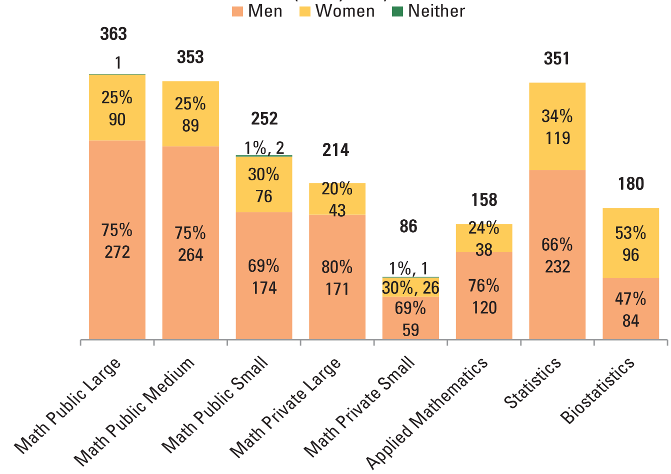


Figure D.2: Citizenship of Doctoral Recipients by Degree-Granting Grouping (n=1,957)

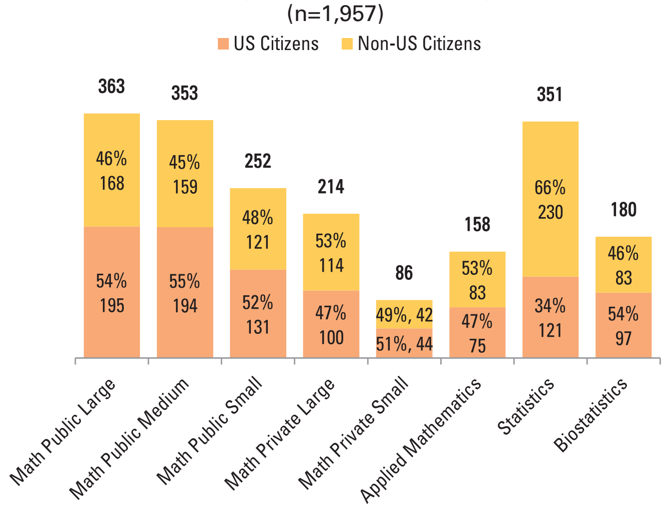


Figure D.3: Gender of US Citizen Doctoral Recipients by Degree-Granting Grouping (n=957)

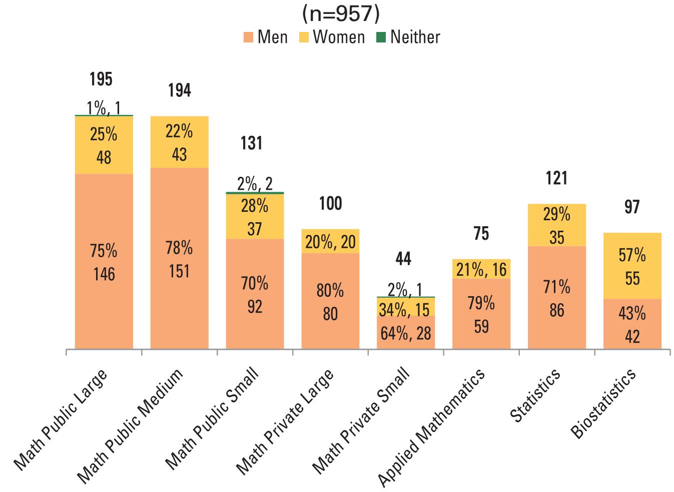
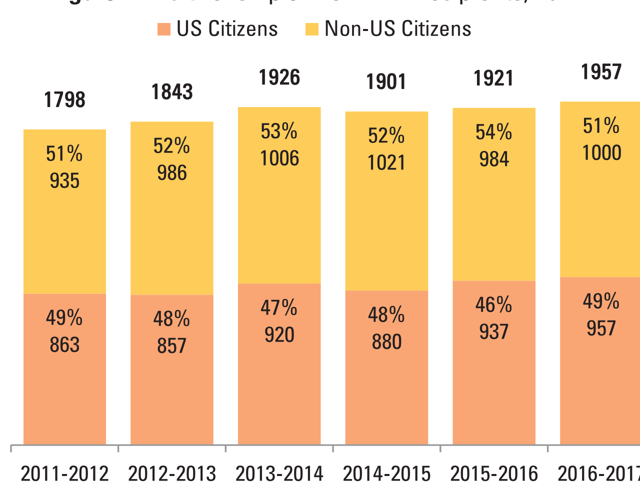


Figure D.4: Citizenship of New PhD Recipients, 2011–17



- Except for departments in Math Private Large, Applied Math, and Statistics, more PhDs were awarded to US citizens.
- 50% of those identifying as men, 46% of those identifying as women, and 100% of those identifying as of neither of these genders were US citizens.
- Among the US citizens earning PhDs, 4 were American Indian or Alaska Native, 113 were Asian, 30 were Black or African American, 33 were Hispanic or Latino, 4 were Native Hawaiian or Other Pacific Islander, 720 were White, and 53 were of unknown race/ethnicity.
- Math Public Large departments awarded 15 PhDs to US citizen minorities, and the Large Private group awarded 2; these are, respectively, the largest and smallest production rates. Departments in the other groups account for the remaining minority PhDs.

Figure F.1: Women as a Percentage of Doctoral Recipients Produced by and Hired by Department Grouping

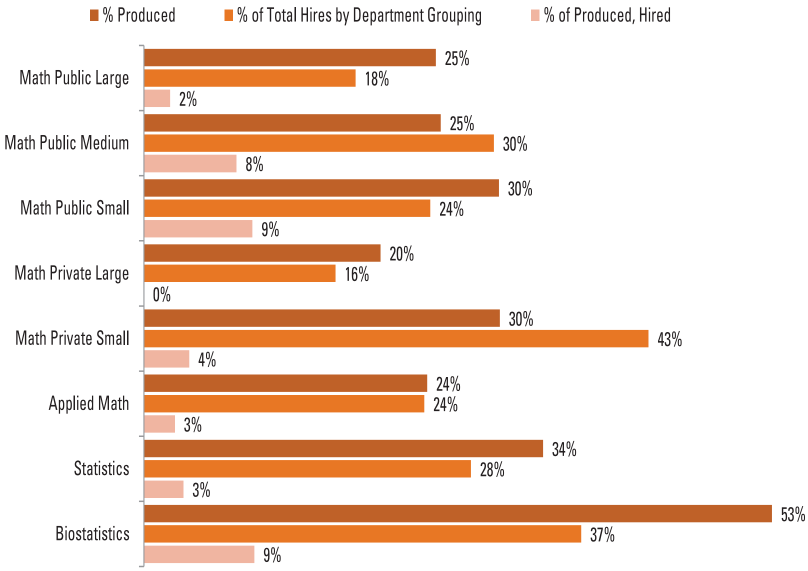
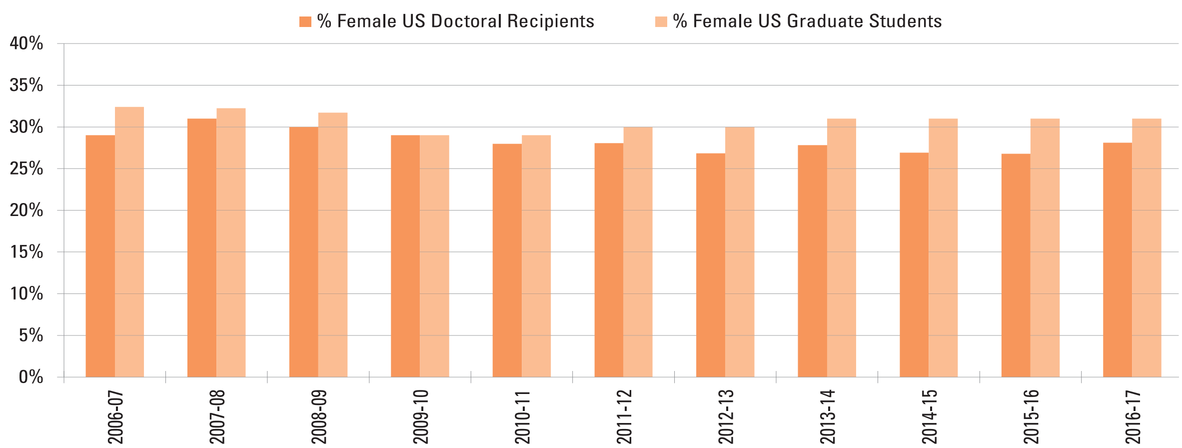


Table F.1: Number of Women Doctorates Produced by and Hired by Department Groupings

	Women		
Department Grouping	Produced	Total Hired	# Hired among women produced
Math Public Large	90	16	2
Math Public Medium	89	22	7
Math Public Small	76	9	7
Math Private Large	43	14	0
Math Private Small	26	9	1
Applied Math	38	5	1
Statistics	19	10	4
Biostatistics	96	13	9
Total	77	98	31

Figure F.2: Women as a Percentage of US Citizen Doctoral Recipients and Graduate Students



Women Doctoral Recipients

Overall, 29% of doctorate recipients were women, a fraction that has fallen by roughly a percentage point a year for the third year in a row. Of the 1,009 PhDs taking academic jobs, 28% (286) were women, and this continues a parallel decline in that percentage. Figure F.2 focuses on the percentage, over time, of US-citizen PhDs and graduate students who are women. Among US-citizen PhDs, the fraction in 2016–17 who are women ticked up by a percentage point or two from 26% in 2014–15 and 2015–16.

Figure F.1 gives some insight to which groups tend to hire their own women graduates. For example the graph shows that in Math Public Small departments, women constituted 30% of PhDs produced, 24% of faculty hired in this group were women from this group, and overall 9% of women produced by this group were hired in this group.

The section on Demographics contains some discussion of the status of women earning PhDs in mathematical sciences.

Figure S.1: PhDs Awarded by Statistics/Biostatistics Departments (n=531)

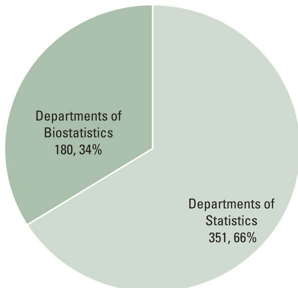


Figure S.2: Gender of PhD Recipients from Statistics/Biostatistics Departments (n=531)

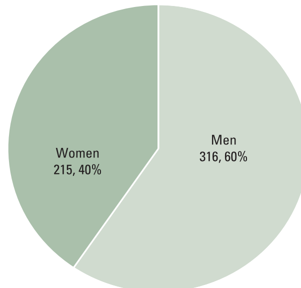


Figure S.3: Citizenship of PhD Recipients from Statistics/Biostatistics Departments (n=531)

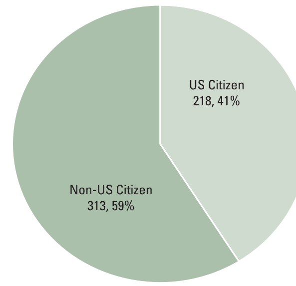


Figure S.4: Employment Status of PhD Recipients from Statistics/Biostatistics Departments (n=531)

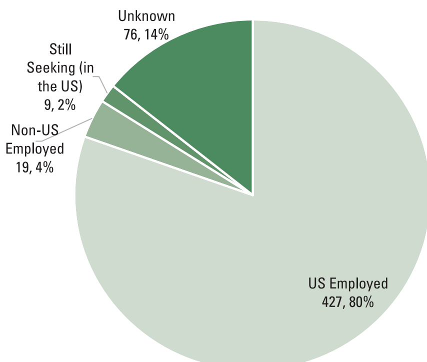
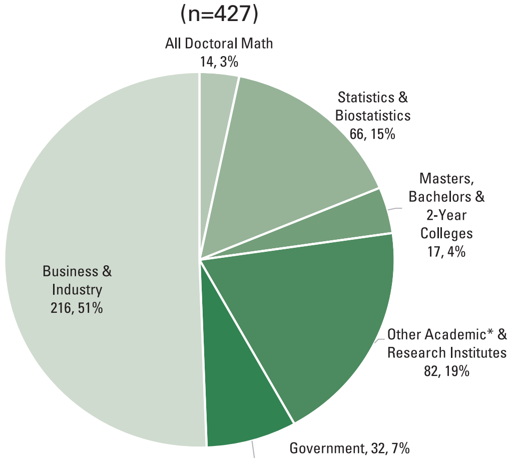


Figure S.5: US-Employed PhD Recipients from Statistics/Biostatistics Departments by Type of Employer (n=427)



* Other Academic consists of departments outside the mathematical sciences including numerous medical-related units.

PhDs Awarded by Statistics and Biostatistics (Stat/Biostat) Departments

One hundred one Stat/Biostat departments (58 of 59 statistics and 43 of 46 biostatistics) responded to this survey. They produced 531 doctorates, all of whom had dissertations in statistics/biostatistics, 14% more than in 2015–16. Figures S.1 through S.5 give breakdowns of these numbers by gender, citizenship, and employment status.

In addition, departments in the Mathematics groups had 123 PhD recipients with dissertations in statistics, so the overall number of PhDs specializing in statistical sciences for 2016–17 was 615, or 31% of the total. For the remainder of this section, the counts and percentages stated refer to doctorates awarded by departments in the Statistics groups.

Here are some attributes of the 2016–17 Stat/Biostat PhDs:

- 27% of all those in mathematical sciences were in the Stat/Biostat group.
- Women accounted for 34% of Statistics and 53% of Biostatistics.
- 41% of the US citizens were women.
- The unemployment rate of 2% is less than half of the corresponding percentage among Math PhDs.
- 32% of 2016–17 mathematical and statistical sciences PhDs hired by Stat/Biostat departments were women.

Departmental Groupings

In this report, *Mathematical and Statistical Sciences* departments are those in four-year institutions in the US that refer to themselves with a name that incorporates (with a few exceptions) "Mathematics" or "Statistics" in some form. For instance, the term includes, but is not limited to, departments of "Mathematics," "Mathematical Sciences," "Mathematics and Statistics," "Mathematics and Computer Science," "Applied Mathematics," "Statistics," and "Biostatistics." Also, *Mathematics (Math)* refers to departments that (with exceptions) have "mathematics" in the name; *Stat/Biostat* refers to departments that incorporate (again, with exceptions) "statistics" or "biostatistics" in the name but do not use "mathematics."

Listings of the actual departments that comprise these groups are available on the AMS website at www.ams.org/annual-survey/groupings.

A department is in Group...	...when its subject area, highest degree offered, and PhD production rate p
Math Public Large	Math PhD, $7.0 \le p$
Math Public Medium	Math PhD, $3.9 \le p < 7.0$
Math Public Small	Math PhD, $p < 3.9$
Math Private Large	Math PhD, $3.9 \le p$
Math Private Small	Math PhD, $p < 3.9$
Applied Math	Applied mathematics, PhD
Statistics	Statistics, PhD
Biostatistics	Biostatistics, PhD
Masters	Math, masters
Bachelors	Math, bachelors
Doctoral Math	Math Public, Math Private, & Applied Math
Stat/Biostat or Stats	Statistics & Biostatistics
Math	All groups except Statistics & Biostatistics

Department Response Rates by Grouping

Group	Received
Math Public Large:	26 of 26 including 0 with no degrees
Math Public Medium:	40 of 40 including 0 with no degrees
Math Public Small:	67 of 68 including 8 with no degrees
Math Private Large:	23 of 24 including 0 with no degrees
Math Private Small:	28 of 28 including 1 with no degrees
Applied Math:	30 of 30 including 2 with no degrees
Statistics:	58 of 59 including 4 with no degrees
Biostatistics:	33 of 46 including 4 with no degrees
Total:	315 of 321 including 4 with no degrees

As of press time for this issue of *Notices*, the following departments had not responded to the survey. Therefore, any PhDs which may have been awarded by these departments are not included in this report.

Mathematics Departments
California Institute of Technology
University of Puerto Rico, Rio Piedras

Statistics Departments
University of Pennsylvania

Biostatistics Departments
Saint Louis University College for Public Health & Social Justice
University of Illinois at Chicago
University of Texas–School of Public Health

Doctoral Degrees Conferred 2016–2017

Supplementary List

The following list supplements the list of thesis titles published in the September 2018 *Notices*, pages 969–999.

CALIFORNIA

Stanford University (26)

Statistics

Choi, Yunjin, Selecting the dimension of a subspace in principal component analysis and canonical correlation analysis.
Dobriban, Edgar, Topics in high-dimensional asymptotics.
Erdogdu, Murat Anil, Stein's Lemma and subsampling in large-scale optimization.
Fukuyama, Julia, Multivariate methods for the analysis of structured data.
Gorham, Jackson, Measuring sample quality with Stein's method.
He, Hera, Efficient permuation P-value estimates for gene set tests.
Huang, Ruojun, Monotone interactions of random walks and graphs.
Janson, Lucas, A model-free approach to high-dimensional inference.
Jiang, Bai, Two parameter inference methods in likelihood-free models: approximate Bayesian computation and contrastive divergence.
Kou, Jiyao, Large-scale inference with block structure.
Kuang, Yuming, Adaptive particle filters in hidden Markov models: A new approach and its application.
Lee, Minyong, Prediction and dimension reduction methods in computer experiments.
Liu, Linxi, Convergence rates of a class of multivariate density estimators based on adaptive partitioning.
Loftus, Joshua, Post-selection inference for models characterized by quadratic constraints.
Michael, Haben, Evaluating diagnostics under dependency.
Pekelis, Leonid, Flase discoveries with dependence, towards an objective inference.
Powers, Scott, Leveraging similarity in statistical learning.
Sen, Subhabrata, Optimization, random graphs, and spin glasses.
Sepehri, Amir, Non-parametric goodness-of-fit testing and applications.
Tian, Xiaoying, Topics in selective inference.
Wager, Stefan, Causal inference with random fields.
Wang, Chaojun, Financial markets and trading networks.
Wang, Jingshu, Factor analysis for high dimensional inference.
Xiang Gao, Katelyn, Scalable estimation and inference for massive linear mixed models with crossed random effects.
Zhao, Qingyuan, Topics in causal and high dimensional inference.
Zheng, Charles Yang, Supervised evaluation of representations.

University of California, Los Angeles (10)

Statistics

Gordon, Joshua Seth, Nonparametric estimation forecasts, and model evaluation of spatial temporal point process models for California seismicity.
Ho, Hao, Integrative analysis of genomic and transcription data in Taiwanese lung and adenocarcinomas.
Lu, Yang, Coupling and learning hierarchical generative and descriptive models for image systems and analysis.
Mao, Junhua, Multimodal learning for vision and language.
Razaee, Zahra, Community detection in networks with node covariates.

Rosario, Ryan Robert, A data augmentation approach to short text classification.
Wang, Jianyu, Modeling objects and parts by compositional relations.
Wang, Peng (Jerry), Joint multiple visual task understanding from a single image via deep learning and conditional random field.
Xia, Fangting, Pose-guided human semantic part segmentation.
Yu, Chengcheng (Joey), Single view 3D reconstruction and parsing using geometric commonsense for scene understanding.

University of California, Merced (5)

School of Natural Sciences

Adhikari, Lasith, Nonconvex sparse recovery methods.
Dark, Julie, A theoretical understanding of circular polarization memory.
Davis, Jason Karl, Mathematical models of prions in S.cerevisiae.
Madushani, R.W.M.A., Parameter inference for stochastic differential equations.
Sandoval, Christopher, Generalized Kubelka-Munk theory—A derivation and extension from radiative transfer.

University of California, Santa Barbara (3)

Statistics & Applied Probability

He, Jingyi, Fixed mixed effects models with big data.
Shi, Jian, Some contributions to smoothing spline density estimation and inference.
Zhu, Ling, Regularization and look-ahead procedures for selection of basic functions from multiple libraries.

COLORADO

Colorado State University (3)

Statistics

Liao, Xiyue, Change-point estimation using shape-restricted regression splines.
Wang, Lulu, Some topics on model-based clustering.
Weller, Zachary, Nonparametric tests of spatial isotropy and calibration-capture-recapture.

CONNECTICUT

Yale University (1)

Statistics and Data Science

Shaham, Uri, Algorithms, applications and theoretical properties of deep neural networks.

DISTRICT OF COLUMBIA

George Washington University (8)

Statistics

Chen , Chen, Advances in urn models and applications to self-similar bipolar networks.
Cheung, Li, Mixture models for left- and interval-censored data and concordance indices for composite survival outcomes.
Feng, Yarong, On fast growth models for random structures.
Huang, Hailin, Semi-parametric and structured nonparametric modeling.
Wang, Cong, Analysis for familial aggregation using recurrence risk for complex survey data.
Yang, Aotian, Constrained maximum entropy models for selecting genotype interactions associated with interval-censored failure times and methods for power calculation in a three-arm four-step clinical bioequivalence study.
Yang, Bioa, Particle and ensemble methods for state space models.
Zhao, Wanying, Adaptive designs utilizing covariates for precision medicine and their statistical inference.

Howard University(1)

Mathematics

Pleasant, Kendra, When Ramsey meets Stone-Cech: Some new results in Ramsey theory.

FLORIDA

University of South Florida (2)

Epidemiology & Biostatistics

Nash, Michelle, Deployment, post-traumatic stress disorder and hypertensive disorders of pregnancy among US active-duty military women.
Sebastião, Yuri Combo Vanda, Racial and ethnic differences in low-risk cesarean deliveries in Florida.

ILLINOIS

Northwestern University (4)

Statistics

Gao, Yi, On a generalization of the Gini correlation for statistical data mining.
Hu, Xiaofei, Volatility estimation for integer-valued financial time series.
Mei, Xuan, Small dispersion asymptotics in stratified models.
Seeskin, Zachary, Topics on official statistics and statistical policy.

KENTUCKY

University of Louisville (2)

Bioinformatics & Biostatistics

Dutta, Sandipan, Some contributions to nonparametric inference for clustered and multistate data.
Shah, Jasmit, Novel statistical approaches for missing values in truncated high-dimensional metabolomics data with a detection threshold.

MISSOURI

University of Missouri–Columbia (3)

Statistics

Cheng, Yuan, Bayesian analysis of fMRI data and RNA-Seq time course experiment data.
Wang, Henan, Bayesian partition models for DNA methylation analysis.
Yu, Guanglei, Regression analysis of panel count data with informative observations and drop-outs.

NEW YORK

Clarkson University (1)

Mathematics & Computer Science

Al Basheer, Aladeen, A mathematical investigation of the effects of cannibalism in two and three species predator-prey systems.

Columbia University (4)

Applied Physics & Applied Mathematics

Dandapani, Aditi, Enlargement of filtration and the strict local Martingale property in stochastic differential equations.
Shaevitz, Daniel, Extreme weather: Subtropical floods and tropical cyclones.
Tian, Xiaochuan, Nonlocal models with a finite range of nonlocal interactions.

Biostatistics

Chen, Yakuan, Methods for functional regression and nonlinear mixed-effects models with applications to PET data.

Cornell University (7)

Biological Statistics & Computational Biology

Dias, Jishnu, Using protein interactome networks to understand human disease and evolution.

Gao, Feng, Utilizing rare and X-linked variants for inference of population size history and association studies of complex diseases.

Huang, Lei, Information topology of kinetic models of metabolism.

Meyer, Michael J., Methods for functional inference in the proteome and interactome.

Ramstetter, Monica, High resolution relative detection via inference of identical by descent sharing of sample ancestors.

Sinclair, David Giles, Model selection results for latent high-dimensional graphical models on binary and count data with applications of fMRI and genomics.

Zawack, Kelson, A comprehensive analysis of the United States' National Resistance Monitoring System.

Rensselaer Polytechnic Institute (3)

Mathematical Sciences

Heath, Emily, Optimization approaches to problems in network mitigation and restoration.

Pickering, William, Solution of urn models by generating functions with applications to social, physical, biological, and network sciences.

Shen, Xin, Complimentary formulations for problems with sparcity objective.

NORTH CAROLINA

North Carolina State University (12)

Statistics

Alfaro Cordoba, Marcela, Variable selection methods with applications to atmospheric sciences.

Choi, Bong Seog, Testing and estimation under hidden activity.

Das, Priyam, Bayesian quantile regression.

Hager, Sarah Rebecca, Optimal dynamic treatment regimes from a classification perspective for two stage studies with survival data.

Kang, Suhyun, Flexible estimation and testing methods for survival data with application in epidemiology and precision medicine.

Li, Yuan, GPU computing in statistics and R solution.

Morris, Samuel Alan, Spatial methods for modeling extreme and rare events.

Park, So Young, Longitudinal functional data analysis with biomedical applications.

Peng, Huimin, Selection and inference for high-dimensional regression with applications in biomedical research.

Peterson, Geoffrey Cohn Lee, Mean-dependent spatial prediction methods with applications to materials sciences.

Wang, Chong, A study of sufficient dimension reduction methods.

Xu, Yingzi, Binormal precision-recall and ROC classification and variable selection.

NORTH DAKOTA

North Dakota State University, Fargo (1)

Statistics

Sattler, Elizabeth, Subfractals induced by subshifts.

PENNSYLVANIA

Carnegie Mellon University (2)

Statistics

Asher, Jana, Methodological innovations in the collection and analysis of human rights violation data.

Chen, Yen-Chi, Statistical inference using geometric features.

Pennsylvania State University (6)

Statistics

Berstein, Jason, Inference of biophysical diffusion with transient binding using particle filters and stochastic EM.

Chu, Wanghuan, Feature screening for ultra-high dimensional longitudinal data.

Hao, Han, Modeling the genetic architecture of complex traits.

Russell, James, Stochastic models for individual and collective animal movement.

Taoufik, Bahaeddine, Functional data based inference for high frequency financial data.

Xu, Zhanxiong, Efficient parameter estimation methods using quantile regression in heteroscedastic methods.

University of Pittsburgh (2)

Statistics

Lee, Sung Won, Analysis of variation structure of high-dimensional multi-block data.

Zhang, Yun, Cluster analysis and network community detection with application to neuroscience.

SOUTH CAROLINA

University of South Carolina (1)

Epidemiology & Biostatistics

Xu, Xinling, Statistical methods for multivariate and correlated data.

VERMONT

University of Vermont (4)

Mathematics & Statistics

Cody, Emily, Mathematical modeling of public opinion using traditional and social media.

McAndrew, Thomas, Weighted networks: Applications from power grid construction to crowd control.

Regan, Andrew, Towards a science of human stories: Using sentiment analysis and emotional arcs to understand the building blocks of complex social systems.

Stephens, Thomas, Topological methods for evolution equations.

VIRGINIA

Virginia Commonwealth University, Medical Center (4)

Biostatistics

Czarnota, Jenna, Modeling spatially varying effects of chemical mixtures.

Evani, Bhanu, Weighted quantile sum regression for analyzing correlated predictors acting through a mediation pathway on a biological outcome.

Ferber, Kyle, Methods for predicting an ordinal response with high-throughput genomic data.

Joshi, Kabita, Finding the cutpoint of a continuous covariate in a parametric survival analysis model.

ANNUAL SURVEY

Acknowledgments

The Annual Survey attempts to provide an accurate appraisal and analysis of various aspects of the academic mathematical sciences scene for the use and benefit of the community and for filling the information needs of the supporting organizations. Every year, college and university departments in the United States are invited to respond, and the Annual Survey relies heavily on the conscientious efforts of the dedicated staff members of these departments. On behalf of the Joint Data Committee and the Annual Survey Staff, we thank the many secretarial and administrative staff in the departments for their cooperation and assistance in responding to the survey questionnaires.

The Annual Survey is co-sponsored by the American Mathematical Society (AMS), American Statistical Association (ASA), Institute for Mathematical Statistics (IMS), Mathematical Association of America (MAA), and Society for Industrial and Applied Mathematics (SIAM).

Report on the 2016–2017 Employment Experiences of the New Doctoral Recipients

Amanda L. Golbeck, Thomas H. Barr, and Colleen A. Rose

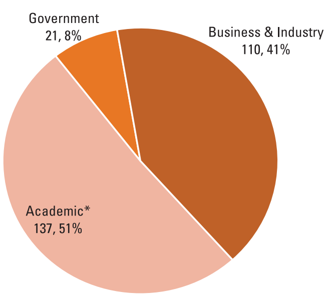
This report focuses on information that comes from the Employment Experiences of New Doctorate Recipients (EENDR) survey of individual PhD recipients regarding their experiences in finding and beginning new jobs. The survey was sent to the 1,419 new PhDs for whom departments provided contact information, and responses were collected during the period July 2017 to October 2018. Six hundred thirty-five (45%) responded. Some of the gross features of the respondents to the EENDR are similar to those of the overall group on which the New Doctorates report is based. For instance, of the 635, 33% were women (29% overall), 60% were US citizens (49% overall), 11% were employed outside the US (9% overall), and 4% were members of underrepresented minority groups (8% overall).

Figure EE.1 shows a breakdown by sector of EENDR respondents working in permanent jobs in the US in the broad sectors academia, business and industry, and government; Figure EE.2 gives the same breakdown for those in temporary jobs. All but 2% of these jobs are full-time. When combined, the information in these two figures can be compared with that in Figure E.2 in the New Doctorates report:

Employment Sector	EENDR Overall % US Employed (n=544)	DR Overall % US Employed (n=1,406)
Academia	72%	61%
Government	7%	6%
Business & Industry	21%	34%

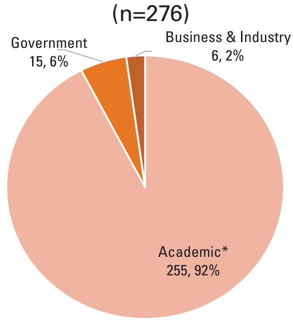
This comparison suggests that 2016–2017 EENDR responses are somewhat biased toward those employed in academia, and thus any conclusions about the entire group of new PhDs based on EENDR responses alone should be made with this qualification. Such bias is not unexpected, since the EENDR responses are not the product of a random sample. The similarities here suggest that estimates based on the EENDR data (e.g., median starting salaries) may not

Figure EE.1: EENDR Respondents Reporting Permanent US Employment by Sector (n=268)



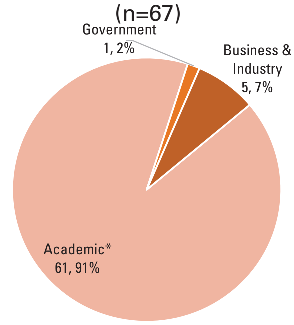
* Includes research institutes and other non-profits.

Figure EE.2: EENDR Respondents Reporting Temporary US Employment by Sector (n=276)



*Includes research institutes and other non-profits.

Figure EE.3. EENDR Respondents Employed Outside the US by Sector (n=67)



* Includes research institutes and other non-profits.

Table EE.1: Number and Percentage of EENDR Respondents Employed in the US by Job Status

					Temporary		Temporary Postdocs				
Year	Perm Total	%	Temp Total	%	Perm Not Avail	% of Temp Total	Total	% of Temp Total	Perm Not Avail	% of Temp Postdocs	#(%) Unknown
Fall 2013	374	53%	335	47%	173	52%	247	74%	106	43%	0
Fall 2014	363	51%	343	49%	148	43%	260	76%	88	34%	0
Fall 2015	357	51%	341	49%	160	47%	258	76%	102	40%	0
Fall 2016	323	52%	298	48%	136	46%	214	72%	82	38%	2 (<1%)
Fall 2017	268	49%	276	51%	134	49%	209	76%	147	70%	5 (1%)

Table EE.2: Percentage of EENDR Respondents Employed in the US by Employment Sector within Job Status

Year	Permanent			Temporary		
	Acad	Govn	B/I	Acad	Govn	B/I
Fall 2013	53%	7%	40%	92%	4%	4%
Fall 2014	54%	6%	40%	92%	5%	3%
Fall 2015	44%	8%	48%	93%	3%	4%
Fall 2016	47%	7%	46%	93%	5%	3%
Fall 2017	51%	8%	41%	92%	5%	2%

be wildly different from the actual values for all of the new PhDs, but the reader should keep these differences in mind.

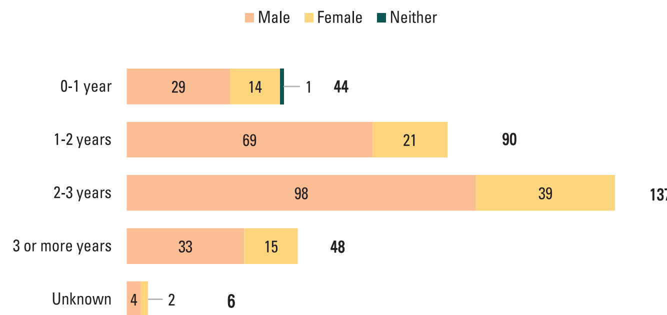
Table EE.1 gives a longitudinal comparison of responses to the EENDR questionnaire from 2013 through 2017. Here are a few features to note:

- 49% of those employed for fall 2017 were in permanent positions.
- The percentage of those in temporary jobs because a permanent one was not available has ranged between 43% and 52% in the years 2013 to 2017, and the 2017 value of 49% is in line with these percentages.
- The percentage of those in temporary jobs who are postdocs has remained consistent over this five-year period, and the 2017 percentage of 76% is the modal value. Also in 2017, of those in postdocs, 70% hold that position because a permanent job was not available.

Table EE.2 compares percentages of PhDs taking employment in various sectors, by job durability. Over the five years shown, the percentages in all of these categories have remained remarkably stable.

Figures EE.5, EE.6, and EE.7 show breakdowns of employment in the broad sectors of education, government, and business and industry. The following table provides

Figure EE.4: Temporary Positions by Duration, Gender, and Count (n=325)



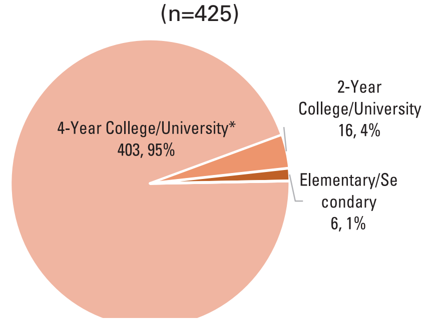
Sector	Number of Responses	% US Citizens	% Women	% Temporary	% Looking
Education	425	59%	32%	67%	35%
Government	38	73%	37%	42%	10%
Business and Industry	121	58%	26%	5%	12%

further insight to these figures by sector, demographics, and job status.

Five hundred eighty-three doctorates provided age information, and Figure EE.8 provides the distribution. The median age was 30, the youngest PhD was 23, and the oldest 55. Almost 60% of these respondents were between the ages of 26 and 30.

Figure EE.9 gives percentages by employment sector of EENDR respondents who identify themselves as Hispanic. The designation "unknown" indicates the respondent did not provide ethnicity data.

Figure EE.5. Employment by Type of Educational Institution (Educ) (n=425)



*Includes research institutes and other non-profits.

Figure EE.6. Employment by Type of Government (Gov) (n=38)

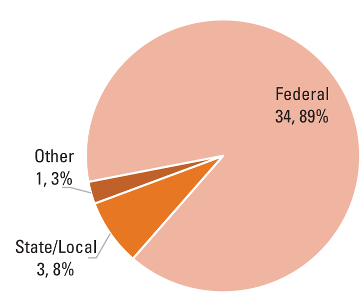
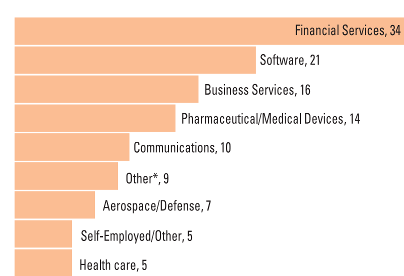


Figure EE.7. Employment by Type of Business/Industry (BI) (n=121)



*Includes Biotechnology (2), Consumer Merchandising (3), Energy (3), and Survey/Market Research (1).

Figure EE.8. Age Distribution of New PhD Respondents (n=583)

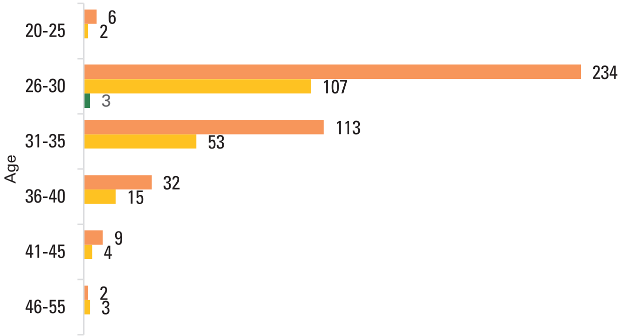
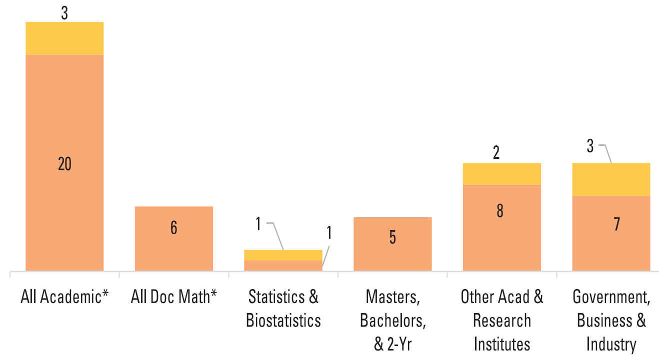


Figure EE.9. Ethnicity of New PhD Respondents by Type of US Employer (n=33)



Nineteen respondents to the EENDR survey were Black or African American, 4 were American Indian or Alaska Native, 3 were Native Hawaiian or Other Pacific Islander, 198 were Asian, 394 were White, and 17 were unknown.

Starting Salaries of the 2016–2017 Doctorate Recipients

The starting salary figures were compiled from information gathered on the EENDR questionnaires sent to 1,419 individuals using addresses provided by the departments granting the degrees; 635 individuals responded between late October 2017 and June 2018. Responses with insufficient data or from individuals who indicated they had part-time or non-US employment were excluded. Numbers of usable responses for each salary category are reported in the tables on page 1164.

Readers are warned that the data in this report are obtained from a self-selected sample, and inferences from them may not be representative of the full population. Detailed information, including boxplots which traditionally appeared in this report, is available on the AMS website at `www.ams.org/annual-survey/survey-reports`.

Remarks on Starting Salaries

Key to Tables and Graphs. Salaries are those reported for the fall immediately following the survey cycle. Years listed denote the survey cycle in which the doctorate was received—for example, survey cycle July 1, 2016–June 30, 2017 is designated as 2017. Salaries reported as 9–10 months exclude stipends fom summer grants teaching, or the equivalent. M and W are men and women, respectively. Separate figures are not provided when the number of salaries available for analysis in a particular category was five or fewer. All categories of "Teaching/Teaching and Research" and "Research Only" contain those recipients employed at academic institutions only.

Graphs. The graphs show standard boxplots summarizing salary distribution information for the years 2010 through 2017. Values plotted for 2010 through 2016 are converted to 2017 dollars using the implicit price deflator prepared annually by the Bureau of Economic Analysis, US Department of Commerce [`https://www.bea.gov`]. The category for each graph is based on a work activity reported in the EENDR. Salaries of postdoctorates are shown separately. They are also included in other academic categories with matching work activities.

For each boxplot the box shows the first quartile (Q1), the median (M), and the third quartile (Q3). Upper whiskers extend from Q3 to the largest data value below Q3+1.5IQR, and lower whiskers from Q1 down to the smallest data value above Q1−1.5IQR. Data points falling between Q3+1.5IQR and Q3+3IQR or Q1−1.5IQR and Q1−3IQR are designated as outliers and plotted as circles (°). Data outside the range Q1−3IQR to Q3+3IQR are designated as extreme outliers and plotted as stars (*).

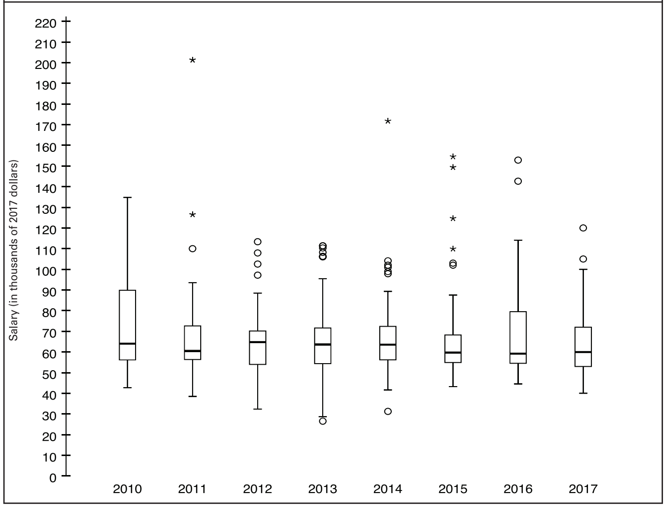
Response Rates

New PhD Recipient Response Rates by Granting Department Grouping

Granting Department Group	Received	Percent
Math Public Large:	139 of 288	48%
Math Public Medium:	122 of 259	47%
Math Public Small:	67 of 180	37%
Math Private Large:	87 of 176	49%
Math Private Small:	34 of 61	56%
Applied Math:	50 of 133	38%
Statistics:	82 of 195	42%
Biostatistics:	54 of 127	43%
Total:	**635 of 1,419**	**45%**

Academic Teaching/Teaching and Research 9–10-Month Starting Salaries* (in thousands of dollars)

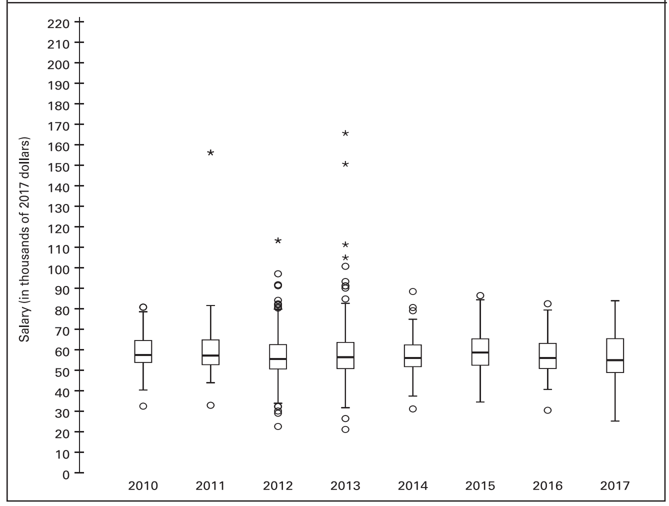
PhD Year	Min	Q_1	Median	Q_3	Max
Total (19 men/4 women/2 neither)					
2017 M	25.3	50.0	56.6	65.0	110.0
2017 W	40.0	55.4	60.3	70.5	110.0
2017 N	too few to report				
One year or less experience (130 men/58 women/2 neither)					
2017 M	25.3	50.0	56.5	65.0	110.0
2017 W	40.0	55.1	60.8	69.9	110.0
2017 N	too few to report				



* Includes postdoctoral salaries.

Academic Postdoctorates Only* 9–10-Month Starting Salaries (in thousands of dollars)

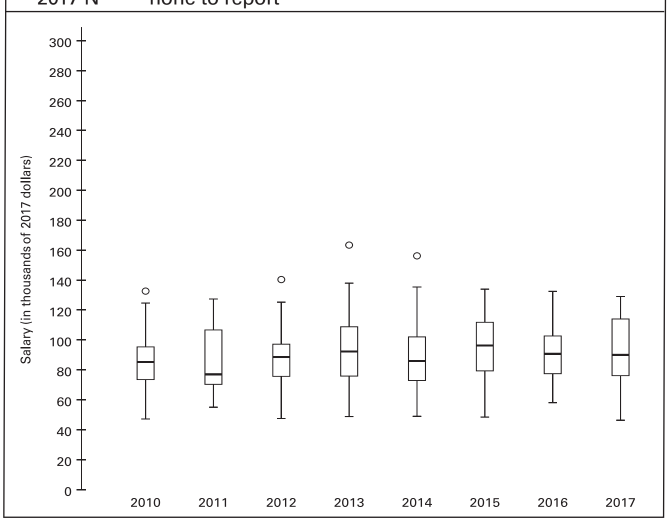
PhD Year	Min	Q_1	Median	Q_3	Max
Total (43 men/11 women/1 neither)					
2017 M	25.3	47.3	53.0	61.5	84.0
2017 W	55.0	60.8	66.0	71.0	82.0
2017 N	none to report				
One year or less experience (43 men/10 women/1 neither)					
2017 M	25.3	47.3	53.0	61.5	84.0
2017 W	55.0	62.8	67.8	71.5	82.0
2017 N	none to report				



* A postdoctoral appointment is a temporary position primarily intended to provide an opportunity to extend graduate training or to further research experience.

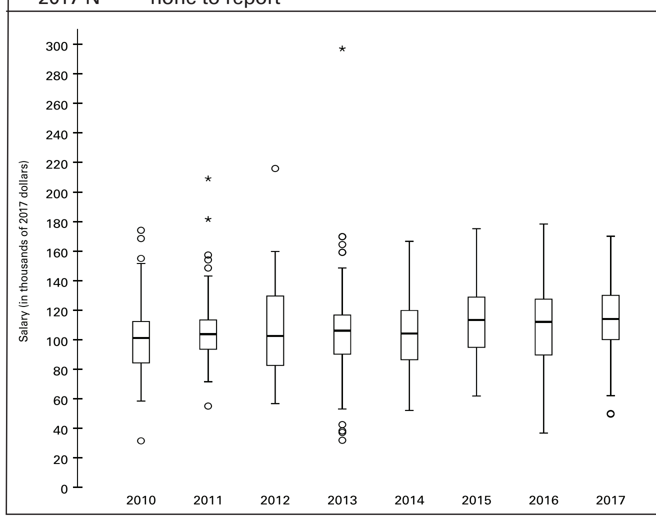
Government 11–12-Month Starting Salaries (in thousands of dollars)

PhD Year	Min	Q_1	Median	Q_3	Max
Total (22 men/14 women/0 neither)					
2017 M	46.3	74.1	94.5	107.0	129.0
2017 W	66.9	80.5	94.3	115.0	125.0
2017 N	none to report				
One year or less experience (18 men/10 women/0 neither)					
2017 M	46.3	60.3	82.4	103.0	129.0
2017 W	66.9	80.5	91.3	115.0	125.0
2017 N	none to report				



Business and Industry 11–12-Month Starting Salaries (in thousands of dollars)

PhD Year	Min	Q_1	Median	Q_3	Max
Total (80 men/29 women/0 neither)					
2017 M	50.0	104.8	118.5	135.0	400.0
2017 W	49.5	87.5	108.5	120.0	160.0
2017 N	none to report				
One year or less experience (67 men/21 women/0 neither)					
2017 M	50.0	100.0	114.0	130.0	400.0
2017 W	70.0	85.0	110.0	120.0	160.0
2017 N	none to report				



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The Annual Survey is co-sponsored by the American Mathematical Society (AMS), American Statistical Association (ASA), Institute for Mathematical Statistics (IMS), Mathematical Association of America (MAA), and Society for Industrial and Applied Mathematics (SIAM).

Distribution of New PhD Recipient Responses by EmployerType

Employer Type	Received	Percent
Math Public Large:	40	6%
Math Public Medium:	32	5%
Math Public Small:	13	2%
Math Private Large:	47	7%
Math Private Small:	9	1%
Applied Math:	10	2%
Statistics:	10	2%
Biostatistics:	11	2%
Masters:	19	3%
Bachelors:	92	14%
Two-Year Institutions:	15	2%
Other Academic:	70	11%
Research Institute/Other Non-profit:	24	4%
Government:	36	6%
Business/Industry:	116	18%
Non-US Academic:	61	10%
Non-US Nonacademic:	6	1%
Not Seeking (US):	3	<1%
Still Seeking (US):	17	3%
Unknown (US):	0	0%
Non-US: Not Seeking, Still Seeking, Unknown:	4	1%
Total:	**635**	**100%**

INTRODUCING AN AMS MEMBER

Chris Castillo

Associate Professor of Mathematics, Cecil College

Member Type: Introductory
AMS Member Since: 2009
Primary Field of Research: Permutation Polynomials over Finite Fields
Dissertation Advisor: Robert S. Coulter
Undergraduate Institution: Towson University
Favorite Non-Math Hobby: Rockhounding
Erdős Number: 4
PhD-Granting Institution: University of Delaware

Favorite Memory From an AMS Conference or Event:

Starting with my first Joint Meetings (New Orleans 2007) and even continuing throughout today, I remember seeing mathematicians giving talks and even walking through the halls whose names I recognized from textbooks and seminal papers – names like Michael Artin, Gilbert Strang, Benedict Gross. For me, it humanizes mathematics knowing that these are not just names but actual people in the flesh.

How would you describe to a non-math person that mathematics (or at least your area of specialty) is beautiful?

I am fascinated by structure. Certainly mathematics gives us a vocabulary to describe structure and relationships in the world around us, but the structure of mathematics is also inherently beautiful. Starting from just a few simple axioms we can build up massive structures that enable us to describe the way numbers behave (Fermat's Last Theorem) or why the object behaves (The Classification of Finite Simple Groups). As a more "accessible" example, the statement of Cauchy's Integral Formula is a surprising result (at first) for a student who has taken calculus, and amply demonstrates the richness of the field of complex numbers. With this last example, we find both structure and simplicity (of a sort) – in short, beauty.

AMS American Mathematical Society
Advancing research. Creating connections.

Mathematics People

Wu Awarded 2019 CAIMS-Fields Industrial Mathematics Prize

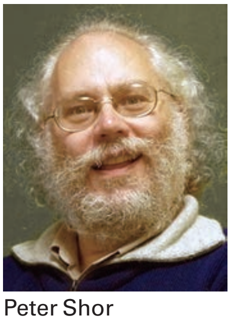
Jianhong Wu

Jianhong Wu of York University has been awarded the 2019 CAIMS-Fields Industrial Mathematics Prize of the Canadian Applied and Industrial Mathematics Society (CAIMS) and the Fields Institute "in recognition of his many contributions to dynamical systems in mathematical epidemiology and in particular his collaborative research with public health professionals in government and industry: applying his expert knowledge to infectious disease mitigation strategies and preparedness." Wu received his PhD from Hunan University in 1987 and joined York University in 1990. His major research interests are nonlinear dynamics and delay differential equations, neural networks and pattern recognition, mathematical ecology and epidemiology, and big data analytics. Wu tells the *Notices*: "I like all team sports in general, particularly basketball and soccer. Whenever possible, during major training and professional events that I have organized, I coordinated efforts to facilitate team sport activities among all participants. I have found these opportunities particularly useful to build interdisciplinary and bilateral collaborations, especially when participants come from different disciplines. Bringing diversified culture and scientific backgrounds together provides a great experience. For these reasons, I enjoy playing cards as well. I also enjoy gardening. I am very proud that my front garden has received multiple annual Outstanding Garden awards from the city (in Toronto)."

—From a CAIMS-Fields announcement

Micius Quantum Prizes

David Deutsch



Peter Shor

Juan Ignacio Cirac

Peter Zoller

The Micius Quantum Foundation has inaugurated the Micius Quantum Prize to recognize scientists who have made outstanding contributions in the field of quantum communications, quantum simulation, quantum computation, and quantum metrology. The prizes for 2018 were awarded in the area of quantum computation. The awardees are **David Deutsch** of the University of Oxford "for his seminal conceptual contribution on quantum Turing machine and quantum algorithms"; **Peter Shor** of the Massachusetts Institute of Technology "for his groundbreaking theoretical work on factoring algorithm and quantum error correction"; and **Juan Ignacio Cirac** of the Max Planck Institute of Quantum Optics and **Peter Zoller** of the University of Innsbruck "for their outstanding theoretical contributions that enabled the scalable implementations of quantum information processing such as quantum computation with trapped ions, quantum simulation with ultracold atoms, and quantum repeaters." The prizes carry cash awards of 1 million Chinese yuan (approximately US$150,000).

—From a Micius Quantum Foundation announcement

Mulholland Awarded PIMS Education Prize

Jamie Mulholland

Jamie Mulholland of Simon Fraser University has been awarded the 2019 PIMS Education Prize of the Pacific Institute of the Mathematical Sciences "in recognition of his important work on mathematics outreach and popularization. With Malgorzata Dubiel, he has organized a series of Student Math Camps and Math Camps for Teachers. In addition, he has produced a large number of high-quality video teaching resources that have been made available through YouTube, making his valuable work accessible to a broad audience." Mulholland received his PhD from the University of British Columbia. He has received Excellence in Teaching awards from Simon Fraser University and the Canadian Mathematical Society. He tells the *Notices*: "I'm married with two children and I spend the vast majority of my spring and summer evenings/weekends on the baseball field coaching my kids' teams. This is something I look forward to doing every year. I try to maintain an active lifestyle: snowboarding in the winter, stand-up paddle boarding in the summer, and jiu jitsu all year long."

—From a PIMS announcement

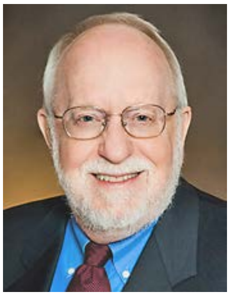
NCTM Lifetime Achievement Awards

The National Council of Teachers of Mathematics (NCTM) has selected three educators to receive Lifetime Achievement Awards for 2019. They are **Margaret (Peg) Smith** of the University of Pittsburgh, **Lee V. Stiff** of North Carolina State University, and **Johnny Lott** of the University of Montana.

Margaret Smith

According to the prize citation, Smith "has had a career thoroughly dedicated to mathematics and furthering the knowledge of other educators." She received her EdD in mathematics education from the University of Pittsburgh in 1995. She has taught in New York, Ohio, Michigan, and Pennsylvania, in addition to her service at the University of Pittsburgh. She has been awarded numerous grants in continuing education and mathematics research. Her awards include the Susan Loucks-Horsley Award and both the Dean's Award for Teaching Excellence and the Chancellor's Distinguished Teaching Award from the University of Pittsburgh. She has served on boards of the Pennsylvania Council of Teachers of Mathematics and the NCTM, as well as many committees and associations, and was one of the foundational editors of the journal *Math Teacher Educator*. With Mary Kay Stein, she authored the book *Five Practices for Orchestrating Productive Mathematics Discussion*.

The citation for Stiff states: "Stiff's work is foundational in providing and encouraging learning to students of color. Equity has played a major role in Dr. Stiff's work and research, often speaking to this subject." Stiff received his PhD in mathematics education in 1978 from the North Carolina State College of Education. He has served as the president of NCTM, as well as a member of and participant in various committees and associations. His work "has positively impacted the quality of science and mathematics education for current and prospective teachers." He is an advocate of including more technology in the learning and use of mathematics. Stiff received the Benjamin Banneker Lifetime Achievement Award in 2015 and the TODOS Iris M. Carl Leadership and Equity Award in 2017.



Johnny Lott

Lott's citation states: "Through his years as a mathematician and educator, Dr. Lott has impacted many careers: providing support, advice, and direction to those around him." Lott received his PhD in mathematics education from Georgia State University in 1973. He has served as president of NCTM and as vice president of the Montana Academy of Science, and he was a board member of the Montana Council of Teachers of Mathematics. He was also professor of mathematics and of education at the University of Mississippi. He has been the recipient of both the Dean Preble Memorial Award and the Meritorious Service Award of the Montana Council of Teachers of Mathematics and of the George and Jane Dennison Faculty Award from the University of Montana, and he has written and contributed to several books and authored a number of journal articles. Lott tells the *Notices*: "I have a few hobbies, including painting watercolors (having taken ten classes since retirement), quilting (which began long before I retired and continue), writing short stories for my personal satisfaction, working many hours every month for the Lafayette County Democratic Party in Mississippi as editor of the biweekly newsletter (among other things), and cohosting a biweekly men's lunch group where progressive ideas are discussed."

—From NCTM announcements

Prizes of the ICA

The Institute of Combinatorics and Its Applications (ICA) has awarded a number of prizes in 2019.

Kai-Uwe Schmidt of the University of Paderborn has been awarded the Hall Medal, given for extensive quality research with substantial international impact. The prize citation reads: "Dr. Kai-Uwe Schmidt has repeatedly made outstanding contributions concerning algebraic, analytic, and probabilistic techniques for combinatorial problems in coding theory and cryptography. He has published about forty papers in the best journals, covering a substantial breadth. The hallmark of Dr. Schmidt's research is the development of novel, fundamental theory that addresses practical applications. In particular, his research on merit factor, exceptional polynomials, configurations in finite projective planes, and vector spaces over finite fields have all been influential in developing deep theory for practical problems. Dr. Schmidt is increasingly recognized as a leader in his field through invitations to speak at numerous international conferences and to join prestigious editorial boards."

Fan Chung of the University of California, San Diego, and **Dieter Jungnickel** of Augsburg University have been awarded the Euler Medals for distinguished lifetime career contributions to combinatorial research. The prize citation for Chung states: "Fan Chung has conducted research across a wide range of problems in theoretical and applied combinatorics. Chung has been a role model and ambassador for combinatorics throughout her distinguished career." Chung received her PhD from the University of Pennsylvania in 1974. She has published around 275 papers, principally in graph theory, algorithm analysis, probability, communications networks, and computation. Her awards and honors include the Allendoerfer Award of the Mathematical Association of America (1990), the AMS Noether Lectureship in 2009, and an invited plenary address at the International Congress of Mathematicians in 1994. She is a Fellow of the AMS and the Society for Industrial and Applied Mathematics (SIAM) and a member of the American Academy of Arts and Sciences. The citation for Jungnickel states: "Dieter Jungnickel's research contributions span a very broad range of areas of discrete mathematics and are characterized by his constant efforts to explore connections among topics." He received his PhD from the Free University of Berlin in 1976 and has published more than 200 research papers in design theory, coding theory, difference sets, finite fields, finite geometry, operations research, and group theory. With Thomas Beth and Hanfried Lenz, he authored the two-volume book *Design Theory*. He is editor-in-chief of the journal *Designs, Codes and Cryptography*. He has organized numerous conferences and workshops and has provided support and mentoring to many junior colleagues.

Shuxing Li of Simon Fraser University and **Alexander Bors** of the University of Western Australia have been awarded Kirkman Medals for excellent research by Fellows or Associate Fellows of ICA early in their research careers. The citation for Li reads in part: "Dr. Shuxing Li has made significant contributions on a number of central problems in algebraic coding theory, discrete geometry, combinatorial design theory, and information theory. He applies deep theory and powerful tools to answer fundamental questions, not only in combinatorics but also in practical digital communications applications such as coding, compressed sensing, and energy minimization." The citation for Bors reads in part: "Dr. Alexander Bors has made outstanding contributions to the understanding of combinatorial and quantitative problems on finite groups. He addresses fundamental theoretical questions, some of which are motivated by practical applications. He seeks characterizations that support efficient algorithmic decidability."

Robin Wilson of Open University has been awarded the Stanton Medal for significant lifetime contributions to promoting the discipline of combinatorics. The prize citation reads: "Dr. Robin Wilson has, for fifty years, been an outstanding ambassador for graph theory to the general public. He has lectured widely (giving some 1,500 public lectures), and extended the reach of his lectures through television, radio, and videotape. He has also published extensively (authoring or editing some forty books) on combinatorial ideas, written in a style that is engaging and accessible. He has provided direction, encouragement, and support to colleagues and students at all levels. His superb talents at conveying the beauty of graph-theoretic ideas and inviting his readers and listeners to join in have enthused many students, teachers, and researchers. Professor Wilson's advocacy and outreach for combinatorics continue to yield many positive impacts that are enjoyed by researchers and nonspecialists alike."

Carsten Thomassen of the Technical University of Denmark was named an honorary fellow for his preeminent contributions to combinatorics and its applications. The citation names him "one of the most influential graph theorists of our time." He was a founding fellow of the ICA and a council member from 1990 through 2015. He is coauthor with B. Mohar of the book *Graphs on Surfaces* and has been editor in chief of the *Journal of Graph Theory* since 1989. He has served on editorial boards of a number of combinatorics journals. He was an invited speaker at the International Congress of Mathematicians in Kyoto in 1990 and received the Lester R. Ford Award from the Mathematical Association of America in 1993.

The Institute of Combinatorics and Its Applications was established in 1990 for the purpose of promoting the development of combinatorics and of encouraging publications and conferences in combinatorics and its applications.

—From ICA announcements

Vybíral Awarded Traub Prize in IBC

Jan Vybíral of Czech Technical University has been named the recipient of the Joseph F. Traub Prize for Achievement in Information-Based Complexity for 2019. He will receive a cash award of US$3,000 and a plaque, to be given at the Dagstuhl Seminar on Algorithms and Complexity for Continuous Problems in August 2019.

—Joseph F. Traub Prize Committee announcement

National Academy of Sciences Elections

Ten individuals in the mathematical sciences are among the 125 new members and foreign associates elected to the National Academy of Sciences (NAS). They are:

- **Christopher S. Bretherton**, University of Washington
- **Russel E. Caflisch**, Courant Institute of Mathematical Sciences, New York University
- **Jennifer T. Chayes**, Microsoft Research
- **Bryna Rebekah Kra**, Northwestern University
- **Kathryn Roeder**, Carnegie Mellon University
- **Scott J. Shenker**, University of California, Berkeley
- **Barry Simon**, California Institute of Technology
- **Karen E. Smith**, University of Michigan

Elected as Foreign Associates were **Artur Avila**, Zurich University, and **David Harel**, Weizmann Institute of Science. Caflisch, Chayes, Kra, and Simon are members of the Inaugural Class of AMS Fellows.

—From an NAS announcement

MathWorks Math Modeling (M3) Challenge

The 2019 MathWorks Math Modeling (M3) Challenge (formerly the Moody's Mega Math Challenge) was held in New York City on April 29, 2019. This year's challenge was to use mathematical modeling to recommend solutions to the spread of substance abuse in the United States.

The Challenge Champions Team Prize of US$20,000 in scholarship money was awarded to a team from High Technology High School, Lincroft, New Jersey. The team members were **Eric Chai**, **Gustav Hansen**, **Emily Jiang**, **Kyle Lui**, and **Jason Yan**. They were coached by Raymond Eng.

The First Runner-Up Team Prize of US$15,000 in scholarship money was awarded to a team from Richard Montgomery High School, Rockville, Maryland. The team members were **Matt Kolodner**, **Clarissa Xia**, **Jack Yang**, **Laura Yao**, and **Lauren Zhou**. They were coached by Matt Davis.

The Third Place Team Prize of US$10,000 in scholarship money was awarded to a team from Nicolet High School, Glendale, Wisconsin. The team members were **Zach Godkin**, **Gabe Guralnick**, **Savir Maskara**, and **Ryan Mortonson**. Their coach was Mike Weidner.

Finalist Team Prizes of US$5,000 in scholarship money were awarded to three teams. The team from the Academy for Science and Design in Nashua, New Hampshire, consisted of **Denver Blake**, **Daniel Bujno**, **Ian Coolidge**, **Frederick Lee**, and **Nathan Yeung**; they were coached by Karen Legault. The team from Adlai E. Stevenson High School in Lincolnshire, Illinois, consisted of **Matthew Jalnos**, **Joey Rivkin**, **Tony Tan**, **Joshua Tsai**, and **Angela Zhang**, and their coach was Paul Kim. The team from Wayzata High School in Plymouth, Minnesota, consisted of **Amanda Chan**, **George Lyu**, **Zachary Xiong**, **Caroline Zeng**, and **Alisha Zhu**; they were coached by William Skerbitz.

The M3 Challenge invites teams of high school juniors and seniors to solve an open-ended, realistic, challenging modeling problem focused on real-world issues. The top five teams receive awards ranging from US$5,000 to US$20,000 in scholarship money. The competition is sponsored by MathWorks, a developer of computing software for engineers and scientists, and is organized by the Society for Industrial and Applied Mathematics (SIAM).

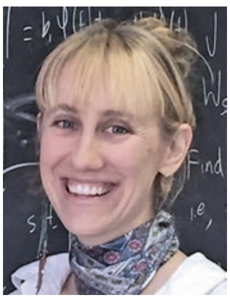
—From a MathWorks/SIAM announcement

Credits

Photo of Jianhong Wu is courtesy of the Faculty of Sciences, York University.
Photo of David Deutsch is courtesy of Lulie Tanett.
Photo of Peter Shor is courtesy of Charles H. Bennett.
Photo of Juan Ignacio Cirac is courtesy of Thorsten Naeser.
Photo of Peter Zoller is courtesy of IQOQI/M. Knabl.
Photo of Jamie Mulholland is courtesy of Dale Northey.
Photo of Margaret Smith is courtesy of Margaret Smith.
Photo of Johnny Lott is courtesy of Bailey Swearingen.

Community Updates

AMS Congressional Fellow Announced



Lucia D. Simonelli has been awarded the 2019–2020 AMS Congressional Fellowship. Simonelli received her PhD in mathematics from the University of Maryland-College Park. She is currently a postdoctoral fellow in the mathematics section at the Abdus Salam International Centre for Theoretical Physics (ICTP) in Trieste, Italy.

Simonelli works predominantly in dynamical systems, specifically to prove properties of parabolic flows using functional analytic techniques. She also recently co-organized the first Latin American School in Applied Mathematics held in Quito, Ecuador (http://indico.ictp.it/event/8691/). The mission of the school is to foster the growth of advanced, rigorous studies and research in physical and mathematical sciences in Latin America. Specifically, the goal of the school is to highlight the applications of mathematics in other disciplines and in industry to demonstrate the usefulness of mathematics as a tool in career choice.

The Congressional Fellowship program is administered by the American Association for the Advancement of Science (AAAS) and provides an opportunity for scientists and engineers to learn about federal policy making while contributing their knowledge and analytical skills to the process. Fellows spend a year working on the staff of a member of Congress or a congressional committee, working as a special legislative assistant in legislative and policy areas requiring scientific and technical input. The fellowship program includes an orientation on congressional and executive branch operations and a year-long professional development program.

The fellowship is designed to provide a unique public policy learning experience to demonstrate the value of science–government interaction and to bring a technical background and external perspective to the decision-making process in Congress.

For more information on the AMS-AAAS Congressional Fellowship, go to http://bit.ly/AMSCongressionalFellowship.

—AMS Office of Government Relations

Fan China Exchange Program Grants Awarded

The Society's Fan China Exchange Program awards grants to support collaborations between Chinese and US or Canadian researchers. Institutions in the United States or Canada apply for the funds to support a visitor from China or vice versa. This funding is made possible through a generous gift made to the AMS by Ky and Yu-Fen Fan in 1999. The 2019 grants follow.

Jinqiao Duan of the Department of Applied Mathematics, Illinois Institute of Technology, received a US$4,500 grant for a one-month stay by **Meihua Yang** of Huazhong University of Science and Technology, Wuhan, China.

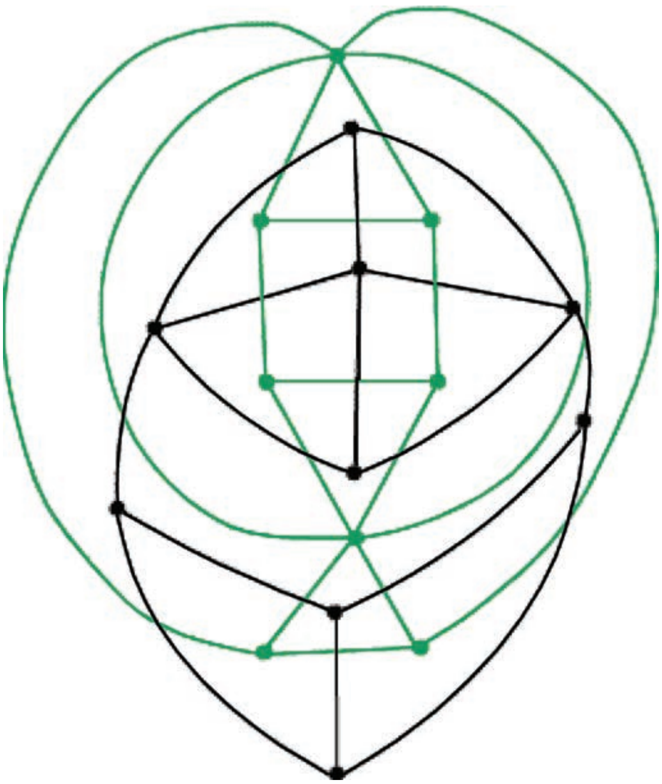
Yong Yang of the Department of Mathematics, Texas State University, was awarded a US$4,000 grant for a one-month stay by **Guohua Qian** of Changshu Institute of Technology, Jiangsu, China.

For information about the Fan China Exchange Program, visit the website http://www.ams.org/programs/travel-grants/china-exchange/china-exchange or contact the AMS Professional Programs Department, email: chinaexchange@ams.org., telephone 401-455-4105 (within the U.S., toll free: 800-321-4267, ext. 4088).

—AMS Membership and Programs Department

From the AMS Public Awareness Office

Mathematical Moments. Steven Strogatz of Cornell University talks about how differential equations helped in controlling HIV, and **Mary Ellen Bushman** of Emory University explains the math she used to understand malarial drug resistance in the Mathematical Moment "Keeping People Alive" at http://www.ams.org/publicoutreach/mathmoments/mm144-hiv-and-malaria-podcast.



Feature Column. Read monthly essays for and from those who enjoy mathematics. Pieces this year include "Topology and Elementary Electric Circuit Theory: Duality," by Tony Phillips; "Pretty as a Picture," by Joe Malkevitch; "Non-Negative Matrix Factorizations," by David Austin; and "Understanding Kepler—Earth's Motion," by Bill Casselman. `https://www.ams.org/featurecolumn`

Math in the Media. This is your portal to media coverage of math and mathematicians. Read about 2019 Abel Prize winner **Karen Uhlenbeck, John Urschel, Moon Duchin,** the effectiveness of remedial math courses, gender in the math profession, and recent research results; and link to reviews of the latest math-related books. `http://www.ams.org/mathmedia`.

—Annette Emerson and Mike Breen
AMS Public Awareness Officers
`pao@ams.org`

Mathematics Opportunities

Listings for upcoming mathematics opportunities to appear in Notices *may be submitted to* notices@ams.org.

Call for Nominations for 2020 Abel Prize

The Norwegian Academy of Science and Letters awards the Abel Prize annually to recognize outstanding scientific work in mathematics, including mathematical aspects of computer science, mathematical physics, probability, numerical analysis and scientific computing, statistics, and also applications in the sciences. Nominations are due by **September 15, 2019**. See www.abelprize.no/c53676/artikkel/vis.html?tid=53705.

—*Norwegian Academy of Science and Letters*

Call for Nominations for Adams Prize

The 2019–2020 Adams Prize of the University of Cambridge will be awarded in the field of algebra. The deadline for nominations is **October 31, 2019**. For more information, see the website https://www.maths.cam.ac.uk/adams-prize.

—*From a University of Cambridge announcement*

Call for Nominations for 2020 Balaguer Prize

The Ferran Sunyer i Balaguer Prize will be awarded for a mathematical monograph of an expository nature presenting the latest developments in an active area of research in mathematics. The prize consists of 15,000 euros (approximately US$17,000). The winning monograph will be published in the Birkhäuser series "Progress in Mathematics." The deadline for nominations for the 2020 prize is **November 29, 2019**. For more information see https://ffsb.espais.iec.cat/en/the-ferran-sunyer-i-balaguer-prize.

—*Institut d'Estudis Catalans announcement*

Call for Nominations for 2020 Popov Prize

The Vasil A. Popov Prize will be awarded in 2020 to a young mathematician for outstanding research contributions in approximation theory and related areas of mathematics. The deadline for nominations is **January 31, 2020**. For more information see https://www.ljll.math.upmc.fr/popov-prize/call-for-nomination-2020.

—*From a Popov Prize announcement*

Call for Nominations for AWM Falconer Lectureship

The Association for Women in Mathematics (AWM) and the Mathematical Association of America (MAA) annually present the Etta Z. Falconer Lecture at MathFest to honor women who have made distinguished contributions to the mathematical sciences or mathematics education. The deadline for nominations is **September 1, 2019**. See https://sites.google.com/site/awmmath/programs/falconer-lectures.

—*From an AWM announcement*

Call for Nominations for AWM Schafer Prize

The Association for Women in Mathematics (AWM) calls for nominations for the Alice T. Schafer Mathematics Prize to be awarded to an undergraduate woman for excellence in mathematics. The nominee must be an undergraduate when nominated. The deadline is **October 1, 2018**. See https://sites.google.com/site/awmmath/programs/schafer-prize.

—*From an AWM announcement*

Call for Nominations for Gerald Sacks Prize

The Association for Symbolic Logic (ASL) invites nominations for the Gerald Sacks Prize for the most outstanding doctoral dissertation in mathematical logic. See the website https://tinyurl.com/y5427feq.

—From an ASL announcement

Research Experiences for Undergraduates

The Research Experiences for Undergraduates (REU) program supports student research in areas funded by the National Science Foundation (NSF) through REU sites and REU supplements. See www.nsf.gov/funding/pgm_summ.jsp?pims_id=5517. The deadline date for proposals from institutions wishing to host REU sites is **August 28, 2019**. Dates for REU supplements vary with the research program (contact the program director for more information). Students apply directly to REU sites; see www.nsf.gov/crssprgm/reu/list_result.jsp?unitid=5044 for active REU sites.

—From an NSF announcement

Joint DMS/NIGMS Initiative to Support Research at the Interface of the Biological and Mathematical Sciences

The National Science Foundation (NSF) and the National Institute of General Medical Sciences (NIGMS) at the National Institutes of Health (NIH) support research in mathematics and statistics on questions in the biological and biomedical sciences. The application period is **September 1–18, 2019**. For more information see https://www.nsf.gov/funding/pgm_summ.jsp?pims_id=5300.

—From an NSF announcement

Call for Nominations for Joseph F. Traub Prize

The Joseph F. Traub Prize for Achievement in Information-Based Complexity is given for work done in a single year, a number of years, or a lifetime. The deadline for nominations for 2020 is **March 31, 2020**. See https://www.journals.elsevier.com/journal-of-complexity/awards/joseph-f-traub-prize.

—Traub Prize Award Committee

News from AIM

The American Institute of Mathematics (AIM) is accepting applications for the following scientific programs.

Focused Workshop Program: These week-long workshops are distinguished by their specific mathematical goals. This may involve making progress on a significant unsolved problem or examining the convergence of two distinct areas of mathematics. Funding for travel and accommodation is provided. Researchers may apply to attend an upcoming AIM workshop or may propose one. The deadline is **November 1, 2019**. A list of upcoming workshops is available at www.aimath.org/workshops/upcoming.

SQuaREs Program: Structured Quartet Research Ensembles (SQuaREs) bring together groups of four to six researchers for a week of focused work on a specific research problem, returning for up to three consecutive years. Application deadline is **November 1, 2019**.

For more information on both programs, see www.aimath.org.

—From an AIM announcement

News from MSRI

The Mathematical Sciences Research Institute (MSRI) invites applications for Research Professors, Research Members, and Postdoctoral Fellows in the following programs:

- Random and Arithmetic Structures in Topology, August 17–December 18, 2020
- Decidability, Definability and Computability in Number Theory, August 17–December 18, 2020
- Mathematical Problems in Fluid Dynamics, January 19–May 28, 2021

Research Professorships are intended for senior researchers who will be making key contributions to a program, including the mentoring of postdoctoral fellows, and who will be in residence for three or more months. Application deadline: **October 1, 2019**.

Research Memberships are intended for researchers who will be making contributions to a program and who will be in residence for one or more months. Application deadline: **December 1, 2019**.

Postdoctoral Fellowships are intended for recent PhDs. Application deadline: **December 1, 2019**.

For more information, see www.msri.org/application.

—From an MSRI announcement

Classified Advertising

Employment Opportunities

CHINA

Tianjin University, China
Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics. Chinese citizenship is not required.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to mathjobs@tju.edu.cn.

For more information, please visit cam.tju.edu.cn or contact Ms. Erica Liu at mathjobs@tju.edu.cn, telephone: 86-22-2740-6039.

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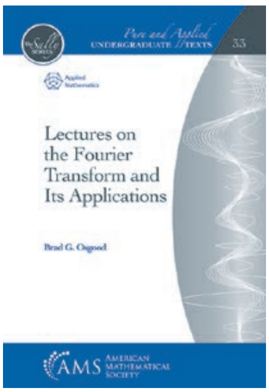
AMS BOOKSHELF

The AMS Book Program serves the mathematical community by publishing books that further mathematical research, awareness, education, and the profession while generating resources that support other Society programs and activities. As a professional society of mathematicians and one of the world's leading publishers of mathematical literature, we publish books that meet the highest standards for their content and production. Visit **bookstore.ams.org** to explore the entire collection of AMS titles.



Lectures on the Fourier Transform and Its Applications

By Brad G. Osgood

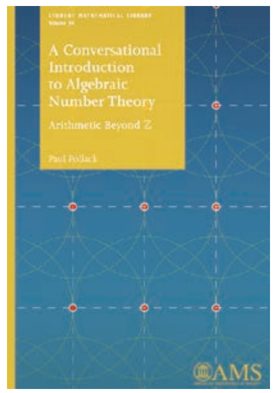
With few pre-requisites beyond multi-variable calculus and linear algebra, Osgood's book puts the theory of Fourier transforms together into a coherent whole for students who may have seen them in a variety of applied contexts. Students will "gain a facility with using the Fourier transform, both specific techniques and general principles, and learning to recognize when, why and how it is used."

A masterful lecturer, Osgood has written this book in a way that conveys the experience of being in his class. He writes as if he is speaking directly to you, and it is not a complete exaggeration to say that the sections of each chapter flow like a musical composition, one theme developing into the next. There is a naturalness and charm to how the subject unfolds in his book. As Tom Körner from University of Cambridge puts it: this is "Fourier Analysis with a swing in its step."

The book has two underlying aims: one is to show engineering students that Fourier Theory is relevant and useful, and the other is to convey the benefits of understanding the deeper mathematical underpinnings of the subject. Osgood succeeds in both: laying out topics of direct interest to engineers like signal processing and medical imaging, and explaining abstract theory in a down to earth way. Here is how Osgood invites students to higher dimensional analysis with the aid of a quote attributed to Euler: "I was just following my pencil."

"...while your initial intuition may be confined to situations where you can draw a picture (two and three dimensions, say), most often the calculations work even when you can't draw a picture (n dimensions). Follow the words. Follow your pencil. Don't underestimate the power of this."

For students and researchers from mathematics, physics, and engineering, this book is a joy to read and a useful resource to have on the bookshelf.



A Conversational Introduction to Algebraic Number Theory: Arithmetic Beyond Z

By Paul Pollack

Pollack's book, aimed at undergraduates with a basic undergraduate algebra and number theory background, introduces students to the field of algebraic number theory in a—true to its title—conversational, well-motivated, and accessible way. In structure and in tone, the book makes you feel like you are hearing a captivating live lecture in real time.

The book focuses on the classical ideas of the eighteenth- and nineteenth-century giants Gauss, Dirichlet, Dedekind, and others that are now fundamental to research in algebraic number theory. Some of the main topics are: unique factorization of ideals, finiteness of the class number, and Dirichlet's unit theorem, and allusions are made to the modern theory in the exercises at the end of each section.

Potential instructors will appreciate the way the book is organized. The titles of chapters and sections work as teasers to spark the students' curiosity and start a train of thought. Chapters, some only five pages long, begin with an example, a general scenario, or a sequence of definitions, expand to a discussion and statements of important results, and finish with useful exercises that fill out the details.

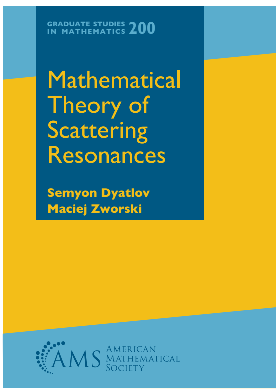
In the first half of the book, Pollack covers the material solely in the context of quadratic field extensions of the rational numbers. He then follows this by a redo in the full setting of finite number fields. This overarching framework provides extra cohesion to the bite-size pieces presented in each chapter, and adds to the book's effectiveness as a textbook for an advanced undergraduate class or as a book for self-study.

Gauss famously called number theory the "Queen of Mathematics." This book provides an inviting entryway to this beautiful subject.

The AMS BookShelf is prepared bimonthly by AMS Senior Editor Eriko Hironaka. Her email address is exh@ams.org.

New Books Offered by the AMS

Differential Equations



Mathematical Theory of Scattering Resonances

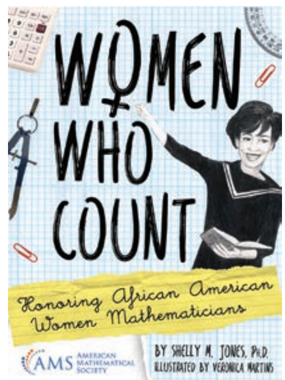
Semyon Dyatlov, *University of California, Berkeley, CA, and MIT, Cambridge, MA*, and **Maciej Zworski**, *University of California, Berkeley, CA*

Scattering resonances replace discrete spectral data for problems in which escape to infinity (scattering) is possible: they are complex numbers with real parts corresponding to rates of oscillation and imaginary parts to rates of decay. This book provides a general introduction in the simplest setting of compactly supported perturbations and is an introduction to modern literature on the subject. *This item will also be of interest to those working in mathematical physics*

Graduate Studies in Mathematics, Volume 200
September 2019, approximately 631 pages, Hardcover, ISBN: 978-1-4704-4366-5, LC 2019006281, 2010 *Mathematics Subject Classification:* 58J50, 35P25, 34L25, 35P20, 35S05, 81U20, 81Q12, 81Q20, List US$95, **AMS members US$76**, **MAA members US$85.50**, Order code GSM/200

https://bookstore.ams.org/gsm-200

General Interest



Women Who Count

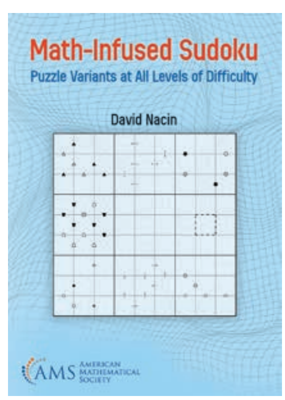
Honoring African American Women Mathematicians

Shelly M. Jones, *Central Connecticut State University, New Britain, CT*

Women Who Count: Honoring African American Women Mathematicians is a children's activity book highlighting the lives and work of 29 African American women mathematicians, including Dr. Christine Darden, Mary Jackson, Katherine Johnson, and Dorothy Vaughan from the award-winning book and movie *Hidden Figures*. It is a must-read for parents and children alike.

August 2019, 143 pages, Softcover, ISBN: 978-1-4704-4889-9, LC 2019013589, 2010 *Mathematics Subject Classification:* 97A30, 97U99, List US$15, **AMS members US$12**, **MAA members US$13.50**, Order code MBK/124

https://bookstore.ams.org/mbk-124



Math-Infused Sudoku

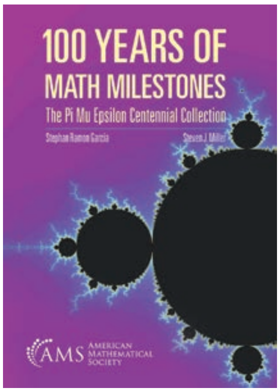
Puzzle Variants at All Levels of Difficulty

David Nacin, *William Paterson University, Wayne, NJ*

Building upon the rules of Sudoku, the puzzles in this volume introduce new challenges by adding clues involving sums, differences, means, divisibility, and more. From beginners trying Sudoku for the first time to experts craving something new, this innovative collection will delight and satisfy puzzle lovers of all ages and levels!

July 2019, 116 pages, Softcover, ISBN: 978-1-4704-5090-8, LC 2019011735, 2010 *Mathematics Subject Classification:* 97A20, 00A08, List US$25, **AMS members US$20**, **MAA members US$22.50**, Order code MBK/123

bookstore.ams.org/mbk-123



100 Years of Math Milestones

The Pi Mu Epsilon Centennial Collection

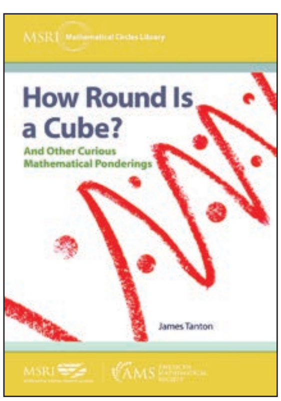
Stephan Ramon Garcia, *Pomona College, Claremont, CA*, and **Steven J. Miller**, *Wiiliams College, Williamstown, MA*

This book is an outgrowth of a collection of 100 problems chosen to celebrate the 100th anniversary of the undergraduate math honor society Pi Mu Epsilon. Each chapter describes a problem or event, the progress made, and connections to entries from other years or other parts of mathematics.

July 2019, 581 pages, Softcover, ISBN: 978-1-4704-3652-0, LC 2019000982, 2010 *Mathematics Subject Classification:* 00A08, 00A30, 00A35, 05–01, 11–01, 30–01, 54–01, 60–01, List US$60, **AMS members US$48**, **MAA members US$54**, Order code MBK/121

https://bookstore.ams.org/mbk-121

Math Education



How Round Is a Cube?

And Other Curious Mathematical Ponderings

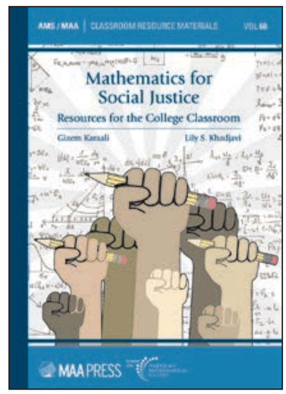
James Tanton, *Mathematical Association of America, Washington, DC*

Intended for the general math enthusiast, this book is a collection of 34 curiosities, each a quirky and delightful gem of mathematics and a shining example of the joy and surprise that mathematics can bring. Each essay begins with an intriguing puzzle, which either springboards into or unravels to become a wondrous piece of thinking. The essays are self-contained and rely only on tools from high-school mathematics (with only a few pieces that ever-so-briefly brush up against high-school calculus).

Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).

MSRI Mathematical Circles Library, Volume 23
August 2019, 262 pages, Softcover, ISBN: 978-1-4704-5115-8, LC 2019004260, 2010 *Mathematics Subject Classification:* 00–XX, List US$25, **AMS members US$20**, **MAA members US$22.50**, Order code MCL/23

https://bookstore.ams.org/mcl-23



Mathematics for Social Justice

Resources for the College Classroom

Gizem Karaali, *Pomona College, Claremont, CA*, and **Lily S. Khadjavi**, *Loyola Marymount University, Los Angeles, CA*, Editors

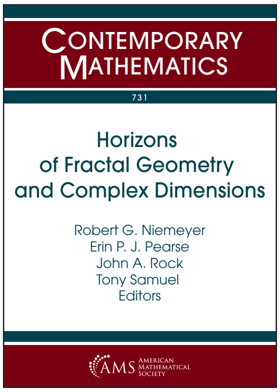
Mathematics for Social Justice offers a collection of resources for mathematics faculty interested in incorporating questions of social justice into their classrooms. The book begins with a series of essays from instructors experienced in integrating social justice themes into their pedagogy. The heart of the book is a collection of 14 classroom-tested modules featuring ready-to-use activities and investigations for the college mathematics classroom.

Classroom Resource Materials, Volume 60
July 2019, 277 pages, Softcover, ISBN: 978-1-4704-4926-1, LC 2019000897, 2010 *Mathematics Subject Classification:* 00–XX, List US$55, **AMS Individual member US$41.25**, AMS Institutional member US$44, **MAA members US$41.25**, Order code CLRM/60

https://bookstore.ams.org/clrm-60

New in Contemporary Mathematics

Analysis



Horizons of Fractal Geometry and Complex Dimensions

Robert G. Niemeyer, *Metropolitan State University of Denver, CO*, **Erin P. J. Pearse**, *California Polytechnic State University, San Luis Obispo, CA*, **John A. Rock**, *California State Polytechnic University, Pomona, CA*, and **Tony Samuel**, *University of Birmingham, Edgbaston, Birmingham, United Kingdom, and California Polytechnic State University, San Luis Obispo, CA*, Editors

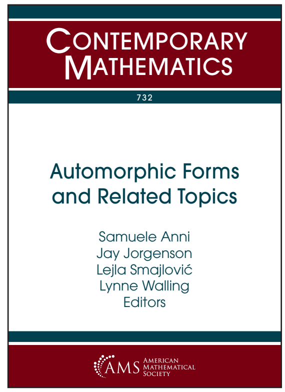
This volume contains the proceedings of the 2016 Summer School on Fractal Geometry and Complex Dimensions, in celebration of Michel L. Lapidus's 60th birthday, held from June 21–29, 2016, at California Polytechnic State University, San Luis Obispo, California.

This item will also be of interest to those working in geometry and topology.

Contemporary Mathematics, Volume 731
July 2019, approximately 304 pages, Softcover, ISBN: 978-1-4704-3581-3, 2010 *Mathematics Subject Classification:* 11M26, 26A30, 28D05, 30D10, 31A10, 35P20, 37B10, 52A39, 52C23, 60G50, List US$117, **AMS members US$93.60**, **MAA members US$105.30**, Order code CONM/731

https://bookstore.ams.org/conm-731

Number Theory



Automorphic Forms and Related Topics

Samuele Anni, *University of Luxembourg, Esch-sur-Alzette, Luxembourg*, **Jay Jorgenson**, *City College of New York, NY*, **Lejla Smajlović**, *University of Sarajevo, Bosnia and Herzegovina*, and **Lynne Walling**, *University of Bristol, United Kingdom*, Editors

This volume contains the proceedings of the Building Bridges: 3rd EU/US Summer School and Workshop on Automorphic Forms and Related Topics, which was held in Sarajevo from July 11–22, 2016. The articles summarize material which was presented during the lectures and speed talks during the workshop.

This item will also be of interest to those working in algebra and algebraic geometry.

Contemporary Mathematics, Volume 732
July 2019, 286 pages, Softcover, ISBN: 978-1-4704-3525-7, LC 2019005230, 2010 *Mathematics Subject Classification:* 11Fxx, 11Mxx, 11Gxx, 14Gxx, 14Kxx, 22E45, 22E50, 22E55, 22E57, 32Nxx, List US$117, **AMS members US$93.60**, **MAA members US$105.30**, Order code CONM/732

https://bookstore.ams.org/conm-732

NEW BOOKS

New in Memoirs of the AMS

Algebra and Algebraic Geometry

Algebraic Geometry over C^∞-Rings

Dominic Joyce, *University of Oxford, United Kingdom*

This item will also be of interest to those working in geometry and topology.

Memoirs of the American Mathematical Society, Volume 260, Number 1256
August 2019, 139 pages, Softcover, ISBN: 978-1-4704-3645-2, 2010 *Mathematics Subject Classification:* 58A40; 14A20, 46E25, 51K10, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/260/1256

https://bookstore.ams.org/memo-260-1256

Moufang Loops and Groups with Triality are Essentially the Same Thing

J. I. Hall, *Michigan State University, East Lansing*

Memoirs of the American Mathematical Society, Volume 260, Number 1252
August 2019, 188 pages, Softcover, ISBN: 978-1-4704-3622-3, 2010 *Mathematics Subject Classification:* 20–XX, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/260/1252

https://bookstore.ams.org/memo-260-1252

Analysis

Matrix Functions of Bounded Type: An Interplay Between Function Theory and Operator Theory

Raúl E. Curto, *University of Iowa, Iowa City*, **In Sung Hwang**, *Sungkyunkwan University, Suwon, Korea*, and **Woo Young Lee**, *Seoul National University, Korea*

Memoirs of the American Mathematical Society, Volume 260, Number 1253
August 2019, 100 pages, Softcover, ISBN: 978-1-4704-3624-7, 2010 *Mathematics Subject Classification:* 30J05, 30H10, 30H15, 47A13, 47A56, 47B20, 47B35; 30J10, 30J15, 30H35, 47A20, 47A57, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/260/1253

https://bookstore.ams.org/memo-260-1253

Differential Equations

On the Stability of Type I Blow Up for the Energy Super Critical Heat Equation

Charles Collot, *Université de Nice-Sophia Antipolis, France*, **Pierre Raphaël**, *Université de Nice-Sophia Antipolis, France*, and **Jeremie Szeftel**, *Université Paris 6, France*

Memoirs of the American Mathematical Society, Volume 260, Number 1255
August 2019, 93 pages, Softcover, ISBN: 978-1-4704-3626-1, 2010 *Mathematics Subject Classification:* 35B32, 35B35, 35B44, 35J61, 35K58, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/260/1255

https://bookstore.ams.org/memo-260-1255

Geometry and Topology

Spectral Invariants With Bulk, Quasi-Morphisms and Lagrangian Floer Theory

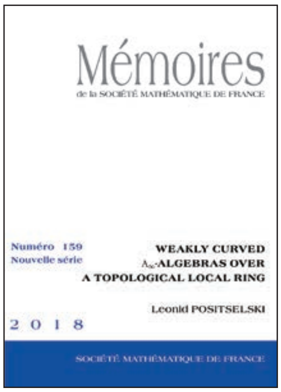
Kenji Fukaya, *Stony Brook University, New York, and Institute for Basic Sciences, Pohang, Korea*, **Yong-Geun Oh**, *Institute for Basic Sciences, Pohang, Korea*, **Hiroshi Ohta**, *Nagoya University, Japan*, and **Kaoru Ono**, *Kyoto University, Japan*

Memoirs of the American Mathematical Society, Volume 260, Number 1254
August 2019, 262 pages, Softcover, ISBN: 978-1-4704-3625-4, 2010 *Mathematics Subject Classification:* 53D40, 53D12, 53D45; 53D20, 14M25, 20F65, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/260/1254

https://bookstore.ams.org/memo-260-1254

New AMS-Distributed Publications

Algebra and Algebraic Geometry



Weakly Curved A_∞-Algebras Over a Topological Local Ring

Leonid Positselski, *Institute for Information Transmission Problems, Moscow, Russia*

The author defines and studies the derived categories of the first kind for curved DG- and A_∞-algebras complete over a pro-Artinian local ring with the curvature elements divisible by the maximal ideal of the local ring. He develops the Koszul duality theory in this setting and deduces the generalizations of the conventional results about A_∞-modules to the weakly curved case. The formalism of contramodules and comodules over pro-Artinian topological rings is used throughout the book. The author's motivation comes from the Floer-Fukaya theory.

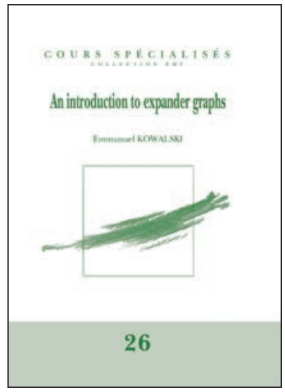
This item will also be of interest to those working in number theory.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Mémoires de la Société Mathématique de France, Number 159
March 2019, 201 pages, Softcover, ISBN: 978-2-85629-899-2, 2010 *Mathematics Subject Classification:* 16E35, 18E30, 13H99, 18G10, 53D37, 13D10, List US$67, **AMS members US$53.60**, Order code SMFMEM/159

https://bookstore.ams.org/smfmem-159

Applications



An Introduction to Expander Graphs

Emmanuel Kowalski, *Swiss Federal Institute of Technologie (ETH), Zürich, Switzerland*

Expander graphs are families of finite graphs that are simultaneously relatively sparse and highly connected. Since their discovery in the late 1960s, they have appeared in many seemingly unrelated areas of mathematics, from theoretical computer science to arithmetic and algebraic geometry, from representation theory to number theory.

The goal of this book is to present the theory of expander graphs and to explore some of these rich connections. Besides a careful exposition of the basic parts of the theory, including the Cheeger constant, random walks and spectral gap characterizations of expander graphs, it contains many different constructions of various families of expander graphs. The applications that are surveyed in the last chapter try to communicate the remarkable reach of expander graphs in modern mathematics.

This item will also be of interest to those working in discrete mathematics and combinatorics.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Cours Spécialisés—Collection SMF, Number 26
April 2019, 276 pages, Hardcover, ISBN: 978-2-85629-898-5, 2010 *Mathematics Subject Classification:* 05Cxx, 05C50, 05C81, 11C20, 11G30, 14H25, 22D10, 60J10, List US$75, **AMS members US$60**, Order code COSP/26

https://bookstore.ams.org/cosp-26

NEW BOOKS

General Interest

Séminaire Bourbaki: Volume 2016/2017 Exposés 1120–1135

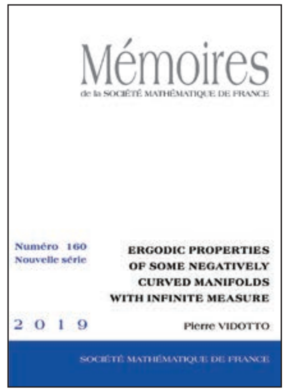
This 69th volume of the Bourbaki Seminar contains the texts of the fifteen survey lectures done during the year 2016/2017. Topics addressed include Langlands correspondence, NIP property in model theory, Navier–Stokes equations, algebraic and complex analytic geometry, algorithmic and geometric questions in knot theory, analytic number theory formal moduli problems, general relativity, sofic entropy, sphere packings, and subriemannian geometry.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 407
March 2019, 588 pages, Softcover, ISBN: 978-2-85629-897-8, 2010 *Mathematics Subject Classification:* 83C05, 57M25, 57N10, 68Q25, 32L20, 32Q20, 32L15, 14E08, 60G15, 68R10, 20B25, 20B15, 05E18, 18D50, 14D15, 11M32, 53D55, 53C23, 49Q05, 53C21, 32C38, 32S60, 18E35, 13F60, 17B10, 82B23, 81R50, 53D05, 11F72, 58J20, 11M36, 35H10, 58J52, 58J65, 37C30, 57R17, 53D15, 52C17, 11H31, 90B80, 11L15, 11L07, 11P55, 35A01, 35D30, 35Q30, List US$105, **AMS members US$84**, Order code AST/407

https://bookstore.ams.org/ast-407

Geometry and Topology



Ergodic Properties of Some Negatively Curved Manifolds with Infinite Measure

Pierre Vidotto, *Laboratoire Jean Leray, Nantes, France*

Let $M=X/\Gamma$ be a geometrically finite negatively curved manifold with fundamental group Γ acting on X by isometries. The purpose of this book is to study the mixing property of the geodesic flow on T^1M, the asymptotic behavior as R – M of the number of closed geodesics on M of length less than R and of the orbital counting function $\sharp\gamma\in\Gamma \,|\, d(\mathbf{o},\gamma\cdot\mathbf{o})\le R$.

These properties are well known when the Bowen-Margulis measure on T^1M is finite. The author considers here Schottky group $\Gamma=\Gamma_1*\Gamma_2*\cdots*\Gamma_k$ whose Bowen-Margulis measure is infinite and ergodic, such that one of the elementary factor Γ_i is parabolic with $\delta_{\Gamma_i}=\delta_\Gamma$.

The author specifies these ergodic properties using a symbolic coding induced by the Schottky group structure.

This item will also be of interest to those working in number theory.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Mémoires de la Société Mathématique de France, Number 160
March 2019, 132 pages, Softcover, ISBN: 978-2-85629-901-2, List US$52, **AMS members US$41.60**, Order code SMFMEM/160

https://bookstore.ams.org/smfmem-160

Meetings & Conferences of the AMS

August Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at: `www.ams.org/meetings`.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to page 127 in the January 2019 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit `www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Georgia Benkart, University of Wisconsin–Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; email: `benkart@math.wisc.edu`; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: `steve.weintraub@lehigh.edu`; telephone: 610-758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: `brian@math.uga.edu`; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; email: `lapidus@math.ucr.edu`; telephone: 951-827-5910.

Meetings in this Issue

See `www.ams.org/meetings` *for the most up-to-date information on the meetings and conferences that we offer.*

Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `https://www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

Madison, Wisconsin

University of Wisconsin–Madison

September 14–15, 2019

Saturday – Sunday

Meeting #1150

Central Section
Associate secretary: Georgia Benkart

Announcement issue of *Notices*: June 2019
Program first available on AMS website: July 23, 2019
Issue of *Abstracts*: Volume 40, Issue 3

Deadlines

For organizers: Expired
For abstracts: July 16, 2019

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Nathan Dunfield, University of Illinois, Urbana-Champaign, *Fun with finite covers of 3-manifolds: connections between topology, geometry, and arithmetic*.

Teena Gerhardt, Michigan State University, *Invariants of rings via equivariant homotopy*.

Lauren Williams, University of California, Berkeley, *Title to be announced* (Erdős Memorial Lecture).

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic and Geometric Combinatorics (Code: SS 12A), **Benjamin Braun**, University of Kentucky, **Marie Meyer**, Lewis University, and **McCabe Olsen**, Ohio State University.

Analysis and Probability on Metric Spaces and Fractals (Code: SS 10A), **Guy C. David**, Ball State University, and **John Dever**, Bowling Green State University.

Applications of Algebra and Geometry (Code: SS 38A), **Shamgar Gurevich** and **Jose Israel Rodriguez**, University of Wisconsin–Madison.

Arithmetic of Shimura Varieties (Code: SS 26A), **Chao Li**, Columbia University, and **Solly Parenti** and **Tonghai Yang**, University of Wisconsin–Madison.

Association Schemes and Related Topics – in Celebration of J.D.H. Smith's 70th Birthday (Code: SS 8A), **Kenneth W. Johnson**, Penn State University Abington, and **Sung Y. Song**, Iowa State University.

Automorphic Forms and L-Functions (Code: SS 16A), **Simon Marshall** and **Ruixiang Zhang**, University of Wisconsin–Madison.

Categorical Gromov-Witten Invariants and Mirror Symmetry (Code: SS 42A), **Andrei Caldararu**, University of Wisconsin–Madison, and **Junwu Tu**, University of Missouri-Columbia and Shanghai Tech University.

Classical and Geophysical Fluid Dynamics: Modeling, Reduction and Simulation (Code: SS 17A), **Nan Chen**, University of Wisconsin–Madison, and **Honghu Liu**, Virginia Tech University.

Combinatorial Algebraic Geometry (Code: SS 21A), **Juliette Bruce** and **Daniel Erman**, University of Wisconsin–Madison, **Chris Eur**, University of California Berkeley, and **Lily Silverstein**, University of California Davis.

Commutative Algebra: in Celebration of the 150th Birthday of Roger and Sylvia Wiegand (Code: SS 22A), **Nicholas Baeth**, Franklin & Marshall College, and **Graham Leuschke**, Syracuse University.

Computability Theory in honor of Steffen Lempp's 60th birthday (Code: SS 6A), **Joseph S. Miller**, **Noah D. Schweber**, and **Mariya I. Soskova**, University of Wisconsin–Madison.

Connecting Network Structure and Behavior of Biological Interaction Systems (Code: SS 31A), **David Anderson**, **Gheorghe Craciun**, and **Abhishek Deshpande**, University of Wisconsin–Madison.

Connections between Noncommutative Algebra and Algebraic Geometry (Code: SS 15A), **Jason Gaddis** and **Dennis Keeler**, Miami University.

Extremal Graph Theory (Code: SS 14A), **Józef Balogh**, University of Illinois, and **Bernard Lidický**, Iowa State University.

Floer Homology in Dimensions 3 and 4 (Code: SS 29A), **Jianfeng Lin**, UC San Diego, and **Christopher Scaduto**, University of Miami.

Fully Nonlinear Elliptic and Parabolic Partial Differential Equations, Local and Nonlocal (Code: SS 25A), **Fernando Charro**, Wayne State University, **Stefania Patrizi**, The University of Texas at Austin, and **Peiyong Wang**, Wayne State University.

Functional Analysis and Its Applications (Code: SS 30A), **Clement Boateng Ampadu**, Boston, MA, and **Waleed Al-Rawashdeh**, Montana Tech University.

Geometry and Topology in Arithmetic (Code: SS 41A), **Rachel Davis**, University of Wisconsin–Madison.

Geometry and Topology of Singularities (Code: SS 13A), **Laurentiu Maxim**, University of Wisconsin–Madison.

Hall Algebras, Cluster Algebras and Representation Theory (Code: SS 27A), **Xueqing Chen**, UW-Whitewater, and **Yiqiang Li**, SUNY at Buffalo.

Hodge Theory in Honor of Donu Arapura's 60th Birthday (Code: SS 11A), **Ajneet Dhillon**, University of Western Ontario, **Kenji Matsuki** and **Deepam Patel**, Purdue University, and **Botong Wang**, University of Wisconsin–Madison.

Homological and Characteristic $p > 0$ Methods in Commutative Algebra (Code: SS 1A), **Michael Brown**, University of Wisconsin–Madison, and **Eric Canton**, University of Michigan.

Homotopy Theory (Code: SS 34A), **Gabe Angelini-Knoll** and **Teena Gerhardt**, Michigan State University, and **Bertrand Guillou**, University of Kentucky.

Large Scale Properties of Interacting Stochastic Systems (Code: SS 33A), **Timo Seppalainen**, **Hao Shen**, and **Benedek Valko**, University of Wisconsin–Madison.

Lie Representation Theory (Code: SS 19A), **Mark Colarusso**, University of South Alabama, **Michael Lau**, Université Laval, and **Matt Ondrus**, Weber State University.

Model Theory (Code: SS 5A), **Uri Andrews** and **Omer Mermelstein**, University of Wisconsin–Madison.

Nonlinear Dispersive Equations and Water Waves (Code: SS 37A), **Mihaela Ifrim**, University of Wisconsin–Madison, and **Daniel Tataru**, University of California, Berkeley.

Number Theory and Cryptography (Code: SS 40A), **Eric Bach**, University of Wisconsin–Madison, and **Jon Sorenson**, Butler University.

Quasigroups and Loops – in honor of J.D.H. Smith's 70th birthday (Code: SS 35A), **J.D. Phillips**, Northern Michigan University, and **Petr Vojtechovsky**, University of Denver.

Recent Developments in Harmonic Analysis (Code: SS 3A), **Theresa Anderson**, Purdue University, and **Joris Roos**, University of Wisconsin–Madison.

Recent Trends in the Mathematics of Data (Code: SS 39A), **Sebastien Roch**, University of Wisconsin–Madison, **David Sivakoff**, Ohio State University, and **Joseph Watkins**, University of Arizona.

Recent Work in the Philosophy of Mathematics (Code: SS 4A), **Thomas Drucker**, University of Wisconsin–Whitewater, and **Dan Sloughter**, Furman University.

Relations Between the History and Pedagogy of Mathematics (Code: SS 32A), **Emily Redman**, University of Massachusetts, Amherst, **Brit Shields**, University of Pennsylvania, and **Rebecca Vinsonhaler**, University of Texas, Austin.

Several Complex Variables (Code: SS 7A), **Hanlong Fang** and **Xianghong Gong**, University of Wisconsin–Madison.

Special Functions and Orthogonal Polynomials (Code: SS 2A), **Sarah Post**, University of Hawaiʻi at Mānoa, and **Paul Terwilliger**, University of Wisconsin–Madison.

Stochastic Partial Differential Equations and Related Fields (Code: SS 28A), **Igor Cialenco**, Illinois Institute of Technology, **Yu Gu**, Carnegie Mellon University, and **Hyun-Jung Kim**, Illinois Institute of Technology.

Supergeometry, Poisson Brackets, and Homotopy Structures (Code: SS 36A), **Ekaterina Shemyakova**, University of Toledo, and **Theodore Voronov**, University of Manchester.

Topics in Graph Theory and Combinatorics (Code: SS 20A), **Songling Shan** and **Papa Sissokho**, Illinois State University.

Topology and Descriptive Set Theory (Code: SS 18A), **Tetsuya Ishiu** and **Paul B. Larson**, Miami University.

Uncertainty Quantification Strategies for Physics Applications (Code: SS 9A), **Qin Li**, University of Wisconsin–Madison, and **Tulin Kaman**, University of Arkansas.

Wave Phenomena in Fluids and Relativity (Code: SS 24A), **Sohrab Shahshahani**, University of Massachusetts, and **Willie W.Y. Wong**, Michigan State University.

Zero Forcing, Propagation, and Throttling (Code: SS 23A), **Josh Carlson**, Iowa State University, and **Nathan Warnberg**, University of Wisconsin–La Crosse.

Binghamton, New York

Binghamton University

October 12–13, 2019

Saturday – Sunday

Meeting #1151

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: August 2019

Program first available on AMS website: August 29, 2019

Issue of *Abstracts*: Volume 40, Issue 3

Deadlines

For organizers: Expired

For abstracts: August 20, 2019

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Richard Kenyon, Brown University, *What polygons can be tiled with squares?*

Tony Pantev, University of Pennsylvania, *Geometry and topology of wild character varieties*.

Lai-Sang Young, New York University, *A dynamical model for controlling of infectious diseases via isolation*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Combinatorics on the Occasion of the 75th Birthday of Thomas Zaslavsky (Code: SS 13A), **Nathan Reff**, State University of New York, The College at Brockport, and **Lucas Rusnak**, Texas State University.

Analysis and Applications of Deterministic and Stochastic Evolution Equations (Code: SS 10A), **Vincent Martinez**, City University of New York, Hunter College, and **Kazuo Yamazaki**, University of Rochester.

Commutative Algebra (Code: SS 9A), **Bethany Kubik**, University of Minnesota, Duluth, and **Denise Rangel Tracy**, Central Connecticut State University.

Effective and Quantitative Advances in Low Dimensional Topology and Geometric Group Theory (Code: SS 3A), **Jenya Sapir**, Binghamton University, and **Edgar Bering**, Temple University.

Group Actions on Manifolds and Related Spaces (Code: SS 12A), **Thomas Koberda**, University of Virginia, **Yash Lodha**, École Polytechnique Féderalé de Lausanne, Switzerland, and **Matt Zaremsky**, University at Albany, State University of New York.

Groups and Their Representations (Code: SS 1A), **Jamison Barsotti** and **Rob Carman**, College of William and Mary, and **Daniel Rossi** and **Hung P. Tong-Viet**, Binghamton University.

Homotopy Theory and Algebraic K-theory (Code: SS 5A), **Cary Malkiewich**, Binghamton University, **Marco Varisco**, University at Albany, and **Inna Zakharevich**, Cornell University.

Invariants of Knots, Links, and Low-dimensional Manifolds (Code: SS 15A), **Moshe Cohen**, Vassar College, **Adam Giambrone**, Elmira College, **Adam Lowrance**, Vassar College, and **Jonathan Williams**, Binghamton University.

Operator Theory and Complex Analysis (Code: SS 17A), **Gabriel T. Prajitura** and **Ruhan Zhao**, College at Brockport, SUNY.

Oriented Matroids and Related Topics (Code: SS 7A), **Laura Anderson**, **Michael Dobbins**, and **Benjamin Schroeter**, Binghamton University.

Percolation, Random Graphs, and Random Geometry (Code: SS 11A), **Shishendu Chatterjee** and **Jack Hanson**, City University of New York, City College.

Recent Trends in Geometrical PDEs and Mathematical Physics (Code: SS 6A), **Xiangjin Xu** and **Gang Zhou**, Binghamton University.

Representations of Lie Algebras, Vertex Operators, and Related Topics (Code: SS 2A), **Alex Feingold**, Binghamton University, and **Christopher Sadowski**, Ursinus College.

Statistics (Code: SS 14A), **Sanjeena Dang**, **Aleksey Polunchenko**, **Xingye Qiao**, and **Anton Schick**, Binghamton University.

Stochastic Evolution of Discrete Structures (Code: SS 8A), **Vladislav Kargin**, Binghamton University.

What's New in Group Theory? (Code: SS 4A), **Luise-Charlotte Kappe**, Binghamton University, and **Justin Lynd** and **Arturo Magidin**, University of Louisiana at Lafayette.

p-adic Analysis in Number Theory (Code: SS 16A), **C. Douglas Haessig**, University of Rochester, and **Rufei Ren**.

Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include an 8% NY state tax and a 5% occupancy tax. Participants must state that they are with the **AMS Fall Eastern Sectional Meeting/American Mathematical Society (AMS) Meeting at Binghamton University** to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. **Hotels have varying cancellation and early checkout penalties; be sure to ask for details.**

Binghamton Fairfield Inn by Marriott, 864 Upper Front Street, Binghamton, NY 13905; (607) 651-1000, `https://www.marriott.com/hotels/travel/bgmfi-fairfield-inn-binghamton`. Rates are US$119 per night for a room with one king bed or two double beds; this rate is applicable for single or double occupancy. To reserve a room at these rates please call the property and identify your affiliation with the American Mathematical Society Fall Eastern Sectional Meeting. Amenities include complimentary wireless Internet, fitness center, indoor pool, refrigerator, on-site convenience store, and complimentary full breakfast. Complimentary parking is available on-site. Check-in is at 3:00 pm, check-out is at noon. This property is located approximately 6 miles from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **September 27**.

Hampton Inn of Binghamton/Johnson City, 630 Field Street, Johnson City, NY 13790; (607) 729-9125; `https://hamptoninn3.hilton.com/en/hotels/new-york/hampton-inn-binghamton-johnson-city-BGMNYHX/index.html`. Rates are US$127 per night for a guest room with double occupancy, with two double or one king sized bed. To reserve a room by phone please call the toll-free reservations number (800) Hampton and identify the group code **AMS**. Reservations can also be made online using this link: `https://hamptoninn.hilton.com/en/hp/groups/personalized/B/BGMNYHX-AMS-20191011/index.jhtml?WT.mc_id=POG`. Amenities include complimentary wireless internet access, indoor pool, fitness center, business center, and complimentary hot breakfast or grab n' go breakfast. Complimentary parking is available on-site. Check-in is at 3:00 pm, check-out is at noon. This property is located approximately 3 miles from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **September 20.**

Hampton Inn & Suites – Binghamton/Vestal, 3708 Vestal Parkway East, Vestal, NY; (607) 797-5000, `https://hamptoninn3.hilton.com/en/hotels/new-york/hampton-inn-and-suites-binghamton-vestal-BGMHSHX/index.html?WT.mc_id=1HX2RE3Hotel4OneTagSol`. Rates are US$119 per night for a room with one king bed or two double beds; this rate is applicable for single or double occupancy. Additional adult guests will be charged at a rate of US$10 per night. To reserve a room over the phone please contact the hotel directly and identify the block of rooms for the **American Mathematical Society Fall Eastern Sectional Meeting**. Reservations can also be made online using this link: `https://secure3.hilton.com/en_US/hp/reservation/book.htm?inputModule=HOTEL&ctyhocn=BGMHSHX&spec_plan=CHHAMC&arrival=20191011&departure=20191013&cid=OM,WW,HILTONLINK,EN,DirectLink&fromId=HILTONLINKDIRECT`. All rooms include a refrigerator, microwave, and coffeemaker. Amenities include free wireless Internet access, fitness center, indoor pool, 24-hour business center, and coin laundry on-property. Complimentary parking is available on-site. A full American breakfast buffet is included in this rate. Check-in is at 3:00 pm, check-out is at 11:00 am. This property is located approximately 1.5 miles from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **September 20.**

Holiday Inn Binghamton, 2–8 Hawley Street, Binghamton, NY, 13901; (607) 722-1212, https://www.holidayinnbinghamton.com. Rates are US$149 per night for a room with a king or two double beds. To reserve a room by phone please call the hotel and identify the block of rooms for the **American Mathematical Society Fall Eastern Sectional Meeting**. Amenities include in-room refrigerator and microwave, complimentary high-speed internet access, fitness center, indoor swimming pool, business center, Cafe Select, on-site restaurant serving breakfast, lunch, and dinner, a game room, and a cocktail lounge. Check-in is at 3:00 pm, check-out is at 11:00 am. This property is located approximately 3.8 miles from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **September 11**.

Quality Inn and Suites, 4105 Vestal Parkway East, Vestal, NY 13850; (607) 729-6371, https://www.qualityinnvestal.com. Rates are US$94.95 for a guest room with two double beds, with single or double occupancy. To reserve a room at this rate, please call the hotel and ask for the group rate for the **American Mathematical Society Fall Eastern Sectional Meeting**. Amenities include microwave and refrigerator, complimentary wireless internet access, fitness center, on-property Fuji San restaurant, on-site pub Johnnie's Tavern, and complimentary breakfast each day. Check-in is at 4:00 pm, check-out is at noon. This property is located less than one mile from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations in this block is **September 11.**

Housing Warning

Please beware of aggressive housing bureaus that target potential attendees of a meeting. They are sometimes called "room poachers" or "room-block pirates" and these companies generally position themselves as a meeting's housing bureau, convincing attendees to unknowingly book outside the official room block. They call people who they think will more likely than not attend a meeting and lure them with room rates that are significantly less than the published group rate—for a limited time only. And people who find this offer tempting may hand over their credit card data, believing they have scored a great rate and their housing is a done deal. Unfortunately, this often turns out to be the start of a long, costly nightmare.

Note that some of these room poachers create fake websites on which they represent themselves as the organizers of the meeting and include links to book rooms, etc. The only official website for this meeting is https://www.ams.org and one that has the official AMS logo.

These housing bureaus are not affiliated with the American Mathematical Society or any of its meetings, in any way. The AMS would never call anyone to solicit reservations for a meeting. The only way to book a room at a rate negotiated for an AMS Sectional Meeting is via a listing on AMS Sectional Meetings pages or *Notices of the AMS*. The AMS cannot be responsible for any damages incurred as a result of hotel bookings made with unofficial housing bureaus.

Food Services

On Campus: There are many options for dining in dining halls and at retail outlets throughout the campus.

The three dining halls located closest in walking distance from the meeting are Hinman Dining Center, Appalachian Collegiate Center, and the College in the Woods Dining Center. The Chenango Champlain Collegiate Center is located slightly further away and features a traditional dining hall as well as Kosher dining options. At the time of publication, all dining halls are set to have hours of operation during the days of the meeting.

- The **MarketPlace** located in University Union offers several outlets for purchasing food. Some options include:
- **Chick-N-Bap** serving a variety of rice bowls (open 4:00 pm–8:00 pm)
- **CopperTop Pizzeria** serving pizza, salads, and pasta (open 12 noon–10:00 pm)
- **Hissho Sushi** serving sushi and Japanese fare (open 11:00 am–8:00 pm)
- Additional options may be available at the **MarketPlace** in the fall of 2019, please check the website for more details.
- Outlets located in **Hinman Dining Hall** on the first floor include Starbucks (open 11:00 am–10:00 pm) and Subway (open 1:00 pm–midnight).

Off Campus: Some dining options in Binghamton include:

- **Cacciatore's**, 119 Harry L Drive, Johnson City, (607) 798-7699; https://www.cacciatores-restaurant.com; Saturday 4:00 pm–10:00 pm, closed Sunday; family-owned Italian restaurant.
- **Copper Top Tavern Vestal**, 4700 Vestal Pkwy E, Vestal, (607) 729-0164, https://www.coppertoptavern.com; Saturday 11:30 am–11:00 pm and Sunday 12 noon–10:00 pm; local restaurant menu that offers an eclectic blend of American dishes with European and Asian influences.

- **Dos Rios Cantina,** 60 Court Street, (607) 235-5322, `https://dosrioscantina.com`; Saturday 11:00 am–3:00 am and Sunday 11:00 am–10:00 pm; a modern twist on traditional Mexican cuisine.
- **Fuji San Japanese Restaurant**, 4105 Vestal Pkwy E, (607) 797-9888; Saturday and Sunday, 12 noon–10:00 pm; serving sushi and Japanese fare.
- **Jane's Diner**, 591 Conklin Rd, (607) 722-3350; Saturday and Sunday 6:00 am–8:00 pm; traditional diner, vegetarian friendly.
- **Jimmy John's**, 3951 Vestal Pkwy E, (607) 304-2380, `https://locations.jimmyjohns.com/ny/vestal`; Saturday and Sunday, 11:00 am–10:00 pm; chain restaurant featuring sandwiches made to order.
- **The Lost Dog Cafe,** 222 Water Street, Ste 101, (607) 771-6063, `https://www.lostdogcafe.net`; Saturday 11:30 am–10:00 pm, Sunday closed; eclectic cafe serving locally sourced options.
- **Lupo's S & S Char Pit**, 6 W State Street, (860) 310-3269, `https://www.lupossscharpit.com`; Saturday 9:00 am–10:00 pm, Sunday 9:00 am–8:00 pm; home of the char-grilled Spiedie sub and sandwiches since 1951.
- **Panera Bread**, 3700 Vestal Pkwy E, Vestal, (607) 797-2163; `https://locations.panerabread.com/ny/vestal/3700-vestal-parkway-east`; Saturday and Sunday, 7:00 am–10:00 pm; Fast casual bakery-cafe serving sandwiches, soups, salads, and bakery items.
- **Social on State**, 201 State Street, (607) 296-4017, `https://socialonstate.com`; Saturday 4:00 pm–3:00 am and Sunday 5:00 pm–1:00 am; serving small plates for sharing featuring American and Spanish fare.
- **Taj Restaurant**, 59 Main St, Ste 2, (607) 723-6454, `https://www.tajbinghamton.com`; Saturday 11:00 am–9:30 pm and Sunday noon–8:30 pm; serving Indian cuisine.
- **Water Street Brewing Co.,** 168 Water Street, (607) 217-4546, `https://www.waterstreetbrewingco.com`; Saturday 12 noon–9:00 pm and Sunday 12 noon–8:00 pm; craft brewpub featuring American fare.

For more information on dining, activities, and events in and around Binghamton please visit the Visit Bing website `https://visitbinghamton.org`.

Registration and Meeting Information

Advance Registration: Advance registration for this meeting will open on **July 22, 2019**. Advance registration fees will be US$71 for AMS members, US$115 for nonmembers, US$13 for students and unemployed mathematicians, and US$15 for emeritus members. Fees will be payable by cash, check, or credit card. Participants may cancel registrations made in advance by emailing `mmsb@ams.org`. 100% refunds will be issued for any advance registrations canceled by the first day of the meeting. After this date, no refunds will be issued.

On-site Information and Registration: The registration desk and coffee service will be located in the lobby of the Lecture Hall building, in front of Lecture Hall 2. The AMS book exhibit will be located in Lecture Hall 3. Special Sessions and Contributed Paper Sessions will take place in classrooms in the Lecture Hall building as well as the Classroom Wing. The Invited Addresses will be held in Lecture Hall 2.

The registration desk will be open on Saturday, October 12, 7:30 am–4:00 pm and Sunday, October 13, 8:00 am–12:00 pm. The same fees apply for on-site registration as for advance registration. Fees are payable on-site via cash, check, or credit card.

Other Activities

Book Sales: Stop by the on-site AMS bookstore to review the newest publications and take advantage of exhibit discounts and free shipping on all on-site orders! AMS and MAA members receive 40% off list price. Nonmembers receive a 25% discount. Not a member? Ask a representative about the benefits of AMS membership.

AMS Editorial Activity: An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you wish to discuss with the AMS, please stop by the book exhibit.

Membership Activities: Renew your AMS membership or join to receive member benefits, such as additional discounts on books purchased at meetings, free subscriptions to *Notices*, *Bulletin*, and Headlines & Deadlines e-newsletter, discounted registration for world-class meetings and conferences, and more!

Complimentary refreshments will be served courtesy, in part, of the AMS Membership Department.

Special Needs

It is the goal of the AMS to ensure that its conferences are accessible to all, regardless of disability. The AMS will strive, unless it is not practicable, to choose venues that are fully accessible to the physically handicapped.

If special needs accommodations are necessary in order for you to participate in an AMS Sectional Meeting, please communicate your needs in advance to the AMS Meetings Department by:

- Registering early for the meeting,
- Checking the appropriate box on the registration form, and
- Sending an email request to the AMS Meetings Department at `mmsb@ams.org` or `meet@ams.org`.

There are 18 buildings housing gender inclusive restrooms on campus. The closest gender inclusive restrooms to the meeting are located in the Library and in the Science 1 building. For a complete list of gender inclusive/gender neutral restrooms please visit the BU website: `https://www.binghamton.edu/multicultural-resource-center/resources/genderneutral.html`.

AMS Policy on a Welcoming Environment

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, or veteran status.

Local Information and Maps

This sectional will take place at Binghamton University, State University of New York, on the Vestal Campus (the main campus) located in the southern tier of upstate New York, just one mile west of the city of Binghamton.

A campus map can be found at `https://www.binghamton.edu/maps/index.html`. Information about the BU Department of Mathematical Sciences can be found at `https://www.binghamton.edu/math`. The meeting will occupy space in two adjoining buildings on campus, Lecture Hall and Classroom Wing.

Parking

On the weekends from 4:00 pm Friday until 7:00 am Monday, no parking permit is needed to park in a valid parking space on campus. Paid spaces and all other parking rules and regulations are still in effect, including permit designation and overnight parking rules. See the visitor parking webpage for more information on short-term parking options: `https://www.binghamton.edu/services/transportation-and-parking/parking/visitor-parking`.

Travel

Binghamton University, State University of New York is located just outside of Binghamton, NY. The Greater Binghamton Airport is the closest airport to the University, however service is limited from this airport. Syracuse Hancock International Airport and Wilkes-Barre/Scranton Airport, although located further from campus, are larger airports, offering more service.

By Air:

The **Greater Binghamton Airport** is the closest airport to Binghamton University. Greater Binghamton Airport (BGM) is located approximately 10 miles from the University in Johnson City, NY. The only airline servicing BGM is Delta Airlines.

Rental cars are available at BGM. Agencies available at this airport include Avis, Budget, and Hertz.

Taxi service is available from the airport. For more information on this airport, please visit the BGM website here: `flybgm.com/index.html`.

Syracuse Hancock International Airport (SYR) is located approximately 75 miles from the University, or approximately 90 minutes driving time. SYR is served by Allegiant Airlines, American Airlines, Delta Airlines, Frontier Airlines, JetBlue, and United Airlines, as well as additional affiliate airlines.

Rentals available at SYR include Avis, Budget, Dollar, Enterprise, Hertz, and National/Alamo. For more information on this airport, please visit the SYR website here: `https://www.syrairport.org`.

Wilkes-Barre/Scranton Airport (AVP) is located approximately 75 miles from the University, or approximately 90 minutes driving time. AVP is served by American Airlines, Delta Airlines, Southern Airways Express, and United Airlines, as well as additional affiliate airlines.

Rentals available at AVP include Avis, Budget, Dollar, Enterprise, Hertz, and National/Alamo. For more information on this airport, please visit the AVP website here: `https://flyavp.com`.

By Bus:

The Binghamton area is served by three bus lines including Trailways, Greyhound, and ShortLine. All bus lines utilize the Binghamton Transportation located at 81 Chenango Street in Binghamton ((800) 776-7548). For more information on Greyhound service please visit their website at `https://www.greyhound.com`. For more information on Trailways service please visit their website at `https://trailwaysny.com`. For more information on ShortLine service please visit their website at `https://web.coachusa.com/shortline/index.asp?nt=0`.

By Car:

Participants driving should use this address when utilizing a GPS: Binghamton University, 4400 Vestal Parkway East, Vestal, NY 13850.

From Boston: Take I-93 S to I-90 W, and I-88 W to NY 201 S in Johnson City. Take exit 70S from NY-17 W. Continue on NY-201 S. Drive to NY-434 E in Vestal. This drive is approximately 311 miles or 5 hours in duration.

From New York City: Take Church Street to I-78 W in Jersey City. Take exit 70S from NY-17 W. Take I-280 W, I-80 W, I-380 N, and I-81 N to NY-201 S in Johnson City. Take exit 70S from NY-17 W. Continue on NY-201 S. Drive to NY-434 E in Vestal. This drive is approximately 189 miles or 3 hours, 15 minutes in duration.

From Albany: Get on I-787 N from N Pearl Street. Follow I-90 W and I-88 W to NY-201 S in Johnson City. Take exit 70S from NY-17 W. Continue on NY-201 S. Drive to NY-434 E in Vestal. This drive is approximately 144 miles or 2 hours, 15 minutes in duration.

Local Transportation

Car Rental: Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at `www.hertz.com`, click on the box "I have a discount," and type in our convention number (CV): 04N30009. You can also call Hertz directly at (800) 654-2240 (US and Canada) or (405) 749-4434 (other countries). At the time of your reservation, the meeting rates will be automatically compared to other Hertz rates and you will be quoted the best comparable rate available. Meeting rates include unlimited mileage and are subject to availability. Advance reservations are recommended, blackout dates may apply.

Taxi Service: Licensed, metered taxis are available in Binghamton. For up-to-date information on cab service in and around BU please visit `https://visitbinghamton.org/taxis`.

Both Lyft and Uber also operate in the Greater Binghamton area.

Bus Service: *B.C. Transit* is the Broome County Department of Public Transportation Bus Line. CTtransit Hartford operates over 18 bus routes in and around the Binghamton metropolitan area. Many local routes operate 7 days a week. For additional information about bus fares and schedules please visit the *B.C. Transit* website at `http://gobroomecounty.com/transit`.

Weather

The average high temperature for October is approximately 55 degrees Fahrenheit and the average low is approximately 38 degrees Fahrenheit. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Social Networking

Attendees and speakers are encouraged to tweet about the meeting using the hashtag #AMSmtg.

Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the US found at `https://travel.state.gov/content/travel/en.html`. If you need a preliminary conference invitation in order to secure a visa, please send your request to `mac@ams.org`.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of "binding" or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- family ties in home country or country of legal permanent residence
- property ownership
- bank accounts
- employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;

* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

* Include a letter of invitation from the meeting organizer or the US host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

Gainesville, Florida

University of Florida

November 2–3, 2019

Saturday – Sunday

Meeting #1152

Southeastern Section
Associate secretary: Brian D. Boe

Announcement issue of *Notices*: September 2019
Program first available on AMS website: September 19, 2019
Issue of *Abstracts*: Volume 40, Issue 4

Deadlines

For organizers: Expired
For abstracts: September 10, 2019

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Jonathan Mattingly, Duke University, *Quantifying Gerrymandering: revealing the geopolitical geometry of redistricting*.

Isabella Novik, University of Washington, *Face numbers: centrally symmetric spheres vs centrally symmetric polytopes*.

Eduardo Teixeira, University of Central Florida, *Geometric regularity theory for diffusive processes and their intrinsic free boundaries*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Cěch-Stone Compactification of Semigroups: Algebra, Topology, Dynamics, and Combinatorics (Code: SS 17A), **Dana Bartošová** and **Jindřich Zapletal**, University of Florida.

Algebras, Analysis and Physics (Code: SS 6A), **Craig A. Nolder**, Florida State University, **Carmen Judith Vanegas Espinoza**, Technical University of Manabi (Ecuador), and **Soren Krausshar**, Universitat Erfurt (Germany).

Analysis of Geometric and Evolutionary PDEs (Code: SS 16A), **Yi Hu**, **Yongki Lee**, **Yuanzhen Shao**, and **Shijun Zheng**, Georgia Southern University.

Applications of Differential Equations in Mathematical Biology (Code: SS 25A), **Nehal Shukla**, Columbus State University.

Applied Topology: Theory and Applications (Code: SS 12A), **Peter Bubenik**, University of Florida, and **Nataša Jonoska**, University of South Florida.

Combinatorial Lie Theory (Code: SS 3A), **Erik Insko**, Florida Gulf Coast University, **Martha Precup**, Washington University in St. Louis, and **Edward Richmond**, Oklahoma State University.

Crystallographic and Highly Symmetric Structures (Code: SS 15A), **Milé Krajčevski** and **Gregory McColm**, University of South Florida.

Effective Equations of Quantum Physics (Code: SS 18A), **Israel Michael Sigal**, University of Toronto, and **Avy Soffer**, Rutgers University.

Experimental Mathematics in Number Theory and Combinatorics (Code: SS 20A), **Hannah Burson**, University of Illinois at Urbana-Champaign, **Tim Huber**, University of Texas, Rio Grande Valley, and **Armin Straub**, University of South Alabama.

Extremal and Probabilistic Combinatorics (Code: SS 19A), **Linyuan Lu**, University of South Carolina, and **Yi Zhao**, Georgia Southern University.

Fractal Geometry and Dynamical Systems (Code: SS 2A), **Mrinal Kanti Roychowdhury**, University of Texas Rio Grande Valley.

Geometric Structures on Manifolds (Code: SS 14A), **Sam Ballas**, Florida State University, **Luca Di Cerbo**, University of Florida, and **Kate Petersen**, Florida State University.

Geometric and Topological Combinatorics (Code: SS 1A), **Bruno Benedetti**, University of Miami, **Steve Klee**, Seattle University, and **Isabella Novik**, University of Washington.

Geometry of Gauge Theoretic Moduli Spaces (Code: SS 23A), **Chris Kottke**, New College of Florida, and **Ákos Nagy**, Duke University.

Homological Methods in Algebra (Code: SS 11A), **Luigi Ferraro** and **W. Frank Moore**, Wake Forest University.

New Developments in Mathematical Biology (Code: SS 21A), **Maia Martcheva**, University of Florida, **Necibe Tuncer**, Florida Atlantic University, and **Libin Rong**, University of Florida.

Nonlinear Elliptic Partial Differential Equations (Code: SS 13A), **Mark Allen**, Brigham Young University, and **Eduardo V. Teixeira**, University of Central Florida.

Nonlinear PDEs in Fluid Dynamics (Code: SS 10A), **Ming Chen**, University of Pittsburgh, **Aseel Farhat**, Florida State University, and **Cheng Yu**, University of Florida.

Nonlinear Solvers and Acceleration Methods (Code: SS 9A), **Sara Pollock**, University of Florida, and **Leo Rebholz**, Clemson University.

Partition Theory and Related Topics (Code: SS 22A), **Dennis Eichhorn**, University of California, Irvine, **Frank Garvan**, University of Florida, and **Brandt Kronholm**, University of Texas, Rio Grande Valley.

Patterns in Permutations (Code: SS 7A), **Miklós Bóna** and **Vince Vatter**, University of Florida.

Probabilistic and Geometric Tools in High-Dimension (Code: SS 4A), **Arnaud Marsiglietti**, University of Florida, and **Artem Zvavitch**, Kent State University.

Recent Progress in Operator Theory (Code: SS 8A), **Mike Jury**, **Scott McCullough**, and **James Pascoe**, University of Florida.

Recent Trends in Extremal Graph Theory (Code: SS 24A), **Theodore Molla** and **Brendan Nagle**, University of South Florida.

Topological Complexity and Related Topics (Code: SS 5A), **Daniel C. Cohen**, Louisiana State University, and **Alexander Dranishnikov** and **Yuli B. Rudyak**, University of Florida.

Riverside, California

University of California, Riverside

November 9–10, 2019

Saturday – Sunday

Meeting #1153

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: September 2019

Program first available on AMS website: September 12, 2019

Issue of *Abstracts*: Volume 40, Issue 4

Deadlines

For organizers: Expired

For abstracts: September 3, 2019

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Robert Boltje, University of California, Santa Cruz, *Title to be announced*.

Jonathan Novak, University of California, San Diego, *Title to be announced*.

Anna Skripka, University of New Mexico, Albuquerque, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

AWM, with Emphasis on Geometry and Dynamics (Code: SS 13A), **Weitao Chen**, **Savanna Gee**, **Paige Helms**, and **Qixuan Wang**, University of California, Riverside.

Advances in Functional Analysis (Code: SS 17A), **Marat Markin**, California State University, Fresno, and **Yunied Puig De Dios**, University of California, Riverside.

Advances in Operator Algebras (Code: SS 31A), **Scott Atkinson**, Vanderbilt University, **Rolando de Santiago**, UCLA, and **Feng Xu**, UC Riverside.

Algebraic and Combinatorial Structures in Knot Theory (Code: SS 6A), **Jieon Kim**, Pusan National University, and **Sam Nelson**, Claremont McKenna College.

Analysis of Nonlinear Partial Differential Equations and Applications (Code: SS 9A), **Nam Q. Le**, Indiana University, Bloomington, and **Connor Mooney**, University of California, Irvine.

Applied Category Theory (Code: SS 12A), **John Baez** and **Joe Moeller**, University of California, Riverside.

Applied Partial Differential Equations and Optimization (Code: SS 24A), **Yat Tin Chow** and **Amir Moradifam**, University of California, Riverside.

Arithmetic Geometry in Finite Characteristic (Code: SS 22A), **Nathan Kaplan** and **Vlad Matei**, University of California Irvine.

Canonical Bases, Cluster Structures and Non-commutative Birational Geometry (Code: SS 11A), **Arkady Berenstein**, University of Oregon, Eugene, **Jacob Greenstein**, University of California, Riverside, and **Vladimir Retakh**, Rutgers University.

Celebrating MM Rao's Many Mathematical Contributions as he Turns 90 Years Old (Code: SS 27A), **Jerome Goldstein**, University of Memphis, and **Michael Green**, **Alan Krinik**, **Randall J. Swift**, and **Jennifer Switkes**, California State Polytechnic University, Pomona.

Computational Methods in Hyperbolic Geometry (Code: SS 35A), **Brian Benson**, University of California, Riverside, and **Jeffrey S. Meyer**, California State University, San Bernardino.

Data Science (Code: SS 16A), **Shuheng Zhou**, University of California, Riverside.

Differential Equation, Differential Geometry and Mathematical General Relativity (Code: SS 26A), **Po-Ning Chen** and **Michael McNulty**, University of California, Riverside.

Dynamical Systems and Ergodic Theory (Code: SS 10A), **Nicolai Haydn**, University of Southern California, **Huyi Hu**, Michigan State University, and **Zhenghe Zhang**, University of California, Riverside.

Fluid Dynamics: from Theory to Numerics (Code: SS 18A), **James P Kelliher** and **Ali Pakzad**, University of California, Riverside.

Fractal Geometry, Dynamical Systems, and Related Topics (Code: SS 30A), **Tim Cobler**, Fullerton College, **Therese Landry**, University of California, Riverside, **Erin Pearse**, California Polytechnic State University, San Luis Obispo, and **Goran Radunovic**, University of Zagreb.

Geometric Methods in Representation Theory (Code: SS 25A), **Mee Seong Im**, United States Military Academy, West Point, **Neal Livesay**, University of California, Riverside, and **Daniel Sage**, Louisiana State University.

Geometric Partial Differential Equations and Variational Methods (Code: SS 4A), **Longzhai Lin**, University of California, Santa Cruz, **Xiangwen Zhang**, University of California, Irvine, and **Xin Zhou**, University of California, Santa Barbara.

Geometry and Representation Theory of Quantum Algebras and Related Topics (Code: SS 19A), **Mee Seong Im**, United States Military Academy, West Point, **Bach Nguyen**, Temple University, **Hans Nordstrom**, University of Portland, and **Karl Schmidt**, University of California, Riverside.

Graph Theory (Code: SS 29A), **Zhanar Berikkyzy** and **Mei-Chu Chang**, University of California, Riverside.

Integrating Forward and Inverse Modeling: Machine Learning and Multiscale, Multiphysics Challenges (Code: SS 21A), **Mark Alber**, University of California, Riverside, and **William Cannon**, Pacific Northwest National Laboratory.

Interactions between Geometric Group Theory and Teichmüller Theory (Code: SS 36A), **Matthew Durham**, University of California, Riverside, and **Thomas Koberda**, University of Virginia.

Invariants of Knots and Spatial Graphs (Code: SS 5A), **Alissa Crans**, **Blake Mellor**, and **Patrick Shanahan**, Loyola Marymount University.

Inverse Problems (Code: SS 3A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

Mathematical Biology: Multi-Scale Modeling of Complex Biological Systems (Code: SS 15A), **Suzanne Sindi** and **Mikahl Banwarth-Kuhn**, University of California, Merced.

Mathematical Modeling in Developmental Biology (Code: SS 14A), **Weitao Chen** and **Qixuan Wang**, University of California, Riverside.

Random Matrices and Related Structures (Code: SS 2A), **Jonathan Novak**, University of California, San Diego, and **Karl Liechty**, De Paul University.

Representations of Finite Groups and Related Topics (Code: SS 7A), **Robert Boltje**, University of California at Santa Cruz, **Klaus Lux**, University of Arizona at Tucson, and **Amanda Schaeffer Fry**, Metropolitan State University of Denver.

Research in Mathematics by Early Career Graduate Students (Code: SS 20A), **Marat Markin** and **Khang Tran**, California State University, Fresno.

Several Complex Variables and Complex Dynamics (Code: SS 34A), **Xin Dong**, University of California, Irvine, and **Sara Lapan** and **Bun Wong**, University of California, Riverside.

Symplectic and Low Dimensional Topology (Code: SS 33A), **Nur Saglam**, University of California, Riverside.

Topics in Algebraic Geometry (Code: SS 23A), **Jose Gonzalez**, **Ziv Ran**, and **Zhixian Zhu**, University of California, Riverside.

Topics in Extremal and Structural Graph Theory (Code: SS 32A), **Andre Kundgen**, California State University San Marcos, and **Craig Timmons**, California State University Sacramento.

Topics in Global Geometric Analysis (Code: SS 8A), **Fred Whilhelm** and **Qi Zhang**, University of California, Riverside.

Topics in Operator Theory (Code: SS 1A), **Anna Skripka** and **Maxim Zinchenko**, University of New Mexico.

Undergraduate Research in Mathematics: Presentations on Research and Mentorship (Code: SS 28A), **David Weisbart**, University of California, Riverside.

Denver, Colorado

Colorado Convention Center

January 15–18, 2020

Wednesday – Saturday

Meeting #1154

Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: October 2019

Program first available on AMS website: November 1, 2019

Issue of *Abstracts*: Volume 41, Issue 1

Deadlines

For organizers: Expired

For abstracts: September 17, 2019

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/national.html`.

Joint Invited Addresses

Michael C Fu, University of Maryland, College Park, *Escalator etiquette: Stand or walk? That is the question* (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).

Skip Garibaldi, Institute For Defense Analyses Center for Communications Research, *Title to be announced* (AMS-MAA Invited Address).

Karen Lange, Wellesley College, *Title to be announced* (AMS-MAA Invited Address).

Birgit Speh, Cornell University, *Title to be announced* (AWM-AMS Noether Lecture).

AMS Invited Addresses

Bonnie Berger, Massachusetts Institute of Technology, *Title to be announced*.

Ingrid Daubechies, Duke University, *Title to be announced* (AMS Colloquium Lectures: Lecture I).

Ingrid Daubechies, Duke University, *Title to be announced* (AMS Colloquium Lectures: Lecture II).

Ingrid Daubechies, Duke University, *Title to be announced* (AMS Colloquium Lectures: Lecture III).

Gregory W. Moore, Rutgers, The State University of New Jersey, *Title to be announced*.

Nancy Reid, University of Toronto, *Title to be announced* (AMS Josiah Willard Gibbs Lecture).

Tatiana Toro, University of Washington, *Title to be announced*.

Anthony Várilly-Alvarado, Rice University, *Title to be announced*.

Charlottesville, Virginia

University of Virginia

March 13–15, 2020

Friday – Sunday

Meeting #1155

Southeastern Section
Associate secretary: Brian D. Boe

Announcement issue of *Notices*: January 2020
Program first available on AMS website: February 4, 2020
Issue of *Abstracts*: Volume 41, Issue 2

Deadlines

For organizers: August 15, 2019
For abstracts: January 21, 2020

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Moon Duchin, Tufts University, *Title to be announced* (Einstein Public Lecture in Mathematics).

Laura Ann Miller, University of North Carolina, *Title to be announced*.

Yusu Wang, Ohio State University, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

"Young Guns" in Arithmetic Geometry and Number Theory (Code: SS 12A), **Evangelia Gazaki** and **Ken Ono**, University of Virginia.

Advances in Difference, Differential, Fractional Differential and Dynamic Equations with Applications (Code: SS 2A), **Muhammad Islam** and **Youssef Raffoul**, University of Dayton.

Advances in Infectious Disease Modeling: From Cells to Populations (Code: SS 4A), **Lauren Childs**, **Stanca Ciupe**, and **Omar Saucedo**, Virginia Tech.

Algebraic Groups: Arithmetic and Geometry (Code: SS 10A), **Raman Parimala**, Emory University, **Andrei Rapinchuk**, University of Virginia, and **Igor Rapinchuk**, Michigan State University.

Categorical Representation Theory and Beyond (Code: SS 11A), **You Qi** and **Liron Speyer**, University of Virginia, and **Joshua Sussan**, CUNY Medgar Evers (AMS-AAAS).

Curves, Jacobians, and Abelian Varieties (Code: SS 1A), **Andrew Obus**, Baruch College (CUNY), **Tony Shaska**, Oakland University, and **Padmavathi Srinivasan**, Georgia Institute of Technology.

Homotopy Theory (Code: SS 6A), **John Berman**, University of Texas at Austin, and **Prasit Bhattacharya**, University of Virginia.

Knots and Links in Low-Dimensional Topology (Code: SS 5A), **Thomas Mark**, University of Virginia, **Allison Moore**, University of California Davis, and **Ziva Myer**, Duke University.

Mathematical String Theory (Code: SS 9A), **Ilarion Melnikov**, James Madison University, **Eric Sharpe**, Virginia Tech, and **Diana Vaman**, University of Virginia (AMS-AAAS).

Numerical Methods for Partial Differential Equations: A Session in Honor of Slimane Adjerid's 65th Birthday (Code: SS 3A), **Mahboub Baccouch**, University of Nebraska at Omaha.

Recent Advances in Graph Theory and Combinatorics (Code: SS 8A), **Neal Bushaw**, Virginia Commonwealth University, and **Martin Rolek** and **Gexin Yu**, College of William and Mary (AMS-AAAS).

Recent Advances in Harmonic Analysis (Code: SS 7A), **Amalia Culiuc**, Amherst College, **Yen Do**, University of Virginia, and **Eyvindur Ari Palsson**, Virginia Tech.

Representation Theory of Algebraic Groups and Quantum Groups: A Tribute to the Work of Cline, Parshall and Scott (CPS) (Code: SS 13A), **Chun-Ju Lai**, University of Georgia, **Daniel K. Nakano**, University of Georgia, and **Weiqiang Wang**, University of Virginia.

Special Sets of Integers in Modern Number Theory (Code: SS 14A), **Cristina Ballantine**, College of the Holy Cross, and **Hester Graves**, Center for the Computing Sciences.

Homotopy Theory (Code: SS 15A), **Julie Bergner**, University of Virginia, and **Nick Kuhn**, University of Virginia.

Motivic Aspects of Topology and Geometry (Code: SS 16A), **Kirsten Wickelgren**, Duke University, and **Inna Zakharevich**, Cornell University.

Medford, Massachusetts

Tufts University

March 21–22, 2020

Saturday – Sunday

Meeting #1156

Eastern Section
Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: January 2020
Program first available on AMS website: February 11, 2020
Issue of *Abstracts*: Volume 41, Issue 2

Deadlines

For organizers: August 22, 2019
For abstracts: January 28, 2020

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Daniela De Silva, Columbia University, *Title to be announced*.
Enrique Pujals, City University of New York, *Title to be announced*.
Chris W. Woodward, Rutgers, The State University of New Jersey, *To be determined*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Analysis on Homogeneous Spaces (Code: SS 6A), **Jens Christensen**, Colgate University, **Matthew Dawson**, CIMAT, Mérida, México, and **Fulton Gonzalez**, Tufts University.

Anomalous Diffusion Processes (Code: SS 3A), **Christoph Borgers**, Tufts University, and **Claude Greengard**, New York University and Foss Hill Partners.

Geometric Dynamics and Billiards (Code: SS 4A), **Boris Hasselblatt** and **Eunice Kim**, Tufts University, **Kathryn Lindsey**, Boston College, and **Zbigniew Nitecki**, Tufts University.

Mathematics of Data Science (Code: SS 5A), **Vasileios Maroulas**, University of Tennessee Knoxville, and **James M. Murphy**, Tufts University.

Modeling and Analysis of Partial Differential Equations in Fluid Dynamics and Related Fields: Geometric and Probabilistic Methods (Code: SS 1A), **Geng Chen**, University of Kansas, **Siran Li**, Rice University and Centre de Recherches Mathématiques, Université de Montréal, and **Kun Zhao**, Tulane University.

Recent Advances in Schubert Calculus and Related Topics (Code: SS 2A), **Christian Lenart** and **Changlong Zhong**, State University of New York at Albany.

West Lafayette, Indiana

Purdue University

April 4–5, 2020
Saturday – Sunday

Meeting #1157
Central Section
Associate secretary: Georgia Benkart

Announcement issue of *Notices*: February 2020
Program first available on AMS website: February 18, 2020
Issue of *Abstracts*: Volume 41, Issue 2

Deadlines

For organizers: September 5, 2019
For abstracts: February 4, 2020

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Harmonic Analysis (Code: SS 2A), **Brian Street** and **Shaoming Guo**, University of Wisconsin–Madison.

Low-dimensional Topology (Code: SS 4A), **Matthew Hedden**, **Katherine Raoux**, and **Lev Tovstopyat-Nelip**, Michigan State University.

Mathematical Methods for Inverse Problems (Code: SS 3A), **Isaac Harris** and **Peijun Li**, Purdue University.

The Interface of Harmonic Analysis and Analytic Number Theory (Code: SS 1A), **Theresa Anderson**, Purdue University, **Robert Lemke Oliver**, Tufts University, and **Eyvindur Palsson**, Virginia Tech University.

Fresno, California

California State University, Fresno

May 2–3, 2020
Saturday – Sunday

Meeting #1158
Western Section
Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: March 2020
Program first available on AMS website: March 19, 2020
Issue of *Abstracts*: Volume 41, Issue 2

Deadlines

For organizers: October 3, 2019
For abstracts: March 3, 2020

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Mathematical Methods in Evolution and Medicine (Code: SS 1A), **Natalia Komarova** and **Jesse Kreger**, University of California, Irvine.

El Paso, Texas

University of Texas at El Paso

September 12–13, 2020
Saturday – Sunday

Meeting #1159
Central Section
Associate secretary: Georgia Benkart

Announcement issue of *Notices*: June 2020
Program first available on AMS website: July 28, 2020
Issue of *Abstracts*: Volume 41, Issue 3

Deadlines
For organizers: February 20, 2020
For abstracts: July 14, 2020

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Structures in Topology, Logic, and Arithmetic (Code: SS 3A), **John Harding**, New Mexico State University, and **Emil Schwab**, The University of Texas at El Paso.

High-Frequency Data Analysis and Applications (Code: SS 1A), **Maria Christina Mariani** and **Michael Pokojovy**, University of Texas at El Paso, and **Ambar Sengupta**, University of Connecticut.

Low-dimensional Topology and Knot Theory (Code: SS 4A), **Mohamed Ait Nouh** and **Luis Valdez-Sanchez**, University of Texas at El Paso.

Nonlinear Analysis and Optimization (Code: SS 2A), **Behzad Djafari-Rouhani**, University of Texas at El Paso, and **Akhtar A. Khan**, Rochester Institute of Technology.

State College, Pennsylvania

Pennsylvania State University, University Park Campus

October 3–4, 2020
Saturday – Sunday

Meeting #1160
Eastern Section
Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: August 2020
Program first available on AMS website: August 25, 2020
Issue of *Abstracts*: Volume 41, Issue 3

Deadlines
For organizers: March 3, 2020
For abstracts: August 11, 2020

Chattanooga, Tennessee

University of Tennessee at Chattanooga

October 10–11, 2020
Saturday – Sunday

Meeting #1161
Southeastern Section
Associate secretary: Brian D. Boe

Announcement issue of *Notices*: August 2020
Program first available on AMS website: September 1, 2020
Issue of *Abstracts*: Volume 41, Issue 4

Deadlines
For organizers: March 10, 2020
For abstracts: August 18, 2020

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Giulia Saccà, Columbia University, *Title to be announced*.
Chad Topaz, Williams College, *Title to be announced*.
Xingxing Yu, Georgia Institute of Technology, *Title to be announced*.

Salt Lake City, Utah

University of Utah

October 24–25, 2020

Saturday – Sunday

Meeting #1162

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: August 2020
Program first available on AMS website: September 17, 2020
Issue of *Abstracts*: Volume 41, Issue 4

Deadlines

For organizers: March 24, 2020
For abstracts: September 1, 2020

Washington, District of Columbia

Walter E. Washington Convention Center

January 6–9, 2021

Wednesday – Saturday
Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Brian D. Boe
Announcement issue of *Notices*: October 2020
Program first available on AMS website: November 1, 2020
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2020
For abstracts: To be announced

San Francisco, California

San Francisco State University

May 1–2, 2021

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Grenoble, France

Université de Grenoble-Alpes

July 5–9, 2021

Monday – Friday
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Buenos Aires, Argentina

The University of Buenos Aires

July 19–23, 2021

Monday – Friday
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Omaha, Nebraska

Creighton University

October 9–10, 2021

Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 5–8, 2022

Wednesday – Saturday
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2021
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Boston, Massachusetts

John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2023

Wednesday – Saturday
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2022
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Call for Nominations

AMS EXEMPLARY PROGRAM AWARD

The AMS Award for Exemplary Program or Achievement in a Mathematics Department is presented annually to a department that has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of the society. Examples might include a department that runs a notable minority outreach program, a department that has instituted an unusually effective industrial mathematics internship program, a department that has promoted mathematics so successfully that a large fraction of its university's undergraduate population majors in mathematics, or a department that has made some form of innovation in its research support to faculty and/or graduate students, or which has created a special and innovative environment for some aspect of mathematics research.

The award amount is $5,000. All departments in North America that offer at least a bachelor's degree in the mathematical sciences are eligible.

The Award Selection Committee requests nominations for this award, which will be announced in Spring 2020. Letters of nomination may be submitted by one or more individuals. Nomination of the writer's own institution is permitted. The letter should describe the specific program(s) for which the department is being nominated as well as the achievements that make the program(s) an outstanding success, and may include any ancillary documents which support the success of the program(s). The letter should not exceed two pages, with supporting documentation not to exceed an additional three pages.

Further information about AMS prizes can be found at the Prizes and Awards website **www.ams.org/prizes-awards**.

Further information and instructions for submitting a nomination can be found at the prize nomination website: **www.ams.org/nominations**.

For questions contact the AMS Secretary at secretary@ams.org.

Deadline for nominations is September 15, 2019.

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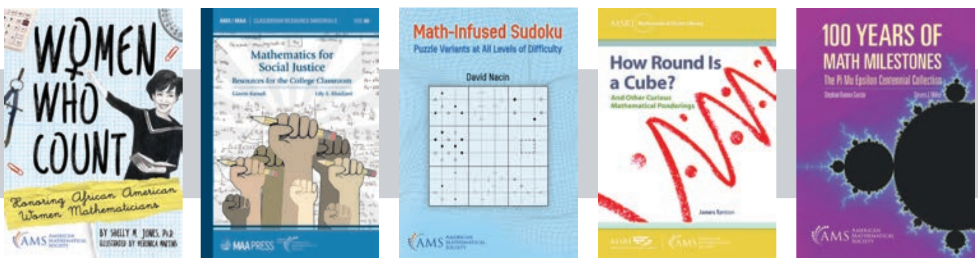
NEW RELEASES
from the AMS



Women Who Count

Honoring African American Women Mathematicians

Shelly M. Jones, *Central Connecticut State University, New Britain*

I applaud the author's creativity! This activity book is a unique way to expose children early to mathematical ideas and to a part of American history that is not readily accessible at a young age.

—Dr. Sylvia T. Bozeman

This children's activity book highlights the lives and work of 29 African American women mathematicians, including Dr. Christine Darden, Mary Jackson, Katherine Johnson, and Dorothy Vaughan from the award-winning book and movie *Hidden Figures*.

2019; 143 pages; Softcover; ISBN: 978-1-4704-4889-9; List US$15; AMS members US$12; MAA members US$13.50; Order code MBK/124

Mathematics for Social Justice

Resources for the College Classroom

Gizem Karaali, *Pomona College, Claremont, CA*, and **Lily S. Khadjavi**, *Loyola Marymount University, Los Angeles, CA*, Editors

This collection of resources for mathematics faculty interested in incorporating questions of social justice into their classrooms includes both a series of essays from instructors experienced in integrating social justice themes into their pedagogy and 14 classroom-tested modules featuring ready-to-use activities and investigations for the college mathematics classroom.

Classroom Resource Materials, Volume 60; 2019; 280 pages; Softcover; ISBN: 978-1-4704-4926-1; List US$55; AMS members US$41.25; MAA members US$41.25; Order code CLRM/60

Math-Infused Sudoku

Puzzle Variants at All Levels of Difficulty

David Nacin, *William Paterson University, Wayne, NJ*

Building upon the rules of Sudoku, the puzzles in this volume introduce new challenges by adding clues involving sums, differences, means, divisibility, and more.

2019; 116 pages; Softcover; ISBN: 978-1-4704-5090-8; List US$25; AMS members US$20; MAA members US$22.50; Order code MBK/123

How Round Is a Cube?

And Other Curious Mathematical Ponderings

James Tanton, *Mathematical Association of America, Washington, DC*

Intended for the general math enthusiast, this book is a collection of 34 curiosities, each a quirky and delightful gem of mathematics and each a shining example of the joy and surprise that mathematics can bring.

Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).

MSRI Mathematical Circles Library, Volume 23; 2019; 262 pages; Softcover; ISBN: 978-1-4704-5115-8; List US$25; AMS members US$20; MAA members US$22.50; Order code MCL/23

100 Years of Math Milestones

The Pi Mu Epsilon Centennial Collection

Stephan Ramon Garcia, *Pomona College, Claremont, CA*, and **Steven J. Miller**, *Williams College, Williamstown, MA*

Each chapter in this book of 100 problems chosen to celebrate the 100th anniversary of the undergraduate math honor society Pi Mu Epsilon describes a problem or event, the progress made, and connections to entries from other years or other parts of mathematics.

2019; 581 pages; Softcover; ISBN: 978-1-4704-3652-0; List US$60; AMS members US$48; MAA members US$54; Order code MBK/121

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