

# On the work and persona of Gilles Lachaud



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# Gilles LACHAUD (26 July 1946 – 21 February 2018)



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[Source: Google Images and the article by Y. Aubry in *Gaz. Math.* **157** (2018), 74–75.]

## Career in Brief

- **Doctorat d'Etat, Univ. Paris 7, 1979** [Advisor: Roger GODEMENT. Thesis on *Analyse spectrale et prolongement analytique: Séries d'Eisenstein, Fonctions Zeta et nombre de solutions d'équations diophantiennes*]

## Gilles LACHAUD: Career in Brief (Contd.)

- Prix Rivoire, 1979
- Held a position with the CNRS and was for most part at IML, Marseille
- Director of **CIRM**, September 1986 – August 1991
- Director (*Responsable*), Jan 2000 – August 2011
- Founder and a strong driving force behind the **AGCT meetings**
- 13 Ph.D. students: Renault DANSET (1983), Bernadette DESHOMMES (1983), Franck WIELONSY (1983), Jean-Pierre CHERDIEU (1985), Marc PERRET (1990), Yves AUBRY (1993), Robert ROLLAND (1995), Didier ALQUIE (1996), Antoine EDOUARD (1998), Cédric CORNUS (2000), François-Régis BLACHE (2000), Alexandre TEMPKINE (2000), and Iman ISLIM (2001).
- Guided Habilitations of: Iwan DUURSMA (2000), Yves AUBRY (2002).
- Conference in honour of his 60th birthday: SAGA-1, Tahiti, May 2007. *Proceedings* published by World Scientific, Singapore, 2008.

# Some Numbers



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## Lachaud, Gilles

MR Author ID: **192094**  
Earliest Indexed Publication: **1973**  
Total Publications: **55**  
Total Related Publications: **5**  
Total Citations: **458**

Published as: Lachaud, G. (2)

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## Co-authors (by number of collaborations)

Aubry, Yves   Castryck, Wouter   **Ghorpade, Sudhir Ramakant**   Haloui, Safia   Iglesias-Zemmour, Patrick   Lucien, Isabelle   Martin-Deschamps, Mireille   Mercier, Dany-Jack   O'Sullivan, Michael E.   Perret, Marc   Ram, Samrith   **Ritzenthaler, Christophe**   Rolland, Robert   Stern, Jacques   Tsfasman, Mikhail   Anatolievich   Vlăduț, Sergei G.   Wolfmann, Jacques   Zykin, A. I.

## Publications (by number in area)

Algebraic geometry   Field theory and polynomials   History and biography   Information and communication, circuits  
Manifolds and cell complexes   Number Theory   **Number theory**   Partial differential equations   Several complex variables and analytic spaces   Topological groups, Lie groups

## Publications (by number of citations)

Algebraic geometry   Information and communication, circuits   Manifolds and cell complexes   Number Theory   **Number theory**

# Major Themes of Work (and some representative papers)

## • Automorphic Forms.

- Spectral analysis of automorphic forms on rank one groups by perturbation methods, in: *Proc. Sympos. Pure Math.*, Vol. XXVI, AMS, 1973, 441–450.
- Analyse spectrale des formes automorphes et séries d'Eisenstein. *Invent. Math.* **46** (1978), 39–79.
- Variations sur un thème de Mahler, *Invent. Math.* **52** (1979), 149–162.
- The distribution of the trace in the compact group of type  $G_2$ , *Contemp. Math.* **722** (2019), 79–103.

## • Curves and Abelian Varieties over Finite Fields.

- Sommes d'Eisenstein et nombre de points de certaines courbes algébriques sur les corps finis, *C. R. Acad. Sci. Paris Sér. I Math.* **305** (1987), 729–732.
- (with M. Martin-Deschamps) Nombre de points des jacobiniennes sur un corps fini, *Acta Arith.* **56** (1990), 329–340.
- Ramanujan modular forms and the Klein quartic, *Mosc. Math. J.* **5** (2005), 829–856.

## Major Themes of Work (and some representative papers) Contd.

- (with C. Ritzenthaler) On some questions of Serre on abelian threefolds, in: *Algebraic Geometry and its Applications*, World Scientific, 2008, 88–115.
- (with C. Ritzenthaler and A. Zykin) Jacobians among abelian threefolds: a formula of Klein and a question of Serre, *Math. Res. Lett.* **17** (2010), 323–333.
- (with Y. Aubry and S. Haloui) On the number of points on abelian and Jacobian varieties over finite fields. *Acta Arith.* **160** (2013), 201–241.
- **Algebraic Varieties and Algebraic Sets over Finite Fields.**
  - (with M. A. Tsfasman) Formules explicites pour le nombre de points des variétés sur un corps fini, *J. Reine Angew. Math.* **493** (1997), 1–60.
  - (with S. R. Ghorpade) Étale cohomology, Lefschetz theorems and number of points of singular varieties over finite fields, *Mosc. Math. J.* **2** (2002), 589–631.
  - (with R. Rolland) On the number of points of algebraic sets over finite fields, *J. Pure Appl. Algebra* **219** (2015), 5117–5136.



## Major Themes of Work (and some representative papers) Contd.

- **Continued Fractions, Sails and Klein Polyhedra.**
  - Polyèdre d'Arnold et voile d'un cne simplicial: analogues du théorème de Lagrange, *C. R. Acad. Sci. Paris Sér. I Math.* **317** (1993), 711–716.
  - Klein polygons & geometric diagrams, *Contemp. Math.* **210** (1998), 365–372.
  - Sails and Klein polyhedra, *Contemp. Math.* **210** (1998), 373–385.
- **Linear Codes and Related Varieties**
  - Les codes géométriques de Goppa, Séminaire Bourbaki, no. 641, 1984/85, *Astérisque* **133-134** (1986), 189–207.
  - (with J. Wolfmann), Sommes de Kloosterman, courbes elliptiques et codes cycliques en caractéristique 2, *C. R. Acad. Sci. Paris Sér. I Math.* **305** (1987), 881–883.
  - (with J. Wolfmann), The weights of the orthogonals of the extended quadratic binary Goppa codes, *IEEE Trans. Inform. Theory* **36** (1990), 686–692.
  - The parameters of projective Reed-Muller codes, *Discrete Math.* **81** (1990), 217–221.

## Major Themes of Work (and some representative papers) Contd.

- **Linear Codes and Related Varieties (Contd.)**
  - Artin-Schreier curves, exponential sums, and the Carlitz-Uchiyama bound for geometric codes. *J. Number Theory* **39** (1991), 18–40.
  - Number of points of plane sections and linear codes defined on algebraic varieties, in: *Arithmetic, geometry and coding theory* (Luminy, 1993), de Gruyter, 1996, 77–104.
  - (with S. R. Ghorpade) Higher weights of Grassmann codes, in: *Coding theory, cryptography and related areas*. Springer, Berlin, 2000, 122–131.
  - (with S. R. Ghorpade) Hyperplane sections of Grassmannians and the number of MDS linear codes, *Finite Fields Appl.* **7** (2001), 468–506.
  - (with Y. Aubry, W. Castryck, S. R. Ghorpade, M. E. O’Sullivan, and S. Ram) Hypersurfaces in weighted projective spaces over finite fields with applications to coding theory, in: *Algebraic geometry for coding theory and cryptography*, Springer, 2017, 25–61.

# A Sampling of the work of Gilles Lachaud

MOSCOW MATHEMATICAL JOURNAL  
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## ÉTALE COHOMOLOGY, LEFSCHETZ THEOREMS AND NUMBER OF POINTS OF SINGULAR VARIETIES OVER FINITE FIELDS

SUDHIR R. GHORPADE AND GILLES LACHAUD

*Dedicated to Professor Yuri Manin for his 65th birthday*

**ABSTRACT.** We prove a general inequality for estimating the number of points of arbitrary complete intersections over a finite field. This extends a result of Deligne for nonsingular complete intersections. For normal complete intersections, this inequality generalizes also the classical Lang-Weil inequality. Moreover, we prove the Lang-Weil inequality for affine, as well as projective, varieties with an explicit description and a bound for the constant appearing therein. We also prove a conjecture of Lang and Weil concerning the Picard varieties and étale cohomology spaces of projective varieties. The general inequality for complete intersections may be viewed as a more precise version of the estimates given by Hooley and Katz. The proof is primarily based on a suitable generalization of the Weak Lefschetz Theorem to singular varieties together with some Bertini-type arguments and the Grothendieck-Lefschetz Trace Formula. We also describe some auxiliary results concerning the étale cohomology spaces and Betti numbers of projective varieties over finite fields, and a conjecture along with some partial results concerning the number of points of projective algebraic sets over finite fields.

2000 MATH. SUBJ. CLASS. 11G25, 14F20, 14G15, 14M10.

**KEY WORDS AND PHRASES.** Étale cohomology, varieties over finite fields, complete intersections, Trace Formula, Betti numbers, zeta functions, Weak Lefschetz Theorem, hyperplane sections, motives, Lang-Weil inequality, Albanese variety.

स्मिन्नीनो किल्लतो रश्मिरोषम् \*

### INTRODUCTION

This paper has roughly a threefold aim. The first is to prove the following inequality for estimating the number of points of complete intersections (in particular,

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\*The first named author supported in part by a 'Career Award' grant from AICTE, New Delhi and an IIT Bombay grant from IIT Bombay.

\* "Their cord was extended across" (Rg Veda X.129).

The quotation from *Rg Veda* (X, 129) meaning "Their cord was extended across" that appears in this paper owes its presence to Gilles Lachaud.

## Some Background:

**Lang-Weil Inequality (1954).** If  $X$  is an **irreducible projective variety** in  $\mathbb{P}^N$  defined over  $\mathbb{F}_q$  and of dimension  $n$  and degree  $d$ , then

$$\left| |X(\mathbb{F}_q)| - p_n \right| \leq (d-1)(d-2)q^{n-(1/2)} + Cq^{n-1},$$

where  $C$  is a constant depending only on  $N$ ,  $n$  and  $d$ .

**Deligne's Inequality for Smooth Complete Intersections (1973).** If  $X$  is a **nonsingular complete intersection** in  $\mathbb{P}^N$  over  $\mathbb{F}_q$  of dimension  $n = N - r$ , then

$$\left| |X(\mathbb{F}_q)| - p_n \right| \leq b'_n q^{n/2}.$$

Here  $b'_n = b_n - \epsilon_n$  is its **primitive  $n$ th Betti number** of  $X$  (where  $\epsilon_n = 1$  if  $n$  is even and  $\epsilon_n = 0$  if  $n$  is odd), and  $p_n := |\mathbb{P}^n(\mathbb{F}_q)| = q^n + q^{n-1} + \dots + q + 1$ .

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We remark that if  $X$  has multidegree  $\mathbf{d} = (d_1, \dots, d_r)$ , then  $b'_n = b'_n(N, \mathbf{d})$  equals

$$(-1)^{n+1}(n+1) + \sum_{c=r}^N (-1)^{N+c} \binom{N+1}{c+1} \sum_{\substack{\nu_1 + \dots + \nu_r = c \\ \nu_i \geq 1 \forall i}} d_1^{\nu_1} \dots d_r^{\nu_r}$$

# Estimates for singular complete intersections

Theorem (Deligne-type inequality for arbitrary complete intersections)

Let  $X$  be an irreducible complete intersection of dimension  $n$  in  $\mathbb{P}_{\mathbb{F}_q}^N$ , defined by  $r = N - n$  equations, with multidegree  $\mathbf{d} = (d_1, \dots, d_r)$ , and let  $s \in \mathbb{Z}$  with  $\dim \text{Sing } X \leq s \leq n - 1$ . Then

$$\left| |X(\mathbb{F}_q)| - p_n \right| \leq b'_{n-s-1}(N - s - 1, \mathbf{d}) q^{(n+s+1)/2} + C_s(X) q^{(n+s)/2},$$

where  $C_s(X)$  is a constant independent of  $q$ . If  $X$  is nonsingular, then  $C_{-1}(X) = 0$ . If  $s \geq 0$ , then

$$C_s(X) \leq 9 \times 2^r \times (r\delta + 3)^{N+1} \quad \text{where} \quad \delta = \max\{d_1, \dots, d_r\}.$$

For normal complete intersections, this may be viewed as a common refinement of Deligne's inequality and the Lang-Weil inequality. Corollaries include previous results of [Aubry and Perret \(1996\)](#), [Shparlinskiĭ and Skorobogatov \(1990\)](#), as well as [Hooley and Katz \(1991\)](#).

# Estimates for irreducible varieties over finite fields

## Theorem (Effective Lang-Weil inequality)

Suppose  $X$  is a projective variety in  $\mathbb{P}_{\mathbb{F}_q}^N$  or an affine variety in  $\mathbb{A}_{\mathbb{F}_q}^N$  defined over  $\mathbb{F}_q$ . Let  $n = \dim X$  and  $d = \deg X$ . Then

$$\left| |X(\mathbb{F}_q)| - p_n \right| \leq (d-1)(d-2)q^{n-(1/2)} + C_+(\bar{X})q^{n-1},$$

where  $C_+(\bar{X})$  is independent of  $q$ . Moreover if  $X$  is of type  $(m, N, \mathbf{d})$ , with  $\mathbf{d} = (d_1, \dots, d_m)$ , and if  $\delta = \max\{d_1, \dots, d_m\}$ , then we have

$$C_+(\bar{X}) \leq \begin{cases} 9 \times 2^m \times (m\delta + 3)^{N+1} & \text{if } X \text{ is projective} \\ 6 \times 2^m \times (m\delta + 3)^{N+1} & \text{if } X \text{ is affine.} \end{cases}$$

As a corollary, one obtains an analogue of a result of [Schmidt \(1974\)](#) on a lower bound for the number of points of irreducible hypersurfaces over  $\mathbb{F}_q$ .

# A Conjecture of Lang and Weil

When **Lang and Weil** proved the inequality , namely,

$$\left| |X(\mathbb{F}_q)| - p_n \right| \leq (d-1)(d-2)q^{n-(1/2)} + Cq^{n-1}, \quad (1)$$

they showed in the same paper that if  $K$  is an algebraic function field of dimension  $n$  over  $k = \mathbb{F}_q$ , then there is a constant  $\gamma$  for which (1) holds with  $(d-1)(d-2)$  replaced by  $\gamma$ , for any model  $X$  of  $K/k$ , and moreover, the smallest such constant  $\gamma$  is a **birational invariant**. Subsequently, Lang and Weil went on to conjecture that this constant  $\gamma$  can be described *algebraically* as being twice the dimension of the associated **Picard variety**  $P$ , at least when  $X$  is nonsingular. They made further *conjectural statements* relating the Weil zeta function of  $X$  and the “**characteristic polynomial**” of  $P$  when  $X$  is projective and nonsingular.

In effect, we show that these conjectures hold in the affirmative provided one uses the “correct” Picard variety.



# Tools and Techniques

Proofs of the above theorems use a variety of techniques from algebraic geometry and topology, and to a lesser extent complex analysis and algebra. These include

- a variant of [Bertini's theorem](#) to successively construct good hyperplane sections.
- a suitable generalization of the [Weak Lefschetz Theorem](#) for singular varieties, which is proved in the paper.
- [Grothendieck-Lefschetz trace formula](#) and [Deligne's Main Theorem](#) for general varieties over finite fields.
- [Katz's estimates](#) for sums of Betti numbers.
- Analysis of zeros and poles of the Weil zeta function and related objects.
- Combinatorial methods to find suitable bounds using the formulae of [Hirzebruch](#) and [Jouanolou](#) for nonsingular complete intersections.

## Applications and Extensions

There have been several applications, some quite surprising. These include:

- Work of [T. Bandman](#), [G.-M. Gruel](#), [F. Grunewald](#), [B. Kunyavskii](#), [G. Pfister](#) and [E. Plotkin](#) (2003 and 2006) and [E. Ribnere](#) (2009) on the characterization of finite solvable groups by two-variable identities
- topics in diophantine equations (Waring's problem in function fields), by [Y.-R. Liu](#) and [T. Wooley](#) (2007)
- the study of Boolean functions by [F. Rodier](#) (2008), [M. Delgado](#) (2017)
- classification of hyperovals in planes by [F. Caullery](#) and [K.-U. Schmidt](#) (2015)
- arithmetic progressions over finite fields, by [B. Cook](#) and [A. Magyar](#) (2010)
- the study of primitive semifields by [R. Gow](#) and [J. Sheekey](#) (2011)
- to coding theory, by [Nakashima](#) (2009), [F. Edoukou](#), [S. Ling](#), and [C. Xing](#) (2009), and also [J. B. Little](#) (2011).

There have also been several extensions and generalizations of some of the results, mainly due to [A. Cafure](#) and [G. Matera](#) (2007-2012).

## A conjecture for algebraic sets over finite field

In the same paper, one can find the following conjecture due to [Lachaud](#).

### Conjecture.

If  $X$  is a complete intersection in  $\mathbb{P}^m$  defined over  $\mathbb{F}_q$  of dimension  $n \geq m/2$  and degree  $d \leq q + 1$ , then

$$|X(\mathbb{F}_q)| \leq dp_n - (d - 1)p_{2n-m} = d(p_n - p_{2n-m}) + p_{2n-m}.$$

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If  $X$  is a hypersurface (so that  $n = m - 1$ ), then this is [Tsfasman's Conjecture](#) or [Serre–Sørensen Inequality \(1989/1991\)](#).

If  $X = V(F)$ , where  $F \in \mathbb{F}_q[x_0, \dots, x_m]_d$ , with  $d \leq q + 1$ , then

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Recently, [A. Couvreur](#) (2016) has proved this conjecture in the affirmative and in fact, proved a more general result. See also: [Lachaud and Rolland](#) (2015).

## Another conjecture for algebraic sets over finite fields

**Notation:**  $p_n := |\mathbb{P}^n(\mathbb{F}_q)| = q^n + \dots + q + 1$  if  $n \geq 0$  and  $p_n := 0$  if  $n < 0$ . Define

$$e_r(d, m) := \max\{|V(F_1, \dots, F_r)(\mathbb{F}_q)| : F_1, \dots, F_r \in \mathbb{F}_q[x_0, \dots, x_m]_d \text{ lin. indep.}\}$$

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### Tsfasman-Boguslavsky Conjecture (TBC)

Assume that  $1 \leq r \leq M$  and  $1 \leq d < q - 1$ . Let  $(\nu_1, \dots, \nu_{m+1})$  be the  $r$ -th element in the descending lexicographic order among  $(m + 1)$ -tuples  $(\alpha_1, \dots, \alpha_{m+1})$  of nonnegative integers satisfying  $\alpha_1 + \dots + \alpha_{m+1} = d$ . Then

$$e_r(d, m) = p_{m-2j} + \sum_{i=j}^m \nu_i (p_{m-i} - p_{m-i-j}) \quad \text{where } j := \min\{i : \nu_i \neq 0\}.$$

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**Example:** Suppose  $d > 1$ . The first  $m+1$  tuples ordered as above look like  $(d, 0, 0, \dots, 0)$ ,  $(d-1, 1, 0, \dots, 0)$ ,  $\dots$ ,  $(d-1, 0, 0, \dots, 1)$ . Hence for  $r \leq m$ ,  $e_r(d, m) = (d-1)q^{m-1} + p_{m-2} + q^{m-r}$  and  $e_{m+1}(m, d) = (d-1)q^{m-1} + p_{m-2}$ .



# Current Status of Tsfasman-Boguslavsky Conjecture

- TBC follows from the Serre-Sørensen Inequality if  $r = 1$ . It is also trivially valid when  $d = 1$  or  $m = 1$ . Note that in general,  $r \leq \binom{m+d}{d}$
- **Boguslavsky (1997)**: TBC holds if  $r = 2$ .
- **Datta – G (2015)**: TBC holds when  $d = 2$  and  $r \leq m + 1$ , but it can be false if  $r > m + 1$ .
- **Datta – G (2017)**: TBC holds for any  $d < q - 1$ , provided  $r \leq m + 1$ .
- **Datta – G (2017)**: A new conjectural formula for  $e_r(d, m)$  proposed if  $r \leq \binom{m+d-1}{d-1}$ . [the “incomplete conjecture”].
- **Beelen – Datta – G (2018)**: Incomplete conjecture established for  $2 < d < q$  and  $r \leq \binom{m+2}{2}$ .
- **Beelen – Datta – G (2018-19)**: A “complete conjecture” proposed for  $e_r(d, m)$ . It is established for several (but not all) values of  $r$ .

## Concluding Remarks

*Gilles Lachaud has made important and lasting contributions to mathematics, especially in the study of algebraic varieties over finite fields and linear codes. His knowledge and interests were deep and wide. When he became interested in some topic, he would usually delve deeper and spend considerable time learning many aspects of it. As far as I have seen, he would never be in a rush to publish quickly, but would prefer to take time and be thorough.*

*Besides his contributions to mathematics, Gilles was an institution builder. He helped nurture an institution like the CIRM. Also, the continuing success of the AGCT conferences owes largely to his vision and efforts.*

## Concluding Remarks

*Other than scientific institutes and conferences, Gilles served as the President of the French Pavilion at Auroville, near Pondicherry, India. He had read or had at least browsed through significant amount of ancient and modern Sanskrit works, including the Vedas, Upanishadas, and the scholarly treatises of Sri Aurobindo.*

*Above all, Gilles was a wonderful human being, generous, warm-hearted, and kind, always willing to help others, especially students and younger colleagues. His untimely demise last year is a great loss to our subject and the community. Personally, it has been a pleasure and honour to have known him. He will certainly be missed.....*