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THORALF ALBERT SKOLEM 1887–1963
A BIOGRAPHICAL SKETCH*

Professor Thoralf Albert Skolem died on 23 March 1963, almost 76 years old. Norway has lost one of its greatest mathematicians of all time. His death was unexpected. He seemed to be as active as ever. He was preparing for a visit to the USA where, as on many previous occasions, he had been invited to lecture at several universities. Age had not seemed to diminish either his research drive or creative ability.

Thoralf Skolem was born on 23 May 1887 in Sandsvær, in the county of Buskerud in Southern Norway. His father, Even Skolem, taught in elementary school, but the family were mainly farming people. Skolem passed his “Examen artium” (which is the concluding examination of the Norwegian Gymnasium) in 1905 and immediately afterwards started to study mathematics and science at the University of Oslo, where in 1913 he obtained his degree in mathematics. For his thesis, *Undersøkelser innenfor logikkens algebra* (*Investigations on the Algebra of Logic*) he obtained the best possible mark. His outstanding achievement was reported, as happens on rare occasions, to the King of Norway.

Upon graduation Skolem became a private assistant to the well-known physicist Kr. Birkeland, and it was on physics that Skolem, in collaboration with Birkeland, published his first scientific papers.

In the winter of 1915–1916 Skolem studied in Göttingen, experiencing fully the difficult living conditions caused by the First World War. From 1916 to 1918 he was a Research Fellow at the University of Oslo, and in 1918 he was appointed to a newly created position as Docent of Mathematics at the same university.

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In 1926, when he was nearly 40 years old, Skolem obtained his doctorate, the thesis entitled *Einige Sätze über ganzzahlige Lösungen gewisser Gleichungen und Ungleichungen.* The somewhat advanced age has the following explanation. In their younger years, Viggo Brun and Skolem agreed that neither of them would bother to obtain the degree of Doctor, probably feeling that, in Norway, it served no useful function in the education of a young scientist. But in the middle twenties a younger generation of Norwegian mathematician emerged. It seems that Skolem then felt he too ought to fulfil the formal requirement of having a doctorate, and he "obtained permission" from Brun to submit a thesis. In 1924 Brun had been a professor in mathematics at the Norwegian Institute of Technology.

On 23 May 1927 Skolem married Edith Wilhelmine Hasvold.

From 1930 to 1938 Skolem was a Research Associate at the Chr. Michelsen's Institute in Bergen. This was a very independent position, equivalent in many respects to a professorship, but not carrying any teaching or administrative duties. One of the conditions of the research grant was that he should live in Bergen. Since there was no university in Bergen at that time, this meant that he was scientifically very isolated. He also had difficulties in obtaining the necessary mathematical literature while living there, which can be inferred from the preface to the monograph *Diophantische Gleichungen*, which he wrote during his Bergen years.

In 1938, at the age of 51, Skolem was appointed professor at the University of Oslo. He conducted the regular graduate courses in algebra and number theory, and rather infrequently lectured on mathematical logic. Thoralf Skolem was very modest and retiring by nature. He did not create any school and had no research students, but through his great accomplishments and research drive he inspired more than one of the younger Norwegian mathematicians.

Skolem retired in 1957, thereafter visiting American universities on several occasions. Professor Skolem always participated actively in the various administrative and organizational duties which is so necessary in any field of science. He was for many years President of the Norwegian Mathematical Association. For a long period he was Editor of *Norsk matematisk tidsskrift*, and continued after the reorganization as Editor of *Mathematica Scandinavica* from 1952. He also served as Associate Editor for many journals. Professor Skolem was a member of several scientific academies, and as early as 1918 became a member of Vitenskaps-Akademiet in Oslo. He was frequently invited to lecture at international congresses and symposia, and received many scientific honours, e.g. in 1962 he was awarded the Gunnerus Medal by Det Kongelige Norske Vitenskabers Selskab. In 1954 he was named a Knight of the 1st Class in the Royal Order of St. Olav by the King of Norway.

It is for his scientific accomplishments, however, that Professor Skolem will be remembered. Teaching and organizational activities were necessary duties, research was his life.

A rough count reveals that he wrote close to 200 papers, some of them opening up new fields of research, most of them of lasting scientific value, and a few of perhaps lesser interest today; for instance, as Editor of *Norsk matematisk tidsskrift*, he would write a paper or two to secure a complete issue as the publication date approached. Some notes on algebra and number theory belong in this category. In the same journal there is the somewhat curious paper "En liten studie i transfinit mekanikk" ("A small study on transfinite mechanics") from 1940. But he also wrote some very important papers in this rather obscure journal, e.g. "Über die mathematische Logik" in 1928. Professor Skolem also wrote many reviews here, not just within the field of mathematics, but also in physics and philosophy, subjects which strongly held his interest. Logician and number-theorist, he was one of the very few Norwegian mathematicians to regularly attend the physicists' meetings.

Professor Skolem published the greater part of his papers in *Videnskapsakademiets Skrifter* and *Avhandlinger*, in *Norsk matematisk tidsskrift*, or in *Det Kongelige Norske Videnskabsselskabs Forhandlinger*. Another rough count shows that approximately two-thirds of his papers are published here, which has often been of great inconvenience to his fellow scientists abroad. The proceedings of an academy, being a haphazard mixture of every field of science, are very often stacked away in some central library, and rarely seen in the special library of a mathematical institute. This means that many of Professor Skolem's most important contributions have been difficult to obtain. A collected edition of his work is therefore desirable. A start on this project was made before he died, and this work will now be continued.

Great mathematicians start to publish young. Professor Skolem was a counterexample. When appointed Docent in 1918, 31 years old, he had written only two papers (disregarding the notes he coauthored with Kr. Birkeland), but from 1920 on he published a series of papers which gave him instant recognition as one of the foremost among logicians: "Untersuchungen über die Axiome des Klassenkalküls und über Produktations- und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen" dates from 1919. The year after, "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit und Beweisbarkeit mathematischen Sätze nebst einem Theoreme über dichte Mengen" appeared. In 1922 he gave a lecture at the 5th Scandinavian Mathematics Congress "Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre", and finally in 1923 the important paper "Begründung der elementären Arithmetik durch die rekurrierende Denkweise ohne Anwendung scheinbarer Veränderlichen mit unendlichem Ausdehnugsbereich" was published. These are classics in the field of mathematical logic, although they remained to a large extent unnoticed and unread by contemporary mathematicians, in particular Scandinavian mathematicians!

It would be impossible to characterize the scientific work of Professor Skolem in just a few words, but we feel that it is essential to articulate a few of the ideas and some of the problems he worked on.

In the paper from 1920 Skolem proved the theorem which is now known as the Löwenheim-Skolem theorem. Briefly, this states that if a finite or countable infinite set of sentences formalized within a first-order predicate calculus is satisfiable (or, in other terminology, has a model), then the sentences are satisfiable within a countable domain. A by-product of the proof is Skolem's well-known normal form for the predicate calculus. In his early papers we also find the technique of using "Skolem functions" to eliminate quantifiers and in the construction of models—a technique of extraordinary importance in contemporary general model theory.

In the 1922 lecture the Löwenheim-Skolem theorem was applied to a formalization of set theory. The result was a relativization of the notion of set, later known as the Skolem paradox: If the axiomatic system (e.g. as presented by Zermelo) is consistent, i.e. if it is at all satisfiable, then it must be satisfiable within a countable "Denkbereich" (domain). But does this not contradict Cantor's theorem of the uncountable, the existence of a never-ending sequence of transfinite powers? The "paradox" of Skolem is no contradiction. Roughly speaking it asserts that there is no complete axiomatization of mathematics, and that certain concepts must be interpreted relative to a given axiomatization and its models and thus have no "absolute" meaning.

In the same lecture Skolem made several other important contributions to the foundation of set theory. He came up with a method, now universally accepted, for eliminating the psychologism inherent in Cantor's description of a set and still present in the axiomatization given by Zermelo. Skolem also extended and thus completed Zermelo's system on an important point by introducing the axiom of replacement. This was also done at about the same time by A. Fraenkel. The Fraenkel contributions became better known, but in all fairness one might claim that what is to-day called the Zermelo-Fraenkel system of axiomatic set theory ought really to be known as the Zermelo-Fraenkel-Skolem system.

Skolem himself did not have much confidence in set theory as a foundation for "real" mathematics, and he was extremely doubtful about the transfinite powers and non-constructive modes of reasoning of set-theoretical mathematics. His own preferences are better represented by the important 1923 paper. Here Skolem tries to build up elementary arithmetic without applying the unrestricted quantifiers "for all" and "there exists" to the infinite completed totality of natural numbers. Historically, this is perhaps "a first" paper in the theory of recursive arithmetic. In a paper from 1924 he also gave a constructive proof for the fundamental theorem of algebra. But, whereas recursion theory has witnessed a rich development, the strictly finitistic construction of mathematics advocated by Skolem has aroused less interest. In a lecture to the 1950 International Congress of Mathematicians he hoped that "the very natural feature of my considerations would convince people that this finitistic treatment of mathematics was not only a possible one but *the* true or correct one". For an ordinary mathematician this is an extreme position, so Skolem hastens to add: "Now I will not be misunderstood. I am no fanatic." To this, everyone who knew Skolem would surely agree.

Skolem's work in logic did not create much interest among his Scandinavian colleagues at the time, and later he indicated that he could not derive much inspiration while his papers remained unread. So, from the beginning of the 1920s, he turned to more traditional and "respectable" fields—algebra and number theory. Skolem has contributed extensively to the theory of Diophantine equations, not so much to the study of particular equations, but in giving results and methods of a more general nature. He is probably best known for the so-called p-adic method, perhaps one of the more significant developments in Diophantine equations in later years. The method can be used quite generally to conclude that certain equations have only a finite number of solutions in rational integers, and in many cases it is possible to give constructive methods for actually finding the solutions. An important paper by Skolem on this method is "Einige Sätze über p-adische Potenzreihen mit Anwendung auf gewisse exponentielle Gleichungen" from 1935. It is not possible here to enter into the details of the method. The reader may refer to an excellent survey by Skolem himself, "The use of p-adic methods in the theory of Diophantine equations" from 1954. In 1938 Skolem wrote a book *Diophantische Gleichungen* in the Ergebnisse series.

Skolem has also written a large number of papers on algebra. He is perhaps best known in this field for a result published in 1927 in

"Zur Theorie der assoziativen Zahlensysteme". The theorem was later rediscovered by Emmy Nöther and R. Brauer, and to-day carries the name "Skolem-Nöther". It characterizes the automorphisms of simple algebras and says in its modern form that there are no other but inner automorphisms of a simple algebra A of finite rank over a field K, where K is the centre of A and supposed invariant under the automorphisms.

Skolem's contribution to algebra and number theory, which is much more extensive than indicated here (e.g. there is a paper from 1944 "Über Nebenkörper und Nebenringe" which has not perhaps been sufficiently exploited), suffices to give him an honourable position in the Mathematicians' hierarchy. In addition, there is his work on logic.

In 1928 he published a lecture "Über die mathematische Logik" which he gave earlier the same year at a meeting of *Norsk Matematisk Forening.* This rather modest title conceals a very important paper. He sets himself the task of showing how the deduction problem for elementary logic can be reduced to an arithmetical-combinatorial problem. Generally, the latter problem is not effectively solvable, but for certain classes of formulas it is possible to obtain an effective decision procedure. The examples given by Skolem are important contributions to the "Entscheidungsproblem" of logic. Skolem wrote several papers on this topic, and, of necessity, they are quite combinatorial in method. Typical in this respect is a short paper from 1933 "Ein kombinatorischer Satz mit Anwendung auf ein logisches Entscheidungsproblem". The combinatorial theorem referred to is the well-known Ramsay's theorem. Skolem gave a slightly sharper version with a simpler proof.

Skolem has written many papers on pure combinatorics. In collaboration with Viggo Brun he edited the second editon of Netto's *Lehrbuch der Kombinatorik*, which was greatly enriched with many original contributions by Brun and Skolem.

In 1929 one of the better known of Skolem's contributions to logic appeared, "Über einige Grundlagenfragen der Mathematik". In it he returns to the proof of the Löwenheim-Skolem theorem and gives two new ones. One is short, but uses the axiom of choice. The second, which is not constructive either, deserves special comment. Skolem did not think of logic as an axiomatically given science. For him the notion of consistency was to be interpreted semantically, i.e. as satisfiability in some non-empty domain of individuals. Today, it is more common to formulate the notion proof-theoretically: there is no formula A such that both A and not-A are provable. When Skolem stated the Löwenheim-Skolem theorem in the following manner: "If a formula A is consistent, then it is satisfiable in a countable domain", one must remember that it is the semantic interpretation of "consistency" which is intended.

On the proof-theoretic interpretation one obtains quite a different theorem: Gödel's completeness theorem for predicate logic. But what merits special comment is the fact that the second 1929 proof of Skolem can, with some additions, be made to serve as proof of Gödel's theorem.

The axiom of choice received an elegant formulation in the 1929 paper. There exists a "global" choice function w such that $x \varepsilon y \to w(y) \varepsilon y$. Skolem deeply mistrusted this axiom. In a lecture from 1932 he stated: "If one works within a completely formalized mathematics, based on a finite number of precisely stated axioms, there is nothing to discuss but questions of consistency and ease of manipulation. But in ordinary mathematical practice, e.g., in the usual studies on continua which are never given by a set of specified formal rules, the axiom of choice is, in my opinion, definitely undesirable—a kind of scientific fraud."

Towards the end of the 1929 paper Skolem expressed some doubts about the complete axiomatizability of mathematical concepts. His scepticism was based on the set-theoretic relativism which follows from the Löwenheim-Skolem theorem. In 1929 he could give only some partial results, but in a paper from 1934 (and a previous one from 1933) "Über die Nichtcharacterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen" he could prove that there is no finite or countably infinite set of sentences in the language of Peano arithmetic which characterizes the natural numbers. Today, this follows as a simple consequence of Gödel's completeness theorem. The technique used by Skolem was a more direct model-theoretic construction. And this technique, suitably refined to the so-called "ultraproduct" construction, has been an important tool in recent work on model theory.

Skolem has also written many papers on recursion theory since 1923. Among his many contributions may be mentioned a short and elegant paper from 1940, "Einfacher Beweis der Unmöglichkeit eines allgemeinen Lösungsverfahren für Aritmetische Probleme". He first shows that every general recursive relation is arithmetical and concludes from this, by a simple application of Cantor's diagonal method, that if one identifies "effective" with general recursive, then there is no effective decision method for arithmetical problems. Perhaps the main unsolved problem in this field is Hilbert's 10th problem, which postulates an effective method for solving Diophantine equations. It is somewhat strange that Skolem, being equally at home in recursion theory and Diophantine equations, never seemed to have worked on this problem. In a paper from 1962 he remarks: "It has been asked whether the recursively enumerable sets are all of them Diophantine sets ... I regret not having had the opportunity to study this question seriously."

As previously remarked, Skolem published most of his papers in Norwegian journals, and they have not always been easy to obtain for mathematicians abroad. Others have thus rediscovered his results. One example is the Skolem-Nöther theorem. Another is Skolem's contributions to lattice theory, which remained completely unnoticed at the time of their publication. In 1936 he even had to write a paper "Über gewisse 'Verbände' oder 'Lattices'" giving a survey of results which he had obtained in 1913 and 1919 and which were rediscovered in the 1930s. Skolem had been the first, for example, to determine the free distributive lattice generated by n elements (1913); he had shown that every implicative lattice is distributive and, as a partical converse, that every finite distributive lattice is implicative (1919). These results are of great interest for the algebraization of mathematical logic.

Age did not diminish Skolem's research activities. From 1948 to 1957, i.e. from his 61st to his 70th year, he published 48 papers. Skolem was a truly creative mathematician. In his greatest papers he started out from simple, but, as we now recognize, fundamental situations. What he then wrote has remained basic for all further work.

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