Critique of Hirsch's Citation Index: A Combinatorial Fermi Problem

Alexander Yong

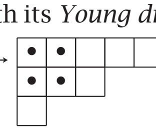
In 2005, physicist J. E. Hirsch [Hi05] proposed the *h*-index to measure the quality of a researcher's output. This metric is the largest integer n such that the person has n papers with at least n citations each, and all other papers have weakly less than n citations. Although the original focus of Hirsch's index was on physicists, the *h*-index is now widely popular. For example, *Google Scholar* and the *Web of Science* highlight the *h*-index, among other metrics such as total citation count, in their profile summaries.

An enticing point made by Hirsch is that the *h*-index is an easy and useful supplement to a citation count ($N_{\texttt{citations}}$), since the latter metric may be skewed by a small number of highly cited papers or textbooks. In his words:

> "I argue that two individuals with similar *h*'s are comparable in terms of their overall scientific impact, even if their total number of papers or their total number of citations is very different. Conversely, comparing two individuals (of the same scientific age) with a similar number of total papers or of total citation count and very different *h* values, the one with the higher *h* is likely to be the more accomplished scientist."

It seems to us that users might tend to eyeball differences of *h*'s and citation counts among individuals during their assessments. Instead, one desires a quantitative baseline for what "comparable," "very different," and "similar" actually mean. Now, while this would appear to be a matter for statisticians, we show how textbook combinatorics sheds some light on the relationship between the *h*-index and $N_{\texttt{citations}}$. We present a simple model that raises specific concerns about potential misuses of the *h*-index.

To begin, think of the list of a researcher's citations per paper in decreasing order $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N_{\text{papers}}})$ as a *partition* of size $N_{\texttt{citations}}$. Graphically, λ is identified with its *Young diagram*.

For example, $\lambda = (5, 3, 1, 0) \leftrightarrow$ .

A combinatorialist will recognize that the *h*-index of λ is the side-length of the **Durfee square** (marked using $\bullet$'s above): this is the largest $h \times h$ square that fits in λ. This simple observation is nothing new, and appears in both the bibliometric and combinatorial literature; see, e.g., [AnHaKi09, FlSe09]. In particular, since the Young diagram of size $N_{\texttt{citations}}$ with maximum *h*-index is roughly a square, we see graphically that $0 \leq h \leq \lfloor \sqrt{N_{\texttt{citations}}} \rfloor$.

Next, consider the following question:

Given $N_{\texttt{citations}}$, what is the estimated range of h?

Taking only $N_{\texttt{citations}}$ as input hardly seems like sufficient information to obtain a meaningful answer. It is exactly for this reason that we call the question a *combinatorial Fermi problem*, by analogy with usual *Fermi problems*; see the section "Combinatorial Fermi Problems."

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Table 1. Confidence intervals for h-index.

$N_{\text{citations}}$	50	100	200	300	400	500	750	1000	1250
Interval for h	[2, 5]	[3, 7]	[5, 9]	[7, 11]	[8, 13]	[9, 14]	[11, 17]	[13, 20]	[15, 22]

1500	1750	2000	2500	3000	3500	4000	4500	5000	5500
[17, 24]	[18, 26]	[20, 28]	[22, 31]	[25, 34]	[27, 36]	[29, 39]	[31, 41]	[34, 43]	[35, 45]

6000	6500	7000	7500	8000	9000	10000
[36, 47]	[37, 49]	[39, 51]	[40, 52]	[42, 54]	[44, 57]	[47, 60]

Table 2. Fields medalists 1998–2010.

Medalist	Award year	$N_{\text{citations}}$	h	Rule of thumb est.	Confidence interval
T. Gowers	1998	1012	15	17.2	[13, 20]
R. Borcherds	1998	1062	14	17.6	[14, 21]
C. McMullen	1998	1738	25	22.5	[18, 26]
M. Kontsevich	1998	2609	23	27.6	[22, 32]
L. Lafforgue	2002	133	5	6.2	[4, 8]
V. Voevodsky	2002	1382	20	20.0	[16, 23]
G. Perelman	2006	362	8	10.0	[7, 12]
W. Werner	2006	1130	19	18.2	[14, 21]
A. Okounkov	2006	1677	24	22.1	[18, 25]
T. Tao	2006	6730	40	44.3	[38, 51]
C. Ngô	2010	228	9	8.2	[5, 10]
E. Lindenstrauss	2010	490	12	12.0	[9, 14]
S. Smirnov	2010	521	12	12.3	[9, 15]
C. Villani	2010	2931	25	29.2	[24, 33]

Since we assume no prior knowledge, we consider each citation profile in an unbiased manner. That is, each partition of $N_{\text{citations}}$ is chosen with equal probability. In fact, there is a beautiful theory concerning the asymptotics of these uniform random partitions. This was largely developed by A. Vershik and his collaborators; see, e.g., the survey [Su10].

Actually, we are interested in "low" (practical) values of $N_{\text{citations}}$ where not all asymptotic results are exactly relevant. Instead, we use generating series and modern desktop computation to calculate the probability that a random λ has Durfee square size h. More specifically, we obtain Table 1 using the **Euler–Gauss identity** for partitions:

$$\prod_{i=1}^{\infty} \frac{1}{1-x^i} = 1 + \sum_{k\geq 1} \frac{x^{k^2}}{\prod_{j=1}^{k}(1-x^j)^2}. \tag{1}$$

The proof of (1) via Durfee squares is regularly taught to undergraduate combinatorics students; it is recapitulated in the section "The Euler–Gauss Identity and its Application to the h-index." The pedagogical aims of this note are elaborated upon in sections "Combinatorial Fermi Problems" and "The Euler–Gauss Identity and its Application to the h-index."

The asymptotic result we use, due to E. R. Canfield–S. Corteel–C. D. Savage [CaCoSa98], gives the mode size of the Durfee square when $N_{\text{citations}} \to \infty$. Since their formula is in line with our computations, even for low $N_{\text{citations}}$, we reinterpret their work as the rule of thumb for h-index:

$$h = \frac{\sqrt{6}\log 2}{\pi}\sqrt{N_{\text{citations}}} \approx 0.54\sqrt{N_{\text{citations}}}.$$

The focus of this paper is on mathematicians. For the vast majority of those tested, the actual h-index computed using *MathSciNet* or *Google Scholar* falls into the confidence intervals. Moreover, we found that the rule of thumb is fairly accurate for pure mathematicians. For example, Table 2 shows this for post-1998 Fields medalists.[1]

[1] *Citations pre-*2000 *in MathSciNet are not complete.* Google scholar *and Thompson Reuters'* Web of Science *also have sources of error. We decided that MathSciNet was our most complete option for analyzing mathematicians. For relatively recent Fields medalists, the effect of lost citations is reduced.*

In [Hi05] it was indicated that the h-index has predictive value for winning the Nobel Prize. However, the relation of the h index to the Fields Medal is, in our opinion, unclear. A number of the medalists' h values above are shared (or exceeded) by noncontenders of similar academic age, or with those who have similar citation counts. Perhaps this reflects a cultural difference between the mathematics and the scientific communities.

In the section "Further Comparisons with Empirical Data," we analyze mathematicians in the National Academy of Sciences, where we show that the correlation between the rule of thumb and actual h-index is $R = 0.94$. After removing book citations, $R = 0.95$. We also discuss Abel Prize winners and associate professors at three research universities.

Ultimately, readers are encouraged to do checks of the estimates themselves.

We discuss three implications/possible applications of our analysis.

Comparing h's when $N_{\text{citations}}$'s Are Very Different

It is understood that the h-index usually grows with $N_{\texttt{citations}}$. However, when are citation counts so different that comparing h's is uninformative? For example, $h_{\text{Tao}} = 40$ (6,730 citations) while $h_{\text{Okounkov}} = 24$ (1,677 citations). The model asserts the probability of $h_{\text{Okounkov}} \geq 32$ is less than 1 in 10 million. Note the *Math Genealogy Project* has fewer than 200,000 mathematicians.

These orders of magnitude predict that no mathematician with 1,677 citations has an h-index of 32, even though *technically* it can be as high as 40. Similarly, one predicts the rarity of pure mathematicians with these citations having "similar" h-index (in the pedestrian sense). This is relevant when comparing (sub)disciplines with vastly different typical citation counts. We have a theoretical caution about "eyeballing."

The Rule of Thumb and the Highly Cited

The model suggests the theoretical behavior of the h-index for highly cited scholars. The extent to which these predictions hold true is informative. This is true not only for individuals, but for entire fields as well.

Actually, Hirsch defined a proportionality constant a by $N_{\texttt{citations}} = ah^2$ and remarked, "I find empirically that a ranges between 3 and 5." This asserts h is between $\sqrt{1/5} \approx 0.45$ and $\sqrt{1/3} \approx 0.58$ times $\sqrt{N_{\texttt{citations}}}$.

One can begin to try to understand the similarity between Hirsch's empirical upper bound and the rule of thumb. A conjecture of E. R. Canfield (private communication, see the section "The Euler–Gauss Identity …") asserts concentration around the mode Durfee square. Thus, *theoretically*, one expects the rule of thumb to be nearly correct for large $N_{\texttt{citations}}$.

Alas, this is empirically not true, even for pure mathematicians. However, we observe something related: $0.54\sqrt{N_{\texttt{citations}}}$ is higher than the actual h for almost every very highly cited ($N_{\texttt{citations}} > 10,000$) scholar in mathematics, physics, computer science and statistics (among others) we considered. On the rare occasion this fails, the estimate is only beat by a small percentage (< 5 percent). The drift in the other direction is often quite large (50 percent or more is not unusual in certain areas of engineering or biology).

Near equality occurs among Abel Prize winners. We also considered all prominent physicists highlighted in [Hi05] (except Cohen and Anderson, due to name conflation in *Web of Science*). The guess is always an upper bound (on average 14–20 percent too high). Near equality is met by D. J. Scalapino (25,881 citations; 1.00), C. Vafa (22,902 citations; 0.99), J. N. Bahcall (27,635 citations; 0.98); we have given the ratio $\frac{\text{true } h}{\text{estimated } h}$.

One reason for highly cited people to have a lower than expected h-index is that they tend to have highly cited textbooks. Also, famous academics often run into the "Matthew effect" (e.g., gratuitous citations of their most well-known articles or books).

Anomalous h-Indices

More generally, our estimates give a way to flag an anomalous h-index for active researchers; i.e., those that are far outside the confidence interval or those for which the rule of thumb is especially inaccurate.

To see what effect book citations have on our estimates, consider the combinatorialist R. P. Stanley. Since Stanley has 6,510 citations, we estimate his index as 43.6. However, $h_{\text{Stanley}} = 35$, a 20 percent error. Now, 3,362 of his citations come from textbooks. Subtracting these, one estimates his h-index as 30.3, while his revised h-index is 32, only a 5 percent error. This kind of phenomenon was not uncommon; see Table 5.

For another example, consider T. Tao's *Google Scholar* profile. Since he has 30,053 citations, the rule of thumb predicts his h-index is 93.6. This is far from his actual h-index of 65. Now, his top five citations (joint with E. Candes on compressed sensing) are applied. Removing the papers on this topic leaves 13,942 citations. His new estimate is therefore 63.7 and his revised h-index is 61.

In many cases we have looked at, once the "skewing" feature of the scholar's profile is removed, the remainder of their profile agrees with the rule of thumb.

Conclusions and Summary

Whether it be Fields medalists, Abel Prize laureates, job promotion, or grant candidates, clearly, the quality of a researcher cannot be fully measured by numerics. However, in reality, the h-index is used, formally or informally, for comparisons. This paper attempts to provide a theoretical and testable framework to quantitatively understand the limits of such evaluations. For mathematicians, the accuracy of the rule of thumb suggests that the differences of the h-index between two mathematicians is strongly influenced by their respective citation counts.

While discussion of celebrated mathematicians and their statistics makes for fun coffee shop chatter, a serious way that the h-index comes up in faculty meetings concerns relatively junior mathematicians. Consider a scholar A with 100 citations and an h-index of 6 and a scholar B who has 50 citations and an h-index of 4. Such numbers are not atypical of math assistant professors going up for tenure. Our model predicts h_A to be a little bigger than h_B. Can one really discern what portion of $h_A - h_B$ is a signal of quality?

The problem becomes larger when A and B are in different subject areas. Citations for major works in applied areas tend to be more numerous than in pure mathematics. In experimental fields, papers may have many coauthors. Since the h-index does not account for authorship order, this tends to affect our estimates for such subjects.

Pure mathematicians have comparatively fewer coauthors, papers, and citations. It is not uncommon for instance, for solutions to longstanding open problems, to have relatively few citations. Thus an explanatory model for pure mathematicians has basic reasons for being divergent for some other fields. Yet, if this is the case, can the h-index really be used universally? This gives us a theoretical reason to question whether one can make simple comparisons across fields, even after a rescaling, as has been suggested in [IgPe07].

Combinatorial Fermi Problems

Usual Fermi Problems

Fermi problems are so-named after E. Fermi, whose ability to obtain good approximate quantitative answers with little data available is legendary. As an illustration, we use the following example [Co]:

How many McDonald's restaurants operate in the United States?

There are ten McDonald's in Champaign county, which has a population of about 200,000. *Assume the number of McDonald's scales with population.* Since the population of the United States is 300 million, a "back-of-the-envelope" calculation estimates the number of McDonald's at 15,000. The actual answer, as of 2012, is 14,157.

Using a simplified assumption like the italicized one above is a feature of a Fermi problem. Clearly the uniform assumption made is not really correct. However, the focus is on good, fast approximations when more careful answers are either too time consuming to determine, or maybe even impossible to carry out. The approximation can then be used in order to guide further work to determine more accurate/better justified answers.

Now, although the estimate is rather close to the actual number, when the estimate is not good, the result is even more interesting, as it helps identify a truly faulty assumption. For instance, analogous analysis predicts that the number of Whole Foods Markets in the United States is 0. Apparently, the presence of that company does not scale by population.

Fermi problems/back-of-the-envelope calculations are a standard part of a physics or engineering education. They are of theoretical value in the construction of mathematical models, and of "real world" value in professions such as management consulting. However, perhaps because the concept is intrinsically nonrigorous, it is not typically part of a (pure) mathematics curriculum. Specifically, this is true for enumerative combinatorics, even though the subject's purpose is to count the number of certain objects—which in the author's experience, many students hope has nontheoretical applicability.

A Combinatorial Analogue

By analogy we define a **combinatorial Fermi problem**:

> Fix $\epsilon > 0$. Let S be a finite set of combinatorial objects and $\omega : S \to \mathbb{Z}_{\geq 0}$ be a statistic on S. Then, we estimate the value of ω on any element to be the **confidence interval** $[a, b]$ where the *uniform* probability of picking an element of S outside of this range has probability $< \epsilon$.

By definition, the (ordinary) **generating series** for the **combinatorial problem** (S, ω) is defined by $G_{(S,\omega)}(x) = \sum_{s\in S} x^{\omega(s)}$. For any k, $\#\{s \in S : \omega(s) = k\} = [x^k]G_{(S,\omega)}(x)$; i.e., the coefficient of x^k in $G_{(S,\omega)}(x)$. Usually textbook work involves extracting the coefficient using formulae valid in the ring of formal power series. However, what is often not emphasized in class is that this coefficient, and $\#S$ itself, can be rapidly extracted using a computer algebra system, allowing for a quick determination of the range $[a, b]$. Since the computer does the work, this is our analogue of a "back-of-the-envelope" calculation.

Table 3. Abel Prize recipients.

Laureate	Award year	$N_{\texttt{citations}}$	h	rule of thumb est.	Estimated range
J. P. Serre	2003	10119	53	54.3	$[47, 60]$
I. Singer	2004	2982	28	29.5	$[24, 34]$
M. Atiyah	2004	6564	40	43.7	$[37, 49]$
P. Lax	2005	4601	30	36.6	$[31, 42]$
L. Carleson	2006	1980	18	24.0	$[19, 28]$
S. R. S. Varadhan	2007	2894	28	29.0	$[24, 33]$
J. Thompson	2008	789	14	15.2	$[11, 18]$
J. Tits	2008	3463	28	31.8	$[27, 36]$
M. Gromov	2009	7671	41	47.3	$[40, 54]$
J. Tate	2010	2979	30	29.5	$[24, 34]$
J. Milnor	2011	7856	48	47.9	$[41, 54]$
E. Szemerédi	2012	2536	26	27.2	$[22, 31]$
P. Deligne	2013	6567	36	43.8	$[37, 50]$

For "reasonable" values of ϵ (such as $\epsilon = 2$ percent), often the range $[a, b]$ is rather tight. In those cases, there may be a theorem of *asymptotic* concentration near a "typical" object. However, even if such theorems are known, this does not solve the finite problem.

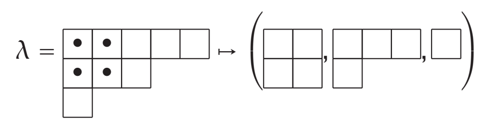
The use of the uniform distribution is a quick way to exactly obtain estimates that can be compared with empirical data. Ultimately, it invites the user to consider other probability distributions and more sophisticated statistical analysis (just as one should with the McDonald's example), using, e.g., Markov Chain Monte Carlo techniques.

We mention another combinatorial Fermi problem we have considered elsewhere: the count of the number of indigenous language families in the Americas [Yo13]. That is a situation where essentially there is no way to know with great certainty the true answer.

The Euler–Gauss Identity and its Application to the h-Index

We apply the perspective of Section 2 to the h-index question, where $S = \mathsf{Par}(n)$ and $\omega : S \to \mathbb{Z}_{\geq 0}$ is the size of a partition's Durfee square. If Par is the set of all partitions and $\sigma : \mathsf{Par} \to \mathbb{Z}_{\geq 0}$ returns the size of a partition, then the generating series for (Par, σ) is $P(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$. That is, $\#\mathsf{Par}(n) = [x^N]P(x)$. A sample textbook reference is [Br10].

Recall the **Euler–Gauss identity** (1) from the introduction. The well-known combinatorial proof is that every Young diagram λ bijectively decomposes into a triple $(D_\lambda, R_\lambda, B_\lambda)$ where D_λ is a $k \times k$ square, R_λ is a Young diagram with at most k rows, and B_λ is a partition with at most k columns. That is, D_λ is the Durfee square, R_λ the shape to the right of the square, and B_λ is the shape below it. For example:



$$= (D_\lambda, R_\lambda, B_\lambda).$$

The generating series for partitions with at most k columns is directly $\prod_{j=1}^{k} \frac{1}{1-x^j}$. Since conjugation (the "transpose") of a shape with at most k rows returns a shape with at most k columns, it follows that the generating series for shapes of the first kind is also $\prod_{j=1}^{k} \frac{1}{1-x^j}$.

From this argument, we see that the generating series for Young diagrams with a Durfee square of size k is $x^{k^2} \prod_{j=1}^{k} (1 - x^j)^{-2}$. We compute, for fixed $h, N_{\texttt{citations}}$:

$$\mathsf{Prob}(\lambda : |\lambda| = N_{\texttt{citations}}, \text{ Durfee square of size } k) = \frac{[x^{N_{\texttt{citations}}}] x^{k^2} \prod_{j=1}^{k} (1 - x^j)^{-2}}{\mathsf{Par}(N_{\texttt{citations}})}.$$

Often textbook analysis ends at the derivation of (1). In a classroom, using a computer to Taylor expand $\sum_{k=a}^{b} x^{k^2} \prod_{j=1}^{k} (1 - x^j)^{-2}$, and comparing the coefficients with the known partition numbers allows the instructor to "physically" demonstrate the identity to the student. Varying a and b shows what range of Durfee square sizes is, e.g., 98 percent likely to occur for partitions of that size. Interpreted in terms of our h-index problem, these same computations are what give us Table 1.[2]

[2] *Actually, our computation of* $\mathsf{Par}(N_{\texttt{citations}})$ *using generating series becomes not so easy on a desktop machine when* $N_{\texttt{citations}}$ *is a few thousand. Instead, one could use the Hardy-Ramanujan approximation* $\mathsf{Par}(N_{\texttt{citations}}) \sim \frac{1}{4N_{\texttt{citations}}\sqrt{3}} e^{\pi\sqrt{\frac{2N_{\texttt{citations}}}{3}}}$. *Even more precisely, one can*

Table 4. Associate professors at three research universities.

	$N_{\text{citations}}$	h	rule of thumb est.	estimated range
Department A				
A1	19	3	2.4	$[1,3]$
A2	80	6	4.8	$[3,6]$
A3	113	6	5.7	$[4,7]$
A4	130	4	6.1	$[4,8]$
A5	202	6	7.7	$[5,10]$
A6	511	11	12.2	$[9,15]$
Department B				
B1	30	3	3.0	$[1,4]$
B2	35	4	3.2	$[2,4]$
B3	56	4	4.0	$[2,5]$
B4	56	5	4.0	$[2,5]$
B5	63	5	4.3	$[3,5]$
B6	63	6	4.3	$[3,5]$
B7	78	3	4.8	$[3,6]$
B8	84	5	4.9	$[3,6]$
B9	88	7	5.1	$[3,6]$
B10	122	8	6.0	$[4,7]$
B11	126	7	6.1	$[4,7]$
B12	133	6	6.2	$[4,8]$
B13	133	7	6.2	$[4,8]$
B14	150	8	6.6	$[4,8]$
B15	163	7	6.9	$[5,8]$
B16	228	10	8.1	$[5,10]$
Department C				
C1	10	2	1.7	$[1,2]$
C2	11	2	1.8	$[1,2]$
C3	25	3	2.7	$[1,3]$
C4	54	4	4.0	$[2,5]$
C5	64	5	4.3	$[3,5]$
C6	64	5	4.3	$[3,5]$
C7	67	6	4.4	$[3,5]$
C8	104	6	5.5	$[4,7]$
C9	144	8	6.5	$[4,8]$
C10	269	5	8.9	$[6,11]$

As we state in the introduction, the work of [CaCoSa98] shows that the model Durfee square size is $\approx 0.54\sqrt{N_{\text{citations}}}$. E. R. Canfield's concentration conjecture states that for each $\epsilon > 0$,

(2)
$$\frac{\#\text{ partitions with } (1-\epsilon)\mu < h < (1+\epsilon)\mu}{\#\mathsf{Par}(N_{\text{citations}})} \to 1,$$

as $N_{\text{citations}} \to \infty$, where $\mu = \frac{\sqrt{6}\log 2}{\pi}\sqrt{N_{\text{citations}}}$. This is consistent with Table 1. Further discussion may appear elsewhere. Also, one would like to examine other distributions on Young diagrams, such as the Plancherel measure, which assigns to the shape λ the probability $(f^\lambda)^2/|\lambda|!$ where f^λ is the number of *standard Young tableaux* of shape λ.

use Wolfram Alpha, which gives the partition numbers for up to a million; this is well beyond our needs.

Further Comparisons with Empirical Data

The National Academy of Sciences

We compared our rule of thumb against all 119 mathematicians of the National Academy of Sciences (see Figures 1 and 2 and the Appendix). The correlation coefficient is $R = 0.93$. After removing books (as identified in *MathSciNet*), $R = 0.95$. A serious concern is that many pre-2000 citations are not tabulated in *MathSciNet*. Nevertheless, in our opinion, the results are still informative. See the comments in "Further Study".

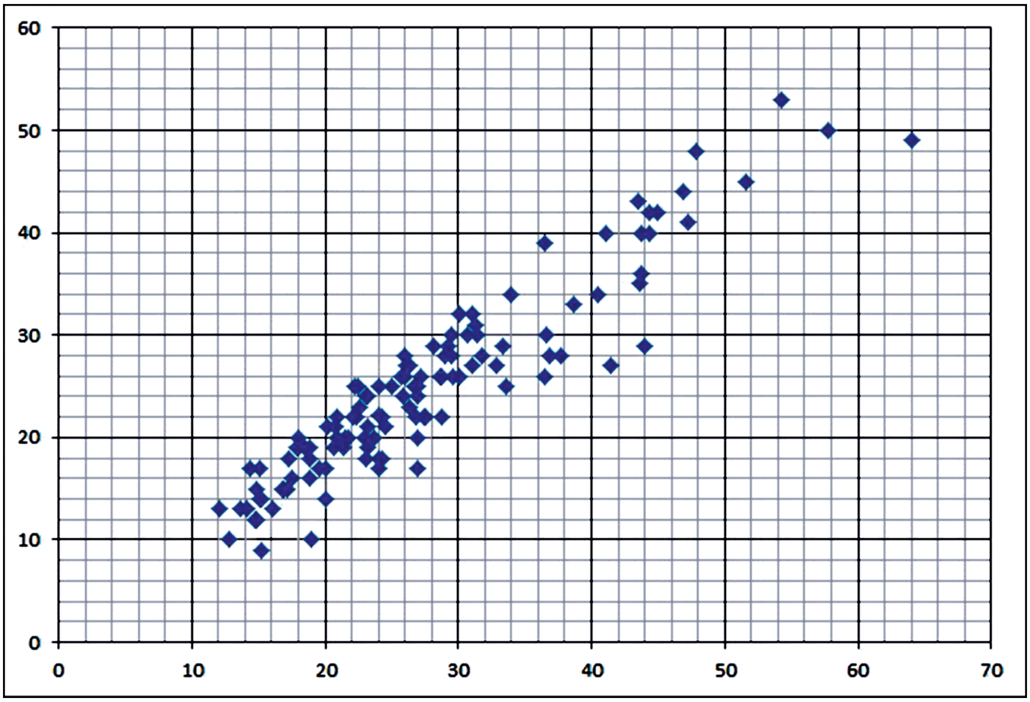


Figure 1. Rule of thumb (x-axis) versus actual h's (y-axis) for mathematics members of the National Academy of Sciences.

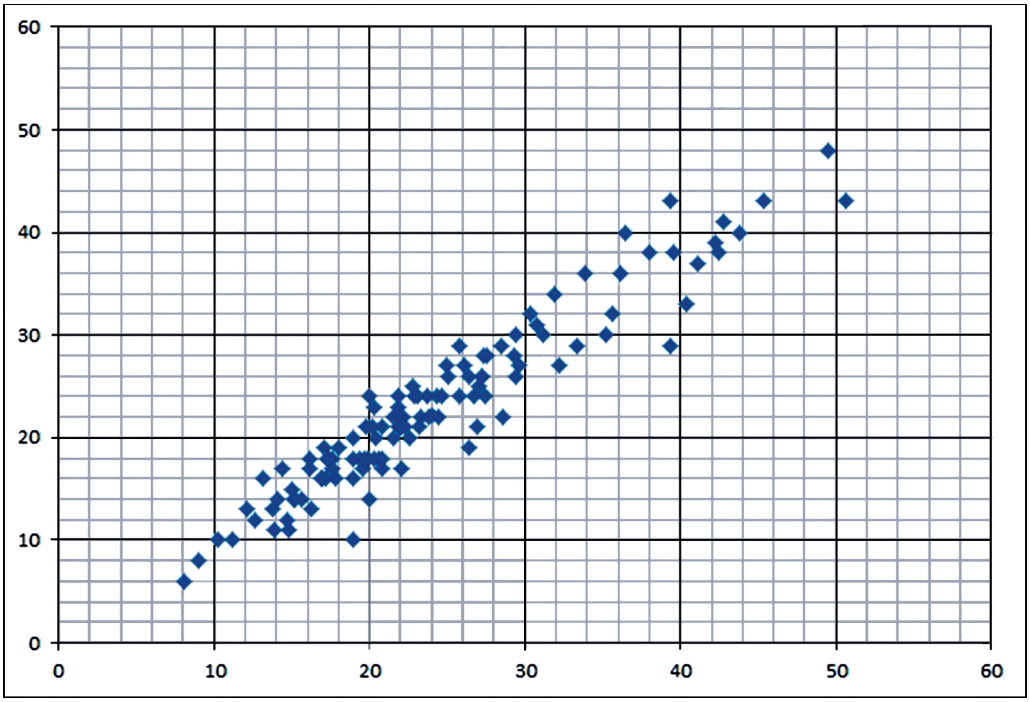


Figure 2. Rule of thumb (x-axis) versus actual h's (y-axis) for mathematics members of the National Academy of Sciences (*with books removed*).

Table 5. Current National Academy of Sciences Members (Mathematics).

Member	$N_{\texttt{citations}}$	Rule of thumb est.	h	non-books only	revised est.	revised h
G. Andrews	4866	37.7	28	2579	27.4	24
M. Artin	2326	26	26	2097	24.7	24
M. Aschbacher	1386	20	17	911	16.3	13
R. Askey	2480	26.9	17	1235	19.0	16
M. Atiyah	6564	43.7	40	5390	39.6	38
H. Bass	2472	26.8	22	1869	23.3	22
E. Berlekamp	764	14.9	12	363	10.3	10
J. Bernstein	2597	27.5	22	2484	26.9	21
S. Bloch	1497	20.9	20	1363	19.9	18
E. Bombieri	1746	22.6	23	1608	21.7	22
J. Bourgain	6919	44.9	42	6590	43.8	40
H. Brezis	11468	57.8	50	8386	49.5	48
F. Browder	2815	28.7	22	2807	28.6	22
W. Browder	646	13.7	13	547	12.6	12
R. Bryant	1489	20.8	21	1228	18.9	20
L. Caffarelli	6745	44.3	42	6280	42.8	41
E. Calabi	1224	18.9	18	1224	18.9	18
L. Carleson	1980	24	18	1484	20.8	17
S.-Y. Alice Chang	1828	23.1	24	1806	22.9	24
J. Cheeger	3387	31.4	30	3348	31.2	30
D. Christodoulou	783	15.1	17	594	13.2	16
A. Connes	6475	43.5	43	5318	39.4	43
I. Daubechies	4674	36.9	28	3002	29.6	27
P. Deift	3004	29.6	26	2545	27.2	26
P. Deligne	6567	43.8	36	5592	40.4	33
P. Diaconis	3233	30.7	30	2970	29.4	30
S. Donaldson	2712	28.1	29	2277	25.8	29
E. Dynkin	1583	21.5	20	1090	17.8	16
Y. Eliashberg	1628	21.8	20	1460	20.6	18
L. Faddeev	1820	23	20	1285	19.4	18
C. Fefferman	3828	33.4	29	3815	33.4	29
M. Freedman	1207	18.8	16	990	17	16
W. Fulton	5890	41.4	27	1424	20.4	20
H. Furstenberg	2064	24.5	21	1650	21.9	21
D. Gabai	1314	19.6	17	1314	19.6	17
J. Glimm	1826	23.1	18	1419	20.3	18
R. Graham	3881	33.6	25	2280	25.8	24
U. Grenander	895	16.1	13	227	8.1	6
P. Griffiths	4581	36.5	26	1692	22.2	22
M. Gromov	7671	47.3	41	6200	42.5	38
B. Gross	1692	22.2	25	1635	21.8	24

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Member	$N_{\texttt{citations}}$	Rule of thumb est.	h	non-books only	revised est.	revised h
V. Guillemin	3710	32.9	27	2035	24.4	22
R. Hamilton	2490	26.9	20	2392	26.4	19
M. Hochster	1727	22.4	22	1657	22	21
H. Hofer	2140	25	25	1928	23.7	24
MJ. Hopkins	714	14.4	17	714	14.4	17
R. Howe	1680	22.1	22	1579	21.5	22
H. Iwaniec	2822	28.7	26	1825	23.1	24
A. Jaffe	794	15.2	9	277	9	8
P. Jones	1112	18	19	1112	18	19
V. Jones	2025	24.3	18	1669	22.1	17
R. Kadison	1922	23.7	20	1042	17.4	18
R. Kalman	558	12.8	10	428	11.2	10
N. Katz	2370	26.3	23	1582	21.5	20
D. Kazhdan	2332	26.1	27	2332	26.1	27
R. Kirby	963	16.8	15	678	14.1	14
S. Klainerman	2324	26	28	2144	25.0	27
J. Kohn	1231	18.9	19	1068	17.6	18
J. Kollár	3100	30.1	26	1947	23.8	22
B. Kostant	2509	27	25	2509	27	25
R. Langlands	1466	20.6	19	773	15.0	15
H.B. Lawson	2576	27.4	22	1846	23.2	21
P. Lax	4601	36.6	30	3560	32.2	27
E. Lieb	5147	38.7	33	4349	35.6	32
T. Liggett	1975	24	17	984	16.9	16
L. Lovasz	5638	40.5	34	4259	35.2	30
G. Lusztig	5786	41.1	40	4945	38.0	38
R. MacPherson	2031	24.3	22	1676	22.1	21
G. Margulis	2267	25.7	26	1788	22.8	25
J. Mather	1399	20.2	21	1399	20.2	21
B. Mazur	2842	28.8	26	2440	26.7	24
D. McDuff	2289	25.8	24	1417	20.3	23
H. McKean	2480	26.9	24	1701	22.3	21
C. McMullen	1738	22.5	25	1368	20.0	24
J. Milnor	7856	47.9	48	4559	36.5	40
J. Morgan	1985	24.1	25	1484	20.8	21
G. Mostow	1180	18.5	19	896	16.2	17
J. Nash	1337	19	10	1337	19	10
E. Nelson	1010	17.2	15	753	14.8	11
L. Nirenberg	9145	51.6	45	8781	50.6	43

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Member	$N_{\texttt{citations}}$	Rule of thumb est.	h	non-books only	revised est.	revised h
S. Novikov	2368	26.3	27	1677	22.1	21
A. Okounkov	1677	22.1	24	1677	22.1	24
D. Ornstein	1100	17.9	19	1022	17.3	18
J. Palis	1570	21.4	19	895	16.2	18
P. Rabinowitz	6633	44	29	5316	39.4	29
M. Ratner	506	12.1	13	506	12.1	13
K. Ribet	1022	17.3	18	1021	17.3	18
P. Sarnak	3114	30.1	32	2780	28.5	29
M. Sato	738	14.7	12	738	14.7	12
R. Schoen	3945	33.9	34	3493	31.9	34
J. Serre	10119	54.3	53	4481	36.1	36
C. Seshadri	984	16.9	15	831	15.6	14
Y. Sinai	3357	31.3	31	2547	27.3	28
I. Singer	2982	29.5	28	2951	29.3	28
Y. Siu	1494	20.9	22	1350	19.8	21
S. Smale	4581	36.5	39	3942	33.9	36
R. Solovay	781	15.1	14	781	15.1	14
J. Spencer	758	14.9	15	1334	19.7	18
R. Stanley	6510	43.6	35	3148	30.3	32
H. Stark	678	14.1	13	653	13.8	13
C. Stein	763	14.9	12	658	13.9	11
E. Stein	14049	64	49	5788	41.1	37
R. Steinberg	1850	23.2	19	1068	17.6	17
S. Sternberg	2438	26.7	25	1476	20.8	18
D. Stroock	3299	31.0	27	2028	24.3	24
D. Sullivan	3307	31.1	32	3248	30.8	31
R. Swan	1109	18	20	998	17.1	19
E. Szemerédi	2536	27.2	26	2536	27.2	26
T. Tao	6730	44.3	40	6214	42.3	39
J. Tate	2979	29.5	30	2612	27.6	28
C. Taubes	1866	23.2	24	1626	21.8	23
J. Thompson	789	15.2	14	789	15.2	14
J. Tits	3463	31.8	28	2958	29.4	26
K. Uhlenbeck	1852	23.2	21	1756	22.6	20
S. Varadhan	2894	29	28	2153	25.1	26
D. Voiculescu	2952	29.3	29	2387	26.4	26
A. Wiles	1387	20	14	1387	20	14
S-T. Yau	7536	46.9	44	7066	45.4	43
E. Zelmanov	1055	17.5	16	1020	17.2	16

Abel Prize Winners

Perhaps a closer analogy to the Nobel Prize than the Fields Medal is the Abel Prize, since the latter does not have an age limit. The fit with the estimated intervals remains decent; the concern about pre-2000 citations remains. See Table 3.

Associate Professors

Finally, in Table 4 we considered all mathematics associate professors at three research universities. Of the thirty-two professors, all but five have their h-index in the estimated range, and all are at most one unit outside this range.

Further Study

It seems to us that the simple model presented describes one force governing the h-index. However, other forces/sources of noise are at play, depending on the field or even the fame of the scholar. Future work seeks to better understand this quantitatively, as one works toward more precise models.

The loss of pre-2000 citations in *MathSciNet* is significant to how we interpret the results for the National Academy members/Abel Prize winners. For example, the rough agreement with the rule of thumb might only reflect an "equilibrium state" that arises years after major results have been published. This concern is partly allayed by the similar agreement for recent Fields medalists (Table 2). However, as *MathSciNet* reaches further back in tabulating citations, one would try to quantify these effects. In the meantime, use of *MathSciNet* has practical justification since, in promotion and grant decision cases, recent productivity is important. So for these purposes, post-2000 data is mostly sufficient.

As a further cross-check, we used the rule of thumb for a broad range of fields using *Google Scholar*. For scholars with a moderate number of citations, the agreement is often similarly good. Also the rule is an upper bound for the vast majority of highly cited scholars (but, as we have said earlier, much less accurate in some fields). However, these checks have an obvious bias as they only consider people who have set up a profile, so we do not formally present these results here.

We propose using the rule of thumb and the confidence intervals as a basis for a systematic study. We suggest that the rule of thumb reflects an "ideal scholar." (This terminology is an allusion to "ideal gas" in statistical mechanics. Indeed, a more conventional use of random partitions concerns the study of Boltzmann statistics on a one-dimensional lattice fermion gas.) Divergence from this ideal is a result of "anomalies." For a choice of field, can one statistically distinguish, on quantifiable grounds, scholars who are close to the rule of thumb (in the sense of confidence intervals) from those who are far from it?

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References

[AnHaKi09] T. Anderson, R. Hankin and P. Killworth, Beyond the Durfee square: Enhancing the h-index to score total publication output, *Scientometrics* **76**, no. 3 (2008), 577–588.

[Br10] R. A. Brualdi, *Introductory Combinatorics*, fifth edition, Pearson Prentice Hall, Upper Saddle River, New Jersey, 2010.

[CaCoSa98] E. R. Canfield, S. Corteel and C. D. Savage, Durfee polynomials, *Elect. J. Comb.* **5** (1998), R32.

[Co] L. Cooper, Making Estimates in Research and Elsewhere, http://physics.illinois.edu/undergrad/SeniorThesis/EstimatesResearch.pdf

[FlSe09] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[Hi05] J. E. Hirsch, An index to quantify an individual's scientific research output, *Proc. Natl. Acad. Sci.* **102**(46) (2005), November 15, pp. 16569–16572.

[IgPe07] J. E. Iglesias and C. Pecharromán, Scaling the h-index for different scientific ISI fields, *Scientometrics* **73**, Issue 3, (2007), December, pp. 303–320.

[Su10] Z. Su, Asymptotic analysis of random partitions, Asymptotic theory in probability and statistics with applications, *Adv. Lect. Math. (ALM)* **2**, Int. Press, Somerville, MA, 2008, pp. 44–79.

[Yo13] A. Yong, The Joseph Greenberg problem: Combinatorics and comparative linguistics, preprint 2013. arxiv:1309.5883