NEGLIGIBLE SUBSETS OF INFINITE-DIMENSIONAL FRÉCHET MANIFOLDS

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1. **Introduction.** K is a *negligible* subset of a topological space X if $X-K$ is homeomorphic to X. Recent results have been reached concerning such sets in manifolds modelled on separable infinite-dimensional Fréchet spaces. This paper generalizes some of these results in an attempt to develop a similar theory for manifolds modelled on nonseparable Fréchet spaces. For Y a complete metric space, a *manifold modelled on* Y will be a paracompact, Hausdorff space, M, such that every point of M has an open neighborhood homeomorphic to Y. With this definition, every manifold is metrizable, and since it is locally completely metrizable, by Corollary 4.3 [**10**] it is completely metrizable. A *Fréchet manifold* will be a manifold modelled on an infinite-dimensional Fréchet space. *Fréchet spaces* are locally convex, completely metrizable topological linear spaces, and thus include all Banach spaces.

K is a *locally closed* subset of X if for each $x\in K$ there exists an open set U such that $x\in U$ and $K\cap U$ is closed in U. This is equivalent, for metric spaces, to K being the difference of two closed subsets of X. K is *locally infinite-deficient* (l.i.d.) if for each $x\in K$, there is an open set U containing x, a metric space Y, an infinite-dimensional Fréchet space $\boldsymbol{F}$, and an imbedding $i: U\to \boldsymbol{F}\times Y$ such that $i(U)$ is open in $\boldsymbol{F}\times Y$ and $i(K\cap U)\subset \{0\}\times Y$. An *invertible isotopy pushing K off X* is a homeomorphism $F:(X\times I)-(K\times\{1\})\to X\times I$ which preserves the second coordinate (i.e. if $(y,t)=F(x,s)$, then $s=t$), and which is the identity on $X\times\{0\}$. The isotopy may also be denoted by $\{H_s\}_{s\in I}$, where for each $s\in I$, H_s is the homeomorphism F restricted to $X\times\{s\}$. Such an isotopy is said to be *limited by G*, a collection of open subsets of X, if for each $x\in X$, either $F(x,s)=(x,s)$ for all $s\in I$, or $\{F(x,s), F^{-1}(x,s)\}_{s\in I}$ is contained in $g\times I$ for some $g\in G$. K is *extractible from X* if for every collection, G, of open subsets of X containing K in their union, there is an invertible isotopy pushing K off X which is limited by G. Finally, K is *locally extractible from X* if for each $x\in K$ there is an open set U containing x such that for any set

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K', a locally closed subset of $U\cap K$, then K' is extractible from U (and hence extractible from any open set containing K').

The main theorem proved in this paper is the following:

THEOREM 1. *Let X be a complete metric space, $K\subset X$ a countable union of locally closed, locally infinite-deficient sets. Then K is extractible from X.*

The most important result is that K is negligible in X, but we are also told that the homeomorphism of $X-K$ onto X may be chosen so as to be isotopic to the identity by an isotopy which moves points as little as is desired.

It follows from Eells and Kuiper [6] that if X is an absolute neighborhood retract and A a l.i.d. subset, then the inclusion map $i\colon X-A\to X$ is a homotopy equivalence.

A closed set K has *Property Z* if for each nonempty and homotopically trivial open set U in X, $U-K$ is nonempty and homotopically trivial. Anderson has shown ([1] and [4]) that Property Z implies infinite deficiency for manifolds modelled on separable Hilbert space. Although it would be nice to prove this for nonseparable cases as well, the proof in the separable case uses the Hilbert cube, and has no obvious generalization. However, most applications (for instance Henderson's theorems in [7] and [8]) use sets which can be shown to be l.i.d. in nonseparable cases. The following corollaries give the results for locally compact sets and locally collared sets, and follow easily from Theorem 1 and Lemmas 1 and 2 of §3.

COROLLARY 1. *Let M be a Fréchet manifold, and let $X\subset M$ be a countable union of locally compact sets. Then X is extractible from M.*

COROLLARY 2. *Let M be a Fréchet manifold. For each $i>0$, suppose M_i is a Fréchet manifold and is collared as a subset of M. If for $i>0$, K_i is a locally closed subset of M_i, then $K=\bigcup K_i$ is extractible from M.*

Following [4], $K\subset X$ is *strongly negligible* if for every open cover, G, of X, there is a homeomorphism $h: X-K\to X$ limited by G. It is easy to see that "extractible" implies "strongly negligible." The following theorem combines Theorem 1 with a recent result of Anderson's to show that they are equivalent for separable Fréchet manifolds:

THEOREM 2. *Let M be a separable Fréchet manifold, and let K be a subset of M. Then the following are equivalent:*

(1) *K is extractible from X,*

(2) *K is strongly negligible in X,*

(3) *K is the countable union of closed sets with Property Z.*

Proof. Lemma 1 of [**4**], which combines results of previous papers, shows that Property Z implies l.i.d. for a closed subset of a separable Fréchet manifold. Hence Theorem 1 shows that (3) implies (1). That (1) implies (2) was noted above, and Theorem 1 of [**2**] states that (2) and (3) are equivalent.

2. **Notation.** Throughout this paper, $\boldsymbol{F}$ will stand for an arbitrary infinite-dimensional Fréchet space and $\boldsymbol{H}$ will stand for separable Hilbert space.

We will use $N_\delta(x)$ to mean the δ-neighborhood of x. If G is a collection of open subsets of X, then $\bigcup G$ will denote the union of the elements of G. If d is a metric for X and x is a point of X, then $d(x; G)$ is defined by

$$d(x;G) = \sup\{\delta \in (0,1] \mid N_\delta(x) \subset \text{some element of } G\} \qquad \text{for } x \in \bigcup G$$

$$= 0 \qquad \text{for } x \in X - \bigcup G.$$

If x and y are points of X, then $|d(x;G)-d(y;G)| \leqq d(x,y)$, and hence $d(x;G)$ is a continuous function from X to the nonnegative reals which is positive on $\bigcup G$. Furthermore, if f is a continuous function from X to the nonnegative reals, then there is a collection of open sets G such that $\bigcup G = X - f^{-1}(0)$, and for any continuous function g limited by G, $d(g(x), x) \leqq f(x)$ for all $x \in X$. Under these conditions we will say that G *refines* the function f.

For X and Y topological spaces, $\pi_1 : X \times Y \to X$ will denote projection onto the first factor.

3. **Several lemmas.** Lemmas 3 and 4 are generalizations of Theorems 9.1 and 4.3 of [**3**].

Lemma 1. *Let M be a manifold modelled on $\boldsymbol{F} \times Y$, where Y is a complete metric space. If K is a locally compact subset of M, then K is l.i.d.*

Proof. Fix $x \in M$. Suppose U is an open neighborhood of x which is homeomorphic to $\boldsymbol{F} \times Y$ and such that $\overline{U} \cap K$ is compact. Using normality of M, let V be an open neighborhood of x such that $\overline{V} \subset U$. Let $K' = \overline{V} \cap K$. We will regard K' and V as subsets of $\boldsymbol{F} \times Y$.

Imbed Y in a Fréchet space F_1, and let $F_2 = \boldsymbol{F} \times F_1$. By a theorem of Corson [**5**], there is an infinite-dimensional closed linear subspace $F_3 \subset \boldsymbol{F}$ such that for any $x \in F_2$, $F_3 + x$ intersects K' in at most one point. By a theorem due to Bartle, Graves and Michael [**11**], it follows that $\boldsymbol{F}/F_3$ can be imbedded (not necessarily linearly) in $\boldsymbol{F}$ such that $h : F_3 \times \boldsymbol{F}/F_3 \to \boldsymbol{F}$ defined by $h((x,y)) = x + y$ is a homeomorphism. Using the Tietze Extension Theorem, K' is part of the graph of a continuous function $f : \boldsymbol{F}/F_3 \times F_1 \to F_3$.

Define a homeomorphism g of $F_3 \times \boldsymbol{F}/F_3 \times Y$ onto itself by $g((x, y, z)) = (x - f(y, z), y, z)$. Then $g(h^{-1} \times \mathrm{id})(K') \subset \{0\} \times \boldsymbol{F}/F_3 \times Y$. Hence, restricting everything to V, the conditions in the definition of l.i.d. will be satisfied.

LEMMA 2. *Let M be a manifold modelled on $\boldsymbol{F} \times Y$, where Y is some complete metric space. Suppose M is collared as a subset of X, and K is a subset of M. Then K is l.i.d. as a subset of X.*

PROOF. Fix $x \in M$. By standard techniques, there is an open set V of X which contains x and such that the pairs $(V, V \cap M)$ and $(\boldsymbol{F} \times Y \times [0, 1), \boldsymbol{F} \times Y \times \{0\})$ are pairwise homeomorphic. By results due to Bartle, Graves, and Michael [11], every infinite-dimensional Fréchet space has a separable infinite-dimensional Fréchet space as a topological factor. Since all separable infinite-dimensional Fréchet spaces are homeomorphic to $\boldsymbol{H}$ (see [3]), then $\boldsymbol{F} \cong \boldsymbol{H} \times W$ for some W.

Klee was the first to show that closed separable Hilbert half-space is homeomorphic to separable Hilbert space (Theorem III (1.3) of [9]). Since the boundary in closed separable Hilbert half-space has Property Z, it follows from Theorem 8.4 of [1] that it is infinitely deficient, i.e. there is a homeomorphism $h : \boldsymbol{H} \times [0, 1) \to \boldsymbol{H} \times \boldsymbol{H}$ such that $h(\boldsymbol{H} \times \{0\}) \subset \{0\} \times \boldsymbol{H}$. Then $\boldsymbol{F} \times Y \times [0, 1) \cong \boldsymbol{H} \times [0, 1) \times (W \times Y) \cong \boldsymbol{H} \times \boldsymbol{H} \times (W \times Y)$, where the second homeomorphism is $h \times \mathrm{id}$, and the resulting homeomorphism takes $V \cap M$ to a subset of $\{0\} \times \boldsymbol{H} \times (W \times Y)$. Hence M, and therefore K, is l.i.d.

LEMMA 3. *Let $K \subset X$ be locally closed and l.i.d. Then K is locally extractible from X.*

PROOF. It suffices to show that for Y a complete metric space and K a locally closed subset of $\{0\} \times Y$, then K is extractible from $\boldsymbol{F} \times Y$. Since $\boldsymbol{F} \cong \boldsymbol{H} \times W$ for some W, we may assume that $\boldsymbol{F} = \boldsymbol{H}$.

We will make use of the invertibly continuous family of invertible isotopies pushing the origin off $\boldsymbol{H}$, denoted ${}_rH_t$, which is defined on pp. 784–786 of [3]. For each $r \in (0, 1]$, $\{{}_rH_t\}_{t \in I}$ is an isotopy pushing the origin off $\boldsymbol{H}$ which is fixed outside the r-neighborhood of the origin. In addition, both ${}_rH_t$ and ${}_rH_t^{-1}$ are simultaneously continuous in r and t.

Let $Y' = \{0\} \times Y$, let G be a collection of open sets of $\boldsymbol{H} \times Y$ such that $K \subset V = \bigcup G$, and choose K_1, K_2 closed subsets of Y' such that $K = K_1 - K_2$. Let $C = K_2 \cup (Y' - (V \cap Y'))$ and $W = Y' - C$. C is a closed subset of Y'. Let $r_1 : Y' \to I$ be a continuous function such that $r_1^{-1}(0) = C$. If d is a metric for X such that $d | \boldsymbol{H} \times \{y\}$ is the usual metric on $\boldsymbol{H}$ for each $y \in Y$, define $r : Y' \to I$ by $r(x) = \min\{r_1(x), d(x; G)\}$. Then r is continuous and $r^{-1}(0) = C$.

Let $t : W \to (0, 1]$ be a continuous function such that $t^{-1}(1)=K$. Finally, define $F : (H\times Y\times I)-(K\times\{1\})\to(H\times Y\times I)$ by

$$F((p, q, s)) = ({}_{r(q)}H_{s\cdot t(q)}(p), q, s) \qquad \text{for } q \in W$$
$$= \text{id} \qquad \text{for } q \in C.$$

It is easily seen that F is 1-1 and onto. By continuity of ${}_rH_t$ and ${}_rH_t^{-1}$ in r, t, and X, F and F^{-1} are continuous at points in $H\times W\times I$. F and F^{-1} are continuous at points of $H\times C\times I$ because $r(q)$ approaches 0 as q approaches C. Hence F is an isotopy pushing K off $H\times Y$. Furthermore, using the definition of r and the fact that ${}_rH_t$ is fixed outside the r-neighborhood of the origin, it is clear that F is limited by G.

LEMMA 4. *Suppose $\{K_i\}_{i>0}$ is a sequence of closed subsets of a complete metric space X. Let $X_i=X-\bigcup_1^{i-1}K_j$, $K_i'=K_i\cap X_i$. If for each i, K_i' is extractible from X_i, then $K=\bigcup K_i$ is extractible from X.*

PROOF. Let G be a collection of open subsets of X such that $K\subset V=\bigcup G$. Let d be a complete metric for X. We inductively define isotopies F_i and collections of open sets H_i satisfying:

(1) $F_i: (X_i\times I)-(K_i'\times\{1\})\to X_i\times I$ is an invertible isotopy pushing K_i' off X_i which is limited by H_i,

(2) $\bigcup H_i=V\cap X_i$,

(3) mesh $H_i\leq 1/2^i$,

(4) for $s\in I$ and $h\in H_i$, diam $F_1\circ\cdots\circ F_{i-1}(h\times\{s\})\leq 1/2^i$,

(5) H_i refines the function $d(x, \bigcup_1^{i-1}K_j)/2^i$,

(6) if G_i is the cover of $V\cap X_i$ gotten by intersecting the elements of G with X_i, then H_i refines the function $d(x; G_i)/2^{i+2}$.

Condition (1) can be satisfied inductively since K_i' is extractible from X_i. By condition (5), F_i can be continuously extended by the identity to a function

$$F_i: (X\times I)-((\textstyle\bigcup_1^i K_j)\times\{1\})\to(X\times I)-((\bigcup_1^{i-1}K_j)\times\{1\}).$$

Then $E_i=F_1\circ F_2\circ\cdots\circ F_i$ is an invertible isotopy pushing $\bigcup_1^i K_j$ off X with inverse $E_i^{-1}=F_i^{-1}\circ\cdots\circ F_1^{-1}$.

It is easy to see that we can find an open cover of $V\cap X_i$ satisfying conditions (3), (5) and (6). Condition (4) can also be satisfied by the following reasoning: For each $x\in V\cap X_i$, define

$$U_x = \{(y, s)\in X\times I \mid d(y, \pi_1(E_{i-1}(x, s))) < 1/2^{i+1}\}.$$

U_x is open in $X\times I$. Now $\{x\}\times I\subset E_{i-1}^{-1}(U_x)$ and hence, by compactness of I, there is a $\delta>0$ such that $N_\delta(x)\times I\subset E_{i-1}^{-1}(U_x)$. Then $N_\delta(x)$ is an open neighborhood of x in X_i satisfying condition (4).

Now define $F:(X\times I)-(K\times\{1\})\to X\times I$ and $F^*: X\times I\to X\times I$ by

$$F(x, s) = \lim_{i\to\infty} \{E_i(x, s)\},$$

$$F^*(x, s) = \lim_{i\to\infty} \{E_i^{-1}(x, s)\}.$$

Let d' be the metric on $X\times I$ defined by $d'((x, s), (y, t))=d(x, y)+|s-t|$. Then conditions (3) and (4) show respectively that

$$d'(E_i^{-1}(x, s), E_{i-1}^{-1}(x, s)) \leqq 1/2^i \quad \text{for } (x, s)\in X\times I,$$

$$d'(E_i(x, s), E_{i-1}(x, s)) \leqq 1/2^i \quad \text{for } (x, s)\in (X\times I) - ((\textstyle\bigcup_1^i K_j)\times\{1\}).$$

Hence $\{E_i(x, s)\}_{i>0}$ and $\{E_i^{-1}(x, s)\}_{i>0}$ are Cauchy sequences in $X\times\{s\}$, and hence converge in $X\times\{s\}$. Hence F and F^* are well defined, and since they are the uniform limit of continuous functions, they are also continuous.

Condition (5) is used to show that $F^*(X\times\{1\})\subset(X-K)\times\{1\}$. For suppose that for some $x\in X$ and $i>0$, $F^*(x, 1)\in K_i\times\{1\}$. Now $d(\pi_1E_i^{-1}(x, 1), K_i)=\delta>0$. Condition (5) is used to show inductively that for $j>i$, $d(\pi_1E_j^{-1}(x, 1), K_i)\geqq(1-(1/2^i-1/2^j))\delta$. Hence $d(\pi_1F^*(x, 1), K_i)\geqq(1-1/2^i)\delta>0$.

Now it is not difficult to show that F and F^* are inverses. Let $(y, s)=F(x, s)=\lim_{i\to\infty}\{E_i(x, s)\}$. We know from condition (3) that $d'(F^*(z, u), E_i^{-1}(z, u))\leqq 1/2^i$ for all (z, u) where F^* is defined. Hence

$$d'(F^*(E_i(x, s)), (x, s)) = d'(F^*(E_i(x, s)), E_i^{-1}(E_i(x, s))) \leqq 1/2^i,$$

and hence by continuity of F^*, $F^*(y, s)=(x, s)$. Similarly, it can be shown that $F\circ F^*=\mathrm{id}$, and therefore $F^*=F^{-1}$. Hence F is an invertible isotopy pushing K off X.

Finally, using condition (6) it can be shown that for each $x\in X$ and $i>0$, $E_i(\{x\}\times I)$ and $E_i^{-1}(\{x\}\times I)$ are contained in $N_{d(x;\,G)/2}(x)\times I$, and hence F is limited by G.

LEMMA 5. *Let $K\subset X$ be closed and locally extractible, where X is a complete metric space. Then K is extractible from X.*

PROOF. Pick a collection of open sets in X which contain K in their union, and such that each of the sets satisfies the requirements in the definition of locally extractible. Add to these the set $X-K$ to get an open cover, H', of X. By a well-known covering lemma due to J. Milnor, there exists an open cover H'' which is a locally finite refinement of H' with sets $\{h_{i\alpha}\}_{\alpha\in B_i}$, $i=1, 2, \cdots$, where $h_{i\alpha}\cap h_{i\beta}=\emptyset$ if $\alpha\neq\beta$. (For a statement and proof of this lemma, see p. 7 of [12].) Using local finiteness of H'', pick open covers of X, $\{h'_{i\alpha}\}_{\alpha\in B_i}$, $\{h''_{i\alpha}\}_{\alpha\in B_i}$ such that $\bar{h}''_{i\alpha}\subset h'_{i\alpha}\subset \bar{h}'_{i\alpha}\subset h_{i\alpha}$. Let $K_{i\alpha}=\bar{h}''_{i\alpha}\cap K$, and $K_i=\bigcup_{\alpha\in B_i} K_{i\alpha}$. Since H'' is locally finite, K_i is closed for each $i>0$.

Now let $X_i=X-\bigcup_1^{i-1} K_j$ and $K_i' =K_i\cap X_i$. Then since $K=\bigcup K_i$, it suffices by Lemma 4 to show that K_i' is extractible from X_i.

Fix $i>0$ and let G be a collection of open sets in X_i which contain K_i' in their union. Pick $\alpha\in B_i$ and suppose that $K_{i\alpha}$ is nonempty. Now $h_{i\alpha}$ refines some element in H' different from $X-K$, hence $K'_{i\alpha}=K_{i\alpha}\cap X_i$ is extractible from $h_{i\alpha}\cap X_i$. Let G' be the collection of open sets in $h_{i\alpha}\cap X_i$ defined by

$$G' = \{(g\cap h'_{i\alpha}\cap X_i) \mid g\in G\}.$$

Let $F_{i\alpha}$ be an invertible isotopy pushing $K'_{i\alpha}$ off $h_{i\alpha}\cap X_i$ which is limited by G'. $F_{i\alpha}$ is the identity outside $h'_{i\alpha}$. For $\alpha\in B_i$ for which $K'_{i\alpha}$ is empty, we take $F_{i\alpha}$ to be the identity.

Finally, define F_i: $(X_i\times I)-(K_i'\times\{1\})\to X_i\times I$ by

$$\begin{aligned} F_i &= F_{i\alpha} && \text{on } (h_{i\alpha}\cap X_i)\times I \quad \text{for each } \alpha\in B_i \\ &= \text{id} && \text{elsewhere.} \end{aligned}$$

Using local finiteness of H'', it follows that F_i is the identity in some neighborhood of any point not contained in $\bigcup_{\alpha\in B_i}(h_{i\alpha}\times I)$. Hence F_i is an invertible isotopy pushing K_i' off X_i, and it is easily seen to be limited by G.

4. **Proof of Theorem** 1. Locally closed sets are countable unions of closed sets. Since a subset of a l.i.d. set is l.i.d., we may assume that $K=\bigcup K_i$ is a countable union of closed l.i.d. sets. Letting $X_i=X-\bigcup_1^{i-1} K_j$ and $K_i' =K_i\cap X_i$, then K_i' is closed and l.i.d. as a subset of X_i. By Lemmas 3 and 5, K_i' is extractible from X_i. Finally, Lemma 4 completes the proof.

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