# *Al-Sijzī's Treatise on* **Geometrical Problem Solving**

A fourth/tenth century text on problem-solving strategies in geometry

Translated and annotated by Jan P. Hogendijk

The text presented here is essentially that published by Fatemi publishing house in Tehran, on the occasion of a conference in Isfahan in June 1996. I have deleted some minor editorial changes made by the publisher without my knowledge.

**Contents**:

vii Introduction to al-Sijzī and his treatise on geometrical problem solving

1 Translation

31 Appendix: Quotations from other texts

34 Some differences between Saidan's edition and the Lahore manuscript

This file includes neither the Persian translation by M. Bagheri, nor the Arabic text, which the Fatemi publishing house in Tehran reprinted from Saidan's edition (including Saidan's own footnotes) without my knowledge.

The numbering of pages and footnotes in this file is exactly the same as in the Fatemi edition. The figures have been redrawn and usually appear on the same pages as in the Fatemi edition.

## Al-Sijzī's Treatise on Geometrical Problem Solving.

*In memory of A.S. Saidan*

### 1. Introduction

This book contains a translation, with commentary, of Al-Sijzī's *Book on Making Easy the Ways of Deriving Geometrical Figures.*[1] An uncritical edition of the Arabic text was published by Saidan in 1983 as 'appendix 3' of his collection of works of Ibrāhīm ibn Sinān (296-335 H. / A.D. 907-946)[2] but the text has never been critically edited or translated into any language. There seems to be only one Arabic manuscript of this text, in a private library in Lahore (Pakistan).

As far as is known, this text is the only treatise by a medieval Islamic geometer on problem-solving strategies in general. There are treatises by other Islamic mathematicians, such as Ibrāhīm ibn Sinān, on the method of analysis used in classical Greek antiquity, but al-Sijzī gives much more. Thus al-Sijzī's treatise resembles G. Polya's well-known book *How to Solve It,* although the works were written in different periods for different audiences and in different mathematical styles. Section 2 of this paper contains some new information on al-Sijzī and an approximate date of the *Book on Making Easy the Ways of Deriving Geometrical Figures.* In Section 3 I discuss al-Sijzī's own summary of the text and I compare his text to the book by Polya.

### 2. The author

Abū Sa$^c$īd Aḥmad ibn Muḥammad ibn $^c$Abdaljalīl al-Sijzī was one of the most prolific Islamic geometers in the fourth century H. / tenth century A.D. Very little is known about his life.[3] The earliest known date in al-Sijzī's life is the month *Rabī$^c$ al-Ākhir* 352 H. (April-May, A.D. 963), when he copied a manuscript of the Arabic translation of the *Introduction to Mechanics* of Pappus of Alexandria.[4] Al-Sijzī was still active at the beginning of Muḥarram 389 H. (end of December A.D. 998), when he wrote a work on the transversal theorem.

The name *Sijzī* indicates that he was from Sijistān, that is present-day Sīstān in South-Eastern Iran. There are various indications that al-Sijzī spent part of his life in this area. Al-Bīrūnī lists in his *Chronology of Ancient Nations*

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[1] كتاب في تسهيل السبل لاستخراج الاشكال الهندسية

[2] A.S. Saidan, *Rasā'il Ibn Sinān*, Kuwayt 1983. Ibrāhīm ibn Sinān's treatise on sundials (GAS V, 294 no. 5) is not included in this work.

[3] Most of the available information has been summarised by Sezgin and Ghorbani; see F. Sezgin, *Geschichte des Arabischen Schrifttums*, Leiden, Brill, Band V (1974), 329-334, Band VI (1978), 224-226 and Band VII (1979), 409-410;
ابو القاسم قربانی، زندگینامه ریاضیدانان دوره اسلامی، تهران، ۲۵۱ - ۲۶۰.

[4] See D.E.P. Jackson, The Arabic translation of a Greek manual of mechanics, *The Islamic Quarterly* **16** (?) 96-103, esp. p. 97.

the names of the months in the calender of Sijistān which had been told him by "the shaykh al-Sijzī".[5] Al-Sijzī says in his *Introduction to Geometry*: "I made in Sijistān a great and important instrument, a model of the whole world, composed of the celestial spheres, the celestial bodies, the orbits of their motions with their sizes, their distances and their bodies, and the form of the earth, the places, towns, mountains, seas and deserts, inside a hollow sphere provided with a grid. I called it 'the configuration of the universe.' "[6] Al-Sijzī was present at astronomical observations made at Shīrāz during the winter solstices of 359 H. /A.D. 969 and 360 H. /A.D. 970.

The manuscript Paris, Bibliothèque Nationale, Fonds Arabe, no. 2457 is of interest in this connection, because many historians[7] believe that it was written by al-Sijzī in this period. This manuscript consists of around fifty mathematical and astronomical treatises by various authors, including al-Sijzī, and according to the colophons, it was written in the period 358-361 H. /A.D. 969-972 by "Aḥmad ibn Muḥammad ibn ᶜAbdaljalīl", i.e. al-Sijzī himself. However, there are also scholars[8] who have argued that the Paris manuscript is a copy, made in the 13th century, of a manuscript written by al-Sijzī. In my view, however, there are two reasons why the Paris manuscript cannot be the handwriting of al-Sijzī. First, there are so many mistakes in the text and the figures that the manuscript cannot have been written by a mathematician of the level which Al-Sijzī must have reached in the years 358-361 H. / A.D. 969-972. Secondly, the manuscript includes a treatise by al-Sijzī entitled "Book of Aḥmad ibn Muḥammad ibn ᶜAbdaljalīl on the Measurement of Spheres by Spheres",[9] which ends with the sentence: "This is the end of what *he* wrote of this book"[10] (emphasis mine).

Al-Sijzī addressed a treatise on parabolic and hyperbolic cupolas to his father Abū'l-Ḥusayn Muḥammad ibn ᶜAbdaljalīl, in the year 340 of the (Persian) Yazdgerd chronology, corresponding to March 971 - March 972 A.D.[11] In his work *On the Selected Problems which were discussed by him and the*

[5] C. Sachau, *The Chronology of ancient nations ... of al-Bîrûnî.* London 1879, p. 52.

[6] واني عملت آلة جليلة خطيرة بسجستان تحكي العالم باسره من الافلاك و الاشخاص العالية ومدارات حركاتها بمقاديرها وكمية ابعادها واجرامها وصورة الارض والبقاع والبلدان والجبال والبحار والرمال في خوف كرة مشبكة وسميتها هيئة الكل.

Ms. Chester Beatty 3562, f. 17b.

[7] Most recently P. Kunitzsch, R. Lorch, A note on codex Paris BN ar. 2457, *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften* **8** (1994), 235-240.

[8] For example F. Sezgin in GAS VI, 192

[9] f. 195a, كتاب احمد بن محمد بن عبد الجليل في مساحة الاكر بالاكر

[10] (f. 198a) هذا آخر ما عمله من هذا الكتاب.

[11] This treatise is listed in GAS V, 331 no. 3 and also no. 5. Two manuscripts survive: Paris, Bibliothèque Nationale, 2457, 137b-139a, and Istanbul, Reşit 1191, 66a-68b. The title is also mentioned in the list of mathematical works of al-Sijzī in the manuscript Dublin, Chester Beatty 3652, 1b-2a.

The Dublin and Istanbul manuscripts say that Abū al-Ḥusayn was 'his (i.e. al-Sijzī's) father'. The Istanbul manuscript says (f. 68b:15) that al-Sijzī finished this text in the year 340 of the Yazdgerd chronology. The colophon of the Paris manuscript Bibliothèque Nationale 2457 says that the manuscript was written on Monday, Rām Rūz of the month Bahman of the year 340 of the Yazdgerd chronology (February 12, A.D. 972).

*geometers of Shirāz and Khorāsān and his annotations*,[12] Al-Sijzī discusses some of his father's solutions to geometrical problems which concern the division of triangles and parallelograms. Therefore al-Sijzī's father must have been a creative mathematician.

Around 40 geometrical treatises of al-Sijzī are known to be extant. In these, al-Sijzī refers to at least 20 more treatises which he wrote but which have not come down to us. Around 20 treatises by al-Sijzī on astronomy and astrology are extant. Only a small part of al-Sijzī's work has been published to date.

The *Book on Making Easy the Ways of Deriving Geometrical Figures* can be dated approximately on the basis of the following arguments. Al-Sijzī cites the work in his treatise on the hyperbola and asymptotes (for details see quotation no. 3 in the appendix to this paper). This text was written in the year 349 Yazdgerd,[13] corresponding to March 980 - March 981 A.D. Less than two years earlier, in the month Dhū'l Ḥijja 368 H. / July 979 A.D., al-Sijzī finished his (hitherto unpublished) *Introduction to Geometry*.[14] In this work, al-Sijzī mentions many theorems which he discusses in more detail in the *Book on Making Easy the Ways of Deriving Geometrical Figures*, but he does not mention that text, although he refers to many other works of his own. It is therefore likely that he wrote the *Book on Making Easy ...* a little later than his *Introduction to Geometry*. Hence the *Book on Making Easy the Ways of Deriving Geometrical Figures* can be dated approximately 370 H. / A.D. 980. Thus the work is the fruit of at least fifteen years of al-Sijzī's own experience with problem solving in geometry.

## 3. Al-Sijzī's own summary of the text

In the following summary and analysis, numbers in square brackets (such as [2]) refer to the paragraphs into which I have divided my translation.

The purpose of the *Book on Making Easy the Ways of Deriving Geometrical Figures* is in al-Sijzī's own words [1]:

*to enumerate ... the rules which will make it easy for the researcher ... to*

[12] كتاب احمد ابن محمّد ابن عبد الجليل في المسائل المختارة التي جرت بينه وبين مهندسي شيراز وخراسان وتعليقاته
In the manuscript Dublin, Chester Beatty 3652, the solutions of al-Sijzī's father begin on on f. 43a:2-3, 50a:9-10, and 50a:27.

[13] The ms. Dublin, Chester Beatty 3652, 61a is the last leaf of his treatise on the hyperbola and the asymptotes. (The rest of the treatise is missing in the Dublin ms. but extant in other undated manuscripts.) The fragment says at the end: "I made this in the year 349 Yazdgerd
عملت في سنة شمط يزدجردية

[14] المدخل في علم الهندسة, see GAS V, 333 no. 20. In the manuscript Dublin, Chester Beatty 3652 the scribe states at the end of this text (f. 17b):

وقد علّقته من خط المصنف وقد فرغ من تصنيفه وكتابه في ذي الحجة سنة ثمان وستين وثلاثمائت

i.e. I copied this from the handwriting of the author, and he finished writing and copying the work in Dhū al-Ḥijja of the year 368.

*derive whatever geometrical constructions he wants.*

Al-Sijzī divided his treatise into an introduction with a general explanation of his 'rules' in [1]-[21], followed by examples in [22]-[50]. In [20] he sums up the discussion with a list of what he calls 'the seven methods for discovery in this art'. I now render these methods in Al-Sijzī's words, with some of my own commentaries and a few quotations from G. Polya's well-known book *How to Solve it.*[15]

1. *Cleverness and intelligence, and bearing in mind the conditions which the proper order (of the problem) makes necessary.* The second aspect is related to the first stage in Polya's scheme for the solution of problems (preface p. xvi):

   "First you have to understand the problem. What is the unknown? What are the data? What is the condition? Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?"

2. *The profound mastery of the (relevant) theorems and preliminaries.*

   Compare Polya, p. 9: "It is hard to have a good idea if we have little knowledge of the subject, and impossible to have it if we have no knowledge. Good ideas are based on past experience and formerly acquired knowledge."

3. ... *But you must combine with that (no. 2) cleverness and guesswork and tricks. The pivotal factor in this art is the application of tricks, and not only (your own) intelligence, but also the thought of the experienced (mathematicians), the skilled, those who use tricks.*

   Compare Polya, pp. 4-5: "Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation and practice. Trying to swim, you imitate what other people do with their hands and feet to keep their heads above water, and, finally, you learn to swim by practising swimming. Trying to solve problems, you have to observe and imitate what other people do when solving problems and, finally, you learn to do problems by doing them."

4. *Information about the common features (of figures), their differences, and their special properties. In this particular approach, the special properties, the resemblance and the opposition are (considered by themselves) without enumeration of the theorems and preliminaries.*

   Two ideas seem to be involved here. In [7], al-Sijzī adds that *there are figures which share one or more special properties, and there are figures which have nothing in common, and there are figures which are more closely or more distantly related, according to configuration, proportionality, and genus.* The word *genus* suggests that Al-Sijzī was thinking of a philosophical

[15] Princeton: Princeton University Press, second edition 1957, I have used the second paperback printing of 1973.

classification of figures, in the way of Aristotle, into *genera*, which are subdivided into *species*. Al-Sijzī may have thought that if two figures (i.e. problems or theorems) are philosophically related, their constructions or proofs are also related.

For al-Sijzī, one of the inherent properties of a problem is its easiness or difficulty. In [22], he calls a certain figure *difficult to derive.* He adds: *This guess is what I have called before (i.e. in [20]) information on the level of easiness or difficulty of them: if there are many unknowns in the figure, it is difficult to find by means of known things.* Hence, philosophically, problems may be divided into easy problems, difficult problems requiring many preliminaries, and even *questions of which the essence is possible in nature, but not for us, or the investigation of them is impossible for us, because of the lack of preliminaries for them. Such is the quadrature of the circle* [14].

G. Polya also speaks about the difficulty of a problem, but he does not make any connection with philosophy. "How difficult is it? ...In answering the question we must rely much more on "feeling" than on clear argument. Still, now and then we arrive at assessing the degree of difficulty of a problem quite well."[16]

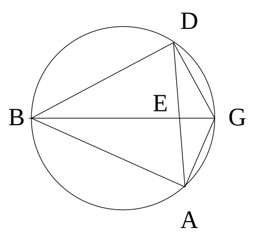


Figure I

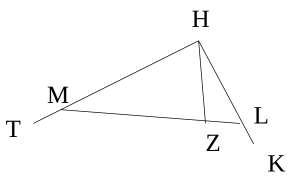


Figure II

5. *Transformation.* Here al-Sijzī means: a particular kind of transformation of one problem to a new problem, in such a way that one or more of the hypotheses of the old problem become unknown in the new problem and one or more of the unknowns of the old problem become hypotheses in the new problem. This particular method is not mentioned by Polya.

   Al-Sijzī gives an example in [22]-[24]. Here I present an easier example from Ibn al-Haytham's *Optics* Book V, prop. 34.[17] I explain the original problem and the *transformation* of this problem to a new problem but not the solution of the new problem, which involves conic sections.

   The original problem is as follows. Suppose that $BG$ and $c$ are two given straight line segments and that $A$ is a given point on the circle whose diameter is $BG$ (Figure I, orginal problem). Required to construct a straight line

[16] G. Polya, *Mathematical Discovery. On understanding, learning and teaching problem solving.* New York etc.: Wiley, 1981, paragraph 14.21, p. 137.

[17] For an English translation see A.I. Sabra, Ibn al-Haytham's lemmas for solving "Alhazen's problem", *Archive for History of Exact Sciences* **26** (1982), 299-324. Sabra calls this particular problem 'Lemma II'.

$AED$ through $A$ which intersects the diameter at $E$ and the circle at $D$ such that $|DE| = c$.

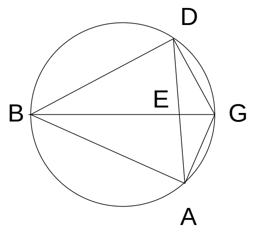


Figure I

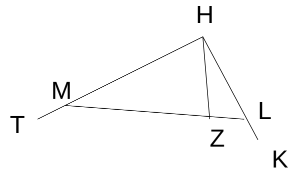


Figure II

First we make the following observations. If we call $\angle BGA = \gamma$, $\angle GBA = \beta$, the angles $\gamma$ and $\beta$ are known and we have by a property of the circle $\angle EDB = \gamma$ and $\angle EDG = \beta$.

Now define the new problem as follows: Draw a line segment $HZ = c$ and draw two lines $HT, HK$ such that $\angle ZHT = \gamma, \angle ZHK = \beta$, as in Figure II. Required to construct a straight segment $MZL$ through $Z$ such that point $M$ is on $HT$, point $L$ is on $HK$, and $|ML| = |BG|$.

The figures in the old and new problem are congruent: points $M, H, Z, L$ in the new problem correspond to points $B, D, E, G$ in the old problem. In the old problem, the position of $BG$ is given and the position of $DE$ is required. In the new problem, the position of $HZ$, corresponding to $DE$, is given and the position of $ML$, corresponding to $BG$, is required. Thus the unknown and one of the data have been exchanged.

In this example, the old and new figures are congruent. In [22]-[24], al-Sijzī discusses a case where the new figure and the old figure are not congruent but similar. From a modern point of view it is natural to emphasize the idea of a geometrical transformation between the old figure and the new figure. However, the modern idea of a mapping between two figures is not mentioned by al-Sijzī or by any other geometer in the Islamic tradition.

The method of *transformation* was used by Abū Sahl al-Kūhī,[18] Ibn al-Haytham and Apollonius.[19] As far as I know, al-Sijzī's explicit description of the method, and his use of a technical term (*transformation*) for it, are unique in the ancient and medieval literature.

6. *Analysis.* This is the ancient Greek method of *analysis* and *synthesis*, which is described by G. Polya in a special article 'Pappus' (pp. 141-148), based on the passage in Book VII of of Pappus of Alexandria's *Mathematical Collection.*[20] This work was not known is the Islamic tradition, but the

[18] See J.L. Berggren, "The correspondence of Abū Sahl al-Kūhī and Abū Isḥāq al-Ṣābī: A translation with commentary," *Journal for History of Arabic Science*, **7** (1983) 39-124, esp. pp. 88-89 (Figure 17 and Figure 18)

[19] See J.P. Hogendijk, *Ibn al-Haytham's Completion of the Conics*, New York 1985, pp. 89-93, 105-113. There I used the term 'reduction to a problem concerning shape' because I did not yet know about the existence of a medieval Arabic term *naql* for the process.

[20] See A. Jones, *Pappus of Alexandria, Book 7 of the Collection*, New York (etc.), Springer-Verlag, 1986, pp. 82-85 and the commentary by Jones on pp. 66-70.

methods of analysis and synthesis were transmitted from Greek into Arabic via other texts. Ibrāhīm ibn Sinān wrote a long treatise on the subject, called *Treatise on the method of analysis and synthesis in the geometrical problems*,[21] which was published by A.S. Saidan in his *Treatises by Ibn Sinān*, and to which he appended his edition of the text of al-Sijzī.

7. *The use of tricks, such as Heron used.*[22] In [19], Al-Sijzī gives some more details: *There were ancient geometers who used subtle tricks, if the discovery of the desired things was difficult, such as the (geometer) whose desired things were connected with ratio, and who used in them numbers and multiplication, or (the geometer) whose problem was to measure a figure, or equality (between figures), and who used in them drawings of them on silk or paper, and weighing them, or who used other similar tricks.* Al-Sijzī must have been quoting from ancient works which are now lost.

   Polya does not mention these 'tricks'. There is no evidence that Al-Sijzī (or any other medieval Islamic geometer) knew the *Method* of Archimedes. In this work Archimedes explains a method for the conjectural determination of the volumes of some solids by dividing them into infinitesimal sections and by weighing these sections on an imaginary balance.[23] The Greek text of this work was discovered in 1906 by Heiberg.

The list of seven rules does not exhaust the interesting topics in al-Sijzī's text, and the reader can find more resemblances between al-Sijzī's text and Polya's books on problem solving than I have given above. I finish this introduction by making some general remarks on the comparison between the works of these two authors.

Polya's book is written with a teaching situation in mind, but al- Sijzī wrote his treatise for the researcher and the scholar. This difference explains many of the features of Polya's book which do not occur in al-Sijzī's text, such as the detailed schemes for problem-solving to be taught by a teacher and used by a student.

Polya's book was written almost ten centuries after the treatise of al-Sijzī and mathematics changed a lot during that period. However, since Polya is concerned with what is now elementary mathematics with emphasis on geometry, his subject overlaps with that of al-Sijzī. In solving geometrical problems, Polya used the algebraic notation ($x$, $y$ etc.) which was developed in the Renaissance and introduced into geometry by René Descartes in 1637, and which was therefore unknown to al-Sijzī. Thus Al-Sijzī does not have a chapter like 'Setting up equations'.[24]

[21] مقالة في طريق التحليل والتركيب في المسائل الهندسية, GAS V, 294 no. 2.

[22] 'Heron' is Heron of Alexandria.

[23] See T.L. Heath, *The works of Archimedes with a supplement , The Method of Archimedes.* Cambridge: Cambridge University Press, 1897, 1912; reprinted in one volume: New York: Dover Reprints, no date.

[24] Polya, *How to Solve It*, p. 174

Unlike Polya, Al-Sijzī did not mention problems in arithmetic and algebra, although such problems were studied in his time and there must have been problem-solving strategies for them also.

Al-Sijzī pays more explicit attention than Polya to the basic structure of mathematics. Thus Al-Sijzī's detailed explanations of the deductive structure of mathematics [32] are not found in Polya. The reason may be twofold: First, al-Sijzī assumes less familiarity with mathematics than does Polya. Second, since al-Sijzī was writing for a scholarly audience, he paid more attention to the axiomatic approach to mathematics as explained by Euclid. Polya was writing for the teacher about a less formalized kind of mathematics which is taught in modern high schools.

Polya's book *How to Solve It* is a short text, but he also wrote a more voluminous work of 432 pages entitled *Mathematical Discovery: On understanding, learning and teaching problem solving.*[25] This work contains much more material than the text of al-Sijzī.

In my opinion, the agreements between Polya and al-Sijzī are more impressive than the differences, considering the large time interval between these authors. Polya would have been very excited to know that he had an Iranian predecessor almost ten centuries earlier. I close the introduction by quoting a passage from Polya's preface to *How to Solve It* which would fit al-Sijzī's text equally well:

"Yes, mathematics has two faces; it is the rigorous science of Euclid, but it is also something else. Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science. Both aspects are as old as the science of mathematics itself. But the second aspect is new in one respect; mathematics "in statu nascendi," in the process of being invented, has never before been presented in quite this manner to the student, or to the teacher himself, or to the general public."[26]

[25] New York etc.: Wiley, 1981.
[26] Polya, *How to Solve It*, p. vii.

*The following translation of Al-Sijzī's* Book On Making Easy the Ways of Deriving Geometrical Figures *is based on the printed edition of the text in Appendix 2 to A.S. Saidan,* Rasā'il Ibn Sinān*, Kuwayt 1983, pp. 339-372. Thanks to Dr. Heinen, München, I have been able to compare this edition to the Lahore manuscript. An appendix to this paper lists all words and passages in which my translation does not follow Saidan's edition because my reading of the manuscript is different from his. Passages between angular brackets* < ... > *have been added to the text by me to restore the original. My own explanatory additions to the translation are in parentheses. Numbers in parentheses such as* (340) *indicate the beginning of a new page in Saidan's edition.*

*I have divided the text into numbered sections* [1], [2] *etc. for easy reference. The numbers appear in the left margin.*

*In my footnotes I often refer to Euclid's* Elements *in the translation of T.L. Heath:* Euclid. The thirteen Books of the Elements*, second edition, Cambridge 1925; reprinted by Dover Publications, New York 1956, 3 volumes. A notation such as Heath 1, 221-232 refers to pages 221-232 of volume 1 of this work.*

## Translation

In the name of God, the merciful, the compassionate. Lord, help us.

**The book of Aḥmad ibn Muḥammad ibn ᶜAbdaljalīl al-Sijzī On Making Easy the Ways of Deriving Geometrical Propositions.**

We want to enumerate, in this book of ours, the rules[27] which will make it easy for the researcher who knows and masters them to derive whatever geometrical constructions he wants. We will mention the methods and ways which will make the mind of the researcher who follows them competent in the different aspects of deriving figures.[28] [1]

Some people think that there is no way of learning the rules for deriving (new propositons) even with much research, practice, study, and lessons in the elements of geometry,[29] unless a man has an innate natural talent which enables him to discover figures, because study and practice are insufficient. But this is not the case. There are people who have a natural aptitude and an excellent ability for deriving figures, but who do not have much knowledge, and who do not work hard to study these things. But there are also people who [2]

[27] Arabic: *qawānīn.*

[28] The Arabic word *shakl* means 'figure' but also 'proposition'. I have translated 'figure' for sake of consistency but the reader should realize that this word can be taken in a wide sense.

[29] Al-Sijzī refers to the basic mathematical knowledge explained in Euclid's *Elements.*

work hard, who study the elements and the methods, but who do not have an excellent natural ability. If a man has an inborn natural talent and if he works hard to study and practise, then he will be first-rate and outstanding. If he does not have a perfect ability, but if he works hard and studies, then he can also become outstanding by means of study. As for (340) someone who has the ability but does not study the elements, and does not devote himself to the constructions of geometry, he will not benefit from it in any way. Since this is as we said, if someone thinks that discovery in geometry proceeds only by means of innate ability and not by study, then he thinks nonsense.

[3] The first thing which is required of a beginner in this art is that he knows the theorems[30] which come after the axioms.[31] This is (also) considered to be part of our aim: the figures which we want to discover. But (here) our aim is the methods (of discovery) which can be found by means the theorems, not only by means of the axioms, which are preliminaries to the theorems.

[4] The discussion of the axioms is very long, but Euclid spared us this in his *Elements* by means of the theorems which he presented and which we have mentioned. As for the theorems which are preliminaries to the aims (i.e. the desired figures), their classification is more difficult than (simply) saying they are "preliminaries", although this is true from the point of view that the parts of geometry are interwoven with one another. For the first of them are preliminaries to the other ones, one by one, as if each one is linked to the one which follows it, until some end (is reached). This here is a difficult matter,[32] but we can summarize the discussion of this in a clear way, as Euclid described in the *Elements*.

[5] If someone says: "If the matter is like this, how can the (i.e. all) theorems be obtained, while the discovery of figures is endless? or: "Why don't we restrict ourselves to the axioms?", then you tell him: "Euclid gave a balanced treatment in his collection. If he had restricted himself to the axioms, then it would be difficult for the researcher to discover (propositions) from the axioms alone, without (having at his disposal) preliminaries among the geometrical theorems, as arranged by Euclid after the axioms. But he did not list too many (theorems) either. (341)

[6] It is necessary for someone who wants to learn this art, to thoroughly

[30] Arabic: *qawānīn*, literally: rules. I have translated *theorems* for sake of clarity, even though al-Sijzī might have included the constructions in Euclid's *Elements* as 'rules'. In the first sentence in [1], the word *rules* is used in a more general sense.

[31] Literally: the 'common notions' (*ᶜulūm mutaᶜārifa*), i.e. certain axioms given by Euclid in Book I of his *Elements*, see Heath 1, 221-232. I have used the translation *axioms* for sake of clarity.

[32] For a modern discussion of the mutual relations between the propositions in Euclid's *Elements* see for example Ian Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's* Elements, Cambridge Mass., London 1981 (MIT Press).

master the theorems which Euclid presented in his *Elements*. For between mastering the thing and the thing itself there is a very deep gap. It is necessary that he has a thorough idea of their genera[33] and their special properties, so if he needs to look for their properties, he is well-prepared to find them. If he has to do any research, then it is necessary for him to study and visualize in his imagination the preliminaries and theorems that are of that genus, or that have (something) in common with it. For example: if we want to derive a figure of the genus of the triangle, then we have to visualize all the properties belonging to triangles, and the theorems which Euclid mentioned, and the angles, arcs, sides, and parallel lines which are involved in the properties of the triangle, so that this may be easy for him[34] (i.e. the researcher), and that he may be prepared for deriving them.

There are figures which share one or more special properties, and there are figures which have nothing in common, and there are figures which are more closely or more distantly related, according to configuration, proportionality, and genus. [7]

If we want to find some figure by means of preliminaries, and we mean by a "preliminary"[35] the figure which comes before it and is a basis for the derivation of it, and if we cannot derive it by means of this preliminary, then we must try to seek (it) by means of a preliminary related to that preliminary - if our search on the basis of that preliminary can be successful. It is a consequence of this affair that if any figure can be derived from some preliminary, then it can also be derived from preliminaries which are related to it in the way we mentioned, or from some of them, according to the extent of the relationship.

Among the special properties of figures is (the fact) that some (figures) can easily be derived by means of many different preliminaries and in many ways, and some can be found by means of one preliminary, and there are some for which no preliminary exists, even though that figure can be imagined, or its correctness is described in nature.[36] This is a consequence (342) of the close relation to the special properties of the preliminaries, or the difference between it (the proposition) and them. [8]

Again, figures can have preliminaries, and their preliminaries can have preliminaries, and their preliminaries can have preliminaries in turn, and those figures can be derived from the preliminaries of the preliminaries.[37] [9]

[33] The notions of 'genus' and 'species' are derived from the philosophy of Aristotle. Here it is sufficient to know that a genus (Arabic: *jins*) is divided into various species (Arabic: *anwā$^c$*). The word *khāṣṣa*, which I have translated as *special property*, is also an Aristotelian technical term.

[34] The sentence starts with 'we', but here the text changes to the third person.

[35] Arabic: *muqaddama*. This word can also be translated as *lemma*.

[36] See [14].

[37] See [32] for an example.

This special property is also from (i.e. based on) the sharing (of special properties) by figures that we have mentioned.

[10] Again, the discovery of figures can be difficult because they require the discovery of a series of preliminaries in succession, from one or two theorems,[38] as we shall explain below,[39] God willing. Sometimes they require many theorems and many preliminaries, not in a succession, but (all) linked (to one another), as we shall also mention,[40] God willing.

[11] Sometimes there appears to the researcher a method by means of which it was easy for him to derive many difficult figures. This (method) is the *transformation*. We shall explain it and give an example of it,[41] God willing.

[12] Another method is easy for the researcher if he follows it: He assumes the desired aim as if it were already constructed, if the aim is a construction, or he assumes that it is true, if the aim is the investigation of a special property. Then he unravels (analyses) it by means of a succession of preliminaries, or by means of (mutually) linked preliminaries, until he ends up with correct and true preliminaries, or with false preliminaries. If he ends up with true preliminaries, the desired thing can be found as a consequence.[42] If he ends up with false preliminaries, the impossibility of the desired thing follows. This is called: analysis by inversion. This method is of more general use than the other methods. We shall give an example of it below, God willing.[43]

Synthesis is the inverse of analysis. That is, synthesis is: following the road (i.e. reasoning) towards the (343) result, by means of the preliminaries. Analysis is following it towards the preliminaries which produce the desired thing.

[13] It is the business of geometry that the unknown becomes constructed or known.[44] Here it (the unknown) is necessarily constructions or special properties.

[14] First the researcher must think about the question and the things which

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[38] Here it is also possible that the word *qawānīn* refers to the rules for discovery mentioned in [1].

[39] Perhaps al-Sijzī is referring to [29].

[40] See [35]-[42].

[41] See [22]-[24].

[42] This is only true if the *desired thing* follows from the preliminaries, and not the other way around. Al-Sijzī was well aware of this fact, as is shown by his careful wording of the analysis in [25],[26].

[43] See for references the footnotes to [20].

[44] The manuscript has: *ma$^c$lūman aw maghlūman*. The word *maghlūman* makes no sense here and Saidan, p. 343 therefore suggests the emendation *maghlūṭan*, meaning: erroneous. I think that the original was *ma$^c$mūlan aw ma$^c$lūman*, because the following sentence explains that the unknown can be a construction or a special property.

are desired. There are questions of which the essence is possible in nature, but not for us, or the investigation of them is impossible for us, because of the lack of preliminaries for them. Such is the quadrature of the circle.[45]

There are (questions) which are indeterminate, the number of their examples (i.e. solutions) cannot be listed. The meaning of (the word) "indeterminate" is that they (the solutions) are not defined by complete conditions, which distinguish them from others. [15]

There are (propositions) that can be discovered but only by means of many preliminaries, such as the figures in the last parts of the *Conics*,[46] for they are not easy without the preliminaries given by Apollonius, or such as the figures in the last parts of the *Book of Circles*.[47] [16]

There are (questions) which require brightness in him (the researcher), that is to say, they require that he visualizes at one moment that many figures are constructed, in addition to the theorems and preliminaries. The general (characteristic) of them is in the investigation of (many) special properties. This man, who researches in this way, is called Archimedes, the culmination of the Greeks, that is to say the *Geometer*.[48] [17]

It is necessary for the researcher, if he wants to discover some figure, that he makes the end of the work the beginning of his thought, and conversely, as we have mentioned before. This is to say that he assumes the desired thing in the beginning of the work, and he considers it as the result of the preliminaries into which it can be unravelled (i.e. analysed). [18]

There were ancient geometers who used subtle tricks if the discovery of the desired things was difficult, such as the (geometer) whose desired things were connected with ratio, and who used in them numbers and multiplication, or (344) (the geometer) whose problem was to measure a figure, or equality (between figures), and who used in them drawings of them on silk or paper, and weighing them, or who used other similar tricks. [19]

These are the methods for discovery in this art. We will enumerate them separately, so that the researcher may visualize them with his mind and master [20]

[45] See for a philosophical discussion of this problem by Ibn al-Haytham the remarks in my *Ibn al-Haytham's Completion of the Conics*, pp. 95-96, and the literature cited there.

[46] The Arabic text of Books V-VII of the *Conics* has been edited in: G.J. Toomer, *Apollonius' Conics Books V to VII, the Arabic translation of the lost Greek original in the version of the Banū Mūsā*, New York: Springer Verlag, 1990, 2 vols.

[47] This is probably Apollonius' work *On Tangencies*, which is now lost but which was known in an Arabic translation. See *Archive for History of Exact Sciences* **35** (1986), pp. 218-223, 248-252.

[48] In a treatise on the regular heptagon al-Sijzī also says that the Greeks called Archimedes *the geometer*. See *Archive for History of Exact Sciences* **30** (1984), pp. 292, 306.

them, if God Most High wills so and grants good success.

First, cleverness and intelligence,[49] and bearing in mind the conditions which the proper order (of the problem) makes necessary.[50]

The second is the profound mastery of the (relevant) theorems and preliminaries.[51]

The third is: following of the methods of them (these theorems and preliminaries) in a profound and successful way, so that you rely not only on the theorems and preliminaries and constructions and their arrangement which we mentioned. But you must combine with that (your own) cleverness and guesswork and tricks. The pivotal factor in this art is the application of tricks, and not only (your own) intelligence, but also the thought of the experienced (mathematicians), the skilled, those who use tricks.[52]

The fourth is: information about the common features (of figures), their differences, and their special properties. In this particular approach, the special properties, the resemblance and the opposition are (considered by themselves) without enumeration of the theorems and preliminaries.[53]

The fifth is the use of transformation.[54]

The sixth is the use of analysis.[55]

The seventh is the use of tricks, such as Heron used.[56]

[21] Since we have presented and mentioned these things in a loose manner, let us now bring for each (345) of them examples, so that the researcher learns their true natures. For one can speak about this art in two different ways: first, abstractly, in a deceiving and illusory manner; and secondly in a profound way, with clear explanations and the presentation of examples, so that it is perceived and understood completely.

Since the discussion of this art is in these two ways, and since we have presented one of them, the general and gesticulating method, we must now present the second way, that is the way of clear explanation, and exhaustive information, with examples and a detailed investigation of them. May God the most High grant success. May He lead to the right path.

[49] See [2].

[50] See [39], [43].

[51] See [3], [4], [5].

[52] See [6], [25], and compare [28], [36]. In the examples [22]-[49], al-Sijzī says many times that 'guesswork', 'intelligence' or 'skill' must be used.

[53] This seems to be is a philosophical classification of problems, which is related to al-Sijzī's philosophy of mathematics as a whole. This is a very interesting subject which merits further research, regardless of the value of al-Sijzī's philosophical insights for modern mathematics. See [7]-[10], [22], [32], [33], [34], [36], [45]-[48]; compare also [13]-[17].

[54] See [11], [22]-[24] and the end of [34].

[55] See [12]-[18], [22], [25], [26], [29], [30], [37], [43], [49].

[56] See [19]. 'Heron' is Heron of Alexandria. Unfortunately, Al-Sijzī does not give examples, and I do not know what tricks he had in mind.

## The examples.

(*Example 1, concerning "transformation"*)

A question on the construction of a figure. [22]

How do we find two lines proportional to two assumed lines, in such a way that one of them is tangent to an assumed circle and the other meets the circle in such a way that its (rectilinear) extension inside the circle passes through its centre?

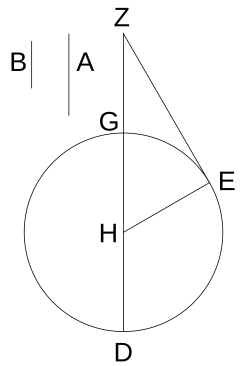


Figure 1

Thus we assume that the figure has been constructed, by way of analysis, so that we may look for its preliminaries. For example (Figure 1): we assume that the ratio is the ratio of $A$ to $B$, that the circle is circle $GD$, that lines $ZE, ZG$ are in the ratio of $A$ to $B$ and that they are the desired lines — as if it (the figure) were already constructed and as if the construction were available to us, as we mentioned before, —- in such a way that if $ZG$ is extended inside the circle towards $D$, $GD$ is a diameter of it.

Then we ask from what construction and what preliminaries its construction could have been found. (346)

Since point $Z$, lines $ZG, ZE$ and the point of contact of (circle) $DG$ at point $E$ are all unknown to us, and again, since the situation of the inclination[57] of angle $Z$ is also unknown, the figure is difficult to derive. This guess is what I have called before:[58] information on the level of easiness of difficulty of them (the propositions): If there are many unknowns in the figure, it is difficult to find by means of known things. (This is) especially (true) if its configuration is such that the components of the figure[59] do not have a relationship as we have mentioned. In this figure neither lines $ZG, ZE$ and the circumference of the circle are closely related, nor angle $Z$ and arc $GE$. Then we also use guesswork and intelligence, and we undertake its construction by means of transformation; for as we mentioned before,[60] it makes the derivation of difficult figures easy.

So we say: How can we place lines $ZG, ZE$ in such a configuration that if a circle is drawn, it is tangent to $ZE$ and meets $ZG$?[61] This is only possible for

[57] The text means that the magnitude of the angle is unknown.

[58] In [20], the fourth method. The reference is not exact.

[59] I have translated the Arabic word *ashkāliha*, literally: its figures, as: the components of the figure.

[60] See [11].

[61] Here it is understood that $ZE : ZG = A : B$ and that $ZG$ extended must pass through the centre of the circle.

us by the assumption of angle $Z$ and the knowledge of it (i.e. its magnitude). Thus we must seek the knowledge of angle $Z$, but we can only know it if we seek something else of the same kind, namely angles. Then, how can we seek (this) from the composition of lines $ZG, ZE$ or $ZE, ZH$ or $ZE, ZD$, for in this figure it is not possible for us (to seek this) from the composition of any other line. Here guesswork and intelligence must be used. If we join $E$ and $G$ (by a straight line), it is sometimes difficult to find it (the angle), and sometimes it is impossible to find it in this way, because the angles which are produced here are also not known in this figure by means of these preliminaries.

Therefore we join $E$ and $H$. Here we have found that angle $E$ of the three angles is known. Then it is necessary that we find the shape of triangle $ZEH$ from the composition of the lines and the angles.

[23] We now seek, after having found them (i.e. these insights) here, (the solution of) another problem.[62] If we have found this (new) problem, by means of it the (original) problem will be solved. It (the new problem) is (as follows): (347) the configuration of triangle $ZEH$ is restricted (i.e. determined) by the fact that it is a right-angled triangle such that the ratio of one side to the hypotenuse minus the other side is as an assumed ratio. Thus our first question has been reduced to this question, by means of this method which we have now followed, in order that it produces what the (first) question requires.

Thus we assume that the triangle is constructed in the usual way (Figure 2): triangle $TKM$ is right-angled, its right angle is angle $K$, but $NM$ is equal to $KM$, and the ratio of $TN$ to $TK$ is as the ratio of $B$ to $A$. Here skill and intelligence must be used, for each time when we seek (the solution of) an original (i.e. new) problem, we must use intelligence and guesswork, not learning.

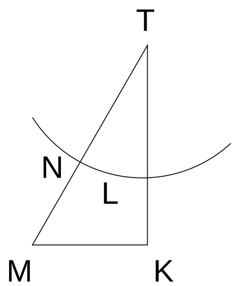


Figure 2

We must ask how we assume $TK$[63] such that the ratio of $TK$ to $TN$ is as the ratio of $A$ to $B$. We extend $KM$ indefinitely, and then we draw $TL$[64] in

[62] This is the "transformation", i.e. the transformation of a problem to another problem. Often, a transformation in the sense of al-Sijzī corresponds to a modern geometric transformation, that is a mapping of a figure (such as $ZEHG$, in which $EH$ is of given length) onto another figure (such as $TKMN$, in which $TK$ is of given length). However, the notion of a mapping is not found in al-Sijzī's text.

[63] Al-Sijzī now assumes that $TK$ is known. Then point $M$ is somewhere on the (known) perpendicular through $K$ to $TK$. Saidan's emendation of "$TK$" in the manuscript to "$N$" shows that he did not understand the mathematics here.

[64] Saidan assumed that point $L$ is on line $TK$, and he reads $TK$ here and elsewhere. In the figure in the manuscript. point $L$ is not on line $TK$.

our imagination (in such a way that) if you extend it to line $KM$, the difference between the moving line $TL$ and its extension[65] to line $KM$ is equal to the line between point $K$ and the junction point on line $KM$. So here is a problem of two unknowns.

So we construct a circle[66] with centre $T$ and radius $TL$ because we imagined line $TL$ moving on point $T$, so that we can be sure that the endpoint $L$ of line $TL$ in the imaginary motion falls always on the circumference of the circle. But the shape of the triangle is placed in front of us so we can see the figure with our eyes at the time of construction in the correct configuration. We seek the centre of a circle which (centre) is common to lines $TM, KM$.[67] So here guesswork (348) and intelligence must be used successfully.

Again, we can only do this by means of an extra construction. So we imagine [24]
this construction: how do we extend $TN$ to $S$ in such a configuration that line $KM$ bisects it and the whole (segment) $NS$ is twice $KM$. So we transfer the problem to another figure, and it is this (Figure 3).

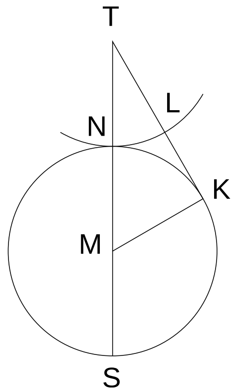


Figure 3

Then we do some thinking here, and we visualize that the aim has been achieved, as we usually do. That is to say: we assume $TNS$ such that $NS$ is twice $KM$, and $NM$ is equal to $KM$. We draw with centre $M$ and radius $MK$ circle $KS$. Then it is clear that $TK$ is tangent to the circle. But (now) we use guesswork and intelligence. If this is so, we must look for a special property of this figure caused by the tangency, (a property) which Euclid established in the *Elements*. The simplest special property of this figure is: the square of $TK$ is equal to $TS$ times $TN$.[68] Thus we have found from this special property in this construction an aid (i.e. an auxiliary construction): we make $NS$ the line such that $TK$ is equal in square[69] to $TS$ times $TN$.[70]

[65] To make mathematical sense, the "extension" should be interpreted as the whole segment between point $T$ and the intersection point with $KM$.

[66] This circle appears in the figures. It is unnecessary in the argument.

[67] The MS. and edition have $KN$.

[68] Euclid would have used the geometrical concept of a rectangle contained by sides equal to $TS$ and $TN$. The medieval Arabic translators of Greek geometrical texts used the arithmetical expressions such as *ḍarb* $TS$ *fī* $TN$, literally 'the product $TS$ times $TN$', or expressions which are a mixture of arithmetic and geometry such as *saṭḥ* $TS$ *fī* $TN$, literally 'the area $TS$ times $TN$'. The word *saṭḥ* is often omitted. I translate the resulting expression $TS$ *fī* $TN$ as '$TS$ times $TN$'.

[69] Arabic: *yaqwā*, literally: equal in power. The terminology was adapted from the Greek.

[70] Since $|TK|$ and $|TN|$ are assumed to be given, $|TS|$ can be found, and hence also $|NS|$.

If we do this, the rest of the construction is easy.[71] That is to say, we have found lines $TN$ and $TK$ and $TS$,[72] so what is left for us is that we find the configuration of $TS$ such that $KM$ bisects $NS$. (349) So we first bisect $NS$ at $M$ and in our imagination we move $TS$ around point $T$ (until) $KM$ bisects $NS$. That is easy to construct by drawing with point $T$ as centre and $TM$ as radius a circle which $KM$ intersects at point $M$.[73] We draw $TMS$ and we draw arc $SKN$. So we have constructed this figure as we wanted.

Then we transform it to the assumed circle by similarity and ratio, and we prove it, and that is what we wanted to explain.

*(Example 2, concerning 'search for special properties')*

[25] Since cleverness in the discovery of special properties is most useful in constructions, we will give an example of the investigation of the special properties of figures. That is to say, we assume triangle $ABG$ and we seek a special property of its angles with the consequence that the sum of the three (angles) is equal to the sum of the angles of an assumed triangle, before we know that they are equal to two right angles.

The method of our investigation of them is, in the first approach, that we assume one angle of it in its (fixed) position, and that we vary its sides, until it appears to us whether the two remaining angles (taken together) are greater or smaller than the two original angles (taken together) or equal to them (taken together).

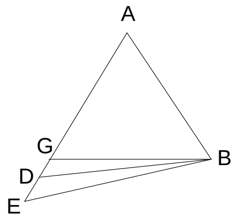


Figure 4

(Figure 4) We have supposed angle $A$ (as a fixed angle), not the other angles, with the following in mind: if we suppose that two angles of an assumed triangle are equal to two angles of another assumed triangle, each one being equal to the corresponding angle, it is necessary that the remaining angle (of one triangle) is equal to the remaining angle (of the other triangle), so we do not[74] obtain the knowledge which we desired.

We extend $AG$ to $D$ and we join $BD$. Then angle $ADB$ is less than angle (350) $AGB$.[75] Then we look at angles $ABG$, $ABD$. Angle $ABD$ is greater than angle $ABG$. Then we repeat this procedure. We extend $AD$ to $E$ and we

[71]Literally: we are close to the easiness of our construction.

[72]That is to say, we have found the magnitudes of these segments, but not their positions yet.

[73]The position of line $TK$ is fixed, so point $M$ is on the perpendicular through $K$ to $TK$. Point $M$ is also on the circle with centre $T$ and radius $|TM|$.

[74]Saidan omitted the word 'not' in the manuscript.

[75]Euclid, *Elements* I:16, Heath 1, 279-280.

join $BE$. Then angle $E$ is less than angle $ADB$ and angle $ABE$ is greater than angle $ABD$. We repeat this procedure constantly. Thus we make the angles which fall on side $AG$ less than before, and we make the angles adjacent to line $AB$ at point $B$ greater than before.

But now we need to test if the increases and decreases are well-balanced in a natural order, that is to say, compensate each other, such as (for example) if there is an increase on one side, there is an equal decrease on the other side. If we have found their order in this way, we have found a special property of general triangles, namely that the (sums of the) three angles are equal to one another.

So in which way do we seek to find the equality of them (the sums)? First we suppose, as usual, that angles $ABG$ (plus) $AGB$ are equal to angles $ABD$ (plus) $ADB$, because we have made this approach a condition at the beginning of the book.[76] If this is as we supposed, it is necessary that angles $GBD$ (plus) $GDB$ are equal to angle $AGB$ because if this is so, the two angles $AGB$ and $GBD$ (plus) $GDB$ are such that to them the two angles $ABG$ are added.[77]

Thus, our problem here is this problem. If we follow our ways (i.e. reasonings) correctly and the outcome is a result which is true, not impossible, then our hypothesis is correct.[78] If an absurdity or impossibility results, it follows that the angles of triangle $ABG$ are not equal to the angles of triangle $ABD$ and not equal to the angles of (351) (any) other triangle except (triangles) similar to it.

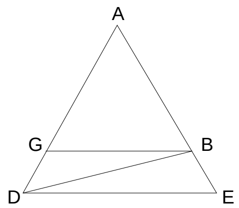


Figure 5

We needed some construction which is more appropriate to it (i.e. the prob- [26]
lem), I mean, with more resemblance to it, or which is of a genus associated to it.[79] So (Figure 5) we draw $DE$ parallel to $BG$ and we join $AE$[80] so that the triangles are similar and equal angles are produced, in order that they meet.[81] For us the consequence will be a result which is either true or false, since we originally supposed that it was true.

[76] This 'approach' is the method of analysis, see [12], [18], [20]. In his treatise on the hyperbola and its asymptotes Al-Sijzī says that he used the method of analysis here (in [25]), see quotation no. 3 in the appendix to this paper.

[77] Thus one obtains the original assumption $\angle ABG + \angle AGB = \angle ABD + \angle ADB$.

[78] Literally: our assumption of what we assumed is correct.

[79] The manuscript is not very legible. I follow Saidan's reading *min jins yaqrinu bihi*. There is no need to delete the word *kāna* as Saidan suggested.

[80] That is, we extend $AB$ to meet $DE$ at point $E$.

[81] The text is not clear to me. Does al-Sijzī mean that $AB$ extended will meet $DE$? Perhaps the word *li-yalqā* 'in order that it meet' (i.e. one of these lines meeting the other) has to be deleted.

Now angle $BDE$ is equal to angle $DBG$ and angles $EDB$ (plus) $BDG$ are equal to angle $EDG$. Therefore angles $BDG, DBG$ taken together are equal to angle $BGA$.[82] So what we sought follows. But we sought the equality of the angles of triangle $ABG$ to the angles of triangle $ABD$. Thus we have found a special property of the angles of the triangle, or more precisely, two special properties, because we found at the end of the investigation that if we extend one of the sides of a triangle, an exterior angle is produced here which is equal to the (sum of the) two interior angles opposite it in the triangle.

[27] Now we seek another special property of them (the angles), after it has become clear to us that the sum of the angles of every triangle is equal to the sum the angles of every other (triangle),[83] and that is that we seek the quantity of these angles (i.e. of the sum). Here we need a measure to measure these angles. This measure must be of their genus, and it is the right angle. So we must assume a triangle and make one of its angles a right angle, for if we make two of its angles right angles, our construction does not produce a triangle but (352) two of its sides will be parallel and do not meet, while the triangle is produced by the meeting of its three sides. So it is necessary that we assume that the two sides cointaining the right angle are equal. (Figure 6) So we assume triangle $ABG$ right-angled and isosceles, and the right angle is angle $A$. Then we use the parallel line because it is more similarly related to this position than other lines. So we draw from point $B$ line $BD$ parallel to line $AG$. Here an angle is produced, and we seek its special properties. We have found angle $DBG$ equal to angle $BGA$, but we have supposed angle $BGA$ equal to angle $ABG$. So angles $ABG$, $DBG$ are equal. But the sum of them is equal to angle $BAG$.[84] Thus it follows that the three angles of triangle $ABG$ are equal to two right angles.

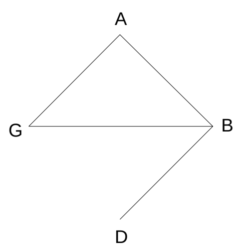


Figure 6

But we have found this special property in a determined (i.e. special) triangle, namely the triangle of which one angle is a right angle and the two sides containing it are equal. But we have mentioned that the (sums of the) angles of determined and general triangles are equal. Thus it has become clear to us that the (sum of) three angles of every triangle are equal to two right angles. That is what we wanted to illustrate.

[82]Since $DE$ is parallel to $BG$, $\angle EDG = \angle BGA$.

[83]To make mathematical sense, I have excised the words *ba$^c$ḍuhā li-ba$^c$ḍin*, meaning (the angles being equal) to one another.

[84]By *Elements* I:29, Heath 1, 311-312, the sum of $\angle DBA$ and $\angle BAG$ is equal to two right angles.

This is one of the methods to seek special properties. So you have to train your understanding and intelligence in this art. In this approach, namely the discovery of figures, the training of the understanding and clarity of the mind are more useful than reading books of geometry which the ancients prescribed, inasmuch as their aim in this was reading books on geometry as an introduction to the remaining books on mathematical philosophy and training of the intelligence. (353) [28]

*(Example 3, concerning analysis)*

Let us give another example concerning another question, so that the student of this art may practise with it, and the questions which are obscure to him may become clear. It is as follows. How can we divide an assumed triangle into three parts according to an assumed ratio? [29]

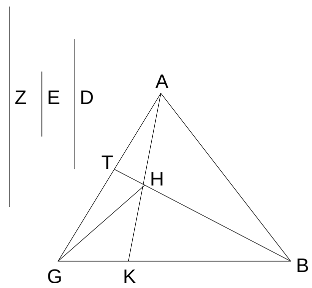


Figure 7

(Figure 7)[85] So let us assume the triangle $ABG$ and the ratio $D$ (to) $E$ (to) $Z$. It is necessary that the configuration of the division consists of three other straight lines which meet in the middle of the triangle. Thus let us assume that the triangle has been divided as we wanted, namely (in) triangles $ABH, AGH, BGH$ in such a way that the ratio of triangle $ABH$ to triangle $AGH$ is as the ratio of $D$ to $E$ and the ratio of triangle $AGH$ to triangle $BGH$ is as the ratio of $E$ to $Z$.

Then we think about looking for a construction which is useful in this problem. We extend $BH$ to $T$ so that it is clear to us that the ratio of triangle $ABH$ to triangle $BGH$ is as the ratio of $AT$ to $GT$.[86] So if we divide side $AG$ in the ratio $D$ to $Z$, the division into the two triangles must coincide with line $BT$. So we divide $AG$ at $T$ in the ratio $D$ to $Z$, and we join $BT$. Then it is necessary that the division point and the production of the angle of the triangle which is adjacent to line $AG$ are on line $BT$.

Thus we must construct a triangle ($AHG$) from side $AG$ and two lines which issue from points $A, G$ and from an angle (i.e. the angular point) which falls on line $BT$, but the ratio of it to one of the remaining triangles is as the ratio of $E$ to $D$ or to $Z$. (354) I use the first construction[87] as a preliminary for it because it is a correct approach. So we do with side $BG$ as we did with

[85] The point $I$ in Saidan's figure and his edition corresponds to point $K$ in the manuscript and the present translation.

[86] By *Elements* VI:1, Heath 2, 191-192, we have $\Delta ABT : \Delta BGT = AT : GT$ and $\Delta AHT : \Delta HGT = AT : GT$.

[87] To make mathematical sense, I omit the word *al-a$^c$māl*, 'the constructions', in the manuscript and Saidan's edition.

side $AG$, and that is, we divide side $BG$ at point $K$ in the ratio of $D$ to $E$, and we join $AK$. Then it is clear that the ratio of triangle $AHB$ to triangle $AGH$ is as the ratio of $D$ to $E$.

We have shown that the ratio of every two triangles of which two sides issue from points $A$, $G$ and meet on line $BT$ is as the ratio of triangles $ABT$, $BTG$.[88] Therefore the three triangles constructed in triangle $ABG$ are in the assumed ratio. That is what we wanted to show.

[30] Another method: We assume that the three triangles have been constructed and we extend $BH$ to $T$ (see Figure 7). It is necessary to seek triangle $AHB$,[89] but we imagine that it has been constructed, as is usual in the derivation of figures by way of analysis.

So we think about it in a mathematical way, and we seek a method for it following an approach close to the first approach, as follows. If we divide $BT$ at point $H$ in such a way that the ratio of triangle $ABH$ to triangle $AHT$ is known, the ratio of triangle $AHT$ to triangle $GTH$ is known to us. But the whole triangles $AGB, AHB$ are missing.[90] If the ratios can be made known, then if we add some of them, it (triangle $ABG$) will come out divided in the assumed ratios, after we know that the ratio of every two triangles which fall as triangles $ABH, GBH$ is known to us. So we seek this method, i.e. if we can find it or not, if (i.e. assuming that) the ratio of $BH$ to $HT$ is known to us, and the ratio of $AT$ to $TG$ is known. After the construction of the triangle the ratio of triangles $AGH, AHB$ is known because that was the assumption.

Now the ratio has been divided at the time of the investigation.[91] So we have to divide one of the proportional lines into (the same) parts in which triangles $AHT, HTG$ are divided. So we divide $E$ into two parts in such a way that the ratio of one of them to the other is as the ratio $D$ to $Z$. We make the ratio of $BH$ to $HT$ as the ratio of $D$ to one of the parts of $E$.[92] We join $AH, GH$. Then the ratio of triangle $ABH$ to triangle $AHT$ is as the ratio of $D$ to one of the parts of $E$, and the ratio of triangle $AHT$ to triangle $HTG$ is as the ratio of one of the parts of $E$ to its remaining part. So the ratio of triangle $ABH$ to triangle $AHG$ is as the ratio of $D$ to $E$. But we have shown that the ratio of triangle $ABH$ to the remaining triangle $BGH$ is as the ratio of $D$ (355) to $Z$. That is what we wanted to show.

[88] The text is puzzling, because Al-Sijzī has not really 'shown' this.

[89] Saidan reads the manuscript as $AGB$, which is also possible.

[90] Arabic: *mafqūdayn*. The text is not clear to me. Perhaps al-Sijzī uses some sort of 'transformation' (*naql*) to a new problem, in which the ratios $BH : HT$ and $AT : TG$ are supposed to be known and the ratios of the triangles are supposed to be unknown for the moment. In the following proof, the Lahore manuscript differs from Saidan's edition because Saidan omitted a few words and two lines by an oversight.

[91] Perhaps al-Sijzī means that the ratio has been divided because triangles $ATH, GTH$ have been introduced.

[92] That is to say, the part of $E$ which corresponds to $D$ in the division of $E$ in the ratio $D : Z$.

Another method of constructing this figure is as follows: (Figure 8) We divide side $AB$ in the ratio between $D$, $E$, $Z$ at (points) $H, T$ and we join lines $GH, GT$. Then it is clear that each of the desired triangles ($AGK, BGK, ABK$ in Figure 8) is equal to one of these triangles ($AGT, TGH, HGB$). In the first approach, this method has been imagined. Then we think and seek the meeting point ($K$) of the lines of the sides of the three (desired) triangles equal to these three triangles which have been constructed. So we draw $TK$ parallel to $AG$, because we know that every triangle equal to triangle $ATG$ and with base $AG$ meets the line parallel to $AG$.[93] In the same way we draw $HK$ parallel to $BG$ for the reason we have mentioned. Then they meet at $K$, so we join $AK, BK, GK$ and we decide that it has been divided as we wanted. This (method) belongs to its methods (of solution) but we have not explained it in full. [31]

There is another method for this figure, but it leads to these two methods which we have mentioned and therefore we have omitted it and have not mentioned it.

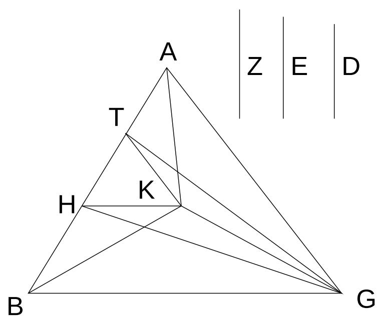


Figure 8

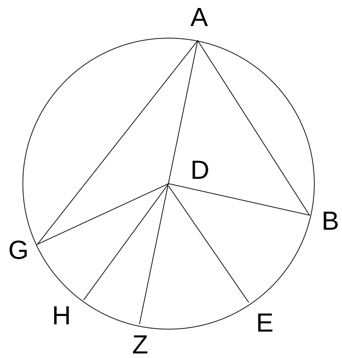


Figure 9

*(Example 4, concerning deductive structure)*

As for what we mean by our statement:[94] "If we have a preliminary, or a theorem, among the preliminaries and theorems, and then for that preliminary or theorem again a preliminary, and then for that preliminary again a preliminary, it will be possible to prove the preliminary or theorem from the preliminary of its preliminary": (Figure 9) let us assume circle (356) $AB$ with centre point $D$. Angle $BAG$ has been placed in arc $BAG$. We join $BD, GD$. I say that angle $BDG$ is twice angle $BAG$. [32]

Euclid proved it[95] by the special property

[93] Euclid, *Elements* I: 39, Heath 1, 336-7.
[94] See [9].
[95] See *Elements* III:20, Heath 2, 46-47.

of the exterior angle of the triangle of which one of the sides is extended. That is the thirty-second figure of the first book of his Book on *Elements*.[96] But the twenty-ninth and the thirty-first figure are preliminaries to that figure. So it is necessary to test if it (the property of the circle) can be derived from the two of them or from one of them or not. So we let pass though point $D$ a line parallel to $BA$, namely $DE$, and another line parallel to $AG$, namely $DZ$, and we extend $AD$ towards $H$. This is the application of the thirty-first figure[97] which he made a preliminary to his preliminary. But the exterior angle $EDH$ is equal to the interior angle $BAD$ and angle $EDB$ is equal to the alternate angle $DBA$.[98] But angle $DBA$ is equal to angle $BAD$. The equality of the two arms which appears in this figure is not a preliminary but a special property of the figure which he (Euclid) imposed in this figure, so let us keep it in this sense.

Thus each of the two angles $BDE, EDH$ is equal to angle $BAD$. Therefore (357) angle $BDH$ is twice angle $BAD$. It is also clear in the same way that angle $HDG$ is twice angle $DAG$. Therefore the entire angle $BDG$ is twice the entire angle $BAG$. This is the application of the twenty-ninth figure.[99] So we have applied the preliminaries of its preliminaries, and we have been able to prove it. That is what we wanted to show.

*(Example 5, concerning properties common to various propositions)*

[33] We will give an example of the common features[100] of figures, using the figures composed of the division of a line in mean and extreme ratio.[101] In general the figures which are composed from that (division) involve the (number) five:[102]

Thus the construction of the equilateral pentagon involves the division of a line in mean and extreme ratio.[103]

From putting together (in a straight line) the radius (of a circle) and the side of the (inscribed) decagon, which is related to the side of the pentagon since it is the chord of half its arc, one obtains a line divided in mean and

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[96] Heath 1, 316-317. In his proof of *Elements* III:20, Euclid uses *Elements* I:32 to the effect that in the notations of Figure 9, $\angle BDH = \angle DBA + \angle DAB$.

[97] Euclid, *Elements* I:31: through a given point to draw a straght line parallel to a given line. Heath 1, 315.

[98] By Euclid, *Elements* I:29: A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles." Heath 1, 311.

[99] As a matter of fact proposition I:29 is used earlier, see the preceding footnote.

[100] See [7].

[101] A line segment $AB$ is divided in mean and extreme ratio at point $C$ if $AB : BC = BC : AC$. In modern mathematics, this division is called the golden section.

[102] Al-Sijzī refers to this passage in his *Treatise on the notion that the figures are all (derived) from the circle*, see appendix, quotation 2b.

[103] *Elements* IV:10-11, Heath 2, 96-101.

extreme ratio.[104]

The two chords which fall in the circle of the pentagon, I mean those that issue from the angles of the pentagon (inscribed) in the circle, divide one another in mean and extreme ratio.[105]

< If a line is divided in mean and extreme ratio > and to the greater part half of the entire line is added, the square of that is five times the square of half the line.[106]

Of any line divided into two parts in this ratio, the square of the entire line is five times the square of the first part.[107]

If any line is divided in mean and extreme ratio and if there is added to the (358) shorter part a line equal to half of the longer part, the square of that (the sum) is five times the square of half of the longer part.[108]

From the addition of the sides of a square figure divided into five equal parts, [34]
and their subtraction, one can obtain a line divided in mean and extreme ratio. I mean by 'addition' the addition of some of the lines to others and their junction in such a way that the result is one straight line, and by 'subtraction' that the longer line is divided into two parts in such a way that one of the parts is equal to the shorter line.

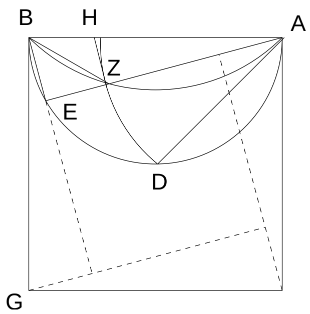


Figure 10

Example: (Figure 10)[109] We assume square $AG$,[110] but angle $E$ is a right angle, so (the sum of) the squares $AE, EB$ is equal to square $AB$. We find another line $AD$ in such a way that twice its square is equal to the square $AB$, and it is equal to $AZ$. Seeking line $AD$[111] is easy: we draw a semicircle $ADB$ and we bisect it at $D$ and we join $AD$. Then twice the square of $AD$ is equal to the square of $AB$. Thus we need to seek line $AE$ such that if $EB$ is drawn, $EB$ is equal to $EZ$ and $ZA$ is equal to $AD$, so that our aim is reached.

To obtain it, we imagine the situation that this line has been derived, and

[104] *Elements* XIII:9, Heath 3, 455.

[105] *Elements* XIII:8, Heath 3, 453.

[106] *Elements* XIII:1, Heath 3, 440.

[107] This theorem is mathematically incorrect, and it is not in the Greek text of the *Elements.* To make sense here, one can try to emend the text thus: If a segment is divided in mean and extreme ratio, then the square of the (sum of the) segment plus the lesser part is five times the square of the first (i.e. greater) part.

[108] *Elements* XIII:3, Heath 3, 445.

[109] As Saidan points out, there is no figure in the manuscript. Figure 10 is my reconstruction. Saidan does not present a figure. The text is problematical, see my footnote at the end of [34].

[110] Text: $AD$.

[111] Text: $AG$.

this means finding $ZE$ equal to $EB$. It is clear that if we draw $AE$ and make at point $B$ of line $EB$ half a right angle, and if we join $BZ$, line $ZE$ will be equal to line $EB$. After this we have to seek the equality of $AZ$ and $AD$. We have to imagine line $AE$ moving around point $A$, so we draw with centre $A$ and distance $AD$ circle $DZ$. It is necessary that this line falls on circle $DZ$. Thus we need to construct an arc containing an angle equal to one and a half times a right angle, like arc $AZB$, because if circle $DZ$[112] cuts it and $AZ$ is extended toward $E$ and $BZ$ is joined, the exterior angle $AZB$ is equal to the two interior angles (359) $E, B$. But it is clear to us that angle $E$ is a right angle, so we conclude that angle $B$ is equal to half a right angle. As a result of this, angle $B$ and angle $Z$ in triangle $ZEB$ are equal, so line $EZ$ is equal to line $EB$ and line $AZ$ is equal to line $AD$. So $AE$ is divided as we wanted. But, by transformation, if we draw $ZH$[113] parallel to $EB$, $AB$ is divided as we wanted.[114] The proof of it is easy. That is what we wanted to show.

*(Example 6. Propositions on the ratios between arcs and their chords and sines.)*

[35] Let us now investigate how we prove the following figure which Ptolemy presented in the *Almagest*: for any two different arcs in an assumed circle, the ratio of the chord of the greater arc to the chord of the lesser arc is less than the ratio of the greater arc to the lesser arc.[115]

[112]The MS. and the edition have $DAZ$.

[113]The manuscript has $ZG$. Saidan omits this sentence.

[114]This proposition presents serious problems. The manuscript does not contain a figure and Saidan presents the text unedited and without figure. Figure 10 is my reconstruction. I now explain my problems with the interpretation of the text.

I take al-Sijzī's statement that $AE$ is 'divided as we wanted' to mean that al-Sijzī wants to divide $AE$ in extreme and mean ratio at $Z$, such that $AE : AZ = AZ : ZE$. Because Al-Sijzī constructs point $Z$ such that $ZE = EB$ and $AZ = AD$, I conclude $AE : AD = AD : BE$, hence $AD^2 = BE \cdot AE$, that is to say that the area of the square with side $AD$ is twice the area of the right-angled triangle $AEB$. Thus the construction of $Z$ can be related to the division of a big square (with side $AB$) into a small square (with side equal to $AD$) and four triangles, as suggested by Figure 10, broken lines. However, by the theorem of Pythagoras $AB^2 = AE^2 + BE^2 = AE^2 + EZ^2$, and since $AE$ is divided in extreme and mean ratio at $Z$ with greater part $AZ$, we have $AE^2 + EZ^2 = 3AZ^2$, as Euclid proved in *Elements* XIII:4 (Heath 3, 447-448), and as the reader can verify algebraically. Thus the division of a line segment in extreme and mean ratio can be related to the division of a big square with side $AB$ into one small square with size one-third of the original square, and four right-angled triangles (such as $AEB$) with size one-sixth of the original square.

There are two problems with this interpretation:

1. Al-Sijzī states unambiguously $AB^2 = 2AD^2 = 2AZ^2$, not $AB^2 = 3AD^2$.

2. The introduction to this proposition discusses the division of a square into *five*, equal parts. By means of such a division of a square one can construct a line segment divided in extreme and mean ratio, but the text does not seem to explain such a construction.

I have no solution to these difficulties so I must leave the problem to the reader. Perhaps there is a lacuna in the text we have.

[115]Al-Sijzī's proof is somewhat different from the proof in Ptolemy's *Almagest* I:10, for which see G.J. Toomer, *Ptolemy's Almagest*, London: Duckworth, 1984, pp. 54-55. Al-Sijzī refers to this part of the text in his *Answer to geometrical questions asked by people from Khorāsān*, see appendix to this paper, quotation 1.

In this problem we need to use intelligence, to visualize complex constructions, and to mix figures. But this (proposition) and similar ones are easy in the sense that the truth of the question is known to us and the constructions by means of which he proved it have also been achieved. From these two points of view this problem and similar ones are easy. Since the proof of this problem is impossible for us unless we add another construction to it, we need another construction such that if we add it to it (our solution), the proof of it is easy by the combination of the two of them. By means of the construction which Ptolemy presented we can easily construct it, (bearing in mind) how he approached the problem and the thing which he added to it in order to prove it. For he added to it triangles composed of straight lines and of arcs and then he proved it by means of those triangles and their angles, arcs and chords.

Here we say something which does not concern this question, but which we need here. We have adopted our approach in this figure from the approach of the ancients, in the sense that the figures have proportionalities and special properties such that if the skilful (mathematician) thinks about them, it becomes clear to him that some of them are interwoven (360) with others, and some are mixed with others, as if they become one essence and one situation. For they have such links and mutual relations[116] that if we imagine that they are different in species but belong to the same genus, it is necessary that the essences of their special properties which are common to them also belong to that same genus.[117] [36]

An example is the intersection of two chords in the circle.[118] Parts of them are proportional to (other) parts of them, so this is an absolute statement belonging to their genus. The way in which they belong to a species and the situation of the position of the two intersecting chords in the circle imply its property which involves the ratio. If one investigates this situation by the intermediary of figures, one arrives at the essential nature of that property. Similarly, the necessity of the proportionality of the lines containing the area to the areas,[119] and the fact that an arc contains equal angles,[120] and the equality of triangles at equal bases and between parallels,[121] — if one investigates these and similar figures, one finds their special properties and their essences, God willing.

[116] Arabic: *madārāt*, literally: circuits.

[117] It seems to me that this explains to al-Sijzī why the proof of Ptolemy's proposition necessarily involves the comparison of a ratio between two circular segments with a ratio between two triangles.

[118] Al-Sijzī returns to this example in [47].

[119] For example *Elements* VI:23: equiangular parallelograms have to one another the ratio compounded of the ratios of their sides, Heath 2, 247-248.

[120] *Elements* III: 27, Heath 2, 58-59.

[121] *Elements* I:38, Heath 1, p. 333.

For this and similar reasons related to the special properties of figures and their order, we rely somewhat on the nature of them (the figures) in the beginning (of our investigation) before we find them (i.e. before we find the propositions).

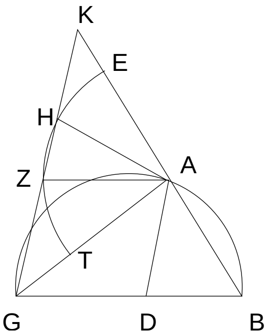


Figure 11

[37] Now we return to what we said (Figure 11): we assume arc $BAG$ and we divide it into two different parts at $A$ such that the longer part is $AG$. We draw chords $AB, AG$. I say that the ratio of arc $AG$ to arc $AB$ is greater than the ratio of chord $AG$ to chord $AB$.

Proof: we join $BG$, we extend $BA$ toward $K$ and we make $AK$ equal to $AG$. We have constructed this in this order because we add to this figure successive constructions for this configuration;[122] no other procedure is possible for us.

Then we join $GK$. Now two triangles have been added to the figure of the proposition[123] which was our original problem, one of them is triangle $ABG$ and the other is triangle $AKG$. But the aim has not yet been achieved by means of these triangles. So we draw $AD$ parallel to $KG$. We draw $AD$ parallel to $KG$ because (361) there is an order in it: either the equality of the angles $DAG <$ $AGK >$ or (of) $DAB$, $AKG$. Then we use our skill here: we draw $AZ$ parallel to $BG$.[124]

We need to construct segments of circles in order to know the angles by means of their arcs and to find the magnitude of the proportionality of the sides of the triangles and the angles of the arcs. Then we wish to find the proportionality between arcs $BA, AG$ and the angles of the segments. So we draw with centre $A$ and distance (i.e. radius) $AZ$ arc $TZHE$. We have made this circle with centre $A$ because the desired (things) which are on arcs $TZHE$ are proportional to the angles at point $A$.

Then we seek the end of the construction, the proportionality between the angles at point $A$ and the angles contained by sides $AB$, $AG$, $BG$, so that our aim is a (logical) consequence.

Since line $ZG$[125] is less than line $ZK$, arc $HE$ is equal to arc $ZT$.[126] We

[122] Al-Sijzī may have had in mind the above-mentioned 'triangles composed of straight lines and of arcs' (such as triangles $AZG, AZH$ and circular segments $AZT, AZH$ in Figure 11) which Ptolemy also used.

[123] Here the word *shakl* means proposition, because al-Sijzī uses the word *ḍsūra* for figure.

[124] Point $Z$ is assumed to be on line $GK$, not on the circle as in the Arabic text, figure 142.

[125] We have $ZG < ZK$ since $GA = AK$ and $\angle ZAG = \angle AGB < \angle ABG = \angle KAZ$. Thus the circle with radius $AZ$ and centre $A$ intersects $ZK$ between $Z$ and $K$ at point $H$. Segment $ZG$ is incorrectly omitted in the figure in the manuscript.

[126] Triangles $HAK$ and $ZAG$ are congruent.

join $AH$. Then $AH$ is equal to $AZ$, and segment $ZTG$ is equal to segment $HEK$.

Then we seek to achieve our aim (i.e. the desired theorem) by means of the proportionality between the segments, the arcs, the triangles and the sides. Here we need to imagine the results (as having been achieved) and to analyse them from the aim (end) towards the beginning (i.e. the hypotheses),[127] then (362) we proceed from the beginning to the aim.[128]

Here guesswork is used. Since segment $AZT$ is equal to segment $AEH$ and segment $HEK$ is equal to segment $ZTG$, taking segment $HZA$ in common and triangle $AZH$, the ratio of triangle $AZG$ to triangle $AZK$ is greater than the ratio of segment $AZT$ to segment $AZE$.[129] So the ratio of line $ZG$ to line $ZK$ is greater than the ratio of angle $TAZ$ to angle $ZAK$. But the ratio of line $ZG$ to line $ZK$ is as the ratio of line $BA$ to line $AG$, since $AG$ is equal to $AK$.

So the ratio of angle $GAZ$ to angle $ZAK$ is less than the ratio of $BA$ to $AG$. But angle $GAZ$ is equal to angle $AGD$ and angle $KAZ$ is equal to angle $ABG$, so the ratio of angle $G$ to angle $B$ is less than the ratio of line $BA$ to line $AG$. Thus the ratio of arc $AG$ to arc $AB$ is greater than the ratio of line $AG$ to line $AB$. That is what we wanted to explain.

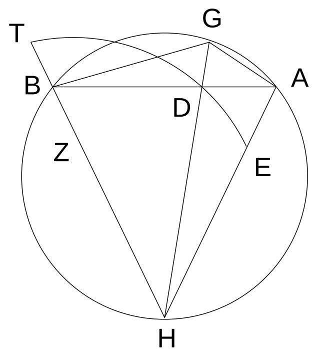


Figure 12a

We seek the proof of it in another way.[130] (Figures 12a, 12b) We assume circle $ABH$[131] and two different arcs $AG, GB$, the greater arc is $GB$, and we say what we said.[132] [38]

We join $AB$ and we bisect angle $G$ by line $GH$. We bisect the angle because line $AB$ is divided (by $GH$) at $D$ in such a way that the ratio of $AD$ to $DB$ is as the ratio of $AG$ to $GB$. Thus line $AB$ becomes an intermediary step for the constructions we need. (363)

Then we need a construction in this circle or outside it which connects the two angles $A, B$ together to one point. The reason is that if we make

[127] This is the analysis.

[128] This is the synthesis.

[129] Since $\Delta AZG >$ sg. $AZT$ and $\Delta AZH <$ sg. $AZH$, we have $\Delta AZG : \Delta AZH >$ sg. $AZT$ : sg. $AZH$. Therefore $\Delta AZG : (\Delta AZH + \Delta AZG) >$ sg. $AZT$ : (sg. $AZH$ + sg. $AZT$). Since $\Delta AZG = \Delta AHK$ and sg. $AZT$ = sg. $AEH$, we have $\Delta AZG : \Delta AZK >$ sg. $AZT$ : sg. $AZE$ as in the text.

[130] The following proof is essentially that of Ptolemy in *Almagest* I:10.

[131] Note that only the circle is given. Point $H$, which is used here to label the circle, will be defined later.

[132] That is to say, arc $BG$ : arc $GA > |BG| : |GA|$.

that point a centre and draw an arc around it with some radius, it will be useful for our aim. These constructions are obscure for us at the beginning of the matter, but this approach is successful.

So we join $AH, BH$. Then angles $AHG, BHG$ come together at point $H$, and they are equal to angles $A, B$. We have joined lines $AH, BH$ at point $H$, and we have not drawn from points $A, G, B$ three lines meeting at any other point of arc $AHB$. For if we achieve our aim by means of this construction, the easiest approach will be to achieve the aim by drawing them to the midpoint of arc $AHB$, because of the proportionality between lines $AD, DB$ and lines $AG, GB$. What is closer to a proportionality and an order is closer to existence (i.e. easier to find).

Then we have to find an arc with centre $H$ and some radius, and now I do not yet know what radius, so that the excess of the proportionality of arc $BG$ to arc $GA$ over chord $BG$ to chord $GA$ will follow by means of the segments, arcs and angles at point $H$, lines $AD, DB$ and triangles $ADH, DHB$.

Here is the place of the following error. If someone says: "We draw with centre $H$ and radius $HD$ arc $EDT$ and we extend $HB$ towards $T$ as in this figure" (Figure 12a) and if he proves it by means of it (the figure),[133] then we say: "This is not possible: since line $AH$ is equal to line $BH$, and the endpoint of the arc falls at point $E$, the other endpoint falls at point $Z$ facing point $E$."[134]

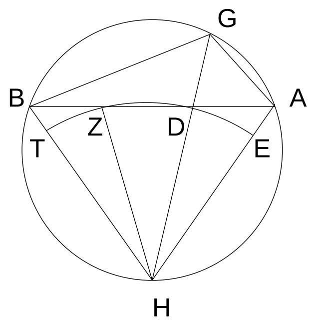


Figure 12b

Since the circle segments and triangles involve point $D$ in this figure in all aspects, as before, we draw with centre $H$ and radius $HD$[135] arc (364) $EDZT$ to see if we can find what we were looking for or not. (Figure 12b)[136] We join $HZ$. $<$ Then the ratio $>$ of segment $ZDH$ to segment $DEH$ is greater than the ratio of triangle $ZDH$ to triangle $HDA$. So, by addition,[137] the ratio of segment $DTH$ to segment $DHE$ is greater than the ratio of triangle $DHB$ to triangle $DHA$. Therefore the ratio of arc $DT$ to arc $DE$

[133] This 'proof' would be very easy: since segment $HDE <$ triangle $HDA$ and segment $HDT >$ triangle $HDB$, $AG : GB = AD : DB = \Delta ADH : \Delta HDB >$ segment $DEH$ : segment $DHT = \angle AHD : \angle DHB =$ arc $AG$ : arc $GB$.

[134] Point $Z$ is located on line $HB$ in the figure in the manuscript.

[135] The manuscript and edition have $D$.

[136] This figure is omitted in the text. The manuscript and edition only have figure 12a, which is drawn for the incorrect case where arc $ED$ meets line $HB$ extended.

[137] This is my translation of the word *tarkīb*, which is used here as a technical term in the theory of proportions of Book V of the *Elements* (Heath translates 'componendo'). Note that in Figure 12b triangles $ZHB$ and $DHA$ are congruent since $ZH = HD$ and $AH = HB$.

is greater than the ratio of line $DB$ < to > line $DA$.[138] But the ratio of arc $DT$ to arc $DE$ is as the ratio of < arc $BG$ to arc $GA$. So the ratio of arc $BG$ to arc $GA$ is greater than the ratio of >[139] chord $BG$ to chord $GA$. That is what we wanted to explain.

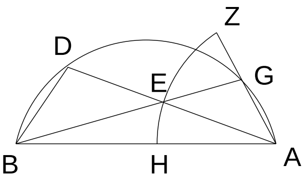


Figure 13

Since the aim of Ptolemy was (arcs of) one degree and half a degree,[140] it is necessary for us (to assume) that the arcs for which the theorem is proved are less than a semicircle. In this problem we need (to discuss) this condition < ... > [141] of it a proof different from the preceding proofs. We follow in it another method, as follows: [39]

(Figure 13) We assume arc $AB$ less than a semicircle and we divide it into two unequal parts at $G$ and the greater (part) is $GB$. We join $AG, GB$, we make $BD$ equal to $AG$, and we join $AD$. We draw with centre $A$ and radius $AE$[142] arc $HEZ$ and we extend $AG$ toward $Z$. It is clear that it ($Z$) falls outside the arc, and it is also clear that < $AE$ is> equal to $EB$. Then the ratio of segment $AZE$ to sgement $AEH$ is greater than the ratio of triangle $AGE$ to triangle $AEB$. By addition, the ratio of segment $AZH$ to segment $AEH$ is greater than the ratio of triangle $AGB$ to triangle $AEB$. But the ratio of arc $ZH$ to arc $EH$ is as the ratio of arc $GB$ to arc $DB$. (365) Therefore the ratio of arc $GB$ to arc $DB$ is greater than the ratio of line $GB$ to line $EB$. But line $EB$ is equal to line

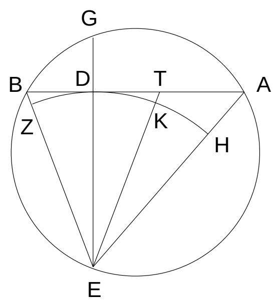


Figure 14

[138] The manuscript and edition have $DE$.

[139] I have restored this sentence to the text to make mathematical sense.

[140] In *Almagest* I:10, Ptolemy applied the theorem twice, to show that $\frac{2}{3}$ Ch($\frac{3}{2}^o$) < Ch($1^o$) < $\frac{4}{3}$ Ch($\frac{3}{4}^o$), here the notation Ch($x$) means the chord of $x$. In modern notation Ch($x$) = $2R\sin\frac{x}{2}$ with $R = 60$. Because Ptolemy had computed Ch($\frac{3}{2}^o$) and Ch($\frac{3}{4}^o$), he could now find an approximation of Ch($1^o$), see tr. Toomer, pp. 55-56. From Ch($1^o$) he computed Ch($\frac{1}{2}^o$) by the formula for the chord of half an arc (tr. Toomer, p. 52), which is unrelated to the theorem presented by al-Sijzī.

[141] The manuscript is smudged and one word (corresponding to Saidan's *ya'tī*) is illegible.

[142] Point $E$ is the intersection of $AD$ and $BG$.

$AE$ and line $AE$ is greater than line $AG$, since we assumed arc $AGB$ less than a semicircle. Therefore the ratio of arc $GB$ to arc $GA$ is much greater than the ratio of chord $GB$ to chord $AG$. That is what we wanted to explain.

[40] (Figure 14) Circle $ABG$ is assumed, and arcs $AG$ and $GB$ are different, $AG$ is greater than $GB$. We draw $AB$ and we drop from point $G$ a perpendicular onto $AB$. I say that the ratio of line $AD$ to line $DB$ is greater than the ratio of arc $AG$ to (arc) $GB$.

Proof: We extend perpendicular $GD$ towards $E$ and we join $EB, EA$. We draw $ET$ equal to $EB$. We draw with centre $E$ and radius $ED$ circle $ZDKH$. Then the ratio of triangle $ADE$ to triangle $DBE$ is greater than the ratio of arc $HKD$ to arc $DZ$ since triangle $ADE$ exceeds the segment by trapezium $ATKH$. Arc $HD$ subtends[143] angle $AED$, and arc $DZ$ subtends angle $DEZ$. Similarly, arcs $AG, GB$ subtend them, so they are proportional. So the ratio of $AD$ to $DB$ is greater than the ratio of (arc) $AG$ to (arc) $GB$. That is what we wanted to explain.[144]

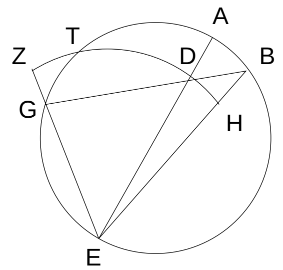


Figure 15a

[41] (Figure 15a,b)[145] Arc $AG$ is greater than arc $AB$. I say that the ratio of the chord of twice the longer arc to the chord of twice the shorter arc is greater than the ratio of the longer arc to the shorter arc.[146]

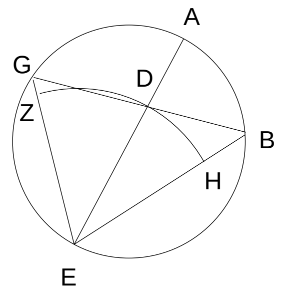


Figure 15b

[143]The text says literally: arc $HD$ is the chord of angle $AED$. The same curious terminology appears in following propositions.

[144]The reasoning can be clarified as follows: Draw an auxiliary circle with centre $E$ and radius $ET$; then $\Delta ATE$ is greater and $\Delta DTE$ is less than the corresponding segment in this new circle. Hence $\Delta ATE : \Delta DTE >$ arc $KH$ : arc $KD$, so we have $\Delta ADE : \Delta DTE >$ arc $DH$ : arc $KD$, so $AD : DB = AD : DT >$ arc $DH$ : arc $KD =$ arc $DH$ : arc $DZ =$ arc $AG$ : arc $GB$.

[145]Saidan's edition does not contain a figure. There is a figure in the manuscript which I render here as Figure 15a.

[146]The text is mathematically wrong. If words 'longer' and 'shorter' are interchanged, the text is correct and consistent with what al-Sijzī proves in this proposition. Since the chord of twice an arc is two times the sine of the arc (if the sine is defined in a circle with the same radius), the theorem says that if $\beta < \gamma, \sin\beta : \sin\gamma > \beta : \gamma$.

Proof: we draw diameter $AE$. We draw $EB$, $EG$, and we draw with centre $E$ and radius $ED$ arc $HDTZ$.[147] Then point $Z$ falls either at point $G$,[148] but it is outside it, for if it falls inside line $GE$,[149] angle $ADG$[150] is a right or acute angle, and this is not the case.[151] Therefore the ratio of triangle $BDE$ to triangle $DGE$ is greater than the ratio of segment $HDE$ to segment $DZE$. So the ratio of line $BD$ to line $DG$ is greater than the ratio of angle $HED$ to angle $DEZ$, and (greater) than the ratio of arc $BA$ to arc $AG$.[152] But the ratio of $BD$ to $DG$ is as the ratio of the chord of twice arc $BA$ to the chord of twice arc $AG$.[153] So the ratio of the chord of twice arc $AB$ to the chord of twice arc $AG$ is greater than the ratio of arc $BA$ to arc $AG$. That is what we wanted to explain.

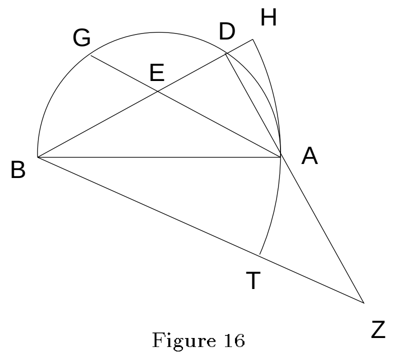


Figure 16

(Figure 16) We assume <semi>circle $ABG$.[154] Two chords $AG, BD$ fall in it and their intersection is at point $E$. I say that the ratio of line $DE$ to line $EB$ is less than the ratio of arc $AD$ to arc $GB$. [42]

Proof of this: We extend $DA$ towards $Z$ and we draw with centre $B$ and radius $BA$ circle $HAT$. We extend $BD$ to $H$ and we draw $BZ$ parallel to $AG$. (367) Then the ratio of triangle $ADB$ to triangle $AZB$ is less than the ratio of segment $AHB$ to segment $ATB$.[155] Similarly, the ratio of line $AD$ to line $AZ$ is less than the ratio of angle $DBA$ to angle $ABZ$. But angle $ABZ$ is equal to angle $GAB$ and the arcs $AD, GB$ subtend[156] them. So the ratio of $AD$ to $AZ$ is less than the ratio of arc $AD$ to arc $GB$. But the ratio of line $AD$ to line $AZ$ is as the

[147] $HDTZ$ is my emendation. The manuscript and edition have $DHZ$.
[148] Al-Sijzī does not list the other alternatives.
[149] The manuscript has $GZ$, the edition $HZ$.
[150] The manuscript has $ABG$, the edition $AZD$.
[151] If my interpretation is correct, al-Sijzī commits an error of the kind which he criticises himself in Figures 12a and b. If line $GDB$ is nearly perpendicular to the radius $ED$ of arc $HDZT$, this arc will intersect segment $DG$ between points $G$ and $D$. Then point $Z$ is between points $G$ and $E$ as in Figure 15b, which is not in the manuscript.
[152] The manuscript has $AD$ and the edited text has $AZ$.
[153] This is proved by Ptolemy in *Almagest* I:13, tr. Toomer p. 65.
[154] The proposition is true if $AB$ is a diameter, so I have assumed that the word *niṣf* ('half') was omitted from the text. This correction is suggested by the fact that the figure in the manuscript displays only a semicircle $ABG$, and also by the fact that the figure in the manuscript (also in the edited text p. 366) shows the circle $HAT$ (with centre $B$ and radius $BA$) externally tangent to the given circle. The two circles are tangent at $A$ only if $BA$ is the diameter of circle $ABG$.
[155] The manuscript and edited text have $AZB$.
[156] As before, the Arabic text uses a curious terminology here: 'the arcs $AD, GB$ are chords of them.'

ratio of line $DE$ to $EB$. So the ratio of line $DE$ to line $EB$ is less than the ratio of arc $AD$ to arc $GB$. That is what we wanted to explain.

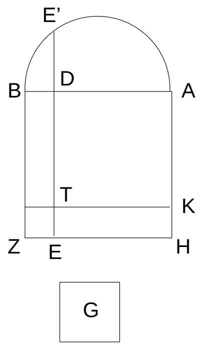


Figure 17

*(7. Two problems on 'geometrical algebra')*

[43] (Figure 17) How do we divide line $AB$ into two parts such that (the sum of) the area contained by the whole line $AB$ and one of its parts, plus the square of the the second part plus the assumed square $G$, is equal to the square of $AB$?

Thus we have to apply to $AB$ a square because we can (then) perceive it and because it is the central issue $<$ in $>$ the end of the construction.

Then we assume that $AB$ is divided as we wanted at point $D$. If this is so, we have to draw $DE$ parallel to $BZ$ so we know that area $DZ$ is the (area) contained by $AB, DB$. So we have to construct a square on line $AD$, so we construct square $AT$. If the square $AT$ plus the area $DZ$ plus the square $G$ is equal to the square of $AB$, it is necessary that area $KE$ is equal to square $G$. (368) But the (i.e. any) two complements are equal,[157] so area $KE$ is equal to area $BT$, since they are complements. But area $BT$ is the (area) contained by lines $AD, DB$.

So if we draw on diameter $AB$ a semicircle $AE'B$,[158] and if we draw $DE'$ perpendicular to $AB$, lines $AD$, $DB$ will be equal in power to square $DE'$.[159] Therefore line $DE'$ becomes the side of square $G$. Thus it is necessary that the side of $G$ is not greater than half line $AB$, since the construction of that is not possible. So another condition has been added to the condition (of the problem).

Synthesis: We construct on $AB$ semicircle $AE'B$ and we let fall in it a perpendicular to $AB$ equal to the side of square $G$, namely $DE'$. We extend it toward $E$. We add to $AD$[160] square $AT$. Then area $AB$ times $BD$ is $DZ$, and the square $G$ is equal to $AD$ times $DB$, that is area $KE$. The square of $AD$ is $AT$. Thus we have divided $AB$ into two parts at $D$ (such that) the area $AB$ times $BD$ plus the square of $DA$ plus the square $G$ is equal to the

[157] *Elements* I:43. 'Complements' are figures obtained by deleting from a parallelogram (such as $ABZH$ in Figure 17) two parallelograms similar to it (such as $ADTK$, $TEZ$) with a common angular point $T$ on the diagonal $AZ$, see Heath 1, 340-341.

[158] Text: $AEB$. As Saidan points out, point $E$ has already been defined. I will call the new point on the circle $E'$. In the manuscript $E$ and $E'$ are written in the same way.

[159] The text means $DE^2 = AD \cdot DB$, which is a consequence of *Elements* III:3 and III:35, Heath 2, 10-11, 71-73.

[160] The manuscript and edited text have $AE$.

square of line $AB$. That is what we wanted to explain.

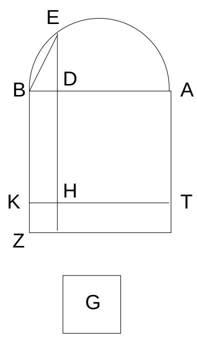


Figure 18

(Figure 18) If we want to divide $AB$ into two parts, for example at $D$, such that the area $AD$ times $DB$ plus the square of $AD$ plus the square $G$ is equal to the square of $AB$, we apply to $AB$ square $AZ$. We make $AD <$ times $BD >$ an area, namely $DK$, and we draw $KT$ parallel to $AD$. Then we show that $AH$ is the square of $AD$. By subtraction, area $TZ$ is equal to the square $G$. (369) But area $TZ$ is $AB$ times $BD$. [44]

Therefore, by way of synthesis, we have to apply to diameter $AB$ a semicircle[161] $AEB$ and we let fall in it the side of square $G$ as a chord in it with endpoint $B$, namely $BE$. We draw $ED$ perpendicular to $AB$. Then it is clear that it is divided as we wanted, since line $EB$ is equal in square to area $TZ$,[162] area $DK$ is area $AD$ times $DB$ and the square of $AD$ is square $AH$. So we have divided $AB$ at $D$ such that $AD$ times $DB$ plus the square of $AD$ plus the square $G$ are equal to the square of $AB$. That is what we wanted to explain.

*(8. Miscellaneous problems)*

Since we have explained these things, let us now finish this book, so the discussion in it does not become too long, the mind of the reader does not become tired, and it does not make him weary. [45]

Since the investigation of the natures of figures and their special properties by means of their essences is in two ways: (1) either we imagine the necessity of their special properties by changing their species in a way which is taken from the sensual perception, or in a way in which the sensual perception participates, (2) or we assume these properties as hypotheses, and we attach them to (properties) which come before them (i.e. preliminaries), or of (properties) which follow them, according to geometrical necessity (i.e. mathematical reasoning) – therefore I am now giving an example of this, as an instruction for someone who is involved in this art.[163]

[161] This semicircle is unnecessary. Segment $DB$ can be found as the side of a rectangle with known side $AB$ and equal in area to the known rectilineal figure $G$ by *Elements* I:45, a construction using straight lines only.

[162] The identity $EB^2 = AB \cdot BD$ is obtained by adding $DB^2$ to both terms of the identity $DE^2 = AD \cdot DB$ mentioned above.

[163] Al-Sijzī refers to this discussion in his *Treatise on the notion that the figures are all (derived) from the circle*, see appendix to this paper, quotation 2a.

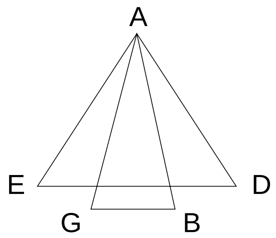


Figure 19

[46] As for the imagination of the necessity of special properties by the change of their species, in a way in which the sensual perception participates, this is as we have explained before in an example, namely that in every triangle the sum of its angles is the same.[164] An(other) example: (Figure 19) (if) two triangles $ABG, ADE$ are isosceles, but side $AD$ is equal to side $AB$, and angle $DAE$ is greater than angle $BAG$, then the base $DE$ is longer than the base $BG$. This special property can also be imagined by participation of the sensual perception. (370) The researcher begins his investigations of special properties in this way.

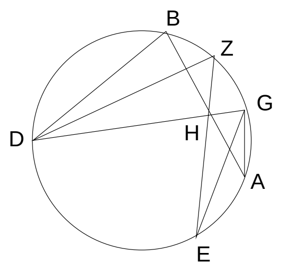


Figure 20

[47] Concerning the second aspect, which makes it necessary for the researcher to investigate them deeply in a geometrical way, so it becomes a training for him, and the visualization of its special properties becomes for him clear and his (mental) possession: I am also giving an example of this, as follows:

(Figure 20) We assume circle $AGB$, and we let fall in it two chords $AB, GD$ which intersect at point $H$. We investigate from what point of view the equality of area $AH, HB$ to area $GH, HD$ is necessary.

Then we join $GA, BD$. So here we have two similar triangles $AGH, BHD$ since equal angles at the circumference of the circle stand on the same arc.[165] So the ratio of $GH$ to $AH$ is as the ratio of $BH$ to $HD$. Similarly if we draw line $EHZ$ and join $GE$, $DZ$, triangles $GEH$, $DHZ$ are similar, so their sides are proportional.[166]

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[164] See [25]-[27].

[165] *Elements* III:26, Heath 2, 56-57. *Elements* III:27 proves the converse, i.e. that angles standing on equal arcs are equal. This is what we need to prove the similarity of triangles $AGH, DBH$.

[166] In *Elements* III: 35 (Heath 2, 71-73) Euclid proves $AH \cdot BH = GH \cdot DH$ in a more complicated way, without proportions and similarity. These notions are introduced in Books V and VI of the *Elements*.

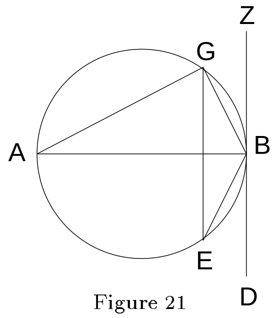


Figure 21

Concerning the investigation of the fact that a segment of a circle contains [48] an angle equal to the angle between the chord of the segment and the tangent: (Figure 21) we draw circle $ABG$ with diameter $AB$. We draw $BD$ tangent to it.

It is clear that the semicircle $AGB$ contains an angle equal to angle $ABD$. So we extend $BD$. (371)

It is necessary to investigate in a natural way the change of the species of this figure and the necessity of its special properties. So we join $BG$, $AG$, $GE$. Because the change of angle $B$ is common in an essential way to the circumference of the circle and lines $AB, DZ$, this property is necessary here. And because triangle $ABG$ is right-angled and arc $GB$ contains equal angles, angle $E$ of triangle $GEB$ is equal to angle $A$ of triangle $AGB$.

But to angle $B$ of triangle $AGB$ there has been added an angle, namely angle $< ABE >$.[167] So it is necessary to subtract from angle $AGB$ the same (amount) which has been added to angle $B$ according to this analogy, and that is angle $AGE$.

Thus the change in the species of this figure and the necessity of its special properties have become clear to us, by the equality of angle $EBD$[168] <to> the angles which arc $EAGB$ contains, to the eyes and according to geometrical reasoning. That is what we wanted to explain.

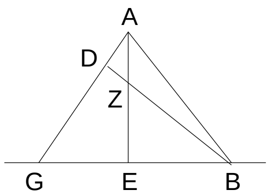


Figure 22

Let us now begin with a figure by way of analysis, so that the beginner may [49] practise by means of it. It is as follows: (Figure 22) Point $A$ and line $BG$ (are given).[169] We want to draw from point $A$ towards line $BG$ lines such as $AB, AG$ which contain the known angle $A$,

[167] I owe this emendation to Saidan.

[168] To make mathematical sense, I have interchanged the positions of $D$ and $Z$ in the figures in the manuscript and Saidan's edition. Line $GE$ is perpendicular to $AB$ in these figures, but this assumption is not necessary.

[169] Saidan adds the word 'given' to make mathematical sense. Note that one is given a point $A$ and a line $\ell$, which is indicated by letters $BG$. The positions of points $B$ and $G$ on line $\ell$ are not given.

I mean: which contain an angle equal to a given angle, and such that (area) $AB$ times $AG$ is known.

Analysis: We make $AB$ times $AG$ contain a known area and a known angle, I mean angle $A$. Let us draw perpendiculars $AE, BD$. (372)

Since $AB$ times $AG$ is known, and triangle $ABD$ is known in shape – since the angles $A, B$ are known[170] – the ratio of $AB$ to $AD$ is known. The ratio of $AB$ times $AG$ to $AD$ times $AG$ is therefore known, because $AG$ is the common term. Therefore $AD$ times $AG$ is known. But $AE$ times $AZ$ is equal to $AD$ times $AG$ because of the similarity of triangles $AZD$, $AGE$, and $AE$ is known, so $AZ$ is known. So if we apply to $AZ$ an arc containing an angle equal to angle $ABD$, then at its intersection of the given line we draw $AB$, and $AG$ will contain with it a known angle.[171]

We make the synthesis and we prove it in the way of the synthesis.

[50] Here we finish the book, because these examples are sufficient for those who practise. These things, which we wanted (to explain) in making easy the ways to deriving geometrical figures, are sufficient for someone who considers them, studies them extensively and practises by following what we have shown him and pointed out to him.

Our success is with God Most High, and on Him we rely. He is enough for us, the Sufficient, the Supporter.

The book is finished, by the praise of God and by the good success which He has given.

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[170] $\angle B = 90^o - \angle A$.

[171] Since $\angle ABD = 90^o - \angle A$, point $B$ is on the circular arc. Once $B$ has been found one may find $DG$ as the perpendicular through point $A$ to line $BZ$. Note that point $Z$ is the orthocentre of the triangle, so $\angle ZGA = 90^o - \angle A$, so point $G$ is also on a circular arc with base $AZ$ and containing an angle equal to $\angle ABD$.

## Appendix. Quotations from other texts.

1. From:*Answer by Aḥmad ibn Muḥammad ibn ᶜAbdaljalīl to geometrical questions asked by people from Khorāsān.* The text is unpublished, I have used the manuscripts Istanbul, Reşit 1191, 118a:19-118b:7, Dublin, Chester Beatty 3652, 57a:39-57b:12. See F. Sezgin, *Geschichte des arabischen Schrifttums*, Band V, Leiden: Brill, 1974, p. 333 no. 22.)

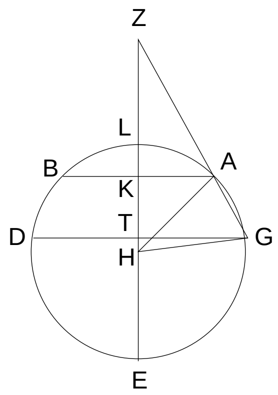


Figure App.1

Problem. We want to prove that the ratio of any greater arc to any lesser arc in the (same) circle is greater than the ratio of the chord of the greater arc to the chord of the lesser arc by a method other than the method of Ptolemy in the book *Almagest*.

I have solved this problem by different methods and easy proofs in the examples which I have given in the *Book on Making Easy the Ways to Deriving Geometrical Figures.*[172] But it has been possible (for me to solve this problem) with a proof different from the methods which I followed in that book. It is as follows. (Figure App. 1) Let arcs $AB, GD$ be different. I say that the ratio of the greater arc $GD$ to the lesser arc $AB$ is greater than the ratio of the chord of $GD$ to the chord of $AB$. Proof of this: Let the chords be parallel. We drop from the centre of the circle perpendicular $HTK$[173] onto the two chords, and we extend it on both sides toward $E, L$. We join $HG$, $HA$. We extend $GA, EL$ to meet at $Z$. Then the ratio of angle $HGA$ to angle $AGL$ is as the ratio of arc $GA$ to arc $AL$, (which is) as the ratio of sector $GAH$ to sector $AHL$,[174] (which is) greater than the ratio of triangle $GAH$ to triangle $AHZ$.[175] (So) by addition, the ratio of sector $GHL$ to sector $AHL$[176] is greater than the ratio of triangle $GHZ$ to triangle $AHZ$. But the ratio of triangle $GHZ$ to triangle $AHZ$ is as the ratio of $GZ$ to $AZ$ and as the ratio of $GT$ to $AK$. So the ratio of arc $GL$ to arc $AL$, I mean $GLD$ to $ALB$, is greater than the ratio of line $GT$ to line $AK$, I mean chord $GD$ to chord $AB$. That is what we wanted to explain.

2. From: *Treatise on the notion that the figures are all (derived) from the circle*, attributed to Naṣr ibn ᶜAbdallāh. The work was in fact written by al-Sijzī,

[172] See [35]-[39] above.
[173] $K$ and $L$ are indistinguishable in Ms. Reşit 1191.
[174] Ms. Reşit 1191: $AGL$.
[175] Ms. Reşit 1191: $AGZ$.
[176] Ms. Reşit 1191: $AGL$.

as is shown by the present quotations and by references to other works by al-Sijzī.[177] The text is edited in *Rasā'ilu'l Mutafarriqa fī'l Hai'at li'l-mutaqaddimīn wa-mu$^c$āsiray il-Bīrūnī. Containing eleven important treatises on astronomy and other subjects contributed by the famous predecessors and contemporaries of Al-Bīrūnī (9th, 10th, 11th century A.D.).* Hyderabad: Osmania Oriental Publications Bureau, 1367 H. / A.D. 1948.

2a. p. 3: (beginning of the text) We have shown in our book which we composed for the Treasury of the Victorious King that all figures are (derived) from the circle in a summarising and concise way, and we have summarised it in only two propositions: that the circle is the cause of the figures and that the figures are all found in it. We have shown[178] in our book *On Making Easy the Ways to the Geometrical Figures* some common features of it and the (other) figures and some special properties of it, and then the way to the knowledge of special properties of figures and their subdivisions, and to their essential natures.[179] Thus one may obtain information (1) in a general way, about the essence of the circle and about the knowledge of the special properties of the figures in the circle, (2) in a special way, (about the fact that) some of them are distinguished from others in the same way as they are distinguished in different ways in the circle.[180]

2b. p. 10 line 15-18:

We have indicated the property of the line divided in mean and extreme ratio (derived) from the (number) five which is found with it, in our book *On Making Easy the Ways of Deriving Geometrical Figures* for the purpose of knowing the common features of figures.[181]

3. From *Treatise by Aḥmad ibn Muḥammad ibn $^c$Abdaljalīl al-Sijzī on how to visualize the two lines which approach but do not meet, even if they are extended indefinitely, which the excellent Apollonius mentioned in the second Book of his* Conics. The text has been edited by R. Rashed in "Al-Sijzī et Maïmonide: commentaire mathématique et philosophique de la proposition II-14 des *Coniques* d'Apollonius", *Archives Internationales d'Histoire des Sciences* **37** (1987), 263-296.

[177] See J.P. Hogendijk, Rearranging the Arabic manuscript Bankipore 2468, *Journal for History of Arabic Science* **6** (1982), 146-147.

[178] The reference is to [45]-[48], compare also [36].

[179] Arabic: *dhawāt $^c$uyūnihā.*

[180] Al-Sijzī discusses this theme in the rest of his treatise.

[181] See [33].

(Arabic: p. 288 lines 27-30, French translation: p. 281.) Concerning the thing which can (only) be imagined after the proof of it (has been given), it is as for example the equality of the three angles of the triangle to two right angles, as we have shown by way of analysis in our *Book on Making Easy the Ways of Deriving Geometrical Figures.*[182]

[182] See [25]-[27].

**Some differences between Saidan's edition and the Lahore manuscript.**

Saidan's edition is sometimes different from the manuscript of al-Sijzī's text contained in the private library of Nabi Khan in Lahore, Pakistan, ff. 2-27. Dr. Heinen, München kindly showed me his copy of this manuscript. Below I list a number of instances where I prefer the reading of the Lahore manuscript over Saidan's edition. The only purpose of this list is to allow the reader to reconstruct the Arabic text which I have translated. The list is of course not a substitute for a critical edition of al-Sijzī's text.

My list does not imply a negative judgement on Saidan's edition. The Lahore manuscript is unvocalized and many diacritical marks are not written. If this manuscript was Saidan's source (as is likely), Saidan deserves great credit for his correct interpretations and emendations of many ambiguous words and expressions. These correct interpretations and emendations are too numerous to be listed below.

A notation such as 339:7 S يريد , L. نريد; means that on line 7 of the relevant page (339) of Saidan's edition, Saidan prints يريد and the Lahore manuscript has نريد . Unless otherwise indicated, I prefer the reading of L. Passages in Saidan's edition between square brackets [ ] are his own addition. I have listed these words in cases where I think they have to be deleted. In all other cases I believe they have to be maintained.

In all short passages in the following list, I have omitted the slashes above letters indicating points in geometrical figures, because they cannot easily be printed in the program ArabTeX which I have used.

339:5 S مقدّمت المؤلّف omitted in L. 339:7 S يريد , L نريد . 339:16 S يمكن , L. يكن .
339:17 S مجتهد , L يجتهد .
340:6 delete [من]. 7 S يفضل , L. نقصد . 340:11 S اعسر , L. اعسر (S's reading may be correct). 340:12 S. عن هذا , MS. لزم? Reading uncertain. 340:14-15 S قدمنا هذا , L. وهذا هنا .
341:6 delete [من]. 341:7 مثلا , L. مثلا انّا .
342:6 S. ولقدماتها مقدمات ايضا , L. ولقدماتها مقدمات ايضا ولقدماتها مقدمات ايضا .
342:9 S. يسهل L. سَهَّلَ (vocalized). 342:18 delete [التركيب] .
343:5 S. بذاته , L ذاته .
344:4 الله , L. الله تعالى . 344:11 S. ظنّ , L. reads ظفه , S. is correct, or should one read ظنّة ?
346:9 S. لا سيما , L. لا فيما .
347:4 S. نعتناه , L. نعتاد . 347:12 S. نفرض ن , L. نفرض طك . 347:13 S. فنخرج , L. و نخرج . 347:13 S. نخرج طن , L. نخرج طل . 347:14 S. خط طن , L خط طل. 347:17 S. ببعد طن , L. ببعد طل . 347:17 S. خط طن , L. خط طل . 347:18 S. ن من خط طن , L. ل من خط طل . 347:19 S. وليكن موضوعا , L. ولكن موضوع .

348:6 S. and L. have العرض, which I prefer to read as الغرض. 348:9 S. فنبين , L. فبين . 348:10 S. لكي , L. لكن . 348:11 and elsewhere S. خاصية , L. خاصة . 348:15 S. طن، طك، طس , L. طن و طك و طس . 349:9 S. معروف من , L. مفروض . 349:11 S. الفصل , L. القصد . 349:19 S. يحصل , L. فلا يحصل . 349:20 S. علم , L. علمه . 349:21 S. فصارت , L. فقد صارت . 350:11 S. فما , L. كما . 350:11 S. ينقص , L. فننقص (correct: فينقص). 350:14 S. فنأتي , L. فبأي . 350:17 S. لانّه , L. وذلك لانّه . 350:17-18 S. زاويتي ج ب د , L. زاويتي ادب حبد , which should be read as زاويتي ا ب ج . زاوية ا ب ج , L. زاويتي ا ج ب ، ج ب د . 350:18 S. زاويتي ا ب ج .

351:5-6 S. ليلتقى بعضها مع , L. ليلقى بعضها من . 352:4 S. المناسب , I prefer التناسب (the ms. is ambiguous). 352:5 S. موازيا اج , L. مواز لاج . 352:20 delete [وهو]. 352:20 S. آمن , L. امر . 353:11 S. فنسبة , L. نسبة . 353:15 S. المثلّثين , L. المثلّثين منها . 353:16 S. على د الى ز , L. على نسبة د الى ه , which should be emended to على نسبة د الى ز . 353:17-18 S. الذي على خط ب ج , L. الذي يلي خط ا ج . 354:1 S. فاقوم , L. فاقدم . Delete the following word الاعمال in L. and S. 354:2 S. نقطة ي, L. نقطة ك , the figure also shows a point ك where Saidan's figure has ي . 354:3 S. ونصل ا ي فتكون نسبة , L. ونصل اك فيبّن ان نسبة . 354:9 S. ا ج ب , L. احب , I read this as ا ح ب . 354:10 S. فنفكر , L. فنتفكر . 354:13 S. نسبة جميع , L. لكن جميع . 354:13 S. ا ح ج ان , L. ا ح ج مفقودين ان . 354:17 S. ط ح, L. ط ج . 354:18 S. ا ج ب , L. احب , which I interpret as ا ح ب . 354:18 S. الغرض , L. الفرض . 354:19 S. بنسبة دون , L. النسبة وقت . 354:20 S. ج ط ح , L. ح ط ج . 354:22 between S. قسمي ه and ونسبة , L. has the following line:

ونصل اح جح فنسبة مثلث اب ح الى مثلث ا ح ط كنسبة د الى احد قسمي ه

354:22 S. ج ط ح , L. ح ط ج . 354:23 between S. مثلث ا ب ح and الى , L. has the following line:

الى مثلث ا ح ج كنسبة د الى ه وقد بينا ان نسبة مثلث ا ب ح

354:23 S. مثلث د ح ج , L. مثلث ب ج ح . 355:3 S. على , L. على نسبة . 355:9 S. الفصل , L. القصد . 355:9 S. نفكر , L. تتفكر . 355:10-13 all points ي in the edition and figure 140 in S. are points ك in the manuscript. 355:11 S. ا ج , L. لاج . 355:12 S. الموازي ا ج , L. الموازي لاج . 355:20 S. يمكن , L. سيمكن. 356:9 S. ج ا , L. لاج. 356:10 S. ز د , L. د ز . 356:10 S د ا , L. ا د . 356:12 S. ح د ه, L. ه د ح . 356:12 S. د ا ب, L. ب ا د . 356:12 S. ب د ه, L. ه د ب . 356:13 S. ا ب د , L. د ب ا (twice). 356:13 S. د ا ب , L. ب ا د . 356:15 S. اوجبها , L. وجبها (S. is probably correct). 356:15 S. بهذا الشكل , L. بهذا المعنى . 357:2 S. ج د ح, L. ح د ج. 357:18 S. قسمين , L. بقسمين . 358:9 S. ز ا , L. ا ز . 358:18 S. ا ب د , L. ا د . 358:19 S. فاذا , L. فاذا, which has to be interpreted as فاذن. (L always writes فاذن as فاذا .) 358:21 S. ا ز ب , L. ا ز ب . 359:3 S. ه د , L. ه ب. 359:4 between على ما اردنا and والبرهان L. has the following line:

لكن بالنقل اذا اخرجنا زح موازٍ لـ هب يصير اب مقسوما على ما اردنا

359:5 S. على أن , L. على . 359:15 S. وأتى بشيء , L. واي شي . 359:15 S. برهن , L. برهنه . 360:3 S. مشاركة , L. مشتركة . 360:5 S. وقوع , L. بوقع (should this be read توقّع ?). 360:7 S. بها على , L. عليه على, I think عليه should be deleted as a scribal error. 360:10 S. احوالها, this is the first word of the first line on f. 19 of the manuscript L. The manuscript has خواصها but at the beginning of this word one can see the trace of the ink of (the mirror image) of the letters كا of the word اشكال, which is the last word on the first line of f. 18 in the manuscript. These letters misled Saidan, who then read the word as احوالها. This misreading proves that Saidan, who did not indicate his source, used manuscript L. 360:17 S. ا جـ , L. لا جـ . 360:22 S. ك جـ , L. لك جـ (twice). 361:1 S. النسق , L. ان فيه النسق. 361:2 S. ب جـ , L. لب جـ . 361:3 S. نفسها , L. بقسيها . 361:12 S. ليكون مرّ لنا , L. ليلزم لنا . 361:13 S. ز ح , L. ز جـ . 361:15 S. ز ط ح , L. ز ط جـ . 362:2 S. ز ط ح , L. ز ط جـ . 362:3 S. ا ز ح , L. ا ز جـ . 362:5 S. ز ح , L. ز جـ . 362:6 S. ز ح , L. ز جـ . 362:9 S. ا جـ ك , L. ا جـ د . 362:16 S. وانقسام , L. وانقسامنا . 363:3 S. مهمة , L. مبهمة . 363:13 S. والزمن , L. والزوايا . 363:14 S. فصل, L. فصل, which I prefer to read as فضل (diacritical mark omitted from the manuscript). 364:5 S. د ب , L. د ب الى . 364:5 between كنسبة and وتر a line is missing in L and S, see my translation. 364:6 S. د جـ , L. ب جـ . 364:9 S. المغالطة , L. الشريطة . 364:18 S. [أن ا ه مثل]: the word مثل is still legible in L. 365:9 S. نسبة, L. نسبة خط . 365:14 S. قوس د ز ك , L. قوس د ز . 366:5 S. أو , L. وانه . 366:6 S. ا ز د , L. ا ب جـ , which should be read ا د جـ . 366:8 S. جـ د ه , L. ح د ه . 366:10 S. ز جـ , L. د جـ. 366:10 S. جـ ه د , L. ح ه د . 366:12 S. ز جـ , L. د جـ . 366:16 delete [مع]. 367:1 S. ا جـ ب , L. ا ح ب . 367:11 S. والمربّع , L. و مربّع . 367:16 S. د ه ، ا ب ز ح , L. د ه موازٍ لـ ب ز . 367:17 S. ليعلم , L. لنعلم which I read as لنعلم . 368:6 S. نصف مربّع , L. نصف . 368:10 S. ا د , L. لا د . 368:19 S. ا د , L. لا د . 368:19 S. فنقول , L. فنبين . 369:5 S. ومربع ا جـ هو مربع ا ح , L. ومربع ا ح هو مربع ا د , which should be read as ومربع ا د هو مربع ا ح . 369:12 S. يوضع , L. نوضع , which I read as نوضع . 369:12 S. ويلزمها , L. ونلزمها , which I read as ونلزمها . 370:15 S. الماس , L. الماس لها . 372:6 S. ا ه معلوم , L. و ا ه معلوم . 372:8 S. يخرج, L. نخرج ,which I read as نخرج . 372:8 S. ا جـ ، ا ب , L. ا ب و ا جـ .