FACTORS OF INFINITE-DIMENSIONAL MANIFOLDS

BY

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1. **Introduction.** Let J^∞ denote the Hilbert cube, i.e. the countable infinite product of closed intervals, and let s denote the countable infinite product of open intervals (or lines as convenient). Specifically, let $J^\infty = \prod_{i>0} J_i$ and let $s = \prod_{i>0} J_i^0$ where for each $i>0$, J_i is the closed interval $[-1, 1]$ and J_i^0 is the open interval $(-1, 1)$. In [1] it was shown that s is homeomorphic to Hilbert space, l_2, and thus, on the basis of results in [6] and [7], to all separable infinite-dimensional Fréchet spaces (and therefore, of course, to all separable infinite-dimensional Banach spaces).

A *Fréchet manifold* (or *F-manifold*) is a separable metric space with an open cover of sets each homeomorphic to an open subset of s. Thus all separable metric Banach manifolds modeled on separable infinite-dimensional Banach spaces are F-manifolds. A Fréchet manifold is known to admit a complete metric and to be nowhere locally compact.

A *Hilbert cube manifold* or *Q-manifold* is a separable metric space with an open cover of sets homeomorphic to open subsets of J^∞. A Q-manifold is known to admit a complete metric and to be locally compact. We could, equivalently, specify that a Q-manifold admits an open cover by sets whose closures are homeomorphic to J^∞ itself.

The following are the principle theorems of this paper.

THEOREM I. *If M is any F-manifold, then $s \times M$ is homeomorphic to M. (For a somewhat stronger version of this theorem see the addendum at the end of this paper.)*

THEOREM II. *If M is any Q-manifold, then $J^\infty \times M$ is homeomorphic to M.*

Since s is known [2] or [5] to be homeomorphic to $s \times J^\infty$, from Theorem I we immediately have the following.

COROLLARY. *If M is any F-manifold, then $J^\infty \times M$ is homeomorphic to M.*

Almost identical proofs of Theorems I and II can be given. We shall explicitly give the proof of Theorem I only. It will be understood that the proof of Theorem I also constitutes a proof of Theorem II with the natural modifications needed such as s replaced by J^∞, F-manifold replaced by Q-manifold and open interval factors of s replaced by closed interval factors of J^∞.

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In the next section we shall reduce the proof of Theorem I to the proof of Lemma B.

2. **Reduction of Theorem I to Lemma B.** We begin with a definition.

Let r be a continuous function of a topological space X into the closed unit interval $I=[0, 1]$. Let $J^0(0)=\{0\}$ and for $t\in(0, 1]$, let $J^0(t)=(-t, t)$. Then

$$J^0\times^r X = \{(y, x)\in J^0\times X : y\in J^0(r(x))\}$$

is the *variable product* of J^0 *by* X (with respect to r). Likewise, let s_0 be the origin of s or, where convenient, the single point set consisting of the origin of s and for $t\in(0, 1]$, let $s_t=\prod_{i>0} J_i^0(t)$ where $J_i^0(t)=(-t, t)$. Note that for each t, $s_t\subset s$. Then

$$s\times^r X = \{(y, x)\in s\times X : y\in s_{r(x)}\}$$

is the *variable product* of s *by* X (with respect to r). If $A\subset X$, let $J^0\times^r A$ (or $s\times^r A$) be the variable product of J^0 (or s) by A (with respect to $r|A$).

We are now in a position to state Lemmas A and B.

LEMMA A. *Let M be an F-manifold, let $V\subset U\subset M$ where V is closed and U is open and is homeomorphic to an open subset of s, and let $s\times^{r_0} M$ be a variable product of s by M. There exists a homeomorphism H of $s\times^{r_0} M$ onto a variable product $s\times^r M$ such that* (1) $r\leqq r_0$, (2) $r(V)=0$, *and* (3) $H|s\times^{r_0}[(M\setminus U)\cup r_0^{-1}(0)]$ *is the identity.*

We now reformulate Lemma A in a somewhat more convenient form.

LEMMA B. *Let U be an open subset of s, let $V\subset W\subset U$ where W is open and V is closed in U, and let $s\times^{r_0} U$ be a variable product of s by U. There exists a homeomorphism H of $s\times^{r_0} U$ onto a variable product $s\times^r U$ such that* (1) $r\leqq r_0$, (2) $r(V)=0$, *and* (3) $H|s\times^{r_0}[(U\setminus W)\cup r_0^{-1}(0)]$ *is the identity.*

Proof that Lemma B implies Lemma A. Letting M, V, U, and $s\times^{r_0} M$ be as in the hypothesis of Lemma A, we may let W be any open set in M such that $V\subset W\subset U$ and the closure of W in M is the subset of U. Regarding such U, V, and W as in the hypothesis of Lemma B and regarding r_0 in Lemma B as $r_0|U$, it follows that any homeomorphism H as in the conclusion of Lemma B has an automatic extension to a homeomorphism satisfying the conditions of Lemma A.

Proof that Lemma A implies Theorem I.

Step 1. As proved in [8] and applied in [3], since M is separable and metric, there exists a countable star-finite open cover G of M with sets homeomorphic to open subsets of s. (By a *star-finite* cover we shall mean a cover where the closure of each element of the cover intersects the closure of only finitely many elements of the cover.)

Step 2. We follow a procedure as used in Theorem 2 of [3]. Let $(g_i)_{i>0}$ be any ordering of the elements of G. Let $E_1=\{g_1\}$ and inductively let E_{i+1} be the set containing the least indexed element g_k which is not in $\bigcup_{j\leqq i} E_i$ together with all the elements of G which are not in $\bigcup_{j\leqq i} E_j$ and which intersect some element of

E_i. Clearly each E_i must be finite. We now order the elements of G as $(U_i)_{i>0}$ by first listing the element of E_1 and then inductively listing the elements of E_{2n+1} followed by those of E_{2n}. It is easy to verify that for any sequence $(H_i)_{i>0}$ of homeomorphisms, it will follow that $(H_i \circ \cdots \circ H_1)_{i>0}$ converges to a homeomorphism of $s \times M$ onto $\bigcap_{i>0} H_i \circ \cdots \circ H_1(s \times M)$ provided (1) H_1 maps $s \times M$ into itself with $H_1|s \times (M \setminus U_1) = \text{identity}$ and (2) for each $i>0$, H_{i+1} maps $H_i \circ \cdots \circ H_1(s \times M)$ into itself with $H_{i+1}|[s \times (M \setminus U_{i+1})] \cap H_i \circ \cdots \circ H_1(s \times M) = \text{identity}$.

Step 3. We now apply Lemma A inductively to define $(H_i)_{i>0}$ as in Step 2 with $\bigcap_{i>0} H_i \circ \cdots \circ H_1(s \times M) = s_0 \times M$ which is homeomorphic to M. For each $i>0$, let $V_i \subset U_i$ where V_i is closed and $\{V_i\}_{i>0}$ covers M. By Lemma A there exists a homeomorphism H_1 of $s \times M$ onto a variable product $s \times^{r_1} M$ such that $r_1(V_1) = 0$ and $H_1|s \times (M \setminus U_1) = \text{identity}$. Inductively, by Lemma A let H_{i+1} be a homeomorphism of $s \times^{r_i} M$ onto a variable product $s \times^{r_{i+1}} M$ where $r_{i+1} \leqq r_i$, $r_{i+1}(V_{i+1}) = 0$, and $H_{i+1}|s \times^{r_i} (M \setminus U_{i+1}) = \text{identity}$. Hence, by Step 2 and by the definition of the H_i we have $(H_i \circ \cdots \circ H_1)_{i>0}$ converging to a homeomorphism of $s \times M$ onto $s_0 \times M$ which is homeomorphic to M. □

3. **Introduction to the proof of Lemma B.** The proof of Lemma B will be given in §§3, 4, 5, and 6. It will be shown that the homeomorphism H of Lemma B can effectively be defined on $s \times U \setminus r_0^{-1}(0)$ instead of on the variable product $s \times^{r_0} U$. In fact, H will ultimately be described by means of various interchanges of coordinates on $(y_1, y_2, \ldots, z_1, z_2, \ldots) \in s \times U \setminus r_0^{-1}(0)$ where $(y_1, y_2, \ldots) \in s$ and $(z_1, z_2, \ldots) \in U \setminus r_0^{-1}(0)$. In order that H be continuous and a variable product we shall also shrink the coordinates in the y_i coordinate places. We now become more explicit in our discussion.

Let $\{\alpha_i\}_{i>0}$ be a collection of disjoint infinite sets of integers whose union is the set of positive integers such that if the elements of each α_i are monotonically ordered as $(i, 1), (i, 2), \ldots$, then $(i, j) < (k, j)$ for $i < k$. The interchangings and shrinkings of coordinates are to occur only within the systems $(y_i, z_{(i,1)}, z_{(i,2)}, \ldots)$ but on all such systems simultaneously. Indeed, the interchanges and shrinkings to be described will be independent of i and thus may be described simply by describing for any $i>0$ the procedure for $(y_i, z_{(i,1)}, z_{(i,2)}, \ldots)$. If $p = (y_1, y_2, \ldots, z_1, z_2, \ldots) \in s \times s$, let $p^i = (y_i, z_{(i,1)}, z_{(i,2)}, \ldots)$ and for notational simplicity we shall refer to such a sequence as $(x_0, x_1, x_2, \ldots)$. Thus, $\{p^i : p \in s \times s\} = J_i^0 \times \prod_{j>0} J_{(i,j)}^0$ and we will denote this by X_0^∞. At our convenience we shall regard X_0^∞ as $\prod_{j \geq 0} X_j$ where $X_0 = J_i^0$ and for $j>0$, $X_j = J_{(i,j)}^0$.

To analyze the types of coordinates of points $(y_1, y_2, \ldots, z_1, z_2, \ldots)$ in $s \times (U \setminus r_0^{-1}(0))$ we note for $i>0$, that y_i ranges over the interval $(-1, 1)$ whereas the z_i's are restricted by the requirement that $(z_1, z_2, \ldots) \in U \setminus r_0^{-1}(0)$. However, since such a set is open in s, for a fixed point $(z_1, z_2, \ldots)$ in $U \setminus r_0^{-1}(0)$ there exists an integer n such that $\{z_1\} \times \cdots \times \{z_n\} \times \prod_{i>n} J_i^0 \subset U \setminus r_0^{-1}(0)$.

We now make some definitions. An open set E of s is an *n-basic open set in s* if $E=E_1\times\cdots\times E_n\times\prod_{i>n}J_i^0$ where each E_i is open in J_i^0 and is a subinterval but not necessarily a proper subinterval of J_i^0. E is a *basic* open set in s if E is an n-basic open set in s for some n. We correspondingly define an *n-basic open set in* J^∞ by replacing s with J^∞ and J_i^0 with J_i in the above. Thus, if E is an n-basic open set in s (or J^∞) and $m\geqq n$ then E is also an m-basic open set in s (or J^∞). If W is a subset of s, let $\pi: s\times W\to W$ be the natural projection onto W and for $n>0$, let π_n be defined on W as follows. For $z=(z_1, z_2,\ldots)\in W$, let $\pi_n(z)=(z_1,\ldots,z_n, 0, 0,\ldots)$. Note that in general π_n does not map into W but in our applications it usually does. Also, if Y is a space and $f\colon W\to Y$ is a function, define $f^*\colon s\times W\to Y$ by $f^*=f\pi$.

DEFINITION. Let W be an open subset of s and let $\{G_i\}$ be a star finite collection of m_i-basic open sets in s (that is, for each i, G_i is m_i-basic) whose union is W. For each $x\in W$, let

$$m_x = \text{minimum}\,\{m_i : x\in G_i\}.$$

Let Y be a topological space. A map, i.e. continuous function, $f\colon W\to Y$ is a *local product map of W* with respect to the G_i and m_i if $f(x)=f(\pi_{m_x}(x))$ for each $x\in W$. If, additionally, $Y=[1,\infty)$ and $f(x)\geqq m_x$ for each $x\in W$, then f is a *local product indicator map of W* with respect to the G_i and m_i.

A special case of Lemma B. We introduce, in a very special case, some of the procedures and notation to be used later. We shall assume the notation and conventions already introduced. For this special case of Lemma B we shall assume that for some $n>0$, (1) the open sets U and W are n-basic open sets in s, (2) V is the closure of an n-basic open set in s, and (3) $r_0(x)=1$ for each $x\in U$. Observe that condition (3) implies that $s\times^{r_0}U=s\times U$. In the general case we shall not have this obvious product structure with respect to U, W, and V and will have to identify suitable local product structures with respect to these sets and $r_0^{-1}(0)$. Now, consider the space X_0^∞ of points $p^i=(y_i, z_{(i,1)}, z_{(i,2)},\ldots)$ that have been relabeled $(x_0, x_1, x_2,\ldots)$. If the procedure to be described for interchanging coordinates in X_0^∞ leaves the $x_1,\ldots,x_n$ coordinates of points in X_0^∞ fixed, then the induced function will carry each point of $s\times U$ to a point of $s\times U$.

The desired homeomorphism H of $s\times U$ onto $s\times^r U$ will be expressed in terms of a map h of $X_0^\infty\times I\times\{n\}$ into X_0^∞ (with $\{n\}$ a single point set for our special case). The map h is to be an isotopy such that for $x=(x_0, x_1,\ldots)\in X_0^\infty$ we have $h(x, 0, n)=x$ and

$$h(x, 1, n) = (0, x_1,\ldots, x_n, -x_0, -x_{n+1}, -x_{n+2},\ldots).$$

To finish the description of h we first describe a map h' of $X_0^\infty\times[0,1)\times\{n\}$ into X_0^∞ and then modify h' so as to produce h. Let $h'(x, 0, n)=x$ and for each integer $i>0$, let

$$h'(x, 1-2^{-i}, n) = (x_{n+i}, x_1,\ldots, x_n, -x_0, -x_{n+1},\ldots, -x_{n+i-1}, x_{n+i+1},\ldots).$$

Note that for $k=1, 2, \ldots, n, n+i+1, n+i+2, \ldots$, the k coordinate is fixed. In order to define h' it is merely necessary to "fill in the gaps", for the various i, for t between $1-2^{-i}$ and $1-2^{-i-1}$. Note that the coordinate formulas for $h'(x, 1-2^{-i}, n)$ and $h'(x, 1-2^{-i-1}, n)$ differ only in the original 0th and $(n+i+1)$th places. In these two places we have x_{n+i} and x_{n+i+1} in the first case and x_{n+i+1} and $-x_{n+i}$ in the second case. Thus we may "rotate" the coordinate space $X_0 \times X_{n+i+1}$ to change these two coordinates as required. Technically, since $X_0 \times X_{n+i+1}$ is an open square, we first contract the square onto the circular disc $x_0^2 + x_{n+i+1}^2 < 1$, rotate the disc clockwise by a quarter turn and then expand it to the open square. For each $t \in [1-2^{-i}, 1-2^{-i-1}]$, the 0th and $(n+i+1)$th coordinates of $h'(x, t, n)$ are to be the induced combination of x_{i+1} and x_{n+i+1} obtained by linearly identifying t with the appropriate stage of the rotation. For $t \in [1-2^{-i}, 1-2^{-i-1}]$ all other coordinates are those of $h'(x, 1-2^{-i}, n)$.

Finally to obtain h from h' for $0<t<1$ we scale down the 0th coordinate by a factor of $(1-t)$. That is, define $\mu: X_0^\infty \times I \to X_0^\infty$ by $\mu(x, t) = ([1-t]x_0, x_1, \ldots)$ and then define h by $h(x, t, n) = \mu(h'(x, t, n), t)$. Such an h is clearly continuous.

With h defined we can now define H. First, let ϕ be a map of U onto I such that $\phi(U \setminus W) = 0$, $\phi(V) = 1$, and $\phi(z) = \phi(\pi_n(z))$ for each $z \in U$. The last condition makes ϕ a special kind of product map and such a ϕ may be defined by first defining a map ϕ' on $\pi_n(U)$ such that $\phi'(\pi_n(z)) = 0$ for $z \in U \setminus W$ and $\phi'(\pi_n(z)) = 1$ for $z \in V$. Then define ϕ by $\phi = \phi' \pi_n$.

We now define $H: s \times U \to s \times U$ as follows. For $p = (y_1, y_2, \ldots, z_1, z_2, \ldots) \in s \times U$, recall that $p^i = (y_i, z_{(i,1)}, z_{(i,2)}, \ldots)$. Let $[H(p)]^i = h(p^i, \phi^*(p), n)$. It is easy to verify that the map H is a homeomorphism of $s \times U$ onto the variable product $s \times^r U$ where $r = 1 - \phi$ and furthermore $H | s \times (U \setminus W)$ is the identity.

4. **Lemma B: The general case.** For the proof of Lemma B in the general case the sets U, V, and W need not be as nice as those used above. In particular, the sets U and W might be infinite unions of sets of the form of U in the special case where the values of n increase without bound. Furthermore the closed set V need not have any obvious product structure. Thus, instead of having a fixed n we have to introduce a new variable into our isotopy function h which will allow us to pick out an n for a particular local product structure and to continuously vary this value as the local product structure changes. Hence, we want to define a map

$$h: X_0^\infty \times I \times [1, \infty) \to X_0^\infty$$

such that for each integer $n \in [1, \infty)$ and each $x = (x_0, x_1, \ldots) \in X_0^\infty$, as t varies from 0 to 1, $h(x, t, n)$ goes through the same motion as described in the special case above taking x at time $t = 1-2^{-i}$ to the point

$$(2^{-i}x_{n+i}, x_1, \ldots, x_n, -x_0, -x_{n+1}, \ldots, -x_{n+i-1}, x_{n+i+1}, \ldots)$$

and to $(0, x_1, \ldots, x_n, -x_0, -x_{n+1}, \ldots)$ at $t=1$. Also we specify that for any $n \leqq u < n+1$ and any $t \in I$, $h(x, t, u)$ has the same x_1 to x_n coordinates as x and that the coordinate in the 0th place has been shrunk by a factor of $1-t$. Finally, for h to be continuous at $t=1$ we specify for $n \leqq u < n+1$ that $h(x, 1, u)$ is the appropriate intermediate value between $h(x, 1, n)$ and $h(x, 1, n+1)$. These two points differ only in the $(n+1)$th and $(n+2)$th coordinate places. In these places we have $-x_0$ and $-x_{n+1}$ in the first case and x_{n+1} and $-x_0$ in the second case and for the intermediate value u we take the appropriate "rotation" on these two coordinates leaving all other coordinates the same.

In the rest of this section we will prove Lemma B except for the proofs of Lemmas C, E, and F. The proof of Lemma C will be postponed to §6 and amounts to the construction of the map h described above. Lemmas E and F will be proved in §5 and will assert the existence of appropriate local product maps ϕ and g that will provide the values for the variables t and u, respectively, in the definition of the homeomorphism H. We are now ready to state

LEMMA C. *There exists a map*

$$h\colon X_0^\infty \times I \times [1, \infty) \to X_0^\infty$$

such that if $t \in I$ *and* $u \in [1, \infty)$ *are fixed where* $n \leqq u$, *then the map* $H\colon s \times s \to s \times s$ *defined by* $[H(p)]^i = h(p^i, t, u)$ *for* $p \in s \times s$ *is a homeomorphism of* $s \times s$ *onto* $s \times^r s$ *where* (1) r *is the constant function* $1-t$, (2) *if* $t=0$, *then* H *is the identity, and* (3) $\pi_n^* = \pi_n^* H$.

We will now modify Lemma C by using local product maps ϕ and g in place of the t and u respectively.

LEMMA D. *Let* W *be any open subset of* s, *let* $s \times^{r_0} W$ *be a variable product of* s *by* W *where* $r_0(x) > 0$ *for each* $x \in W$, *and let* $\{G_i\}$ *be a star finite collection of* m_i*-basic open sets covering* W. *Let* $\phi\colon W \to I$ *and* $g\colon W \to [1, \infty)$ *be local product and local product indicator maps of* W, *respectively, with respect to the* G_i *and* m_i. *There exists an onto homeomorphism*

$$H\colon s \times^{r_0} W \to s \times^r W$$

where (1) $r = (1-\phi)r_0$ *and* (2) $H | s \times^{r_0} \phi^{-1}(0)$ *is the identity.*

Proof. Clearly the map k from $s \times^{r_0} W$ to $s \times W$ defined by $k(y, z) = (y r_0^{-1}(z), z)$ is an onto homeomorphism. Let h be the map of Lemma C and define $H_1\colon s \times W \to s \times s$ by $[H_1(p)]^i = h(p^i, \phi^*(p), g^*(p))$ for $i > 0$ and $p \in s \times W$. By condition (3) of Lemma C together with the local product map properties with respect to the G_i and m_i we observe (1) that H_1 maps $s \times W$ onto $s \times^{r_1} W$ where $r_1 = 1-\phi$ and (2) that $\phi^* = \phi^* H_1$ and $g^* = g^* H_1$. Property (1) follows since if $p \in s \times W$ where $n \leqq g^*(p)$, then H_1 carries the set $(\pi_n^*)^{-1}\pi_n^*(p)$ onto itself and (2) follows since $\phi^*(p) = \phi\pi(p) = \phi\pi_n\pi(p) = \phi\pi_n^*(p) = \phi\pi_n^* H_1(p) = \phi^* H_1(p)$ and similarly, for g^*. We now show that H_1 has a continuous inverse.

Define G from $s\times^{r_1} W$ to $s\times W$ as follows. Let $b\in s\times^{r_1} W$ and let $H_b\colon s\times W \to s\times^{r_b} s$ be defined by

$$[H_b(p)]^i = h(p^i, \phi^*(b), g^*(b)) \quad \text{for } i>0 \text{ and } p\in s\times W,$$

where $r_b = 1-\phi^*(b)$. Let $x=G(b)=H_b^{-1}(b)$. Since $\phi^*=\phi^* H_1$ and $g^*=g^* H_1$ we have $H_1(x)=b$. Hence, it is clear that G is the inverse of H_1 and G is continuous since h, ϕ, and g are. Thus, H_1 is a homeomorphism. We now observe that H defined on $s\times^{r_0} W$ by $k^{-1}H_1 k$ is a homeomorphism onto $s\times^{r} W$ where $r=(1-\phi)r_0$ and that condition (2) of Lemma D is clear. □

The next lemma will guarantee us that the proper kind of ϕ function can be constructed.

LEMMA E. *Let U be an open subset of s and let $V\subset W\subset U$ and $A\subset U$ where W is open and V and A are closed relative to U. There exists a countable star finite collection $\{G_i\}$ of m_i-basic open sets in s whose union is $W\backslash A$ and a map $\phi\colon U\backslash A\to I$ such that $\phi(V\backslash A)=1$, $\phi((U\backslash W)\backslash A)=0$ and $\phi|W\backslash A$ is a local product map of $W\backslash A$ with respect to the G_i and m_i.*

Proof. (See §5.)

The next lemma asserts the existence of the proper kind of g function. Assume the same hypothesis as in Lemma E.

LEMMA F. *The collections $\{G_i\}$ and $\{m_i\}$ of Lemma* E *can be chosen such that there exists a local product indicator map $g\colon W\backslash A\to[1,\infty)$ with respect to the G_i and m_i where g is unbounded near A, that is, for any $x\in A\cap \mathrm{Cl}\,(W\backslash A)$ and any $n>0$, there is a neighborhood $B(x)$ such that $g|(W\backslash A)\cap B(x)>n$. (By* Cl *$C$ we mean the closure of C.)*

Proof. (See §5.)

We now restate Lemma B and prove it on the basis of the apparatus we have set up.

LEMMA B. *Let U be an open subset of s, let $V\subset W\subset U$ where W is open and V is closed in U, and let $s\times^{r_0} U$ be a variable product of s by U. There exists a homeomorphism H of $s\times^{r_0} U$ onto a variable product $s\times^{r} U$ such that* (1) $r\leqq r_0$, (2) $r(V)=0$, *and* (3) *$H|s\times^{r_0}[(U\backslash W)\cup r_0^{-1}(0)]$ is the identity.*

Proof. By Lemmas E and F take a star finite collection $\{G_i\}$ of m_i-basic open sets and maps ϕ and g satisfying the conditions of the lemmas for the case where $A=r_0^{-1}(0)$.

Now let H' be the homeomorphism H of Lemma D defined with respect to $W\backslash r_0^{-1}(0)$, ϕ, and g. Define H on $s\times^{r_0} U$ by extending H' to the rest of $s\times^{r_0} U$ with the identity function. Thus, if this extension is continuous, then H will be a homeomorphism of $s\times^{r_0} U$ onto $s\times^{r} U$ where $r=(1-\phi)r_0$ on $W\backslash r_0^{-1}(0)$ and $r=r_0$ on $(U\backslash W)\cup r_0^{-1}(0)$. We now show that this extension is continuous. Since

$\phi((U\backslash W)\backslash r_0^{-1}(0))=0$, by condition (2) of Lemma D the identity map on $s\times^{r_0}[(U\backslash W)\backslash r_0^{-1}(0)]$ and $H|s\times(W\backslash r_0^{-1}(0))$ are compatible. To show that these are compatible with the identity on $s\times^{r_0}r_0^{-1}(0)$ we will check the coordinate-wise continuity of H. The continuity of r_0 gives the continuity of H on the first, or s, coordinate and g becoming unbounded near $r_0^{-1}(0)$ yields the continuity of H on the second, or U, coordinate. Since conditions 1 and 2 of the statement of Lemma B are clear the proof is complete. □

5. **Proofs of Lemmas E and F.** Before proving Lemmas E and F we shall need a definition and another lemma.

Let E be a basic open set in s and let $n=\min\{i : E \text{ is } i\text{-basic}\}$. Then $E=E_1\times\cdots\times E_n\times\prod_{i>n}J_i^0$ where each E_i is an open subinterval of J_i^0. We say that $E^+=E_1\times\cdots\times E_n\times\prod_{i>n}J_i$ is the n-basic open set in J^∞ *associated* with E. Note that $E=E^+\cap s$.

LEMMA G. *Let U be an open subset of s and let $V\subset W\subset U$ and $A\subset U$ where W is open and V and A are closed in U. There exist countable collections $\{W_i\}_{i\geq 0}$ and $\{G_i\}_{i>0}$ of open sets in U such that*

(1) *$V\subset W_0$, $\text{Cl } W_i\subset W_{i+1}\subset W$ for each $i\geq 0$ and the union of the W_i's is W.*

(2) *$G=\{G_i\}_{i>0}$ is a star finite collection of basic open sets in s whose union is $W\backslash A$ such that for each i, $\text{Cl } G_i\subset W\backslash A$ and if for some j, $\text{Cl } G_j$ intersects $(\text{Cl } W_i)\backslash W_{i-1}$, then $\text{Cl } G_j\subset W_{i+1}\backslash\text{Cl } W_{i-2}$.*

Proof. We assert the existence of the W_i on the basis of standard elementary techniques of point-set topology. If B is a subset of U, let B' denote $B\backslash A$. Now, for each $x\in(\text{Cl } W_1)'$ let G_x be a basic open set in s contained in W_2'. For $i>1$ and for each $x\in(\text{Cl } W_i)'\backslash W_{i-1}$ let G_x be a basic open set in s contained in $W_{i+1}'\backslash\text{Cl } W_{i-2}$. Let H be the collection of all the G_x. The space s is naturally imbedded in J^∞. For each G_x let G_x^+ be the basic open set in J^∞ associated with G_x and let W^+ be the union of all such G_x^+. Note that $W\backslash A=W^+\cap s$. Let

$$F=\{B \mid B \text{ is basic open in } J^\infty \text{ and for some } G_x\in H,\ \text{Cl } B\subset G_x^+\}$$

and let $\{M_i\}_{i\geq 0}$ be a sequence of compact subsets of J^∞ such that for each $i\geq 0$, $M_i\subset M_{i+1}^0$ and $\bigcup_{i\geq 0}M_i=W^+$. Cover M_1 with elements of F whose closures are contained in M_2^0 and by compactness extract a finite subcover F_1. For $i>1$, cover $\text{Cl }(M_i\backslash M_{i-1})$ with elements of F whose closures are contained in $M_{i+1}^0\backslash M_{i-2}$ and by compactness extract a finite subcover F_i. Thus, one can find a countable star finite open cover G^+ of W^+ with basic open sets in J^∞. Now let G be the collection of intersections of the elements of G^+ and s. Note that the closure of each element of G is contained in $W\backslash A$. □

We are now ready for the

Proof of Lemma E. Let $\{W_i\}_{i\geq 0}$ and $\{G_i\}_{i>0}$ be as in the conclusion of Lemma G.

Let

$$B_1 = \{G_i : \mathrm{Cl}\, G_i \cap V \neq \varnothing\}$$

and for $k > 1$, let

$$B_k = \{G_i : G_i \notin \bigcup_{q=1}^{k-1} B_q \text{ and there exists } G_j \in B_{k-1} \text{ such that } (\mathrm{Cl}\, G_i) \cap (\mathrm{Cl}\, G_j) \neq \varnothing\}.$$

For each G_i, let $n(G_i) = \min\{n : G_i \text{ is } n\text{-basic}\}$ and $m(G_i) = \max\{n(G_j) : \mathrm{Cl}\, G_i \cap \mathrm{Cl}\, G_j \neq \varnothing\}$. Now, let $\{G_{2,i}\}$ be an enumeration of B_2 and inductively pick $m_{2,i} \geqq m(G_{2,i})$ such that $m_{2,i+1} \geqq m_{2,i}$. For $k > 2$ let $\{G_{k,i}\}$ be an enumeration of B_k and pick $m_{k,i} \geqq m(G_{k,i})$ such that $m_{k,i+1} \geqq m_{k,i}$ and if $\mathrm{Cl}\, G_{k,i} \cap \mathrm{Cl}\, G_{k-1,j} \neq \varnothing$, then $m_{k,i} \geqq m_{k-1,j}$. Now, if $G_i = G_{k,j}$ for some k and j, let $m_i = m_{k,j}$ and if $G_i \notin \bigcup_{j>1} B_j$, let $m_i = m(G_i)$.

Define ϕ to be 1 on the union of the closures of the elements of B_1 and extend ϕ to the union of the closures of the elements of B_2 by induction on the i in $\{G_{2,i}\}$. The induction step is the same as the first step so we shall only give the induction step. For $i = k$, use the Tietze extension theorem to extend ϕ to $\mathrm{Cl}\, G_{2,k}$ such that if $x \in \mathrm{Cl}\, G_{2,k} \cap \mathrm{Cl}\, G_{3,j}$, for some j, then $\phi(x) = \frac{1}{2}$, and if $x \in G_{2,k}$ then $1/2 \leqq \phi(x) \leqq 1$ and $\phi(x) = \phi(\pi_{m_{2,k}}(x))$. This is possible since you first do it on the finite dimensional cell $\pi_{m_{2,k}}(\mathrm{Cl}\, G_{2,k})$ and then extend to all of $\mathrm{Cl}\, G_{2,k}$ using the product structure. Extend ϕ to the rest of $\bigcup_{k>1,i>0} \mathrm{Cl}\, G_{k,i}$ such that if $x \in \mathrm{Cl}\, G_{k,i} \cap \mathrm{Cl}\, G_{k+1,j}$, for some j, then $\phi(x) = 2^{-k+1}$, and if $x \in \mathrm{Cl}\, G_{k,i}$, then $2^{-k+1} \leqq \phi(x) \leqq 2^{-k+2}$ and $\phi(x) = \phi(\pi_{m_{k,i}}(x))$. Let $\phi(x) = 0$ for $x \in (U \setminus A) \setminus \bigcup_{k>0,i>0} \mathrm{Cl}\, G_{k,i}$. Condition (2) of Lemma G guarantees, except for the case when $V = W = U$, that ϕ is continuous. If $V = U$, then ϕ is the constant function 1. Thus, in any case we have satisfied the conditions of the lemma. □

Proof of Lemma F. In the proof of the last lemma we could just as well have chosen the m_i such that for each $i > 0$, $m_{i+1} \geqq m_i$ and that the m_i increase without bound. Assume this had been done. We now construct a local product indicator map $g: W \setminus A \to [1, \infty)$. For each $i > 0$, let C_i denote the closure of G_i in J^∞, recalling that $G_i \subset W \setminus A \subset s \subset J^\infty$. As a matter of convenience we shall define g on the union of the C_i and then restrict the function to $W \setminus A$ which is the common part of s and this union.

For each $i > 0$, let $r_i = \text{maximum}\{m_j : C_j \cap C_i \neq \varnothing\}$. Define g to be r_1 on C_1. We now extend to the rest of the C_i by induction. Assume that $k > 1$ is an integer and that g has been extended to $\bigcup_{i=1}^{k-1} C_i$ such that if $x \in \bigcup_{i=1}^{k-1} C_i$ and $m_j = \min\{m_i : x \in C_i, i = 1, \ldots, k-1\}$, then $g(x) = g(\pi_{m_j}(x))$ and $g(x) \geqq r_j \geqq m_j$. To extend g to C_k we have three cases: (1) g has been defined on C_k, (2) g has been defined for no point of C_k and (3) g has been defined for a proper nonvoid subset of C_k. If (1), proceed to C_{k+1}. If (2), define g to be r_k on C_k. If (3), proceed as follows. Let H_k be the set of all C_i that intersect C_k and such that g has been defined for no point of this intersection. Define g to be r_k on $C_i \cap C_k$, for each $C_i \in H_k$. Now take the set of all C_i not in H_k, $i \geqq k$, that intersect C_k and such that g has been defined for some point of this intersection and let D_k be the set of all intersections of C_k with intersections of such C_i. Each element B of D_k is the intersection of a certain maximum number of the C_i; call this number the *index* of B. List the elements of D_k as $B_1, \ldots, B_n$, according to nonincreasing index. It should be noted that $B_n = C_k$ and that, from the definition of the m_i, each B_i is m_k-basic. We now extend g inductively to the B_i. If g has been defined on all of B_1, proceed to B_2. If g has been defined for no point of B_1, let g be equal to r_k on B_1, and if g has been defined on a proper nonvoid subset of B_1, extend g as follows. Use the Tietze extension theorem to extend g to $\pi_{m_k}(B_1)$ where the range of g is the interval bounded by the maximum and minimum of the functional values assigned to points of B_1. Now, extend g to the rest of B_1 using the product structure. Extend g inductively to the rest of the B_i which, at the last stage, includes C_k. The induction step is the same as the first step.

Thus, by induction we have extended g to the union of the C_i. We now restrict g to $W \backslash A$. By construction, g is a local product indicator map with respect to the G_i and m_i and since the closure in s of each G_i is contained in $W \backslash A$ and since the m_i are unbounded, then g is unbounded near A. □

6. **Proof of Lemma C.** The proof of Lemma C consists of constructing the map

$$h\colon X_0^\infty \times I \times [1, \infty) \to X_0^\infty$$

with the required properties. In §3 we have already described $h|X_0^\infty \times I \times \{n\}$ for $n \geqq 1$. Thus in defining h our problem will be that of extending the function we already have to $h|X_0^\infty \times I \times [n-1, n]$ for $n \geqq 2$. Let us denote this restriction of h by h_n. Following the pattern of §3 we shall first describe h'_n and then modify it to produce h_n. According to our description of h' in §3 we have the following chart for $x = (x_0, x_1, \ldots) \in X_0^\infty$, $t \in I$ and $u \in [n-1, n]$ where y denotes $(x_1, \ldots, x_{n-1})$.

$u \backslash t$	0	1/2	3/4	7/8
n	x	$(x_{n+1}, y, x_n, -x_0, x_{n+2}, \ldots)$	$(x_{n+2}, y, x_n, -x_0, -x_{n+1}, x_{n+3}, \ldots)$	$(x_{n+3}, y, x_n, -x_0, -x_{n+1}, -x_{n+2}, x_{n+4}, \ldots)$
$n-1$	x	$(x_n, y, -x_0, x_{n+1}, x_{n+2}, \ldots)$	$(x_{n+1}, y, -x_0, -x_n, x_{n+2}, x_{n+3}, \ldots)$	$(x_{n+2}, y, -x_0, -x_n, -x_{n+1}, x_{n+3}, x_{n+4}, \ldots)$

FIGURE 1

We will describe three maps

(1) $h'_{n,0}\colon X_0^\infty \times [0, 1/2] \times [n-1, n] \to X_0^\infty$,

(2) $h'_{n,1}\colon X_0^\infty \times [1/2, 3/4] \times [n-1, n] \to X_0^\infty$ and

(3) $h'_{n,2}\colon X_0^\infty \times [3/4, 7/8] \times [n-1, n] \to X_0^\infty$,

and then show how to appropriately modify $h'_{n,2}$ to form

$$h'_{n,i}: X_0^\infty \times [1-2^{-i}, 1-2^{-i-1}] \times [n-1, n] \to X_0^\infty$$

for $i > 3$, where the $h'_{n,j}$ patched together will yield h'_n.

Description of $h'_{n,0}$. Let ρ'_n be the homeomorphism that takes $X_0 \times X_n \times X_{n+1}$ onto the unit 3-ball $B_n = \{(x_0, x_n, x_{n+1}) : x_0^2 + x_n^2 + x_{n+1}^2 < 1\}$ by shrinking linearly along radii. Let $\rho_n: X_0^\infty \to X_0^\infty$ be defined by $\rho_n = \rho'_n \times \mathrm{id}$ where id is the identity function of $\prod_{i \neq 0,n,n+1} X_i$. We now define a map

$$h''_{n,0}: B_n \times [0, 1/2] \times [n-1, n] \to B_n$$

as follows. Consider t and u to be elements of $[0, 1/2]$ and $[n-1, n]$ respectively. For $t=0$ and $n-1 \leqq u \leqq n$ let $h''_{n,0}$ be the identity. For $u=n-1$, as t goes from 0 to 1/2 let the x_0 and x_n coordinates be rotated (using sine and cosine functions) so that (x_0, x_n, x_{n+1}) is rotated to $(x_n, -x_0, x_{n+1})$. For $u=n$, as t goes from 0 to 1/2 let the x_0 and x_{n+1} coordinates be rotated so that (x_0, x_n, x_{n+1}) is rotated to $(x_{n+1}, x_n, -x_0)$.

Fill in the rest of the isotopy as follows. For $n-1 < u < n$ take a new axis x_u that is proportionately between x_n and x_{n+1}, see Figure 3, and then rotate x_u and x_0.

We will describe the resultant rotation for $t = 1/2$ and as u goes from $n-1$ to n. See Figure 2. This rotation is a double rotation, that is, starting with $(x_n, -x_0,$

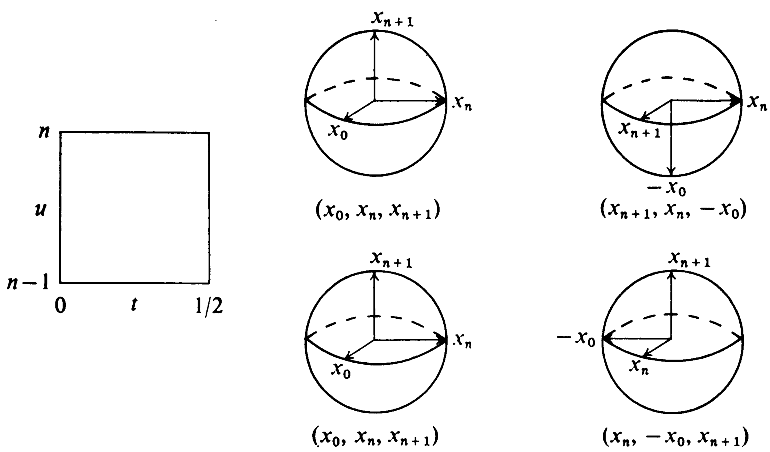

FIGURE 2

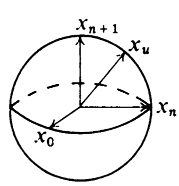

FIGURE 3

x_{n+1}), as x_{n+1} and x_n are being rotated, so are $-x_0$ and x_n. Thus, $(x_n, -x_0, x_{n+1})$ ends up as $(x_{n+1}, x_n, -x_0)$. It should be noted that an explicit formula for $h''_{n,0}$ can be displayed as appropriate combinations of the sine and cosine functions.

Now, let $\hat{h}_{n,0}=h''_{n,0}\times \text{id}$ where id is the identity function of $\prod_{i\neq 0,n,n+1} X_i$. Then $h'_{n,0}=\rho_n^{-1}\hat{h}_{n,0}\rho_n$.

Description of $h'_{n,1}$. Let σ'_n be the homeomorphism that takes $X_0\times X_n\times X_{n+1}\times X_{n+2}$ onto the unit 4-ball

$$C_n = \{(x_0, x_n, x_{n+1}, x_{n+2}) : x_0^2+x_n^2+x_{n+1}^2+x_{n+2}^2 < 1\}$$

by shrinking linearly along radii. Let σ_n: $X_0^\infty \to X_0^\infty$ be defined by $\sigma_n=\sigma'_n\times \text{id}$ where id is the identity function of $\prod_{i\neq 0,n,n+1,n+2} X_i$. We now define

$$h''_{n,1}: C_n\times [1/2, 3/4]\times [n-1, n]\to C_n$$

as follows. For $x=(x_0, x_n, x_{n+1}, x_{n+2})\in C_n$, let $h''_{n,1}(x, 1/2, n-1)=(x_n, -x_0, x_{n+1}, x_{n+2})$ and for $t=1/2$ as u varies from $n-1$ to n let $(x_n, -x_0, x_{n+1}, x_{n+2})$ go through the double rotation to $(x_{n+1}, x_n, -x_0, x_{n+2})$ as described in the definition of $h''_{n,0}$ (applied to the first three coordinates of x, keeping the x_{n+2} coordinates fixed). See Figure 4. For $u=n-1$ and as t goes from 1/2 to 3/4 let x_n and x_{n+1} be rotated

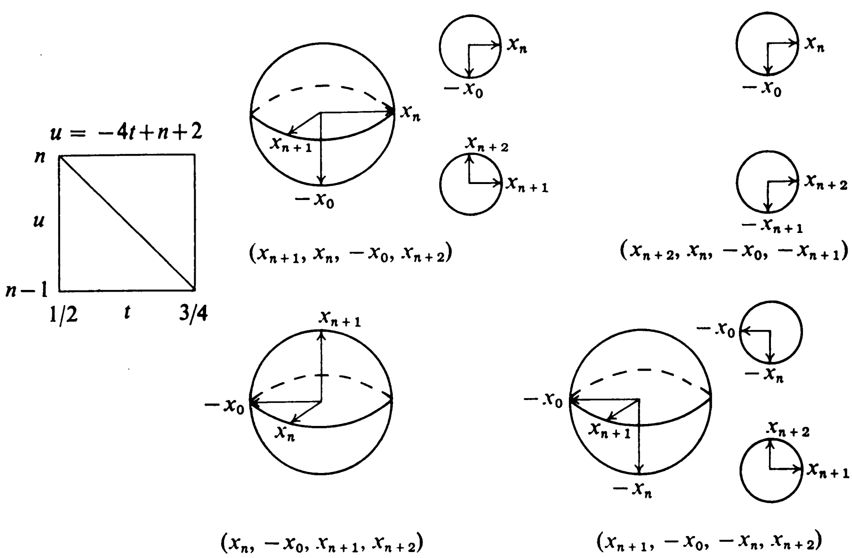

FIGURE 4

such that $(x_n, -x_0, x_{n+1}, x_{n+2})$ is rotated to $(x_{n+1}, -x_0, -x_n, x_{n+2})$. We now describe a rotation corresponding to the diagonal $\{(t, u) : 1/2\leqq t\leqq 3/4,\ u=-4t+n+2\}$ of $[1/2, 3/4]\times[n-1, n]$. As (t, u), where $u=-4t+n+2$, varies from

$(1/2, n)$ to $(3/4, n-1)$, the x_n and $-x_0$ coordinates are rotated so that $(x_{n+1}, x_n, -x_0, x_{n+2})$ is rotated to $(x_{n+1}, -x_0, -x_n, x_{n+2})$. Note that the double rotation corresponding to the path determined by $1/2\times[n-1, n]$ is a combination of the rotations corresponding to the path determined by $[1/2, 3/4]\times\{n-1\}$ and the diagonal. Hence, the map $h''_{n,1}$ extends to the interior of the corresponding triangle. For $u=n$ as t varies from 1/2 to 3/4, rotate the x_{n+1} and x_{n+2} coordinates so that $(x_{n+1}, x_n, -x_0, x_{n+2})$ rotates to $(x_{n+2}, x_n, -x_0, -x_{n+1})$. For $t=3/4$ as u varies from $n-1$ to n we have a double rotation; $-x_0$ and $-x_n$ are rotated and independently but at the same time $-x_{n+1}$ and x_{n+2} are rotated so that $(x_{n+1}, -x_0, -x_n, x_{n+2})$ is rotated to $(x_{n+2}, x_n, -x_0, -x_{n+1})$. Note, for the triangle above the diagonal, that the rotations break up into pairs corresponding to Figure 5. Hence

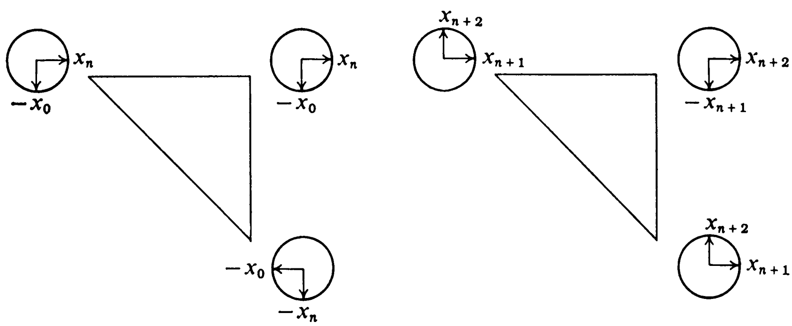

FIGURE 5

we can extend $h''_{n,1}$ to this triangle and the definition of $h''_{n,1}$ is complete. Now let $\hat{h}_{n,1}=h''_{n,1}\times\text{id}$ where id is the identity function of $\prod_{i\neq 0,n,n+1,n+2} X_i$. Then $h'_{n,1}=\sigma_n^{-1}\hat{h}_{n,1}\sigma_n$.

Description of $h'_{n,2}$. Let τ'_n be the homeomorphism of $X_0\times X_n\times X_{n+1}\times X_{n+2}\times X_{n+3}$ onto

$$D_n = \{(x_0, x_n, x_{n+1}, x_{n+2}, x_{n+3}) : x_n^2+x_{n+1}^2 < 1 \quad\text{and}\quad x_0^2+x_{n+2}^2+x_{n+3}^2 < 1\}$$

by shrinking linearly along radii. Note that D_n is homeomorphic to the cartesian product of the 2-ball $E_n=\{(x_n, x_{n+1}) : x_n^2+x_{n+1}^2<1\}$ and the 3-ball

$$F_n = \{(x_0, x_{n+2}, x_{n+3}) : x_0^2+x_{n+2}^2+x_{n+3}^2 < 1\}.$$

At our convenience we may think of D_n as it is defined or as $E_n\times F_n$. Let $\tau_n\colon X_0^\infty\to X_0^\infty$ be defined by $\tau_n=\tau'_n\times\text{id}$ where id is the identity function on $\prod_{i\neq 0,n,n+1,n+2,n+3} X_i$. We now define a map

$$h''_{n,2}\colon D_n\times[3/4, 7/8]\times[n-1, n]\to D_n$$

as follows. For $x=(x_0, x_n, x_{n+1}, x_{n+2}, x_{n+3}) \in D_n$, let

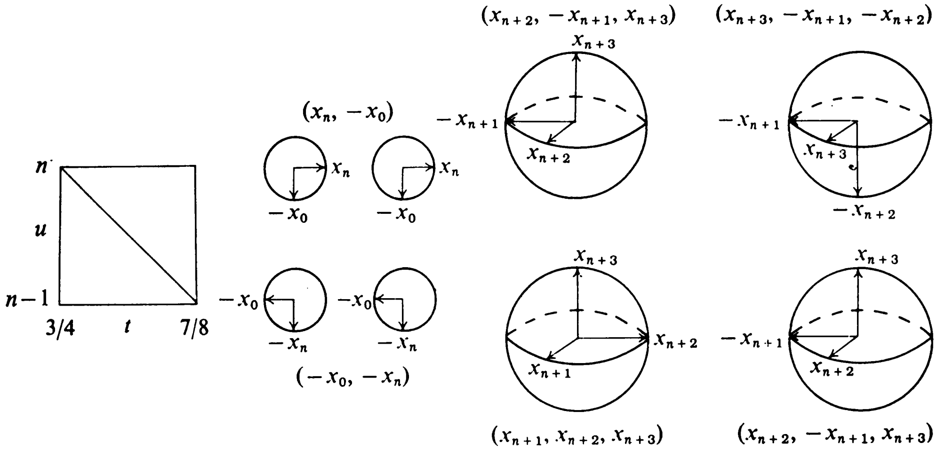

FIGURE 6

$h''_{n,2}(x, 3/4, 0) = (x_{n+1}, -x_0, -x_n, x_{n+2}, x_{n+3}) = ((-x_0, -x_n), (x_{n+1}, x_{n+2}, x_{n,3}))$ and for $t=3/4$, as u varies from $n-1$ to n, take the double rotation described in the definition of $h''_{n,1}$. That is $(x_{n+1}, -x_0, -x_n, x_{n+2}, x_{n+3})$ is rotated to $(x_{n+2}, x_n, -x_0, -x_{n+1}, x_{n+3})$. Note that the rotations are done independently in the 2-ball and 3-ball respectively, and in the 3-ball the rotation is done only in the x_{n+1}, x_{n+2}-plane leaving the x_{n+3} coordinate fixed. We now describe the rotations on the 2-ball. If $3/4 \leqq t \leqq 7/8$, as u varies from $n-1$ to n rotate $(-x_0, -x_n)$ to $(x_n, -x_0)$. That is, for each value of t, as u varies from $n-1$ to n we have the same rotation. We next describe the rotations on the 3-ball. For $u=n-1$, as t varies from 3/4 to 7/8 we rotate $(x_{n+1}, x_{n+2}, x_{n+3})$ to $(x_{n+2}, -x_{n+1}, x_{n+3})$. Note that this is the same rotation we have for $t=3/4$ as u varies from $n-1$ to n. We now fill in the other two legs of the rectangle. That is, for $u=n$, as t varies from 3/4 to 7/8 we have $(x_{n+2}, -x_{n+1}, x_{n+3})$ rotated to $(x_{n+3}, -x_{n+1}, -x_{n+2})$ and for $t=3/4$, as u varies from $n-1$ to n we have the same rotation. Since both paths from $(3/4, n-1)$ to $(7/8, n)$ induce the same rotations, the map $h''_{n,2}$ is extendible to the interior of the rectangle.

Let $\hat{h}_{n,2}=h''_{n,2}\times \text{id}$ where id is the identity function on $\prod_{i\neq 0,n,n+1,n+2,n+3} X_i$. Then $h'_{n,2}=\tau_n^{-1}\hat{h}_{n,1}\tau_n$.

We shall now describe what we mean by appropriately modified copies of $h'_{n,2}$ for the subintervals $[1-2^{-i}, 1-2^{-i-1}]$, $i\geqq 3$. We describe the required rotation. For the nth and $(n+1)$th coordinate places we want $(-x_0, -x_n)$ rotated to $(x_n, -x_0)$ as u varies from $n-1$ to n. Otherwise we want to act on the 0th, $(n+i)$th and $(n+i+1)$th coordinate places where the points

$$(x_{n+i}, -x_{n+i-1}, x_{n+i+1}) \quad (x_{n+i+1}, -x_{n+i-1}, -x_{n+i})$$
$$(x_{n+i-1}, x_{n+i}, x_{n+i+1}) \quad (x_{n+i}, -x_{n+i-1}, x_{n+i+1})$$

correspond respectively to the lattice points of $[1-2^{-i}, 1-2^{-i-1}]\times[n-1, n]$. These rotations for the case $i=2$ are precisely what $h''_{n,2}$ was designed to do. Indeed, in defining $h'_{n,2}$ we could just as well have defined, for $i>2$,

$$h'_{n,i}\colon X_0^\infty\times[1-2^{-i}, 1-2^{-i-1}]\times[n-1, n]\to X_0^\infty.$$

It should be noted for $t=1-2^{-i}$ ($i\geqq 3$), $x\in X_0^\infty$ and $n-1\leqq u\leqq n$, that $h'_{n,i-1}(x, t, u)=h'_{n,i}(x, t, u)$.

The map h_n. Let

$$h'_n\colon X_0^\infty\times[0, 1)\times[n-1, n]\to X_0^\infty$$

be the map obtained by patching together the $h'_{n,j}$ for $j\geqq 0$. To scale down the 0th coordinate we define $\mu\colon X_0^\infty\times I\to X_0^\infty$ as follows. For $x=(x_0, x_1, \ldots)\in X_0^\infty$ and $t\in I$, let $\mu(x, t)=([1-t]x_0, x_1, \ldots)$. From h'_n we define

$$h_n\colon X_0^\infty\times I\times[n-1, n]\to X_0^\infty$$

for $n>1$ as follows. For $x\in X_0^\infty$, $t\in[0, 1)$, and $u\in[n-1, n]$, let $h_n(x, t, u)=\mu(h'_n(x, t, u), t)$ and for $t=1$, as u varies from $n-1$ to n let $(0, x_1, \ldots, x_{n-1}, -x_0, -x_n, -x_{n+1}, \ldots)$ be rotated to $(0, x_1, \ldots, x_{n-1}, x_n, -x_0, -x_{n+1}, \ldots)$. That is, in the nth and $(n+1)$th coordinate places we have $(-x_0, -x_n)$ rotated to $(x_n, -x_0)$ leaving the other coordinates fixed. This yields a continuous h_n since in the definition of $h'_{n,i}$, for $i>1$, we used this same rotation on the nth and $(n+1)$th coordinate places together with a rotation involving the 0th, $(n+i)$th, and $(n+i+1)$th coordinates. The $1-t$ factor in the 0th coordinate place and the fact that $i\to\infty$ as $t\to 1$ imply continuity for h_n.

Thus, h_n is continuous and for $x\in X_0^\infty$, $t\in I$, and for $n>2$ we have $h_{n-1}(x, t, n-1)=h_n(x, t, n-1)$. Hence we may patch the h_n together to form

$$h\colon X_0^\infty\times I\times[1, \infty)\to X_0^\infty.$$

It is clear that if $t\in I$ and $u\in[1, \infty)$ where $n\leqq u$, then $h_0\colon X_0^\infty\to X_0^\infty$ defined by $h_0(x)=h(x, t, u)$ is a homeomorphism of X_0^∞ onto $X_0\times X_1^\infty$ such that if $t=0$, then h_0 is the identity and that the x_1 to x_n coordinates of a point $x\in X_0^\infty$ remain fixed. Hence, the map H of Lemma C is defined coordinate-wise in terms of various copies of h_0 considered with respect to disjoint sets of coordinates and thus inherits the required properties. □

ADDENDUM. In [7] David W. Henderson used Theorem I of this paper to prove that any F-manifold can be embedded as an open subset of s. With the aid of this result we obtain the following stronger version of Theorem I.

THEOREM. *Let M be any F-manifold and let U be any open cover of M. There exists a homeomorphism $h\colon s\times M\to M$ such that for each $(x, y)\in s\times M$, there exists $u\in U$ such that y and $h(x, y)$ are elements of u.*

Proof. By [7] let f be an open embedding of M into s. Let G be a countable star-finite collection of basic open sets in s whose union is $f(M)$ such that G

refines $f(U)$. Let $g: f(M) \to [1, \infty]$ be a local product indicator map of $f(M)$ with respect to G. Consider the map H of Lemma D defined with respect to $r_0 = 1$, $\phi = 1$, and g. It follows that H is a homeomorphism of $s \times f(M)$ onto $f(M)$ such that for each $(x, z) \in s \times f(M)$ we have $H(x, z)$ belonging to each element of G that also contains z. Thus $h: s \times M \to M$ defined by $h = f^{-1} \circ H \circ (\mathrm{id} \times f)$ is the required homeomorphism.

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