LONDON MATHEMATICAL SOCIETY
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NEWSLETTER

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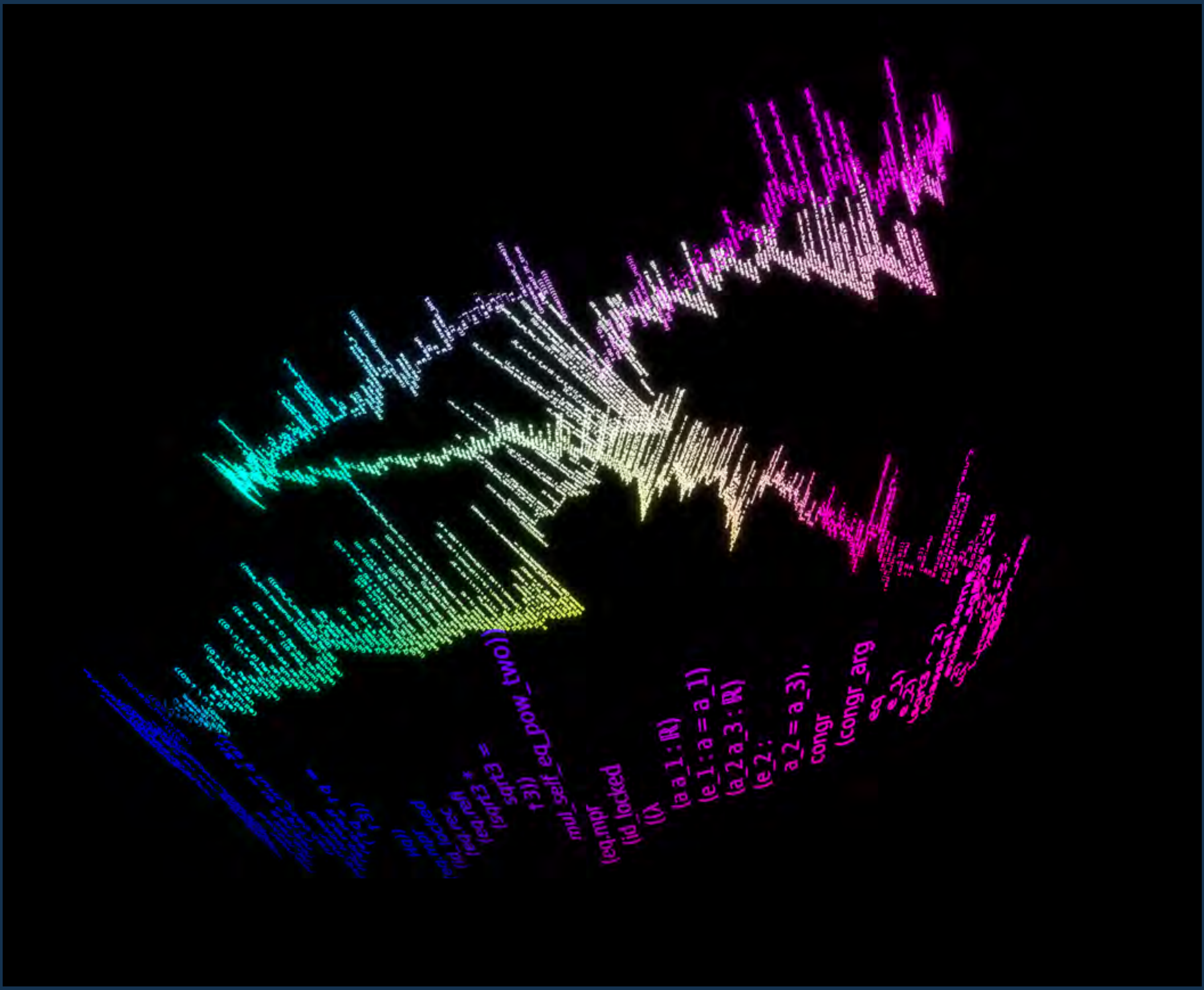

ETHICS IN MATHEMATICS

DRINFELD MODULES AND FACTORISATION

COMPUTERS AND PROOF

EDITORIAL OFFICE

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COVER IMAGE

A formal proof that $\sqrt{3}$ is irrational. See the feature starting on page 32. (Image courtesy of Zak Buzzard.)

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NEWSLETTER WEBSITE

The Newsletter is freely available electronically at lms.ac.uk/publications/lms-newsletter.

MEMBERSHIP

Joining the LMS is a straightforward process. For membership details see lms.ac.uk/membership.

SUBMISSIONS

The Newsletter welcomes submissions of feature content, including mathematical articles, career related articles, and microtheses from members and non-members. Submission guidelines and LaTeX templates can be found at lms.ac.uk/publications/submit-to-the-lms-newsletter.

Feature content should be submitted to the editor-in-chief at iain.moffatt@rhul.ac.uk.

News items should be sent to newsletter@lms.ac.uk.

Notices of events should be prepared using the template at lms.ac.uk/publications/lms-newsletter and sent to calendar@lms.ac.uk.

For advertising rates and guidelines see lms.ac.uk/publications/advertise-in-the-lms-newsletter.

CONTENTS

LMS Elections and Annual General Meeting 2019

Voting for the LMS Elections for Council and Nominating Committee will open on 25 October 2019. The slate of candidates can be found at lms.ac.uk/about/council/lms-elections and an online forum for discussion is available at discussions.lms.ac.uk/lmselections.

In addition, members are to be asked to vote on changes to the LMS Standing Orders — the Royal Charter, Statutes, and By-Laws. The current Standing Orders, together with the proposed changes, can be found on the LMS website at tinyurl.com/lmsstandingorders.

A vote on the proposed changes will be taken at the Annual General Meeting, but LMS Council has agreed that members who are unable to attend will be able to place an online proxy vote from 25 October to 25 November, through the independent Electoral Reform Services, or if preferred by completing a hard copy proxy voting form that will be available from the LMS website.

Instructions for each of the ballots i.e. a) the elections to Council and Nominating Committee and b) by proxy on the changes to the Standing Orders, will be sent to members by email or post before the ballots open. Members are encouraged to check that their contact details are up to date at lms.ac.uk/user.

This year the AGM will be held at Goodenough College, Mecklenburgh Square, London WC1N 2AB on Friday 29 November at 2.30 pm. Please note the change of date to the end of the month and the slightly earlier start time. All those attending the AGM will be required to register on the day. The registration desk will be open from 1.30 pm to allow enough time for members to verify their details and receive voting cards prior to the start of the meeting. The results of the Council and Nominating Committee elections will be announced at the meeting, as will the results of the vote on the changes to the Standing Orders.

Fiona Nixon
Executive Secretary

Donating to the LMS

Making a donation to the LMS has just become easier! There is now a 'donate' button on the main menu bar near the top of the LMS webpage (lms.ac.uk), enabling anyone to make an online donation to the Society.

The LMS gives grants totalling in excess of £680,000 per year to support many mathematical activities. By far the greatest part of our income currently comes from publications, which one can view as a way of recycling money from everyone's work as authors, editors and referees into the community. However, as the publishing industry moves to various forms of open access, the income we get from this resource is likely significantly to diminish. The threat to publications income, combined with the current unpredictability of the financial sector, mean that this aspect of our income has become crucial. Additional donations will help the Society maintain and if possible increase its level of support for all of its objectives.

In the past the Society has benefitted from donations from many individuals. Two in particular stand out: £1,000 from Lord Rayleigh in 1874, which supported the printing of the *LMS Proceedings* and rescued the Society from collapse, and the bequest of £50,000 by G.H. Hardy in 1963 which was completely transformative for the Society.

Alongside the suggested amounts for donations, you will also see a line for a 'De Morgan Donation' of £1865 or more (no prizes for working out where this notable number comes from!) This is a new venture which it is hoped will encourage anyone with the resources to do so to support the Society with a substantial donation. In recognition of their support, De Morgan Donors will from time to time receive invitations to lectures and other special events.

For those in a position to do so, you are also encouraged to think about leaving a legacy to the LMS in your will. Advice on how to do this can also be found on the web page.

Of course, a donation can also be added when you pay the annual membership fee. Whether your donation is large or small, the Society really does value your support.

Stop Press! DeMorgan@21

Unfortunately the big People's Vote march in London recently changed its date to October 19th which means that transport in and out of London is likely to be very crowded on that day. Those planning to come to DeMorgan@21 are therefore advised to reserve train seats and hotels well in advance. We apologise to anyone who was planning to go on the march.

Registration for this event has re-opened with space for another 20 participants. These will be allocated on a first-come-first-served basis. If you are interested please register at tinyurl.com/y4r935w9.

Survey of Mathematics Postdocs

The LMS has recently published the results of its 2017 Survey of Postdoctoral Researchers in the Mathematical Sciences in the UK. This survey was undertaken by the LMS Research Policy Committee in recognition of the fact that there was little overall understanding of postdoctoral activity in the UK, in terms of the size of the population, its distribution in subject area and geographical terms, its origin and its source of funding.

The survey was conducted in autumn 2017 with a census date of 31 October 2017. Individual heads of department were asked to supply information about the postdoctoral researchers in their department. They were asked for data about gender, nationality, home department, country of undergraduate degree, field of interest and source of funding, with reassurances about the confidentiality of detailed information. The initial request was followed up with several reminders and then collated and processed in 2018.

Geography: the survey reported 756 postdoctoral researchers in total — a much larger figure than was initially expected. A few departments did not participate in the survey, so the total number will in fact be higher than this figure. 25% of these were from the UK, and 43% were from the EU outside of the UK. This proportion is remarkably uniform across the 32 departments with EU postdocs. The results showed that there was a greater percentage of UK nationals in the postdoc population in northern areas of the UK than in southern areas.

Funding source: sources of funding were gathered into 8 groups. The diversity of sources was rather encouraging. As expected, EPSRC was by far the largest single funding source at 34% (being twice as big as any other category); the EU, other Research Councils, and the postdoc's own institution were the other main categories, at about 17% each. In several of the larger departments the amount of EU and EPSRC funding was about the same. The effect of losing EU funding is likely to be significant. The survey also identified the proportion of funding to different fields, both overall and per funding source. Generally, the patterns here conformed to expectations.

Gender: the survey looked at gender balance both by department and by field. The departmental balance showed that the overall average percentage of female postdocs was 23%. The survey identified a large variation, from 38% female in industrial mathematics to 11% in PDEs and analysis, 10% in mathematical physics and integrable systems, and 6% in number theory.

Postdocs and the REF: As expected there is a strong relationship between REF performance and numbers of postdocs. It seems that there are typically very few postdocs in departments with less than 20 FTE REF returns, and beyond that range a ratio of 1 postdoc per 2.4 research active staff look like a rather general trend across all sizes.

The full results of the survey can be accessed at tinyurl.com/yyrh7fnv. We are very grateful to departments for their cooperation in helping us to carry out this valuable exercise. It is intended to repeat the survey at regular intervals. Comments and queries are of course very welcome.

John Greenlees
LMS Vice-President

LMS Honorary Members 2019

LMS Honorary Members Ed Witten (left) and Don Zagier

At the Society meeting on 28 June 2019, the LMS elected Professor Edward Witten (Institute for Advanced Study, Princeton) and Professor Don Zagier (Max Planck Institute for Mathematics) as honorary members of the Society.

Edward Witten occupies an unrivalled position amongst contemporary mathematical and theoretical physicists. He has made profound contributions to the development of contemporary physics, including topological quantum field theory, string theory, M-theory and quantum gravity.

Don Zagier is an outstanding mathematician who has made major contributions in number theory, particularly to the theory of modular forms, and in its interactions with other areas of mathematics and mathematical physics.

2019 LMS prize winners

The Society extends its congratulations to the following 2019 LMS prize winners and thanks to all the nominators, referees and members of the Prizes Committee for their contributions to the Committee's work this year.

A De Morgan Medal is awarded to **Professor Sir Andrew Wiles FRS** of the University of Oxford for his seminal contributions to number theory and for his resolution of 'Fermat's Last Theorem' in particular, as well as for his numerous activities promoting mathematics in general.

Professor Ben Green FRS, of the University of Oxford, is awarded a Senior Whitehead Prize for his ground breaking results in additive combinatorics, analytic number theory and group theory.

Professor Nicholas Higham FRS, of the University of Manchester, is awarded a Naylor Prize and Lectureship in Applied Mathematics for his leadership in numerical linear algebra, numerical stability analysis, and communication of mathematics.

Dr Alexandr Buryak of the University of Leeds is awarded a Whitehead Prize in recognition of his outstanding contributions to the study of moduli of curves and integrable systems.

Professor David Conlon of the University of Oxford is awarded a Whitehead Prize in recognition of his many contributions to combinatorics. His particular expertise is Ramsey theory, where he has made fundamental contributions to both the arithmetic and graph-theoretic sides of the subject.

Dr Toby Cubitt of University College London is awarded a Whitehead Prize in recognition of his outstanding contributions to mathematical physics, in particular the interconnections between quantum information, computational complexity, and many-body physics.

Dr Anders Hansen of Cambridge University is awarded a Whitehead Prize for his contributions to computational mathematics, especially his development of the solvability complexity index and its corresponding classification hierarchy.

Professor William Parnell of the University of Manchester is awarded a Whitehead Prize for highly novel and extensive research contributions in the fields of acoustic and elastodynamic metamaterials and theoretical solid mechanics, as well as excellence in the promotion of mathematics in industry.

Dr Nick Sheridan of the University of Edinburgh is awarded a Whitehead Prize for his ground breaking contributions to homological mirror symmetry and the structure of Fukaya categories.

The Berwick Prize is awarded to **Dr Clark Barwick** of the University of Edinburgh, for his paper *On the algebraic K-theory of higher categories*, published in the Journal of Topology in 2016, which proves that Waldhausen's algebraic K-theory is the universal homology theory for ∞-categories, and uses this universality to reprove the major fundamental theorems of the subject in this new context.

Dr Eva-Maria Graefe of Imperial College London is awarded an Anne Bennett Prize in recognition of her outstanding research in quantum theory and the inspirational role she has played among female students and early career researchers in mathematics and physics.

2019 LMS prize winners

Sir Andrew Wiles
De Morgan Medal

Ben Green
Senior Whitehead Prize

Nick Higham
Naylor Prize

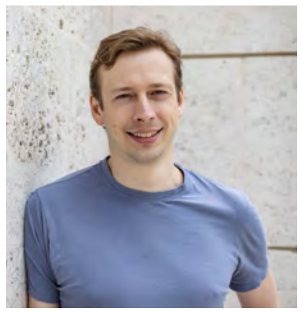
Alexandr Buryak
Whitehead Prize

David Conlon
Whitehead Prize

Anders Hansen
Whitehead Prize

Nick Sheridan
Whitehead Prize

William Parnell
Whitehead Prize



Toby Cubitt
Whitehead Prize

Clark Barwick
Berwick Prize

Eva-Maria Graefe
Anne Bennett Prize

Ken Brown awarded the 2019 David Crighton Medal

The LMS and IMA have awarded the 2019 David Crighton Medal to Ken Brown, Professor of Mathematics at the University of Glasgow, for his seminal contributions to noncommutative algebra and for his remarkable record of service and dedication to the UK mathematics community.

Ken has the rare ability to apply subtle ring-theoretic ideas to solve important problems in related areas. In the 1970s he solved the zero-divisor question for abelian-by-finite groups, introducing the key homological techniques which would form the basis of all later major progress in this area; according to Formanek, he made “the most important and original contribution to the problem since Higman’s [1940] work.” In the 1980s he introduced the class of homologically homogeneous rings: in the last five years, these have been key in the study of noncommutative geometry, derived categories, and moduli spaces. His focus then shifted to the theory of quantum groups and Hopf algebra where he harnessed the combination of Hopf algebras and homological algebra to confirm important conjectures of Kac-Weisfeiler and DeConcini-Kac-Procesi in representation theory. In the last decade he has proved core results in several topics: Noetherian Hopf algebras; number theory through Iwasawa algebras; Poisson geometry in Lie theory; and symplectic reflection algebras. Ken wishes to acknowledge his many collaborators in his research. He was elected a Fellow of the Royal Society of Edinburgh in 1993.

Ken has mirrored his distinguished international mathematical career with extraordinary service to the UK Mathematical Sciences community. He sat on the London Mathematical Society Council for almost two decades, including terms as Vice-President from 1997-99 and 2009-17. During this time he was instrumental in the development of the voice of the Council for the Mathematical Sciences, providing critical input to consultations and leading a variety of task forces, particularly helping highlight the important issues that affect the Mathematical Sciences people pipeline. Beyond this long-term involvement with the LMS, Ken has made numerous further contributions to the UK Mathematical Sciences.

Beyond this long-term involvement with the LMS, Ken has made numerous further contributions to the UK Mathematical Sciences, including as a member of the Research Assessment Exercise (RAE) Mathematics subpanel in 1996, as Vice-Chair of this panel in 2001, and then as Chair of the RAE Pure Mathematics Subpanel in 2008. He was also a member of the Research Excellence Framework (REF) expert advisory group in 2008-09. He has been a member of the EPSRC College since 1996 and served on the EPSRC Mathematical Sciences Strategic Advisory Team (SAT) first as a member from 2013 to 2015 and subsequently as its Chair from 2015 to 2017.

Read the full citation at tinyurl.com/y5ubvpg3.

Calderon Prize

LMS member Carola Schönlieb (University of Cambridge) has been awarded the 2019 Calderon Prize for her work in image processing and partial differential equations. The Inverse Problems International Association awards the Calderon Prize to a researcher under the age of 40 who has made distinguished contributions to the field of inverse problems broadly defined. Carola is the first female mathematician to receive this award. She won an LMS Whitehead prize in 2016 and is the current leader of European Women in Mathematics.

Forthcoming LMS Events

The following events will take place in the next three months:

Prospects in Mathematics Meeting: 6–7 September, Lancaster (tinyurl.com/y4c9aaxk).

Midlands Regional Meeting: 11 September, Nottingham (tinyurl.com/y5vtaytx).

Popular Lectures: 19 September, University of Birmingham (tinyurl.com/hu58wjk).

DeMorgan@21: 19 October, London.

Joint Meeting with the IMA: 21 November, Reading (tinyurl.com/y4sdm74b).

Computer Science Colloquium: 13 November, London (tinyurl.com/cscoll19).

LMS/BCS-FACS Evening Seminar: 21 November, London (tinyurl.com/yyc9oyse).

Graduate Student Meeting: 29 November, London (tinyurl.com/yy58t78v).

Society Meeting and AGM: 29 November, London (tinyurl.com/yy58t78v).

A full listing of upcoming LMS events can be found on page 54.

OTHER NEWS

Alan Turing honoured on new £50 banknote

Alan Turing banknote concept

An announcement made by the Bank of England on 15 July 2019 has confirmed that Alan Turing will be the character to feature on the reverse side of the new £50 banknote, which will come into circulation in 2021.

Alan Mathison Turing OBE FRS (1912–1954) was a mathematician, computer scientist, logician, cryptanalyst, philosopher and theoretical biologist who was instrumental in formalising the concepts of algorithm and computation. Turing worked as a code-breaker during the second world war and is widely accredited with having helped bring an earlier end to the war. The story of his life has had wide implications for changes in political, legal and social attitudes towards human diversity and homosexuality.

In his article 'On computable numbers, with an application to the Entscheidungsproblem' (submitted 28 May 1936, published in *Proceedings of the London Mathematical Society* 42 (1937) 230–265; tinyurl.com/y3vlud4j), Turing presented a first model for a general-purpose computer, later to become known as a 'Turing machine'.

The London Mathematical Society welcomes this great exposure and boost to the public appreciation of mathematics, and is delighted to have contributed to the design of the banknote by giving approval and permission for two mathematical excerpts from this Turing article to be included on the new banknote.

The first excerpt is a table from page 240 which provides a schema for succinctly representing Turing machines. The table gives a complete description of how to specify such machines and therefore can be thought of as one of the first examples of a programming language.

m-config.	*Symbol*	*Operations*	*Final m-config.*	
q_i	S_j	PS_k, L	q_m	(N_1)
q_i	S_j	PS_k, R	q_m	(N_2)
q_i	S_j	PS_k	q_m	(N_3)

The second excerpt, from page 241, is a sequence of Turing machine transitions that helps explain how to encode a Turing machine as a number. The more modern analogue of what Turing describes is how to take an abstract representation of a computer program and convert it into a binary sequence of 0s and 1s so that it can be stored on a disc or in the memory of a computer. The idea that a program can be stored as a number, and used as data (by an operating system) in order to execute the program, is hugely important.

$$q_1 S_0 S_1 R q_2;\ q_2 S_0 S_0 R q_3;\ q_3 S_0 S_2 R q_4;\ q_4 S_0 S_0 R q_1;.$$

Turing went on further in his article to describe large classes of real numbers whose binary expansions are computable by his machines; to describe a 'universal machine' that could serve the purpose of an operating system; and to describe the theoretical limits of his machines. Ultimately, Turing showed that there can be no algorithmic method for determining whether or not a given mathematical statement can be proved in a certain axiomatic system. This proved that David Hilbert's famous *Entscheidungsproblem* has no solution (which was also proved independently by Alonzo Church).

Alan Turing is the second mathematician to appear on a Bank of England banknote. A £1 banknote in circulation between 1978 and 1988 depicted Sir Isaac Newton. Famous Britons have featured on the reverse of Bank of England banknotes since 1970.

Paul Shafer, University of Leeds
Ola Törnkvist, LMS Editorial Manager

MATHEMATICS POLICY DIGEST

Brexit threat to research and innovation

Royal Society President, Sir Venki Ramakrishnan wrote a letter in July warning of the impact a no-deal Brexit will have on research and innovation. More information and the letter are available at tinyurl.com/yyas8lz3.

Exploring the workplace for LGBT+ physical scientists

The Institute of Physics, the Royal Society of Chemistry and the Royal Astronomical Society, have conducted a comprehensive survey to gather data from across the community — giving new insights into the current workplace environment for LGBT+ physical scientists. The full report is available at tinyurl.com/y38x9kxh.

New Director General, Industrial Strategy and Innovation

Jo Shanmugalingam became Director General, Industrial Strategy, Science and Innovation at the Department for Business, Energy and Industrial Strategy (BEIS) on 15 July 2019. More information about the new Director General is available at tinyurl.com/y2hch6cn.

Digest prepared by Dr John Johnston
Society Communications Officer

Note: items included in the Mathematics Policy Digest are not necessarily endorsed by the Editorial Board or the LMS.

EUROPEAN MATHEMATICAL SOCIETY NEWS

From the EMS President

The last months have been full of activities. Here I would like to mention a few. A highlight was the award of the Abel Prize to Karen Uhlenbeck, the first female recipient, on 21 May. This was followed by a day of talks related to her work. These can be viewed at tinyurl.com/yx8p4v6f.

The restructuring of the EMS publishing house is underway. The new organization, in the form of a limited company owned by the EMS, was established at the end of March. The hiring process for the new management is in progress. The new publishing house will immediately face a major challenge, with the increasing importance of open access publishing, in particular Plan S of the European Commission. The EMS has reacted to this plan, pointing the possible consequences for small publishers and the mathematics community. See tinyurl.com/y54uw828.

Another major concern is research funding for mathematics in Europe, which is constantly decreasing even within the European Research Council. See the next item, appealing to EMS members to become more active on all levels.

Funding for Mathematical Research

Volker Mehrmann (President of the EMS) writes: As President of the European Mathematical Society I would like to point out a very urgent and unfavourable situation for the funding of mathematics in Europe. The European Research Council (ERC) budget for each discipline is allocated each year in proportion to the number of proposals and the requested budget received. It has been observed that, since the founding of the ERC, the budget for mathematics in the three funding streams (advanced, consolidator, and starting grants) has dropped to almost half, because there are not enough applications.

There may be several reasons for this decline in applications, e.g. low acceptance rate, the feeling that certain subfields of mathematics have small chances, or the fact that for interdisciplinary research of mathematics with other sciences it is difficult to get funding. Also in mathematics there are often complaints that the maximal possible budgets are too large.

All this is partially right, but not submitting applications leads to a vicious cycle, and further decline of mathematics funding. How can we counteract this unfortunate development? First of all, there is no reason to apply for the full possible budget if this is not appropriate for a research project, smaller proposals are very welcome, and second we mathematicians should be more self-confident in writing proposals. It is not a wasted time, even if one is not funded. In several European countries there is even financial support for proposals that make it to the second round but do not get funded due to budget restrictions.

It is very important that applications are encouraged throughout the mathematical community and the EMS is planning to create an initiative to support applicants. So please distribute this information within your community.

Volker Mehrmann
President of the EMS

Academia–industry roundtable

The EMS Applied Mathematics Committee notes with interest the minutes of an academia-industry roundtable organised by the ministry of economy in Japan (March 2019) (tinyurl.com/y3o54aud) which claims "we have identified the top three science priorities in order for Japan to lead the fourth industrial revolution and to even go beyond its limits: mathematics, mathematics, and mathematics!"

Bernoulli Center (CIB)

The CIB in Lausanne has announced the following schedule of research programmes: *Dynamics with Structures* (1 July – 31 December 2019); *Functional Data Analysis* (1 January – 30 June 2020); *Locally Compact Groups Acting on Discrete Structures* (1 July – 31 December 2020); *Dynamics, Transfer Operators, and Spectra* (1 January – 30 June 2021). For more details see http://cib.epfl.ch.

European Congress of Mathematics

The preparations for the 8th European Congress of Mathematics in Portorož, Slovenia, from 5 to 11 July 2020 are proceeding energetically. A full list of plenary and invited speakers at for the 8th European Congress of Mathematics in Portorož, Slovenia, is available at 8ecm.si/program.

EMS News prepared by David Chillingworth
LMS/EMS Correspondent

OPPORTUNITIES

EMS Prizes

Calls for nominations of candidates for ten European Mathematical Society Prizes as well as The Felix Klein Prize and The Otto Neugebauer Prize for the History of Mathematics are still open: details can be found at the European Mathematical Society website.

EMS Call for Proposals

The European Mathematical Society (EMS) Meetings Committee is calling for nominations or proposals for speakers and scientific events in 2020. The EMS is willing to provide support to cover the cost of EMS Lecturers and Distinguished Speakers, and to give partial support to the organization of EMS Weekends and of EMS Summer Schools of high scientific quality and relevance. The EMS is committed to increasing the participation of women in research in mathematics and its applications. Efforts to give opportunities to mathematicians of both genders and to integrating early career mathematicians will be particularly appreciated. Events within Europe and speakers with strong relations to European academic institutions will be given preference. Closing date for applications is 30 September 2019. For further details and nomination forms go to the EMS website and click on *Scientific Activities*.

Maryam Mirzakhani Prize in Mathematics: call for nominations

In recognition of Dr Mirzakhani's remarkable life and achievements, the National Academy of Sciences has established a newly named Maryam Mirzakhani Prize in Mathematics (formerly the NAS Award in Mathematics, established in 1988 by the American Mathematical Society in honour of its centennial). The prize will be awarded biennially for exceptional contributions to the mathematical sciences by a mid-career mathematician. Nominations for the inaugural Mirzakhani Prize are due by 7 October 2019. For more information see tinyurl.com/yykobras.

Women in Mathematics and other Diversity Events

A call for expressions of interest in running Women in Mathematics Days, Girls in Mathematics Days and a new event for 2019/20, Diversity in Mathematics Day, is now open. The deadline for submitting expressions of interest is Friday 20 September 2019.

More details about Women in Mathematics Days are available at tinyurl.com/y4wy8px3, Girls in Mathematics Days at tinyurl.com/y44t4mdt, and Diversity in Mathematics Day at tinyurl.com/y2a3pmbg.

ICME-14 Bursaries

The Joint Mathematical Council of the UK has launched the ICME bursaries scheme, which is partly funded by the LMS. The scheme will provide nine bursaries of £500 each to attend the 14th International Congress on Mathematical Education to be held from 12 to 19 July 2020 in Shanghai, China (tinyurl.com/yyxbvxth).

The bursaries can fund travel, subsistence, childcare, registration fee or preparation of a presentation. They cannot fund salary or related costs. Applications are encouraged from the full range of those working in all areas of mathematics and mathematics education, including teachers, researchers and postgraduate students who can demonstrate a benefit from attending ICME-14.

Download the application form at tinyurl.com/yyagyo23. Email the completed application form to Professor Chris Budd, University of Bath (mascjb@bath.ac.uk). The deadline is 30 November 2019.

Clay Research Fellowships: call for nominations

The Clay Mathematics Institute calls for nominations for its competition for the 2020 Clay Research Fellowships. Fellows are selected for their research achievements and their potential to become leaders in research mathematics. All are recent PhDs, and most are selected as they complete their thesis work. Terms range from one to five years, with most given in the upper range of this interval. Fellows are employed by the Clay Mathematics Institute, which is a US charitable foundation, but may hold their fellowships anywhere in the USA, Europe, or elsewhere in the world. Nominations should be received by 16 November 2019. To nominate a candidate see tinyurl.com/y6stxs6o.

VISITS

Visit of Grigory Belousov

Dr Grigory Belousov (Plekhanov Russian University of Economics, Bauman Moscow State Technical University) will visit the UK from 1 to 17 November 2019. During his visit, he will give lectures at the University of Liverpool, Loughborough University and the University of Edinburgh. His research revolves around Singular del Pezzo Surfaces. For further details contact Nivedita Viswanathan (Nivedita.Viswanathan@ed.ac.uk). Supported by an LMS Scheme 2 grant.

Visit of Murray Elder

Dr Murray Elder (University of Technology Sydney, Australia) will visit the UK from 4 November to 4 December 2019. His research interests include geometric group theory, complexity theory, automata and formal language theory, enumerative combinatorics and pattern-avoiding permutations. While in the UK he will visit and give talks in Newcastle, Heriot Watt and St Andrews Universities. For further details contact andrew.duncan@ncl.ac.uk. Supported by an LMS Scheme 2 grant.

Visit of Albert Visser

Professor Albert Visser (Universiteit Utrecht) will visit the University of Cambridge from 7 to 29 October 2019. He is professor emeritus of logic and a member of the Royal Netherlands Academy of Arts and Sciences; he is particularly well-known for his research contributions on weak systems of arithmetic. In addition to seminar presentations in Cambridge, talks will be given at the Universities of Oxford (14 October) and Leeds (16 October). For further information contact Benedikt Löwe (b.loewe@dpmms.cam.ac.uk). Supported by an LMS Scheme 2 grant.

LMS Council Diary — A Personal View

Council met at De Morgan House on Friday, 28 June, a little earlier than usual, due to the LMS General Meeting being held that same afternoon. As usual, the meeting began with an update on the President's activities since the last Council meeting, which included attendance at the LMS Reps Day and attendance at several of the Society's committees. She also reported that three members of the LMS received honours in the Queen's birthday list: Kenneth Brown received a CBE, Peter Ransom an MBE, and Peter Donnelly received a Knighthood. Vice President Hobbs reported from the European Mathematical Society Presidents' meeting, which was held in March 2019 in Berlin. She reported that diversity took up some part of the discussion. With the calls for EMS prizes now published, the LMS was encouraged to promote the nominations process, particularly to address the need for diversity of nominations. Council agreed that the free EMS membership for PhD students should be advertised.

The Education Secretary then reported that a working group had been set up in December 2018 to address the issue of the shortage of qualified mathematics teachers. It was felt that the group should be formalised as a sub-committee of Education Committee, which Council approved. Council also agreed to extend the Teachers CPD to allow universities to propose events for HE teaching and learning events.

The General Secretary reported that the latest version of the document containing the proposed changes to the Standing Orders was now available to view on the website, having received informal approval from the Privy Council.

The Chair of the Women in Mathematics Committee reported on the foreword to the National Benchmarking Survey Report, which is the LMS's sole contribution to the report; this foreword was approved. Council also agreed on the LMS Statement on Diversity in Mathematics.

Council noted with pleasure that the Irish Mathematical Society had agreed to a Reciprocity Agreement with the LMS.

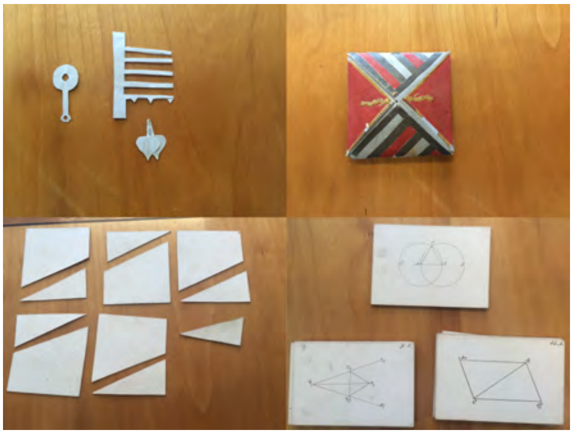
There was some discussion regarding the amount given to individual LMS Undergraduate Bursaries — some Council members felt that it would be detrimental to the Society's reputation, and the diversity of applicants, if recipients of LMS bursaries were disadvantaged in comparison to those funded elsewhere. Council agreed that this item should be deferred for further discussion at the October 2019 meeting.

We also heard reports from the IT Resources Committee and from the Publication Secretary. Furthermore, we received the Third Quarterly Financial Review. Following a report from the Prizes Committee Council agreed that Professor Kenneth Brown be confirmed as the Crighton Medal recipient in 2019.

We then decamped to the nearby Mary Ward House for the General Meeting and Aitken Lecture 2019.

Brita Nucinkis

Perigal Artefacts



The London Mathematical Society has been by given by Daniel Miskow some items which once belonged to Henry Perigal, an LMS member famous for his proof of Pythagoras's Theorem by dissection. The items include a collection of index cards containing diagrams from Euclid's Elements, and a beautifully-decorated envelope made by folding paper. The envelope contains six cardboard triangles and five quadrilaterals, as well as three pieces of blueish paper, carefully cut and folded. The members of the Library Committee have been unable to deduce their purposes and would welcome elucidation from members who are welcome to contact the LMS Librarian (librarian@lms.ac.uk).

Mark McCartney
LMS Librarian

LMS Committee Membership

The detailed business of the LMS is run by about 23 committees and working groups, each usually consisting of about 10 people. Altogether this comes to a large number of people, to whom the Society is extremely grateful for this vital work.

It is Council's responsibility to make the appointments to all these committees and to turn their membership over regularly, so that (a) the broadest possible spectrum of our membership is represented, and (b) the committees remain fresh and energetic. Of course, when forming a committee, account has to be taken of many things, such as maintaining subject and demographic balance, which means that on a given occasion otherwise very strong candidates may not always be able to be appointed.

So we are always looking for new people! See lms.ac.uk/about/committees for a list of committees.

If you are interested, or would like to recommend a colleague, please contact James Taylor at james.taylor@lms.ac.uk in order that Council can maintain a good list of potential members of its various committees. It is not necessary to specify a particular committee. If you would like to know what is involved, you could in the first instance ask your LMS Departmental Representative.

Stephen Huggett
General Secretary

Annual LMS Subscription 2019-20

Members are reminded that their annual subscription, including payment for additional subscriptions, for the period November 2019 – October 2020 is due on 1 November 2019 and payment should be received by 1 December 2019. Payments received after this date may result in a delay in journal subscriptions being renewed.

LMS membership subscription rates 2019–20

Ordinary membership	£88.00	US$176.00
Reciprocity	£44.00	US$88.00
Career break or part-time working	£22.00	US$44.00
Associate membership	£22.00	US$44.00
Associate (undergrad) membership	£11.00	US$22.00

Access to LMS Journals

The Society offers free online access to the *Bulletin, Journal* and *Proceedings of the London Mathematical Society* and to *Nonlinearity* for personal use only. If you would like to receive free electronic access to these journals, please indicate your choices either on your online membership record under the 'Journal Subscription' tab or on the LMS subscription form. The relevant publisher will then contact members with further details about their subscription.

Subscribing to the EMS and JEMS via the LMS

Members also have the option to pay their European Mathematical Society subscription via the LMS and subscribe to the *Journal of the EMS*. If you would like to subscribe to the EMS and JEMS via the LMS, indicate either on your online membership record under the 'Journal Subscription' tab or on the LMS subscription form.

Payment of membership fees for EWM

LMS members who are also members of European Women in Mathematics may pay for their EWM fees when renewing their LMS membership. You decide your category of fees: high, normal, low. Indicate your category of fee either on your online membership record under the 'Journal Subscription' tab or on the LMS subscription form. To join EWM, register at tinyurl.com/y9ffpl73. It is not possible to join the EWM through the LMS.

Online renewal and payment

Members can log in to their LMS user account (lms.ac.uk/user) and make changes to their contact details and journal subscriptions under the 'My LMS Membership' tab. Members can also renew their subscription by completing the subscription form and including a cheque either in GBP or USD. We do not accept payment by cheques in Euros.

LMS member benefits

An LMS annual subscription includes the following benefits: voting in the LMS elections, free online access to selected journals, the bi-monthly Newsletter, use of the Verblunsky Members' Room at De Morgan House in Russell Square, London and use of the Society's Library at UCL. For a full list of member benefits, see lms.ac.uk/membership/member-benefits.

Elizabeth Fisher
Membership & Grants Manager

LMS Grant Schemes

For full details of these grant schemes, and for information on how to submit an application form, visit www.lms.ac.uk/content/research-grants.

MATHEMATICS RESEARCH GRANTS

The deadline is 15 September 2019 for applications for the following grants, to be considered by the Research Grants Committee at its October meeting.

Conferences Grants (Scheme 1): Grants of up to £7,000 are available to provide partial support for conferences held in the United Kingdom. Awards are made to support the travel, accommodation, subsistence and caring costs for principal speakers, UK-based research students and participants from Scheme 5 eligible countries.

Visits to the UK (Scheme 2): Grants of up to £1,500 are available to provide partial support for a visitor to the UK, who will give lectures in at least three separate institutions. Awards are made to the host towards the travel, accommodation and subsistence costs of the visitor. It is expected the host institutions will contribute to the costs of the visitor.

Joint Research Groups in the UK (Scheme 3): Grants of up to £4,000 are available to support joint research meetings held by mathematicians who have a common research interest and who wish to engage in collaborative activities, working in at least three different locations (of which at least two must be in the UK). Potential applicants should note that the grant award covers two years, and it is expected that a maximum of four meetings (or an equivalent level of activity) will be held per academic year.

Research in Pairs (Scheme 4): For those mathematicians inviting a collaborator to the UK, grants of up to £1,200 are available to support a visit for collaborative research either by the grant holder to another institution abroad, or by a named mathematician from abroad to the home base of the grant holder. For those mathematicians collaborating with another UK-based mathematician, grants of up to £600 are available to support a visit for collaborative research.

Collaborations with Developing Countries (Scheme 5): For those mathematicians inviting a collaborator to the UK, grants of up to £3,000 are available to support a visit for collaborative research, by a named mathematician from a country in which mathematics could be considered to be in a disadvantaged position, to the home base of the grant holder. For those mathematicians going to their collaborator's institution, grants of up to £2,000 are available to support a visit for collaborative research by the grant holder to a country in which mathematics could be considered to be in a disadvantaged position.

Research Workshop Grants: Grants of between £3,000 - £5,000 are available to provide support for Research Workshops held in the United Kingdom, the Isle of Man and the Channel Islands.

AMMSI African Mathematics Millennium Science Initiative: Grants of up to £2,000 are available to support the attendance of postgraduate students at conferences in Africa organised or supported by AMMSI. Application forms for LMS-AMMSI grants are available at ammsi.africa.

MATHEMATICS/COMPUTER SCIENCE RESEARCH GRANTS

Computer Science Small Grants (Scheme 7): Grants of up to £1,000 are available to support visits for collaborative research at the interface of Mathematics and Computer Science, either by the grant holder to another institution within the UK or abroad, or by a named mathematician from within the UK or abroad to the home base of the grant holder. Deadline: 15 October.

GRANTS FOR EARLY CAREER RESEARCHERS

The deadline is 15 October 2019 for applications for the following grants, to be considered by the Early Career Research Committee in November.

Postgraduate Research Conferences (Scheme 8): Grants of up to £4,000 are available to provide partial support for conferences held in the United Kingdom, which are organised by and are for postgraduate research students. The grant award will be used to cover the costs of participants.

Celebrating New Appointments (Scheme 9): Grants of up to £600 are available to provide partial support for meetings held in the United Kingdom to celebrate the new appointment of a lecturer at a UK university.

Travel Grants for Early Career Researchers: Grants of up to £500 are available to provide partial travel and/or accommodation support for UK-based Early Career Researchers to attend conferences or undertake research visits either in the UK or overseas.

Report: LMS Invited Lectures 2019

Professor Søren Asmussen

The LMS Invited Lectures 2019 were delivered by Professor Søren Asmussen (Aarhus University) on 20-24 May at the International Centre for Mathematical Sciences in Edinburgh, on advanced topics in life insurance mathematics. This course was based on parts of Professor Asmussen's forthcoming book *Risk and Insurance: A Graduate Text*, co-authored with Mogens Steffensen (University of Copenhagen).

Professor Asmussen gave a fascinating account of inhomogeneous Markov models, semi-Markov models, and their applications in life insurance. Topics covered in detail included unit-linked insurance, interest rate and mortality rate modelling, and dividends and bonuses. The course consisted of nine lectures and two tutorials, and was complemented by an opening lecture by Takis Konstantopoulos (University of Liverpool) and a closing lecture by Andrew Cairns (Heriot-Watt University).

Participants (particularly early-career researchers) were given the opportunity to present some of their own work. During the week we heard interesting presentations from Jason Anquandah (University of Leeds) on *Optimal Stopping in a Simple Model of Unemployment Insurance,* Abdul-Lateef Haji-Ali (Heriot-Watt University) on *Multilevel Monte Carlo for Efficient Risk Estimation,* and Lewis Ramsden (University of Hertfordshire) on *The Time to Ruin for a Dependent Delayed Capital Injection Risk Model*.

In addition, a workshop on *Stochastic Processes in Finance and Beyond* was held on the afternoon of Thursday 23 May, organised on behalf of the Applied Probability Section of the Royal Statistical Society. Professor Asmussen presented applications of phase-type distributions in life insurance, building from the graduate-level material he discussed earlier in the week to the state of the art in the area. Additional, complementary presentations were given by Burak Buke (University of Edinburgh), Ayalvadi Ganesh (University of Bristol) and Ronnie Loeffen (University of Manchester) on various aspects of the theory of stochastic processes, with applications to financial, biological and other systems.

A relaxed schedule and the hospitable environment offered by the ICMS allowed for plenty of interaction between participants at various career stages throughout the meeting, and also time to enjoy the city of Edinburgh in one of the better weeks of weather this summer has given us.

Fraser Daly and Sergey Foss
Heriot-Watt University

Report: LMS Popular Lectures 2019

Peter Higgins

On 26 June 2019 the Institute of Education, London hosted the 2019 LMS Popular Lectures. The audience enjoyed an evening of mathematics with a journey through a life in mathematics and a look at the complexity of the human brain and some of the basic principles at work in shaping our brains.

Peter Higgins (University of Essex) presented the first lecture and took the audience through his personal mathematical journey. Going back to his roots and recounting personal episodes that led to him becoming a professional mathematician. He went on to tell the story of how he invented circular Sudoku. Higgins recognised from his work that the knowledge he had accumulated could be applied to mathematical problems in this case relating to Burrows–Wheeler transform, which is an algorithm that rearranges character strings into runs of similar characters and is important in data compression.

Alain Goriely

The second lecture was presented by Alain Goriely (University of Oxford). The outer surface of the brain (cerebral cortex) is very convoluted (wrinkled) so that a maximum amount of gray matter (brain neurons) can fit inside the skull, but how do these convolutions emerge and how is the brain's geometry related to function? Professor Goriely began by looking at the relative sizes of the brains of some famous scientists and mathematicians, with some interesting results. Einstein in particular had a small brain compared with Babbage, Helmholtz, Gauss and Dirichlet. He went on to describe comparisons in the convolutions of Gauss and De Morgan and discussed how the brain develops its shape by using mechanics. Gauss had complex convolutions and De Morgan had convolutions that, although 'voluminous', were less intricate. Professor Goriely explained how different parts of the brain are connected to each other in brain networks and how neurodegenerative diseases develop. In essence he showed how some of the basic principles at work in shaping the brain can be explained using geometry, scaling laws and network topology.

If you weren't able to attend the London lectures, there is another chance to attend the Lecture in Birmingham on Thursday 19 September. You can register online at lms.ac.uk/events/popular-lectures.

John Johnston
LMS Communications Officer

Report: LMS Undergraduate Summer Schools

The LMS Undergraduate Summer Schools are annual two-week courses, which are held every summer at a UK university. The aim is to introduce modern mathematics to the best UK undergraduates who will enter their final year the following autumn, and encourage them to think about an academic career in mathematics. The LMS Undergraduate Summer School consists of a combination of short lecture courses with problem-solving sessions and colloquium-style talks from leading mathematicians. 50 places are available to students per year. Gwyn Bellamy, organiser of the 2018 LMS Undergraduate Summer School at Glasgow, reports on the LMS Undergraduate Summer School 2018. The 2019 LMS Undergraduate Summer School took place at the University of Leeds from 14–26 July and the next LMS Undergraduate Summer School will take place at the University of Swansea from 12-24 July 2020. Nominations for places will open in October 2019.

In July 2018, 53 extremely eager students from across the United Kingdom descended on Glasgow for a jam-packed two weeks of mathematical activities which is the LMS Undergraduate Summer School. Each mathematics department in the UK elects up to three applicants to attend, and from these we select approximately 50 participants. Almost the entire cost of the two weeks is covered by the LMS.

In the first week, lecture courses were given by Professor Mike Prest (University of Manchester) on *The Compactness Theorem*, by Dr Robert Gray (University of East Anglia) on *The Word Problem in Combinatorial Group and Semigroup Theory* and by Professor Shaun Stevens (University of East Anglia) on *Local–Global Principles in Number Theory*. In the second week, lecture courses were given by Dr Martina Lanini (Università di Roma Tor Vergata) on *Introduction to Schubert Calculus*, by Dr Derek Harland (University of Leeds) on *Fun with Solitons* and by Dr David Bourne (Heriot-Watt University) on *Optimal Transport Theory*. As one would expect, the lectures were uniformly fascinating and engaging, and a real pleasure to listen to. It was clear that the lecturers had spent a great deal of time and energy in preparing their lectures. Moreover, each lecturer had prepared a series of exercises to go with the lectures. Together with PhD and MSc students from the University of Glasgow, the lecturers guided the students through the exercises during the exercise sessions.

In addition to the lecture series, there were colloquium talks later in the afternoon on most days. These were:

- *Arithmetic and Geometry of Conway Rivers* by Professor Alexander Veselov
- *Chasing the Dragon: Tidal Bores in the UK and Elsewhere; Quantum and Hawking Radiation Analogies* by Sir Michael Berry
- *Faster than Fourier (pre)revisited: Vorticulture, Noise, Fractals, Escape...* by Sir Michael Berry
- *Unexpected Connections...* by Dr Tom Leinster
- *A Tour of the Mandelbrot Set* by Dr Holly Krieger
- *The Birch and Swinnerton-Dyer Conjecture: From the Ancient Greeks to a 1 Million Dollar Problem of the 21st Century* by Dr Alex Bartel
- *Mathematics, Magic and the Electric Guitar* by Dr David Acheson
- *Random Games with Finite Groups* by Professor Colva Roney-Dougal

As one can guess from the above list, these wonderful talks took the students on a journey through a huge swath of exciting mathematics, all of which was beautifully illustrated (and in one case accompanied by live music!)

On the Sunday between the two weeks of lectures, there was an organised bus trip to Balmaha on the shores of Loch Lomond. All the students braved the weather to climb the nearby Conic Hill, where there were spectacular views of the loch and nearby Ben Lomond; at least this was the case until about half way up, the summit itself was unfortunately shrouded in mist for the day. Afterwards we retired to the pub at the base of the hill for a well-deserved drink.

As organiser, by far the most enjoyable aspect of the summer school was interacting with the participants. It was energizing being surrounded by so many bright, enthusiastic and hardworking young people! They really gave it their all, and I think this ensured that they got the most out of the two weeks. It was particularly gratifying to see them very quickly bond, seeming to be old friends by the end of the first week. It was also a great experience to be in exercises classes where the students were asking questions throughout and racing to try and solve the problems. To provide motivation for the students to work on mathematics questions (as it turned out, this was not required at all!), a series of mathematical problems were set, with prizes for the best solutions. Surprisingly, around 10 of the students managed to solve all the problems.

There is a real hunger amongst undergraduate mathematics students in the UK to be exposed to research level mathematics. The LMS Undergraduate Summer School is one of the few ways in which we can effectively meet this need. Though it is, as one can imagine, a great deal of work to organize an LMS Undergraduate Summer School, it is also an extremely rewarding experience and an effective way to make a significant contribution to undergraduate mathematics in the UK. For those LMS members in a position to apply to organize a summer school, I would strongly encourage you to do so.

I sincerely hope that the summer school will have had a positive influence on each student's long-term relationship with the mathematical sciences, and that we'll see many of them studying for a PhD in the near future. I would also like to give my thanks to the lecturers and colloquium speakers for all their hard work in preparing their talks, and their enthusiasm in delivery.

Gwyn Bellamy
University of Glasgow

Report: LMS Northern Regional Meeting and Workshop

Conference participants

An LMS Northern Regional Meeting took place in the afternoon of Tuesday 28 May 2019, followed by a wine reception and dinner, and a Workshop on *Higher Dimensional Homological Algebra* was held on Wednesday 29 May 2019 at Newcastle University.

The LMS Northern Regional Meeting and the workshop brought to the UK international experts in the representation theory of finite dimensional algebras to present their work on the most recent developments and the speakers included many of the most renowned experts in the area. The workshop also gave early career mathematicians such as PhD students and postdocs an opportunity to present their work. In total there were nine speakers, of whom one was a PhD student and three were postdocs.

There were two general talks at the Northern Regional Meeting on the latest developments in the field. These talks were given by Karin Baur (University of Graz and University of Leeds) on *CM Modules for Grassmannians* and by Sibylle Schroll (University of Leicester) on *Geometric Models and Derived Invariants for Gentle Algebras*.

The workshop was focused on *Higher Dimensional Homological Algebra*. This area concerns d-cluster tilting subcategories, which are structures found at the cusp of algebra and combinatorics. They have a rich algebraic structure, investigated by homological methods, and there are many examples relating to higher dimensional structures in combinatorics, such as cyclic polytopes.

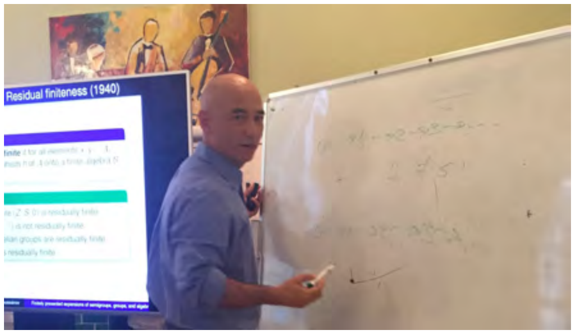
There were seven talks at the workshop, two of which were expository talks by Steffen Oppermann (Norwegian University of Science and Technology) on $(d+2)$*-angulated cluster categories* and by Hugh Thomas (Universite du Quebec a Montreal) on *Tropical Coefficient Dynamics for Higher-Dimensional Cluster Categories*. Talks focusing on recent developments of higher homological algebra were by Francesca Fedele (Newcastle University) on *A* $(d+2)$*-angulated generalization of a theorem by Bruning*, by Martin Herschend (Uppsala University) on *Wide subcategories of* d*-Cluster tilting subcategories for higher Auslander algebras*, by Karin Marie Jacobsen (NTNU) on *d-abelian Quotients of* $d+2$*-angulated Categories*, by Gustavo Jasso (Universitat Bonn) on *Generalised BGP Reflection Functors via Recollements* and by Sondre Kvamme (Université Paris-Sud) on d*-abelian Categories are* d*-cluster Tilting*.

The atmosphere during the meeting was very pleasant and convivial. The frequent coffee breaks, the wine reception and dinner followed by the Northern Regional Meeting and the lunch at the workshop provided great opportunities for discussions, informal exchanges of ideas and for establishing professional contacts. There were 31 participants, 10 of whom were PhD students.

The workshop was organised by Ilke Canakci and Peter Jorgensen. The organisers thank the LMS and the School of Mathematics, Statistics and Physics at Newcastle University for financial support. The website of the workshop can be found at tinyurl.com/y4qc2gwt.

Ilke Canakci
Newcastle University

Report: LMS Graduate Student Meeting and the General Meeting of the Society & Aitken Lecture

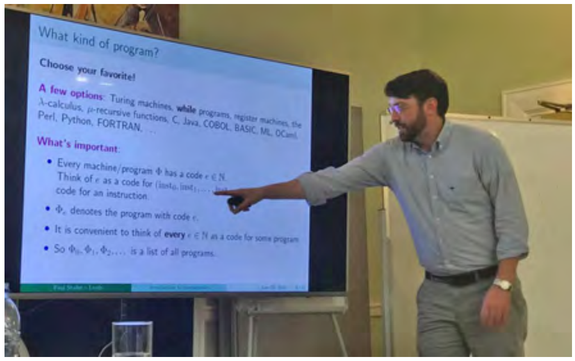


Aitken Lecturer Bakh Khoussainov

The LMS Graduate Student Meeting and the General Meeting of the Society & Aitken Lecture were held on Friday 28 June 2019, a warm and sunny summer's day, at Mary Ward House in London. The morning session was kicked off by Dr Robert D. Gray, of the University of East Anglia, whose lecture served as an introduction to the material which would be presented in lectures later in the day. Ideas surrounding finite presentations of algebraic structures, recursive enumerability, and notions such as the word problem were all introduced. This lecture was followed by a number of graduate student presentations, where the broad selection of topics ranged from climate models, modular representation theory, and polyhedral combinatorics, among others. By popular vote, two graduate speakers were then each awarded a prize: Aras Asaad, for a presentation on using homology to detect computer generated faces, and Carl-Fredrik Nyberg Brodda, for a presentation on decision problems for groups and semigroups.

After the morning's Graduate Student Meeting, the afternoon's General Meeting was opened by Professor Caroline Series. While the morning meeting had seen a respectable audience, the room was quickly filled up by new attendees at the beginning of the General Meeting. Two new Honorary Members of the Society were appointed, and the announcement of the 2019 LMS Prize Winners followed. Following tradition, several new members of the Society were then given the opportunity to sign their names in the Members' Book. To serve as an introduction to the later Aitken Lecture, Dr Paul Shafer, of the University of Leeds, presented a thorough lecture on the topic of computability, which served well to extend the material presented earlier in the day.

As could be expected on a warm day in June, the room — which at this point had become filled to the brim with mathematicians — had grown hot, to the point that a collective sigh of relief could be heard when the windows were finally opened. Refreshed by this flood of fresh air, the audience members all seemed sharpened and ready to enjoy the final event of the General Meeting; this was the Aitken Lecture, given by Professor Bakh Khoussainov of the University of Auckland. A central idea therein, for which the audience was now more than prepared as a result of the earlier lectures, was the question of in which circumstances one can (or, indeed, cannot) find finitely presentable expansions of a given algebra. This question, and the context in which it is asked, touches on a number of different topics, including immune algebras, residual finiteness of algebras, and the generalised Burnside problem for groups.



Paul Shafer

While many new definitions were planted in the minds of the audience members during the course of the presentation, there was no lack of prepared ground in which to place them owing to the earlier lectures; the end result was an excellent insight into this fascinating field of research laying at the intersection of many different areas.

Finally, to finish off the day, a wine reception followed by a conference dinner were both held at the Ambassadors Bloomsbury Hotel, rounding off a most informative and enjoyable day in June.

Carl-Fredrik Nyberg Brodda
University of East Anglia

Records of Proceedings at LMS meetings
Northern Regional Meeting at Newcastle University: 28 May 2019

The meeting was held at the Herschel Building, Newcastle University, as part of the workshop on *Higher Dimensional Homological Algebra*. Over 30 members and guests were present for all or part of the meeting. The meeting began at 3.10 pm, with the Vice-President, Professor Cathy Hobbs, in the Chair.
There were 31 members elected to Membership at this Society Meeting. Four members signed the Members' Book and were admitted to the Society by the Vice-President during the meeting. There were no Records of Proceedings signed at this meeting of the Society.
Dr Ilke Canakci introduced the first lecture, given by Dr Karin Baur (ETH Zurich/Leeds) on *CM-Modules for Grassmannians*.
Dr Ilke Canakci then introduced the second lecture, given by Dr Sibylle Schroll (Leicester) on *Geometric Models and Derived Invariants for Gentle Algebras*.
The Vice-President thanked the speakers for their talks, and further extended her warm thanks to the local organisers, Dr Ilke Canakci and Professor Peter Jorgensen, for hosting such a well-structured and interesting meeting, and additionally thanked the attendees for coming to the Northern Regional Meeting.
Dr Ilke Canakci thanked the speakers, and invited all those present to attend a wine reception held in the Penthouse of the Herschel Building. A Society Dinner was held following the wine reception in the Herschel Building.

Records of Proceedings at LMS meetings
General Meeting: 28 June 2019

The meeting was held at Mary Ward House, Tavistock Square, London. Over 60 members and visitors were present for all or part of the meeting.

The meeting began at 3.30 pm with the President, Professor Caroline Series, FRS, in the Chair. On a recommendation from Council, it was agreed to elect Professor Charles Goldie and Professor Chris Lance as scrutineers in the forthcoming Council elections. The President invited members to vote, by a show of hands, to ratify Council's recommendation. The recommendation was ratified unanimously.

The President, on Council's behalf, proposed that following people be elected to Honorary Membership of the Society: Professor Ed Witten, of the Institute for Advanced Study at Princeton University and Professor Don Zagier, of the University of Bonn. This was approved by acclaim. The President read a short version of the citations, which would be published in full in the *Bulletin of the London Mathematical Society*.

The President then introduced the General Secretary who gave a report on the review of the Charter, Statutes and By-Laws.

The President then announced the awards of the prizes for 2019:

De Morgan Medal: Professor Sir Andrew Wiles (University of Oxford); Senior Whitehead Prize: Professor Ben Green (University of Oxford); Naylor Prize & Lectureship in Applied Mathematics: Professor Nicholas Higham (University of Manchester); Whitehead Prizes: Dr Alexandr Buryak (University of Leeds), Professor David Conlon (University of Oxford), Dr Toby Cubitt (University College London), Dr Anders Hansen (University of Cambridge), Professor William Parnell (University of Manchester), and Dr Nick Sheridan (University of Edinburgh); Berwick Prize: Dr Clark Barwick (University of Edinburgh); Anne Bennett Prize: Dr Eva-Maria Graefe (Imperial College London).

Fourteen people were elected to Ordinary Membership: Dr Alla Detinko, Professor Johny Doctor, Mr Dalebe Gnandi, Dr Marina Iliopoulou, Dr David Kimsey, Dr Tomasz Lukowski, Dr Jesu Martinez Garcia, Professor James Maynard, Mr Luthais McCash, Professor Monica Musso, Dr Louis Theran, Dr Sofya Titarenko, Mr Bunonyo Wilcox and Dr Gordon Woo.

Twelve people were elected to Associate Membership: Mr Isarinade Ayodeji Felix, Mr Eduard Campillo-Funollet, Mr Carl Dawson, Mr Fabio Ferri. Dr Maciej Matuszewski, Mr Dimitris Michailidis, Dr Isaiah Odero, Mr Wasim Rehman, Mr Mark Scott, Dr Joni Ter äväinen, Mr Nicholas Williams and Dr Mehdi Yazdi.

Three people were elected to Reciprocity Membership: Mr Rizwan Kassamally, Dr Kwara Nantomah and Dr Alan Sola.

Three members signed the book and were admitted to the Society.

The President announced the dates of the next two Society Meetings to be held on 11 September in Nottingham as part of the Midlands Regional Meeting and on 21 November in Reading as part of the Joint Meeting with the Institute of Mathematics and its Applications. The President also reminded members that the date of the Annual General Meeting had moved from 15 November to 29 November.

The President announced that, to celebrate the 21st Anniversary of the Society's move into De Morgan House, there would an event on 19 October 2019.

The President further announced that the Society was seeking to recruit a new Editor in Chief for the LMS Newsletter.

The President introduced a lecture given by Dr Paul Shafer (University of Leeds) on *An Introduction to Computable Functions and Computable Structures*.

Following a break for tea, the President introduced the Aitken Lecture by Professor Bakh Khoussainov (University of Auckland) on *Semigroups, Groups, Algebras, and their Finitely Presented Expansions*.

At the end of the meeting, the President thanked both speakers for their brilliant lectures.

The President also thanked Robert Gray (University of East Anglia), who gave the main talk at the Graduate Student Meeting in the morning. Aras Assad (University of Buckingham) and Carl-Fredrik Nyberg Brodda (University of East Anglia) were also congratulated on winning the prizes for the best Graduate Student talks.

After the meeting, a reception was held at the Ambassador Bloomsbury Hotel in the Enterprise Suite, followed by a dinner at the Number Twelve restaurant in the Ambassador Bloomsbury Hotel.

The Importance of Ethics in Mathematics

MAURICE CHIODO AND TOBY CLIFTON

Mathematics is useful because we can find things to do with it. With this utility comes ethical issues relating to how mathematics impacts the world. Now, more than ever, we mathematicians need to be aware of these, as our mathematics, and our students, are changing society. In the first of a two-part series on Ethics in Mathematics, we address why, as mathematicians, we need to consider the ethics of what we do.

Mathematics and the world

We study one of the most abstract areas of human knowledge: mathematics, the pursuit of absolute truth. It has unquestionable authority. But, in some sense, absolute truths have absolutely no meaning. The statement "$2 + 3 = 5$" is an absolute truth, but what does it mean? Its meaning and utility are added later when people who understand the statement reconcile it with the physical world. It is the mathematically-trained who interpret and apply mathematics to the real world and thereby assign it meaning; through this it becomes useful.

Indeed, it is clear that mathematics is one of the most useful and refined tools ever developed. When something is useful, however, it can often also be harmful; this can be either through deliberate misuse or ignorance. The humble knife provides an illustration of the principle; in order to use such a tool responsibly, one must be made aware — often by those who first introduce you to it — of the potential dangers. If a primitive tool like a knife can be so useful and harmful at the same time, then what is mathematics capable of? Mathematics has many more applications, and by the same reasoning must also have a greater potential to do ill. As mathematicians we are seldom warned of this. Other disciplines such as law, medicine, and engineering have, for a long time, addressed the potential for harm within their field. We, as the practitioners and wielders of mathematics, need to be similarly aware, and adjust our actions accordingly. Otherwise we can, and sometimes do, cause harm with our work. But how could mathematics possibly be harmful, and what exactly might this harm be?

In this article our emphasis is on the experiences of pure mathematicians, although our arguments apply equally to applied mathematicians, statisticians and computer scientists. Many of us (although certainly not all) are motivated to study mathematics by its beauty and intrinsic interest, rather than its applications in science and industry. It's as though we are studying a form of abstract art; far from real world impact or considerations, and only fully appreciated by a small number. Despite all this, government and industry pay for our work; one suspects they don't just do this for the sake of our intellectual stimulation. If our work is completely abstract and detached — an art form, so to speak — then shouldn't we seek funding from those who fund abstract art? So what might be the value that science councils and industry see in what we do? It is not only the mathematical results that we produce, but also the mathematicians we train. Our mathematics makes a difference, and our students go out and do real things with their training. If it is the case that our work is being funded because it has perceived impact, then surely we should query and understand why we are being paid to do it.

There is already much discussion of ethics in the mathematical community. However, these discussions usually focus exclusively on issues *within* the community. These are important and many of us are already familiar with them: from improving diversity and inclusivity, to widening participation in mathematics, to addressing instances of plagiarism and publishing irregularities. These are pressing concerns. Every discipline engages with such *intrinsic* ethical issues. These, however, are not our focus in this article. Mathematics is one of the few disciplines that fails to address *extrinsic* ethical issues; those concerning how the community impacts wider society. This particularly includes ethical implications relating to the applications of mathematics and work of mathematicians. It is these extrinsic ethical issues that we are trying to raise awareness of. Our concern is not so much that mathematicians are deliberately malign, but instead that they fail to recognise these extrinsic ethical issues. Indeed, most of the mathematicians we have come across would baulk at the thought of acting unethically; the problem, instead, is that many do not recognise that mathematical work *can* have such an effect.

Some case studies

Having recognised that mathematics is useful because it can be applied, and that with these applications come extrinsic ethical issues, we now consider two concrete examples: the global financial crisis (GFC) of 2007–8 and targeted advertising.

The GFC was one of the defining events that shaped the modern global economy. Its repercussions have been felt around the world, with many suffering a decline in living standards. The causes of the GFC are complex, however, there is consensus that mathematical work played a vital role. An important factor is thought to have been the misuse of Collateralised Debt Obligations (CDOs). These saw mathematicians pool large collections of interest-bearing assets (mostly mortgages), then 'cut the pool into pieces' to form a collection of interest-bearing products. Mathematically these products had less overall risk and thus higher value than the original assets. They were traded wildly. The mathematics behind their construction is highly non-trivial, requiring stochastic calculus, differential equations, etc. Research mathematicians, beginning with the work of Black and Scholes, and later Li, derived a model and pricing formula for CDOs. Though it took a deep understanding of mathematics to derive these models, only a more superficial understanding (at the undergraduate level) was required to apply and to trade them. As a result, their users may not have fully appreciated their limitations or inner workings. Mathematics — which by itself is sure and certain — seemed to explain their value, and so most were happy. Unfortunately, some of the assumptions did not hold. For example, the model assumed there wasn't tail dependence in the default risk of underlying assets, but there was; for instance when two mortgaged houses were on the same street. In the end, the risk was not properly accounted for, and when house prices declined it led to the write-down of $700billion of CDO value from 2007 to 2008. The rest is history.

Our next example is targeted advertising. Adverts have always been placed so as to catch the eye of their desired audience. However, now that people possess portable internet-connected devices and social media accounts, it has become possible to target adverts at the individual level. Nowadays, these can be tuned to fit very specific demographics, and as such it's now possible to specify who *doesn't* see an advert. This allows advertising campaigns that are selective, that contain adverts that contradict each other, and that are impossible to externally scrutinise. In short, adverts can now be used to manipulate individual people. This becomes particularly dangerous when applied to political advertising. Using large data sets obtained through social media, it is possible to profile the political persuasions and preferences of an individual. Machine learning has become the main tool of the trade here, and it is the mathematically trained doing it [3]. These adverts can even deceive by appearing non-partisan. For instance, one can send an advert saying "Voting is important; make sure you vote" only to those who might be inclined to vote for your party. Whatever the strategy, these types of adverts are increasingly prevalent, and it is thought that such tactics influenced the 2016 US election and UK referendum on EU membership. It is we mathematicians who make all this possible. Cambridge Analytica, one of the organisations alleged to have been involved in such advertising, had a small team of no more than 100 data scientists [4], some of whom were trained mathematicians. Regardless of one's political persuasion, it is clear that this sort of work is deceptive and dangerous, and that mathematicians are enabling it. Ultimately, it is mathematicians who make up part of the teams specifying how such targeting works and carrying it out.

The impact of mathematicians

As a result of the pace and scale at which modern technology operates, through use of internet connectivity and readily-available fast computation, the consequences of the actions of mathematicians are more quickly realised and far-reaching than ever before. A mathematician in a big tech company can modify an algorithm, and then have it deployed almost immediately over a user base of possibly billions of people. Even on a smaller scale, we have seen that a small number of mathematicians, despite limited resources, can have a vast impact on the world; targeted advertising exemplifies this.

If you model a physical system, such as gravity, then your model is falsifiable. If the model does not accurately reflect the physical system, then on application it clearly fails — your rocket doesn't launch properly. You know when a model was good because the rocket makes it to the moon and back. Modelling a financial system is more difficult, as the system is affected by the application of the model. A pricing algorithm, if widely-used to buy or sell a product, influences the market for the product in question. How does a model model its own impact?

So now what happens if you are modelling the future behaviour of people by predicting something like 'How likely is a particular individual charged with a crime to reoffend with a serious offence, a non-serious offence, or not reoffend, in the next 24 months?' Furthermore, what if that is being used to determine what prosecution and sentencing mechanisms are applied to that person?[1] If you predict that a person will reoffend seriously in 24 months, and they don't (after being released or acquitted), then you might observe that. But what if they are found guilty and sentenced to 25 months, with the choice of judicial process based on your prediction? How do you test whether your prediction was correct? Now we have a serious ethical issue: we are using mathematical reasoning to make decisions about people that impact their lives, and in many of these cases we can never know whether the decisions made were desirable or appropriate. Is it right to use mathematics in such a way without careful reflection?

We now face an ethical dilemma. Do we limit ourselves to falsifiable claims, or do we allow ourselves to make claims, make decisions, and initiate actions that are unfalsifiable? We are of course entitled to do the latter, however we should then bear in mind that we have lost mathematical certainty. Furthermore, if we do this, we should broaden our perspective and training so that we can incorporate as many aspects of society as possible.

Concerns for the future

So what is on the horizon for mathematicians? Is it sufficient to simply look at the above list of cases and avoid those specific actions or industries entirely? Unfortunately not; new mathematics produces new ethical issues every day. Such a future example may lie in alternative credit scoring. This is starting to be done by new companies who lack access to standard datasets that established credit-scoring agencies have (such as financial records, bill payment history, etc.). They instead use different datasets such as social media profiles, in some cases requesting full access to social media accounts by asking for login credentials [2]. While this sounds undesirable to the point that most people will not be interested, it must be remembered that some people will be sufficiently desperate for credit that there will always be some takers. These companies scrape an applicant's social media looking for actions they perceive to reflect creditworthiness. These could include places the person visits, the hours they sleep, the 'quality' of the friends they have, and so on. This approach is unfalsifiable, lacks proper regulation, and has the potential to harm society since the extension of credit is a mechanism of social mobility. If such a process, one that is enabled by mathematically-trained people, starts having negative impact, who is accountable? Ultimately, we must live in the world that we and our students create, and we must ponder whether there is a sense in which we are partially responsible.

Do these ethical issues arise in academia?

But what about mathematicians working in academia; are any of these ethical issues relevant to them? Consider a pure mathematician, a number theorist, say. Suppose they develop an algorithm for fast factorisation. Should they publish it? If so, when, where, and how? If not, what should they do? Should they have thought about it beforehand? We have asked many mathematicians this exact question, and a typical response is "I would publish it on arXiv immediately. It's my right to publish whatever mathematical work I do." (Not all mathematicians give such a response, but many do.) When pressed on the consequences of publishing such an algorithm in that way — for instance the breaking of RSA encryption in a chaotic manner and the ensuing collapse of internet commerce and the global economy that would follow — one explained "Well, it's their fault for using RSA. It's not my problem." Of course, responsible disclosure is a complicated topic, and one that is heavily debated by security researchers. But with an example like this, ethics has crept into the world of the pure mathematics researcher in academia. If an area as abstract as number theory is not 'safe' from ethical considerations, is there any mathematical work that is? Can a pure mathematician hide from ethical issues in academia? What about a statistician, or an applied mathematician? Or do ethical questions arise for all mathematicians regardless of where we do our mathematical work?

[1] The Harm Assessment Risk Tool (HART) developed by the Durham Constabulary, which uses random forest machine learning, is used to make such predictions, and then determine if an accused criminal is to be offered the opportunity of going through the *Checkpoint* program (tinyurl.com/y4vxrd77) which is an alternative to criminal prosecution aimed at reducing re-offending.

Why management can't guide us

Some mathematicians (academic or industrial) may think that, since they are not directly involved in the application of their work, they need not consider its extrinsic ethical implications. After all, we just do the maths, and so it's 'not our problem'. This oft-held belief is generally associated with the perception that there exist people and structures above us (managers, supervisors, advisory boards, etc.) who will intervene to prevent us from doing anything that we ought not to. *We* work on the abstract problems, *they* worry about why. But can we rely on management to do this effectively? Will they vet our work, to ensure that its use is aligned with the values of society? At each stage of separation from mathematical work, some understanding of it is lost. It is difficult for a manager to understand *all* of the mathematical work we do, and its limitations when applied and used. It is the nature of management that managers will only have partial knowledge of the work being done. There would be no point in a manager reproducing the work of all of the people under them, and mathematics is such that if you don't 'do it for yourself' then there is a chance you may not fully understand it. Given this fact, there is always an onus on the individual mathematician to consider the ethical implications of what is being done. Of course, it must also be considered that managers might have other values, perhaps more aligned with the objectives of the organisation than of wider society. We should understand and anticipate this.

Some managers may go so far as to try to manipulate us. For instance, if we voice objection at what we have been asked to do, they may try to quash it with the classic argument: "if you don't do it, then someone else will". At a first glance, this seems convincing, however, it fails on two counts when referring to mathematical work. First, there are not that many mathematicians in the world. We possess a unique set of skills and abilities, and it requires years of training to produce a good mathematician, even when starting with someone who has the right interests and reflexes. Given the scarcity of mathematicians, this argument fails in practice. Moreover, as mathematicians we understand its contrapositive; the original statement is equivalent to "If no-one else does it, then you will". This is, of course, absurd. The argument has as the implicit underlying assumption that the task being requested will definitely be completed. If no-one else builds me a nuclear bomb, then will you? What we should really be considering here is the argument "If you don't do it, then someone else *might*". True, someone else might, but they may not be easy to find, or even exist at all. Now the power of meaningful objection has returned to the mathematician. Whether you choose to take the pragmatic perspective that there are not many mathematicians, or the logical perspective of the contrapositive, your objection means something. Some mathematicians take this idea even further, and make a conscious decision to take a seat at the table of power, effecting positive change from the managerial level. This happens in various areas: in academia, in industry, and even in politics. This is discussed in more detail in [1] as 'the third level of ethical engagement'.

Why the law can't guide us

The problem extends beyond management. We may think that the law provides a clear description of what is and is not acceptable to society, and thereby presumably what is and is not ethical. However, this misses the point for several reasons. Firstly, the law is not an axiomatised system; it is interpreted by courts rather than by machines. This is a type of system with whose details mathematicians are generally not familiar. Furthermore, there is the problem that the law will always lag behind technological development; we cannot expect lawmakers to have done our mathematics before we do it ourselves. Additionally, the processes by which laws are made are (deliberately) slow, requiring public consultation, votes, and implementation periods. Consider the case of the General Data Protection Regulation (GDPR). It started to be written in 2011, only came in to effect in 2018, and is thought by many to be already out of date. Finally, it can be the case that lawmakers lack a full understanding of the fine details of the subject at hand. For example, a member of the UK Science and Technology Select Committee, Stephen Metcalfe, declared at a public outreach event that "one solution to algorithmic bias is the use of algorithms to check algorithms, and the use of algorithms to check training data". Ultimately, the law is not there to serve as moral advice; there are plenty of immoral things one can do that do not break any laws. As such, it is not well-suited as a source of ethical advice.

Thus, if we can't rely on management and we can't rely on lawmakers and regulators, then who can we rely on? The answer is as obvious as it is difficult to admit: ourselves. The only way mathematicians can try to prevent their work from being used to do harm is if they think about it themselves. No one else can, so we must.

A growing awareness of Ethics in Mathematics

Awareness that mathematicians need to consider extrinsic ethical issues is building in the community. In 2018 the head of mathematics at Oxford, Professor Mike Giles, commented at a panel discussion event: "Cambridge Analytica is interesting from one point of view in that, if you'd asked me 20 years ago whether mathematicians at the PhD level needed to be exposed to ideas of ethics, I would have said 'Clearly, that is irrelevant to mathematicians'. Now I really think that this is something we have to think about. In the same way that engineers have courses looking at 'What it means to be a professional engineer', and 'Ethics, and your responsibilities as an engineer', I think that is something that we have to think about as mathematicians now." Moreover, arxiv.org is currently revising the description of their mathematics tag *History and Overview* to include "Ethics in Mathematics" as a sub-category.

As part of their formal training, few mathematicians have ever been told about extrinsic ethics before. Previous generations of mathematicians have evaded this crucial point, and in the process have possibly let society down. It rests on the current and upcoming generations to pick up this idea, before it's too late. Mathematicians always take a generation or more to accept a new and fundamental idea about the nature of their subject; debates about the admissibility of zero as a number provide such an example. We're at a similar juncture again. Now some say "Surely there's no use in considering ethical issues in mathematics", but by the time our students are professors and industry leaders, they may well be saying "*Of course* we should be considering ethical issues in mathematics!" But why hasn't the mathematical community taken this on board already? Why wasn't this done 100 years ago, by the likes of Gödel and Russell? Two reasons come to mind. Firstly, the dangers were less proximate, since much of today's technology simply did not exist. Secondly, every mathematics undergraduate was already exposed to philosophy, as it formed part of every university education. Thus, exposure to Ethics in Mathematics, in its own right, was less urgently needed.

So if Ethics in Mathematics has become so important to mathematicians, then how might we teach it to them in a relevant and useful way, without foisting an entire philosophy degree upon them? Disciplines such as law, medicine, and engineering have long taught their undergraduate students about extrinsic ethics in their respective fields. In the next issue of the *Newsletter*, we'll further explore why such teaching has not yet occurred in mathematics, and outline how one might go about giving such directed teaching of Ethics in Mathematics.

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FURTHER READING

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Polynomial Factorisation Using Drinfeld Modules

ANAND KUMAR NARAYANAN

The arithmetic of Drinfeld modules has recently yielded novel algorithms for factoring polynomials over finite fields; a computational problem with applications to digital communication, cryptography and complexity theory. We offer a gentle invitation to these developments, assuming no prior knowledge of Drinfeld modules.

Factoring polynomials modulo a prime

Let $\mathbb{F}_q$ denote the finite field of integers modulo an odd prime number q. Polynomials $\mathbb{F}_q[x]$ over $\mathbb{F}_q$ share striking analogies with integers, yet we begin with an algorithmic distinction. While factoring integers remains a notoriously difficult problem, factoring polynomials in $\mathbb{F}_q[x]$ is long known to be easy, at least with access to randomness.

Polynomial factorisation over finite fields is not a mere curiosity, but has many applications. In number theory, finite fields arise as residue fields of global fields such as number fields. While determining the splitting of a prime in a number field, one factors a polynomial defining the number field modulo the prime. Several instances of polynomial factorisation appear while factoring integers using quadratic/number field sieve algorithms or while performing index calculus to compute discrete logarithms, both foundational problems in analysing the security of cryptographic systems. In digital communication, polynomial factorisation aids the construction of certain error correcting codes (BCH and cyclic redundancy codes), structures vital to reliable transmission of information in the presence of noise.

Let us recount a simple polynomial factorisation algorithm. We are given a monic $\mathfrak{f}(x) \in \mathbb{F}_q[x]$ of degree n whose factorisation into irreducible polynomials $\mathfrak{f}(x) = \prod_{i=1}^{m} \mathfrak{p}_i(x)$ is sought. Assume $\mathfrak{f}(x)$ is square free, that is the $\mathfrak{p}_i(x)$ are distinct. This is without loss of generality for there are algorithms to rapidly reduce to this special case. It takes $n \log_2 q$ bits to write down $\mathfrak{f}(x)$ and we seek algorithms that run in time polynomial in n and $\log q$. Berlekamp was the first to show there is a randomized polynomial time algorithm, but we follow a different two step process.

Distinct degree factorisation

The first step, known as distinct degree factorisation, decomposes $\mathfrak{f}(x)$ into factors each of which is a product of irreducible polynomials of the same degree. By Fermat's little theorem, $x^q - x = \prod_{a \in \mathbb{F}_q} x - a$. To extract the product of linear factors of $\mathfrak{f}(x)$, take the greatest common divisor $\gcd(x^q - x, \mathfrak{f}(x))$. To extract products of degree two factors, degree three factors and so on iteratively, look to the succinct expression

$$x^{q^d} - x = \prod_{\mathfrak{p}:\deg(\mathfrak{p})|d} \mathfrak{p}(x)$$

for the product of monic irreducible polynomials $\mathfrak{p}(x)$ of degree dividing d. At the d^{th} iteration, with smaller degree factors already removed, $\gcd(x^{q^d} - x, \mathfrak{f}(x))$ yields the product of degree d irreducible factors.

Care in handling x^{q^d} is required for its degree is exponential in $n \log q$. All we need is $x^{q^d} \mod \mathfrak{f}(x)$, easily accomplished by a sequence of q^{th} powers modulo $\mathfrak{f}(x)$, each performed by repeated squaring. Better still, $x^{q^d} \mod \mathfrak{f}(x)$ can be rapidly computed with a fast algorithm to compose two polynomials modulo $\mathfrak{f}(x)$ in concert with q^{th} powers. Kaltofen and Shoup devised an ingenious improvement over this naive iteration resulting, in a significant speed up. The Kaltofen–Shoup algorithm implemented using the modular composition algorithm of Kedlaya–Umans performs distinct degree factorisation with run time exponent $3/2$ in the degree n and 2 in $\log q$.

Equal degree factorisation

Distinct degree factorization leaves us with the problem of factoring polynomials all of whose irreducible factors are of the same known degree d. All known algorithms for this task with polynomial runtime in $\log q$ are randomized. Even for the simplest case of factoring a quadratic polynomial into two linear factors, no unconditional deterministic polynomial time algorithms are known. It is closely related to the problem of finding a quadratic nonresidue modulo a given large prime.

The following randomized algorithm can be traced backed to ideas of Gauss and Legendre. For a uniformly random $\mathfrak{a}(x)$ of degree less than n,

$$\gcd\left(\mathfrak{a}(x)^{\frac{q^d-1}{2}}-1\,,\ \mathfrak{f}(x)\right)$$

gives a random factorisation of degree d irreducible factors. This follows since raising $\mathfrak{a}(x)$ to the $(q^d-1)/2$-th power modulo a degree d irreducible polynomial results in either 1 or -1, depending on whether $\mathfrak{a}(x)$ reduces to a quadratic residue or not. Remarkably, this computation can be performed with run time exponent 1 in the degree n using an algorithm of von zur Gathen and Shoup implemented with the aforementioned Kedlaya–Umans modular composition.

In summary, the best known polynomial factorisation algorithms have run time exponent $3/2$ in the degree with the bottleneck being distinct degree factorisation. To lower this exponent is an outstanding problem. In fact, to lower this exponent, it suffices for there to be an algorithm that merely estimates the degree of some irreducible factor.

Drinfeld modules and polynomial factorisation

The use of Drinfeld modules to factor polynomials over finite fields originated with Panchishkin and Potemine [3]. Drawing inspiration from Lenstra's elliptic curve integer factorization, they recast the role of the group of rational points on random elliptic curves modulo primes with random finite Drinfeld modules.

We describe three Drinfeld module based algorithms for polynomial factorisation. The first two were devised in [2] and the third in [1]. The first estimates factor degrees using Euler–Poincaré characteristics in hopes of speeding up distinct degree factorisation. The second is a Drinfeld analogue of Lenstra's algorithm, closely related to the aforementioned algorithm of Panchishkin and Potemine [3]. Our exposition begins with a short account of finite Drinfeld modules followed by Euler–Poincaré the characteristic and Frobenius distributions, important ingredients in the first two algorithms. The third algorithm involves Drinfeld modules with complex multiplication with an analogue of Deligne's congruence playing a vital role. It is also the fastest of the three algorithms, with runtime complexity matching the best known algorithms, in theory and practice. The hope is, the rich arithmetic of Drinfeld modules will inform new algorithms to beat the $3/2$ exponent barrier.

Finite Drinfeld modules

Drinfeld introduced the modules bearing his name as an analogue of elliptic curve complex multiplication theory. He in fact called them elliptic modules. Drinfeld modules and their generalisations have played a crucial role in the class field theory of function fields and in proving the global Langlands conjecture over function fields for GL_n. We settle for a concrete simple notion of Drinfeld modules sufficient for our context. Throw the q^{th} power Frobenius σ into $\mathbb{F}_q[x]$ resulting in $\mathbb{F}_q(x)\langle\sigma\rangle$, the skew polynomial ring with the commutation rule $\sigma\,\mathfrak{u}(x) = \mathfrak{u}\,(x)^q\sigma$, for all $\mathfrak{u}(x) \in \mathbb{F}_q[x]$. A rank-2 Drinfeld module over $\mathbb{F}_q(x)$ is (the $\mathbb{F}_q[x]$ module structure on the additive group scheme over $\mathbb{F}_q(x)$ given by) a ring homomorphism

$$\begin{aligned}\phi : \mathbb{F}_q[x] &\longrightarrow \mathbb{F}_q(x)\langle\sigma\rangle\\ x &\longmapsto x + \mathfrak{g}_\phi(x)\,\sigma + \Delta_\phi(x)\,\sigma^2\end{aligned}$$

for some $\mathfrak{g}_\phi(x) \in \mathbb{F}_q[x]$ and nonzero $\Delta_\phi(x) \in \mathbb{F}_q[x]$.

To better understand the map, it is instructive to compute by hand to where x^2, x^3 and so on, get mapped. By design, $\mathfrak{b}(x)$ maps to a polynomial in σ with constant term $\mathfrak{b}(x)$,

$$\mathfrak{b}(x) \longmapsto \phi_{\mathfrak{b}} := \mathfrak{b}(x) + \sum_{i=1}^{2\deg(\mathfrak{b})} \phi_{\mathfrak{b},i}(x)\sigma^i.$$

Consider an $\mathbb{F}_q[x]$ algebra M. In our algorithms to factor $\mathfrak{f}(x)$, M will often turn out to be $M = \mathbb{F}_q[x]/(\mathfrak{f}(x))$. One way to make an $\mathbb{F}_q[x]$ algebra M into an $\mathbb{F}_q[x]$ module is to retain the addition and scalar multiplication but simply forget the multiplication. The Drinfeld module ϕ endows a new $\mathbb{F}_q[x]$ module structure to M by twisting the scalar multiplication. For $\mathfrak{b}(x) \in \mathbb{F}_q[x]$ and $a \in M$, define the scalar multiplication

$$\mathfrak{b}(x) \star a := \phi_{\mathfrak{b}}(a) = \mathfrak{b}(x)a + \sum_{i=1}^{2\deg(\mathfrak{b})} \phi_{\mathfrak{b},i}(x)a^{q^i},$$

where the arithmetic on the right is performed in the $\mathbb{F}_q[x]$ algebra M. Let $\phi(M)$ denote the new $\mathbb{F}_q[x]$ module structure thus endowed to M.

Euler–Poincaré characterisitic

Cardinality is an integer valued measure of the size of a finite abelian group (equivalently, a finite $\mathbb{Z}$-module). A convoluted definition is to assign as the cardinality of a cyclic group of prime order the corresponding prime, and for cardinality of finite abelian groups that sit in an exact sequence to be multiplicative. The

Euler–Poincaré characteristic χ is an $\mathbb{F}_q[x]$-valued cardinality measure of a finite $\mathbb{F}_q[x]$ module defined completely analogously. For a finite $\mathbb{F}_q[x]$ module A, $\chi(A) \in \mathbb{F}_q[x]$ is the monic polynomial such that:

- If $A \cong \mathbb{F}_q[x]/(\mathfrak{p}(x))$ for a monic irreducible $\mathfrak{p}(x)$, then $\chi(A) = \mathfrak{p}(x)$.
- If $0 \to A_1 \to A \to A_2 \to 0$ is exact, then $\chi(A) = \chi(A_1)\chi(A_2)$.

For the $\mathbb{F}_q[x]$ module $\phi(\mathbb{F}_q[x]/(\mathfrak{f}(x)))$ featuring in our algorithms, the Euler–Poincaré characteristic $\chi(\phi(\mathbb{F}_q[x]/(\mathfrak{f}(x))))$ has a simple linear algebraic interpretation: the characteristic polynomial of the linear map ϕ_x on $\mathbb{F}_q[x]/(\mathfrak{f}(x))$. In particular, it is a degree n polynomial that can be computed efficiently.

Frobenius distribution of Drinfeld modules

Let us put our newly defined $\mathbb{F}_q[x]$ modules and cardinality measure χ to use. Take an elliptic curve E over the rational numbers and reduce it at a prime p. The $\mathbb{F}_p$-rational points $E(\mathbb{F}_p)$ famously form a finite abelian group with cardinality $p+1$ up to an error determined by the Frobenius trace $t_{E,p}$. The Hasse bound, considered the Riemann hypothesis for elliptic curves over finite fields, asserts that $|t_{E,p}| \leq 2\sqrt{p}$. Thereby,

$$\#(E(\mathbb{Z}/(p))) = p + 1 - \underbrace{t_{E,p}}_{-2\sqrt{p}\leq\ \leq 2\sqrt{p}}.$$

Gekeler established the following Drinfeld module analogue. Take a Drinfeld module ϕ, a monic irreducible polynomial $\mathfrak{p}(x)$, and consider the resulting $\mathbb{F}_q[x]$ module $\phi(\mathbb{F}_q[x]/(\mathfrak{p}(x)))$. Its Euler–Poincaré characteristic equals

$$\chi\left(\phi\left(\mathbb{F}_q[x]\Big/(\mathfrak{p}(x))\right)\right) = \mathfrak{p}(x) + \underbrace{t_{\phi,\mathfrak{p}}(x)}_{\leq \deg(\mathfrak{p})/2},$$

which is $\mathfrak{p}(x)$ plus an error determined by the Frobenius trace $t_{\phi,\mathfrak{p}}(x)$ of degree at most half that of $\mathfrak{p}(x)$. The analogy with the Hasse bound is striking. The error in each case takes roughly half the number of bits as the estimate.

Factor degree by Euler–Poincaré characteristic

Gekeler's bound concerns Drinfeld modules ϕ at an irreducible $\mathfrak{p}(x)$. What happens at $\mathfrak{f}(x) = \prod_i \mathfrak{p}_i(x)$, our polynomial to factor? The multiplicativity of the Euler–Poincaré characteristic implies

$$\begin{aligned}\chi\left(\phi\left(\mathbb{F}_q[x]\Big/(\mathfrak{f}(x))\right)\right) &= \prod_i \chi\left(\phi\left(\mathbb{F}_q[x]\Big/(\mathfrak{p}_i(x))\right)\right)\\ &= \prod_i \left(\mathfrak{p}_i(x) + \mathfrak{t}_{\phi,\mathfrak{p}_i}(x)\right) = \mathfrak{f}(x) + \mathfrak{t}_{\phi,\mathfrak{f}}(x),\end{aligned}$$

for some $\mathfrak{t}_{\phi,\mathfrak{f}}(x)$ of degree at most $s_f/2$, where $s_\mathfrak{f}$ denotes the degree of the smallest degree factor of $\mathfrak{f}(x)$. Thus, we have an extension of Gekeler's bound to reducible polynomials

$$\chi\left(\phi\left(\mathbb{F}_q[x]\Big/(\mathfrak{f}(x))\right)\right) = \mathfrak{f}(x) + \underbrace{\mathfrak{t}_{\phi,\mathfrak{f}}(x)}_{\leq s_f/2},$$

implying $\mathfrak{f}(x)$ and $\chi(\phi(\mathbb{F}_q[x]/(\mathfrak{f}(x))))$ agree at the high degree coefficients! The number of agreements tells us information about the smallest factor degree.

For a randomly chosen ϕ, $\mathfrak{t}_{\phi,\mathfrak{f}}(x)$ likely has degree exactly $\lfloor s_f/2 \rfloor$ (with probability at least $1/4$). The number of agreements does not merely bound but determines the degree of the smallest factor. To claim this probability, one needs to prove for a randomly chosen ϕ, the Frobenius traces corresponding to the irreducibles of smallest degree do not conspire yielding cancellations. To this end, we seek equidistribution formulae for the Frobenius traces. Analogous to elliptic curves, there is a correspondence between the number of isomorphism classes of Drinfeld modules with a given trace and Gauss class numbers in certain imaginary quadratic orders. The latter can be computed using analytic class number formulae.

An algorithm to estimate the degree of the smallest degree factor of a given $\mathfrak{f}(x)$ is now apparent. Pick a Drinfeld module ϕ (by choosing $\mathfrak{g}_\phi, \Delta_\phi$ at random of degree less than n). Compute the Euler–Poincaré characteristic $\chi(\phi(\mathbb{F}_q[x]/(\mathfrak{f}(x))))$ and count the number of high degree coefficients it agrees in with $\mathfrak{f}(x)$.

Drinfeld module analogue of Lenstra's algorithm

It is instructive to begin with Lenstra's elliptic curve integer factorisation algorithm before seeing its Drinfeld module incarnation. Pollard designed his *p-1* algorithm to factor an integer that has a prime factor modulo which the multiplicative group has smooth order. But this smoothness condition is rarely met. Lenstra recast the role of the multiplicative group with the additive group associated with a random elliptic curve. If the integer has a prime factor modulo which the randomly chosen elliptic curve has smooth order, the algorithm succeeds in extracting that factor. For a random elliptic curve, this smoothness condition is met with a probability depending sub-exponentially on the size of the smallest prime factor. Consequently, it is among the most popular algorithms for integer factorisation, particularly as an initial step to extract small factors.

Pollard's $p-1$ algorithm

Fix a positive integer B as the smoothness bound and denote by m, the product of all prime powers at most B. Given an N to factor, choose a positive integer $a < N$ at random. Assume a is prime to N for otherwise $\gcd(a, N)$ is a nontrivial factor of N. If N has a prime factor p with every prime power factor of $p-1$ at most B,

$$a^m - 1 = (a^{p-1})^{m/(p-1)} - 1 \cong 0 \mod p \implies p \mid a^m - 1$$

and $\gcd(a^m - 1, N)$ is likely a nontrivial factor of N. The running time is exponential in the size of B. For typical N, B needs to be as big as the smallest factor of N and thus the running time is typically exponential in the size of the smallest factor of N.

Lenstra's algorithm

Lenstra's elliptic curve factorization algorithm factors every N in (heuristic) expected time sub-exponential in the size of the smallest factor p of N. A key insight of Lenstra was to substitute the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ in Pollard's $p-1$ algorithm with the group $E(\mathbb{F}_p)$ of $\mathbb{F}_p$ rational points of a random elliptic curve E over $\mathbb{F}_p$. The running time depends on the smoothness of the group order $|E(\mathbb{F}_p)|$ for a randomly chosen E. The Hasse bound guarantees that $||E(\mathbb{F}_p)| - (p+1)| \le 2\sqrt{p}$ and Lenstra proved that his algorithm runs in expected time sub-exponential in the size of p assuming a heuristic on the probability that a random integer in the interval $[p+1-2\sqrt{p}, p+1+2\sqrt{p}]$ is smooth.

Drinfeld module analogue

In the ensuing Drinfeld module version, random elliptic curves will be recast with random Drinfeld modules to factor polynomials. As before, let $\mathfrak{f}(x) \in \mathbb{F}_q[x]$ denote the polynomial to factor. Pick a Drinfeld module ϕ at random and a random element α in the $\mathbb{F}_q[x]$ module $\phi(\mathbb{F}_q[x]/(\mathfrak{f}(x)))$. The order, $\mathrm{Ord}(\alpha)$, of α is the smallest degree polynomial $\mathfrak{b}(x) \in \mathbb{F}_q(x)$ that annihilates α, that is $\phi_{\mathfrak{b}}(\alpha) = 0$. Extract and divide away the linear factors of $\mathrm{Ord}(\alpha)$,

$$\mathfrak{r}(x) := \mathrm{Ord}(\alpha)/\gcd(\mathrm{Ord}(\alpha), x^q - x)$$

and apply the Drinfeld action $\phi_{\mathfrak{r}}(\alpha)$ on α. It is likely that $\phi_{\mathfrak{r}}(a)$ is a zero at some but not all factors $\mathfrak{p}_i(x)$ of $\mathfrak{f}(x)$ and $\gcd(\phi_{\mathfrak{r}}(a), \mathfrak{f})$ gives a random factorisation.

A brief outline of why this is the case follows. As with groups, the order of an element divides the cardinality of an $\mathbb{F}_q[x]$ module. That is, $\mathrm{Ord}(\alpha)$ divides the Euler–Poincaré characteristic of $\phi(\mathbb{F}_q[x]/(\mathfrak{f}(x)))$. In fact, with high probability, $\mathrm{Ord}(\alpha)$ equals the Euler–Poincaré characteristic

$$\mathrm{Ord}(\alpha) = \prod_i \left(\mathfrak{p}_i(x) + \mathfrak{t}_{\phi,\mathfrak{p}_i}(x)\right).$$

If the factors on the right have/don't have a linear factor independently and roughly uniformly at random, then the algorithm yields a random factorisation. This is indeed the case!

The factors on the right lie in the short intervals I_i centred at $\mathfrak{p}_i(x)$ with interval degree bounded by $\deg(p_i)/2$. The Frobenius trace distribution assures a certain semi-circular equidistribution of $\mathfrak{p}_i(x) + \mathfrak{t}_{\phi,\mathfrak{p}_i}(x)$ in the short interval I_i. Remarkably, unlike over integers, factorisation patterns in the short intervals I_i are unconditionally proven to be random enough. In summary, for random ϕ, the factorisation patterns of each $\mathfrak{p}_i(x) + \mathfrak{t}_{\phi,\mathfrak{p}_i}(x)$ is like that of a random degree $\deg(\mathfrak{p}_i)$ polynomial. Further, they are independent.

The computation of $\mathrm{Ord}(\alpha)$ dominates the runtime and can be performed efficiently through linear algebra. This is in stark contrast to integers, where finding the order of an element in the multiplicative group modulo a composite appears hard.

Drinfeld modules with complex multiplication:

Our last algorithm is distinguished in that it samples from Drinfeld modules with complex multiplication. A Drinfeld module ϕ has complex multiplication if its endomorphism ring

$$\mathrm{End}_{\mathbb{F}_q(x)}(\phi) \otimes_{\mathbb{F}_q[x]} (\mathbb{F}_q(x))$$

is isomorphic to a quadratic extension of $\mathbb{F}_q(x)$. A typical Drinfeld module has an endomorphism ring only isomorphic to $\mathbb{F}_q(x)$. Complex multiplication is the rare case where the Drinfeld module is more symmetric than typical. As with reducing elliptic curves over rational numbers at primes, Drinfeld modules can be reduced at irreducible polynomials. The reduction is deemed supersingular if the endomorphism ring is noncommutative, and ordinary otherwise. Every Drinfeld module with complex multiplication has the remarkable feature that the density of irreducible polynomials where it is supersingular is roughly half.

To factor a given polynomial $\mathfrak{f}(x)$, the strategy is to choose a random Drinfeld module with complex multiplication. Using explicit formulae, construct a Drinfeld module ϕ with complex multiplication by the quadratic extension $\mathbb{F}_q(x)(\sqrt{x-c})$ with $c \in \mathbb{F}_q$ chosen at random. Then attempt to separate out the irreducible factors of $\mathfrak{f}(x)$ where ϕ is supersingular from the ordinary. This likely results in a random factorisation of $\mathfrak{f}(x)$, which is recursively factored to obtain the complete factorisation.

To separate the supersingular factors, we look to the Hasse invariant, an indicator of supersingularity. The Hasse invariant $h_{\phi,\mathfrak{p}} \in \mathbb{F}_q[x]/(\mathfrak{p}(x))$ of ϕ at an irreducible $\mathfrak{p}(x)$ vanishes if and only if ϕ is supersingular at $\mathfrak{p}(x)$. For the chosen ϕ, we construct a polynomial that is a simultaneous lift of Hasse invariants at all irreducible polynomials of degree at most that of $\mathfrak{f}(x)$. The efficient construction of this lift relies critically on a Drinfeld module analogue of Deligne's congruence due to Gekeler. The common irreducible factors of this lift and $\mathfrak{f}(x)$ are precisely the irreducible factors of $\mathfrak{f}(x)$ where ϕ is supersingular. The GCD of $\mathfrak{f}(x)$ and the lift separates out the supersingular factors from the ordinary, as desired.

Hasse Invariants and Deligne's congruence:

For a Drinfeld module ϕ with defining coefficients $(\mathfrak{g}_\phi, \Delta_\phi)$, we now construct the aforementioned lift of Hasse invariants. Consider the sequence $\mathfrak{r}_{\phi,k}(x) \in \mathbb{F}_q[x]$ of polynomials indexed by k starting with $\mathfrak{r}_{\phi,0}(x) := 1$, $\mathfrak{r}_{\phi,1}(x) := \mathfrak{g}_\phi(x)$ and for $m > 1$,

$$\mathfrak{r}_{\phi,m}(x) := \left(\mathfrak{g}_\phi(x)\right)^{q^{m-1}} \mathfrak{r}_{\phi,m-1}(x) \\ -(x^{q^{m-1}}-x)\left(\Delta_\phi(x)\right)^{q^{m-2}} \mathfrak{r}_{\phi,m-2}(x).$$

Gekeler showed that $\mathfrak{r}_{\phi,m}(x)$ is the value of the normalized Eisenstein series of weight q^m-1 on ϕ and established Deligne's congruence for Drinfeld modules, which ascertains for any $\mathfrak{p}$ of degree $k \geq 1$ with $\Delta_\phi(x) \neq 0 \mod \mathfrak{p}$ that $h_{\phi,\mathfrak{p}} = \mathfrak{r}_{\phi,k}(x) \mod \mathfrak{p}(x)$.

Hence $\mathfrak{r}_{\phi,k}(x)$ is a lift to $\mathbb{F}_q[x]$ of the Hasse invariants of ϕ at not just one but all irreducible polynomials of degree k. Further, $\mathfrak{r}_{\phi,k}(x), \mathfrak{r}_{\phi,k+1}(x)$ are both zero precisely modulo the supersingular $\mathfrak{p}(x)$ of degree at most k. Since a factor of $\mathfrak{f}(x)$ is of degree at most n, take

$$\gcd(\mathfrak{r}_{\phi,n}(x), \mathfrak{r}_{\phi,n+1}(x))$$

as the Hasse invariant lift.

To show the algorithm indeed works, it remains to demonstrate that with constant probability, our random choice of ϕ with complex multiplication yielding a random factorization of $\mathfrak{f}(x)$. By complex multiplication theory and Carlitz reciprocity, this probability is identified with splitting probabilities in a certain hyperelliptic extension of $\mathbb{F}_q(x)$, and duly bounded. The overall runtime is dictated by the time taken to compute the Hasse invariant lift. An intricate algorithm for this task is devised in [1] using a fast procedure to compute its defining recurrence. Remarkably, the runtime exponent matches the best known factorisation algorithm and is comparable in practice to the fastest existing implementations. In light of this, thorough further investigation of Drinfeld module inspired polynomial factorisation is warranted!

Acknowledgements

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FURTHER READING

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Computers and Mathematics

KEVIN BUZZARD

Mathematicians currently use computers to do tedious calculations which would be unfeasible to do by hand. In the future, could they be helping us to prove theorems, or to teach students how to write proofs?

Mathematics from the future

Take a look at the following piece of computer code.

```
lemma continuous_iff_is_closed
 {f : α → β} :
 continuous f ↔ (∀s, is_closed s →
   is_closed (f ⁻¹' s)) :=
⟨assume hf s hs, hf (−s) hs,
 assume hf s, by rw [←is_closed_compl_iff,
   ←is_closed_compl_iff]; exact hf _⟩
```

This is a proof that a function is continuous if and only if the preimage of every closed set is closed. It is written in a computer language called Lean, a programming language developed by Leonardo de Moura at Microsoft Research. There is a ":=" somewhere in the middle of the code; the statement of the theorem is before this token, and it is basically human-readable. The proof of the theorem is after the token, and is incomprehensible if you do not know how to speak Lean. A computer with Lean installed would have no trouble checking that this is a valid proof, and a human mathematician who can speak Lean would vouch for the fact that the proof is just the obvious proof.

We see from this example that computers can now check proofs of simple theorems in pure mathematics, as long as you are prepared to learn how to talk to them in their own specialised language. In fact this has been the case for decades, and evidence has appeared more recently that computers can check quite complicated proofs. One observation which I find particularly striking is that even though these computer proof verification systems give us a very cool new way of doing mathematics, which also has applications to teaching and outreach (indeed I now use it for both), the number of people in mathematics departments across the UK who can use this sort of software is extremely small. Moreover, most of them are doing research in foundational areas such as category theory or type theory rather than more "mainstream" areas in pure mathematics such as analysis, topology and so on. Were software like this to be adopted by a broader class of mathematicians, we might see a future where these systems start to become useful for a broader class of researchers too.

In this article we will see an overview of why these systems exist and what they are currently capable of. They are getting better, faster, and smarter every year, and I believe that it is only a matter of time until mathematicans will be forced to sit up and take notice. Note however that computers will not be proving theorems by themselves any time soon. All of the mathematics I have done in Lean myself is mathematics which I could have done (in many cases more easily) with pen and paper. The key difference of course is that with the pen and paper method, I could have made slips, or forgotten to check everything that needs checking; Lean will make sure everything is correct before it will compile your proof. There are typecheckers written in other languages that can check Lean proofs, so the chances of errors slipping through the system nowadays are remote — certainly an order of magnitude less than the chance that a human will make a mistake. As I get older, I find I value this certainty more. Perhaps this is some kind of mid-life crisis — but Lean is free and open source software, and hence much cheaper than buying a Ferrari.

Proofs and computer programs

Most of us now rely on computer code to do all sorts of things. Bugs in code can cause chaos, allowing hackers into systems or just making stuff stop working. It is hence not surprising that software developers have made tools which can check computer programs for bugs. But what is a computer program? It is just is a sequence of statements following a precise syntax, which performs a certain task when the statements are interpreted within the framework of a particular programming language (say, Python or C++). A mathematical proof is also a sequence of statements following a precise syntax, which performs a certain task when the statements are interpreted within the logic and language of pure mathematics. This observation is more than an analogy and goes back to Kolmogorov.

Initially it was observed that certain kinds of mathematical proofs (namely, proofs using something called "constructive mathematics") really are exactly the same thing as certain kinds of computer programs. As a consequence, the tools developed by computer scientists to verify that code has no bugs can also be used to show that a proof in constructive mathematics has no gaps or errors.

However, often mathematicians do not work purely in constructive mathematics. One key difference between a proof and a program is that proofs are allowed to use non-constructive things such as the axiom of choice, which can be viewed in this optic as a function for which there is no algorithm, i.e., no program. But some of the newer computer proof checking tools (such as Lean) have been designed with mathematics in mind, and have simply added things like the axiom of choice as an axiom of the system. This means that tools like Lean can now check "normal" mathematical proofs. In short, if you can prove something using classical logic and ZFC set theory (which is what most pure mathematicians just call "mathematics") then you can, in theory, prove it in Lean.

What mathematics is formalized?

What however can we prove *in practice* in Lean or in these other systems? Which results are already available to us? Let us start at the top, and go through some of the traditional breakthrough results.

The systems *HOL Light* and *Isabelle/HOL* were used in the formally verified proof of the Kepler conjecture ("HOL" stands for Higher Order Logic). The formal verification was done by a team led by Tom Hales. Hales' 2017 talk at the Big Proof workshop at the Newton Institute is available on the INI website (see tinyurl.com/y4avuzj9) and tells the interesting story of how he ended up having to formalize the paper proof which he and Ferguson had constructed in the late 1990s. The argument involved, at some point, checking thousands and thousands of nonlinear inequalities, and the human referees had expressed concern that this part of the argument was not human-checkable. They have now all been formally verified by computer.

The system *Coq* was the system used in George Gonthier's formal proof of the four colour theorem, and Gonthier also led the team which in 2013 formalized (again in Coq) the proof of the Feit–Thompson theorem that every group of odd order is solvable. These two theorems are of different natures. The four colour theorem is another example of a theorem where the standard argument ends up having to go through thousands of tedious case checks, but this is not what the Feit–Thompson proof looks like; the latter proof is a long group-theoretic argument for which Thompson recieved a Fields Medal in 1970. The modern mathematical proof of the odd order theorem is written up in two books by Peterfalvi and Bender–Glauberman, both published in the LMS lecture note series [1, 3]. The Coq proof was created by a team of 15 people over a period of 6 years, during which time they formalized the 328 pages of research level mathematics in the two books as well as all the prerequisites. Perhaps surprisingly, over two thirds of the code base is the prerequisites for the books rather than the books themselves. The prerequisites covered basic results in the representation theory of finite groups, Galois theory, some commutative algebra, and a little number theory. Moreover, most of the people involved did not work anywhere near the full six years that the project lasted. Ultimately there were under five lines of computer code corresponding to each line of the book on average, a figure which perhaps better reflects the effort, although it should be said that writing a correct line of code in one of these languages can take a long time.

The work of Hales et al., and Gonthier et al., are hence evidence that these computer systems are now capable of handling very long and complex proofs about relatively simple objects such as finite groups or spheres.

In 2016, Ellenberg and Gijswijt established a new upper bound for the cap set problem, resolving the cap set conjecture. The cap set conjecture is a combinatorial question about the growth of the size of a certain sequence related to an n-dimensional version of the card game "Set", as n increases. The Ellenberg–Gijswijt proof was published in the *Annals of Mathematics*. In 2019 Sander Dahmen, Johannes Hölzl and Rob Lewis formalized the proof in Lean. Dahmen is a number theorist; Hölzl's background is in computer science and Lewis's is in logic. The mathematical proof was remarkable in the sense that many of the methods used were elementary, making the result a very good candidate for formalisation. The Lean proof is modern mathematics being formalized in a realistic time frame.

The next example I will speak about here is of a different nature; it is the work of Sébastien Gouëzel and Vladimir Shchur from [2]. The Morse Lemma is a result about the Hausdorff distance between a geodesic and a quasi-geodesic with the same endpoints in a Gromov-hyperbolic space. In 2013 Shchur gave a quantitative (i.e., with explict constants) proof of the lemma, which was published in the *Journal of Functional Analysis*. Gouëzel decided to formalize the proof in Isabelle/HOL, and during the process he found a serious error missed by both author and referee. A completely new argument was found by Gouëzel and Shchur to fix the problem, and Gouëzel formalized the new proof in Isabelle/HOL. The result was written up as a joint paper which was published in the *Journal of Functional Analysis* earlier this year. Again this is modern mathematics being formalized in a realistic time frame, but this time the objects are more complex.

Last year I got interested in exactly how complex the objects handled by this sort of software could be. Perfectoid spaces were discovered by Peter Scholze in 2011 and his applications of the theory to fundamental questions in the Langlands program and arithmetic geometry won him a Fields Medal in 2018. Earlier this year Patrick Massot, Johan Commelin and myself formalized the definition of a perfectoid space in Lean. The formalisation process took around ten months in total, and we wrote well over 10,000 lines of code in order to formalize this definition, involving (for example) formally proving many of the theorems in Bourbaki's "Topologie Générale". We have as yet proved nothing at all about perfectoid spaces and furthermore cannot yet produce any non-trivial examples, because we still need to formalize the general theory of discrete valuations on finite extensions of the p-adic numbers. Formalising this theory would be an ideal problem for a PhD student who wanted to learn the material in an innovative way, however it would be far less efficient in terms of time.

Undergraduate level formalized mathematics

As well as these "complex" formalisations, there are formalisions of much of the pure material which is typically taught in a UK undergraduate mathematics degree. Formalising undergraduate level mathematics is a very different ball game. It takes far less time, and can be worked on by undergraduates who are learning about the subject for the first time. All of the systems I have mentioned so far contain a lot of undergraduate-level mathematics, but one cannot easily port theorems from one system to another (the problem is analogous to porting computer programs from one language to another), so when new systems come along these libraries need to be built from scratch. Each system contains some, but not all, of the pure mathematics component of a typical UK undergraduate degree; the union of all the systems might well contain most of it nowadays, however because of the portability issue no one system has access to all of it.

I started learning Lean by formalising the questions and solutions in Imperial's first year "introduction to proof" course, a course which I was lecturing at the time. I typically got stuck several times per question, I asked hundreds of questions in the Lean chatroom on Zulip and they were answered promptly and politely, often by computer science PhD students and post-docs. When I had got the hang of things a little more, I started encouraging Imperial's undergraduates to join in, and we have made good progress. Over the last two years, mathematics undergraduates at Imperial have been formalising results which they have been taught during their degree, or learnt independently. For example the theorem of quadratic reciprocity, Sylow's theorems, the fundamental theorem of algebra, matrices, bilinear maps, the theory of localisation of rings, the sine, cosine and exponential functions, tensor products of modules and many other things have been implemented by Imperial's mathematics undergraduates in Lean. Chris Hughes and Kenny Lau in particular are two students who have formalized many results from our undergraduate curriculum. There is more to come; we are working on the Galois theory course and after that we will start on the representation theory course. One possible future project would be to turn some of these formalisation projects into some sort of formally verified book or website; imagine a book on Galois theory which was guaranteed to contain no errors, and which every proof which the reader did not follow could be "unfolded" more and more until the penny dropped.

In 2018–19 I also supervised BSc student Anca Ciobanu, who formalized the basics of group cohomology in Lean, and MSc student Ramon Mir who formalized Grothendieck's notion of a scheme in

Lean. I was very surprised to discover that schemes had not been done in any other formal system. As a result it seems that Mir's work is of a publishable standard. Nowadays we see students typing up their lecture notes in LaTeX; it would be wonderful to see, in the future, students typing up their course notes in Lean or one of these other systems.

Using Lean for teaching.

In the 2018–19 academic year, I taught Imperial's introduction to proof course again, this time using Lean live in the lectures for some of the proofs. For example, I proved that a composite of surjective functions was surjective in Lean, I proved various results about equivalence relations in Lean, and I proved Cantor's diagonal argument in Lean. I wrote the code live during the lectures, projecting my laptop onto the screen, but I also circulated "traditional" proofs in the course notes afterwards. All of this was done under the watchful eye of Athina Thoma, a specialist in mathematical education, who interviewed and surveyed the students and is writing up her conclusions for a forthcoming publication with Paola Iannone, another such specialist based in Loughborough. This approach was completely new to both me and the students, and I note with interest that their performance on the (conceptually complicated) question about binary relations on the end of year exam was done very well. Of course one has to be careful here; not all the students will want to see mathematics being done with computers, and of course these students should not be disadvantaged by such an approach. However I did think it was important that the students learnt, as early as possible, what a logical argument looked like, and how a logical argument is something which can be explicitly checked by an algorithm. This is not how mathematics is taught in schools but it seems to me to be a crucial message which an introduction to proof course in a mathematics department should convey. Patrick Massot is also using Lean in the mathematics department at Orsay, in his introduction to proof course.

On Thursday evenings during term time I run a Lean club at Imperial called the Xena Project (with associated blog at xenaproject.wordpress.com), where undergraduates learn to formalize the mathematics which they are learning in lectures, and any other mathematics of interest to them. One purpose of the project is to make knowledge of formalisation more prevalent amongst the undergraduate community, just as knowledge of LaTeX now is. Another purpose is an ongoing project to completely digitise the pure mathematics part of Imperial's undergraduate curriculum.

I have also used Lean for outreach, showing schoolkids how to prove things like $2 + 2 = 4$ from the axioms of Peano arithmetic and then raising the question of how much more mathematics can be done like this.

Using Lean to help with research?

Currently, the only thing that Lean can offer the researcher is a "certificate" that any theorem they have formalized in Lean is indeed correct. Many humans feel that they do not need this certificate. I find it interesting that more and more mathematics relies on unpublished work; talk to any pure mathematician in any big area and they will know of some theorems which rely on results in the "grey literature" (a letter or an email from X to Y) or even results which are not in the literature at all ("a forthcoming paper", sometimes announced years ago and which never came forth). Humans are good at forgetting to go back and fill in these details. Lean will not let you forget. The computer-generated red squiggly line underneath your theorem reminds you that your proof is not complete, even if your work relies on results announced by others and hence "it is not your job to complete it". During our work formalizing perfectoid spaces, it got to the point where I knew "we had done it", but until the definition compiled we knew we had not convinced Lean. Another issue is the fact that formalising the proof of a big theorem like Fermat's Last Theorem, given the state of the art, would surely take more than 100 person-years. Getting money for this would involve a grant proposal which no grant agency would fund, and which would yield a result which no mathematician would care about anyway — we all know Fermat's Last Theorem is true, because we know the experts have accepted the result.

Because Lean is offering something that pure mathematics researchers do not generally want (a fully formalized verification of their theorems) at a price which researchers generally will not pay (the time taken to learn how to use the software, and then the time taken to formalize their own proofs), in practice one has to rule this out as a viable practical application of these tools at this point in time. Things might change in the future, but rather than speculate on how good computers can or will become at understanding natural language (i.e., arXiv preprints), let me simply observe that currently they are nowhere near good enough, and instead turn to other things.

Tom Hales has suggested a completely different use of these tools, and has won a multi-million dollar Sloan grant to implement his idea, which is called FABSTRACTS (Formal Abstracts). Hales suggests that we should create a mathematical database containing not proofs but formalized *statements* of theorems, cross-referenced to the traditional literature (journal articles etc). He proposes that Lean be used for this database, and indeed that is why I chose Lean. The reason this idea is a game-changer is that if all the definitions are there, then formalising a statement of a new theorem can be *easy*. The fact that Lean can handle the definition of a perfectoid space is surely evidence that it can handle any pure mathematical definition which we can throw at it. There will also be a natural language component to the project; the current plan is that theorems will be presented in a controlled natural language, i.e., in human-readable form, understandable to a mathematician with no Lean training. One can imagine in the near future such a database being constructed, initially concentrating on the areas of speciality of the mathematicians already involved. What would be the potential uses of such a database?

The first obvious use would be search. Say a researcher is wondering what is known about a mathematical statement. They formalize the statement, look it up in the database, and get either a perfect hit (a reference to the literature) or perhaps some partial results (several references, perhaps to variants of the statement proved under certain hypotheses). If we can train PhD students and post-docs to learn enough about Lean to *state* their results, then this database could grow quickly.

The second obvious use would be AI. Already computer scientists are trying to develop tools which can actually *do* mathematics better than humans. They want to train their AI on databases — but these databases are few and far between, and several of the databases which are used in practice are databases of solutions to gigantic logic puzzles which bear no relation to modern mathematics. I believe that it is up to us as a community to explain to computer scientists *what we are actually doing*, in a format which is more useful to them than tens of thousands of arXiv preprints where we use words like "normal" and "complete" to mean five different things and where we write in the idiomatic and bizarre English language, thus adding another layer of obfuscation to the material. Formalisation solves this problem — it makes us say what we mean in a clear and coherent manner. This is what these people need to make tools for us — and we are not giving it to them. Mathematicians do not seem to have any desire to formalise a proof of Fermat's Last Theorem, however computer scientists are desperate to train their AI's on a formalised proof, to see how much further they can take things. If any mathematician reading this decides that they would be interested in taking up the challenge of formalising the statement of one of their theorems in Lean, then they can find me at the Zulip lean chat.

Computer scientists do not understand a lot of what we as mathematical researchers do, or how we do it. Let's tell them in their own language, and see what happens next. Computer scientists have developed tools which can eat mathematics, but until these tools get more self-sufficient they will need to be hand-fed. It will be very interesting to see what these tools become as they grow.

FURTHER READING

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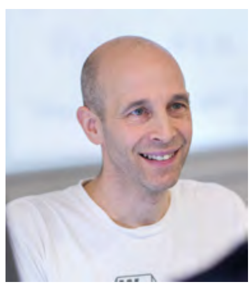

Kevin Buzzard

Kevin Buzzard is an algebraic number theorist and a professor of pure mathematics at Imperial College London. He tweets occasionally at @XenaProject.

Reciprocal Societies: Belgian Mathematical Society

The Belgian Mathematical Society (BMS) was founded on 14 March 1921 (π-day!) at the Free University of Brussels; such a choice of date is perhaps an early testimony to the celebrated Belgian sense of surrealism. Of the nine founders of the society, the best known are Théophile De Donder, Lucien Godeaux and Alfred Errera. The aim of the society was the same in those days as it is now:

> to contribute to the development and diffusion of all forms of mathematics in Belgium. [The BMS] is concerned with mathematics, pure and applied, in the broadest sense. It will try to establish a permanent link between secondary school and university.

A founding father Lucien Godeaux (engraving by J. Bonvoisin 1947, courtesy of the Royal Library of Belgium).

At the beginning, and over the course of several decades, the society organized monthly meetings during which an extremely varied array of subjects were considered. Indeed, we find lectures on mathematical physics (with e.g. De Donder), astrophysics (Lemaître), algebraic geometry (Godeaux), analysis (De la Vallée Poussin, Lepage), engineering (van den Dungen), mathematics of insurance, and secondary school mathematics (with A. Mineur, sharply remembered by generations of Belgian school children for his treatise on descriptive geometry). Amongst foreign speakers, let us just pick two curios. On top of various lectures on integration, Lebesgue gave a talk in 1925 on ruler and compass constructions. In 1922, Millikan gave a lecture, comparing his ideas on the electron with those of Planck and De Donder.

From 1947 onwards, and under the impetus of topologist Guy Hirsch, the BMS started editing its own journal called the *Bulletin de la Société Mathématique de Belgique*. The *Bulletin* underwent several transformations in the following decades and merged with the journal *Simon Stevin* in 1994. It is now addressed to a generalist audience, with a broad editorial board covering a wide range of fields. Since 2003, it is available electronically via Project Euclid and all issues of the new *Bulletin* older than five years are made available with free access.

Attendance at the monthly meetings eventually dwindled and the habit was abandoned in the seventies. In the eighties an annual meeting was launched, with great success at first. However, after a few years, attendance diminished as the pertinence of such a non-specialized national meeting found less priority in the already hectic lives of academics. Since those days, the BMS has diversified the array of mathematical events in which it takes part: co-organizing joint meetings with other National Mathematical Societies (including a joint meeting with the London Mathematical Society organized in Brussels in 1999); organizing joint meetings with the high school teachers' associations; organizing the bi-annual "Ph.D. Day" during which Ph.D. students or young researchers have the opportunity to present their work through posters and short communications; "thematic afternoons" during which the works of primed mathematicians (Fields medalists, Abel prize winners, ...) are explained broadly by experts of the different fields. The BMS also funds the Lucien Godeaux prize and is a sponsor of several mathematical initiatives aimed at younger audiences.

On π-day 2021 our Society will celebrate its 100$^{\text{th}}$ anniversary. The various mutations that it has gone through were necessary and helped the Society to serve the mathematical community over time and remain relevant to the 21$^{\text{st}}$ century Belgian mathematical community.

Yvik Swan
BMS President

Editor's note: the LMS and the BMS have a reciprocity agreement meaning members of either society may benefit from discounted membership of the other.

Microtheses and Nanotheses provide space in the Newsletter for current and recent research students to communicate their research findings with the community. We welcome submissions of micro- and nano-theses from current and recent research students. See newsletter.lms.ac.uk for preparation and submission guidance.

Microthesis: Hypergraph Saturation Irregularities

NATALIE C. BEHAGUE

A graph is F-saturated if it doesn't contain a copy of a graph F but adding any edge creates a copy of F. The maximum number of edges an F-saturated graph can have is well-studied and called a Turán number. Our topic here is the minimum number of edges an F-saturated graph can have, which behaves quite differently.

What is saturation?

A graph is F-*free* if it does not contain a copy of the graph F. Given a fixed graph F and a positive integer n, *Turán's extremal number*, $\mathrm{ex}(F, n)$, is the maximum number of edges an F-free graph G on n vertices can have (see figure 1). What if we were to ask about the minimum number of edges?

If we just replace the word 'maximum' with 'minimum' in the definition above we get a trivial function, since a graph with no edges will be F-free. But notice that in any maximal F-free graph, adding any edge creates a copy of F. This inspires the following definition: we say a graph G is F-*saturated* if it does not contain a copy of F as a subgraph but adding any edge creates a copy of F.

Now we have an equivalent definition of Turán's extremal number:

$$\mathrm{ex}(F, n) = \max\{e(G) : |G| = n \text{ and } G \text{ is } F\text{-saturated}\}$$

and if we replace max with min we get the *saturation number*:

$$\mathrm{sat}(F, n) = \min\{e(G) : |G| = n \text{ and } G \text{ is } F\text{-saturated}\}.$$

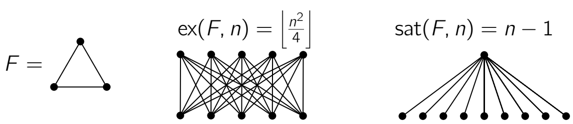

Figure 1. Extremal and saturation numbers for the triangle

For example, if F is the triangle then the maximal triangle-saturated graph has $\lfloor \frac{n^2}{4} \rfloor$ edges. In contrast, the minimal triangle-saturated graph is a star, which has only $n-1$ edges.

Asymptotics

Turán's Theorem and the Erdős–Stone Theorem tell us that for a (non-empty) graph F,

$$\mathrm{ex}(F, n) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right)\binom{n}{2},$$

where $\chi(F)$ is the chromatic number of F. In particular, $\mathrm{ex}(F, n)/\binom{n}{2}$ converges to a limit as n tends to infinity.

Can we say anything similar about the saturation number? Kászonyi and Tuza [2] proved that $\mathrm{sat}(F, n) = O(n)$. Tuza [5] went on to conjecture that for every graph F the limit $\lim_{n\to\infty} \frac{\mathrm{sat}(F,n)}{n}$ exists.

Forbidden families

The definition of saturation can be extended to families of graphs. For a family $\mathcal{F}$ of graphs (called a *forbidden family*), a graph G is $\mathcal{F}$-*saturated* if it does not contain any graph in $\mathcal{F}$ as a subgraph, but adding any edge creates a copy of some graph $F \in \mathcal{F}$ as a subgraph of G. We define the saturation number in the same way as before:

$$\mathrm{sat}(\mathcal{F}, n) = \min\{e(G) : |G| = n \text{ and } G \text{ is } \mathcal{F}\text{-saturated}\}.$$

For a family of graphs $\mathcal{F}$ we have $\mathrm{sat}(\mathcal{F}, n) = O(n)$ [2], just as we did for single graphs. However, the generalisation of Tuza's conjecture to finite families of graphs is not true, as shown by Pikhurko in [4]:

Theorem 1. *There exists a finite family $\mathcal{F}$ of graphs such that $sat(\mathcal{F}, n)/n$ does not tend to a limit as n tends to infinity.*

In Pikhurko's construction, the graphs in $\mathcal{F}$ depend on some fixed constant k. For n divisible by k, one can construct an $\mathcal{F}$-saturated graph on n vertices that uses relatively few edges. For n not divisible by k, there is no such 'nice' construction and an $\mathcal{F}$-saturated graph on n vertices is forced to contain many extra edges. Pikhurko asked whether a similar construction was possible for families of hypergraphs.

Saturation for r-graphs

An *r-graph H*, often referred to as an r-uniform hypergraph, is a pair, (V, E), of *vertices* and *edges* where the edge set E is a collection of r-element subsets of the vertex set V. Note that a 2-graph is exactly a graph.

The definitions of saturation and saturation numbers transfer immediately to the context of r-graphs — the only difference is that the edges contain r vertices rather than 2.

For a family $\mathcal{F}$ of r-graphs, it was shown by Pikhurko [3] that $\mathrm{sat}(\mathcal{F}, n) = O(n^{r-1})$ when the family contains only a finite number of graphs. This leads to the following generalisation of Tuza's conjecture to r-graphs, first posed by Pikhurko [3].

Conjecture 1. For every r-graph F the limit $\lim_{n\to\infty} \frac{\mathrm{sat}(F,n)}{n^{r-1}}$ exists.

As in the 2-graph case we can further generalise this conjecture by replacing the single r-graph F with a finite family of r-graphs $\mathcal{F}$. Pikhurko's question now becomes "Does there exist a family $\mathcal{F}$ of r-graphs such that $\mathrm{sat}(\mathcal{F}, n)/n^{r-1}$ does not converge to a limit as n tends to infinity?" I have shown in [1] that the answer to this question is yes.

Theorem 2. *For all $r \geq 2$ there exists a family $\mathcal{F}$ of r-graphs and a constant $k \in \mathbb{N}$ such that*

$$sat(\mathcal{F}, n) = \begin{cases} O(n) & \text{if } k \mid n \\ \Omega(n^{r-1}) & \text{if } k \nmid n \end{cases}$$

In particular, $\frac{sat(\mathcal{F},n)}{n^{r-1}}$ does not converge.

The proof uses a similar approach to Pikhurko's construction for graphs. The forbidden family used contains two types of r-graph: a pair of intersecting complete r-graphs; or some disjoint complete r-graphs together with a 'bridge' edge between them.

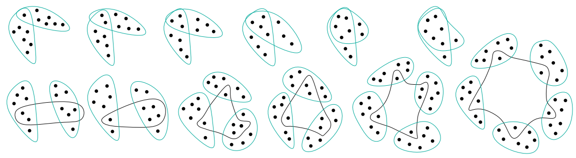

Figure 2. The family $\mathcal{F}$ of r-graphs for $r = 5$ and $k = 7$

An example can be seen in figure 2, where the vertices surrounded by a blue line represent a copy of the complete r-graph on k vertices $K_k^{(r)}$, and vertices grouped by a black line represent a bridge edge.

Tuza's conjecture itself remains open. It seems difficult to reduce the family above to a single hypergraph while keeping the non-convergent behaviour: I have only been able to reduce the family to one of size four. It could be that such a large gap in the asymptotics is not possible for a single hypergraph, and one must look at the coefficients.

FURTHER READING

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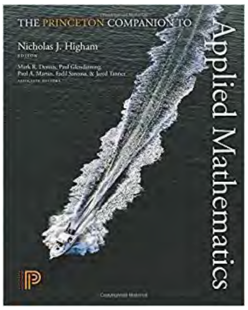
Natalie Behague

Natalie Behague is a PhD student at Queen Mary University of London under the supervision of Robert Johnson, researching problems in extremal combinatorics. She enjoys boardgames, baking, and playing tag rugby quite badly.

The Princeton Companion to Applied Mathematics

by Nichloas J. Higham (editor), Princeton University Press, 2015, £77, US$ 99.50, ISBN: 978-0691150390

Review by David I Graham



The first question that comes to mind when reviewing a book like this is "Why?" A Google search for "Applied Mathematics" finds 382 million pages related to the subject, many of which will be very detailed and with access to animations and relevant computer code as well as links to related work. The editors are, of course, well aware of this and try to answer the question themselves. In the Preface they claim that the distinguishing feature of the book is that it is "self-contained, structured reference work giving a consistent treatment of the subject".

The book is in eight parts and runs to almost 1000 pages. There is certainly a serious attempt to be self-contained, with the first part containing the longest articles in the book, defining basic language and terms from coordinate systems through calculus up to operators and stability. The second part then briefly reminds the reader of essential concepts from asymptotics to wave phenomena. These are arranged in alphabetical order, occasionally meaning that there is no natural flow from one contribution to the next. The structure becomes more free-form as we get further into the book. This is inevitable given the great variety in topics covered. With over 160 authors contributing to the articles, consistency was always going to be difficult to achieve and there is considerable variation in the later parts, especially in the level of detail and follow-up information. One contribution in Part V has a reference list of one item, namely a book written by the contributor himself. Conversely, the exemplary contribution on financial mathematics in Part V provides not only an excellent reference list but also a discussion of what to look for in each of the references. Overall, the reference lists seem to be generally good. In terms of self-containedness, some of the contributions in the later parts require prior knowledge not fully detailed in the introductory parts, though more detailed investigations utilising the reference lists should mean an interested reader would be able to fill the gaps.

Returning to the material covered, Part IV is really the heart of the book. It describes in considerable detail (over 400 pages), forty "Areas of Applied Mathematics" including straightforward choices such as various flavours of mechanics, differential equations and numerical methods but also less obvious areas such as algebraic geometry. There is a very readable contribution on "Symmetry in Applied Mathematics", which starts from the symmetries of plane figures and goes as far as symmetry breaking, with much discussion related to the various symmetries seen in Taylor–Couette fluid flow between co-rotating cylinders. The author confesses that the article "barely scratches the surface". A typical example for this part is the ten page contribution on "Fluid Dynamics", which rattles along at great pace, covering everything from 2-dimensional streamlines through flight aerodynamics up to flow instability — enough material to fill a decent course module. Similarly, the nineteen page contribution on "Numerical Linear Algebra and Matrix Analysis" is a comprehensive collection of the main results relating to matrix computations and notes some useful 'must have' references. I have to confess that some of the other contributions are rather dry for my taste, though the articles generally represent excellent starting points for further investigation — which is one of the great strengths of the book as a resource.

Parts V ("Modelling"), VI ("Example Problems") and VII ("Application Areas") together give us 64 different examples, averaging at around five pages in length. Several of these — including a contribution on "Sport" that strangely covers only sailing, rowing and swimming — are related to fluid dynamics, meaning that there is some repetition between articles. The article on "A Symmetric Framework with Many Applications" outlines a nice example of a unifying viewpoint for minimisation problems of various kinds. As a computational applied mathematician, I welcomed the contributions in Part VII related to historical developments in programming languages (and the confusion to which different conventions can lead) and the future in terms of high-performance computing.

A particularly interesting feature of the book is the last Part (VIII), which offers some "Final Perspectives". I found many of these contributions to be extremely thoughtful and useful, including advice on how to read Mathematics articles or to write articles or even general interest books. As an author of reasonably large fortran codes myself, I found the sections on "Reproducible Research" and "Experimental Mathematics" to be especially thought-provoking.

My final thoughts are that this book is an excellent resource for any mathematics departmental library. The articles cover a vast array of different application of mathematics and are generally well-written with useful reference lists. In particular, the book represents an excellent launch point for individuals such as project students looking for an area of Applied Mathematics to investigate in greater depth.

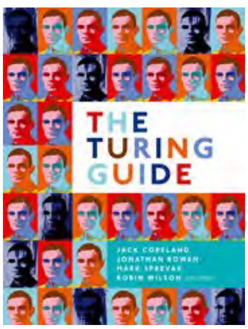
David Graham

David is a Senior Lecturer in Applied Mathematics at the University of Plymouth. His main research interests are in developing and using numerical methods for fluid dynamics. David is a keen footballer and, in decreasing order of competence he also plays guitar, banjo, ukulele and bouzouki.

The Turing Guide

by Jack Copeland, Jonathan Bowen, Mark Sprevak and Robin Wilson, Oxford University Press, 2017, £19.99, US$ 29.95, ISBN: 978-0198747833

Review by David Glass



For several decades after his tragic and untimely death in 1954 at the age of just 41, very little was known about Alan Turing's important work at Bletchley Park during the second world war. However, as the full scale of his achievements in codebreaking and the relevance of the wartime effort to the history of computing became clearer, Turing's reputation has increased dramatically. By providing a very wide-ranging and yet accessible account of Turing's life and work, *The Turing Guide* is an excellent contribution to this development and the growing literature on Turing.

The book consists of forty-two chapters divided into eight sections, with the first section providing biographical material. The first chapter provides a brief sketch of Turing's life, with a helpful timeline of key events, while in the second chapter, entitled 'The man with the terrible trousers', Turing's nephew, Sir John Dermot Turing, provides a unique family perspective. The third chapter is a compilation of extracts from papers and reminiscences of the late Peter Hilton, who worked with Turing for the last 12 years of Turing's life. According to Hilton, Turing was a 'warm, friendly human being', 'an approachable and friendly genius' and he gives a sense of Turing's 'quirky and infectious sense of humour'. In addition to providing insights into Turing's early life, school-days, and working life, the first section also discusses his conviction for homosexuality, the terrible chemical treatment he had to undergo as a result, the subsequent apology from Gordon Brown in 2009 and royal pardon in 2013, as well as the circumstances of his death. As for whether Turing committed suicide, 'We shall most probably never know.'

Those interested in the history of computing and the codebreaking work at Bletchley Park will find a lot of useful material in sections two to four. In Chapter 6, Jack Copeland provides an introduction to Turing's greatest contribution: the universal Turing machine. He also addresses the origin of the concept of the stored program computer, an issue which is further explored in Chapter 20 in the context of the development of Baby, the first electronic universal stored program computer, in 1948. Chapters 7 and 37 provide very readable introductions to Turing's important work on Hilbert's famous *Entscheidungsproblem*, while Chapter 24 explores congruences between Turing's work and the earlier ideas of Charles Babbage and Ada Lovelace.

The Enigma machine is the subject of Chapter 10 with Turing's bombes (special purpose electromechanical computers used to decipher Enigma messages) the subject of Chapter 12. Both of these chapters provide helpful explanations about how the machines worked while Chapter 14 explores Tunny, the encryption machine used by Hitler later in the war. Turing's contribution, as well as that of Bill Tutte, Max Newman and Tommy Flowers, including the essential role of Flowers' Colossus computer, are all discussed. A number of people who worked as codebreakers or operated Colossus or the bombes provide fascinating accounts of their time at Bletchley Park.

Section five explores Turing's contributions to artificial intelligence and the mind. These chapters include discussions of topics such as the Turing Test, his use of heuristic search techniques in the bombe, his early work on connectionist models, child machines, his contribution to cognitive science, and his chess program. Particularly interesting is Diane Proudfoot's discussion of Turing's concept of intelligence (Chapter 28). She argues that Turing thought of intelligence as an emotional concept that is partly in the eye of the beholder and she also argues against the common view that Turing was a behaviourist. Perhaps the most surprising contribution is a chapter on the paranormal (Chapter 32), which provides context to Turing's equally surprising comments about the powers of extra-sensory perception in his famous article in *Mind* in 1950.

Sections six and seven look at Turing's contributions to biology and mathematics, with the former providing an introduction to his fascinating work on morphogenesis in which he used reaction-diffusion equations to model the development of biological structures. In Chapter 33, Margaret Boden explores the influence of Turing's work on research in self-organisation, artificial life, and structuralism in biology. Interestingly, she argues that Turing's innovative ideas were 'out of line with the biological orthodoxy' of neo-Darwinism and associates his approach with that of the structuralist D'Arcy Thompson. In Chapter 35, Bernard Richards describes how he carried out work to validate Turing's theory by showing how solutions to his equations in three dimensions could explain the shapes of marine creatures known as Radiolaria. He had arranged to meet Turing on 8 June 1954 to show him his intriguing findings, but sadly Turing died the previous day.

Chapter 36 provides a very helpful overview of Turing's work on the central limit theorem, group theory and the Riemann hypothesis, while other chapters in this section explore further topics such as Turing's work on randomness, which 'anticipated by nearly thirty years the basic ideas behind the theory of algorithmic randomness'. In Chapter 38 (and also 13), Edward Simpson, who worked as a cryptanalyst at Bletchley Park (and after whom Simpson's paradox is named), provides a very interesting and insightful explanation of Banburismus, a process for reducing the work the bombes had to do to decipher Enigma messages. He also gives a nice account of Turing's use of Bayes factors and logarithmic scoring to weigh evidence.

The final section has a very interesting chapter on whether the universe is computable. This wide-ranging chapter covers topics such as computability, Turing's thesis (or the Church–Turing thesis) and misconceptions about it, and the physical computability thesis. It ends by arguing that Turing was not committed to the idea that the universe or even the brain is computable. The final chapter highlights Turing's impressive legacy both within science and beyond.

The Turing Guide is extremely informative, highly readable, and well produced with many photographs and useful figures to aid exposition (though some colour would have been nice). The preface states the book was 'written for general readers, and Turing's scientific and mathematical concepts are explained in an accessible way'. This has been achieved with great success. However, those working in a range of fields will also benefit a lot from articles written by experts and pointers to the extended literature. In other words, it does exactly what a good guide should do. Alan Turing's legacy has grown enormously in recent years and *The Turing Guide* will serve to further enhance understanding of his remarkable achievements.

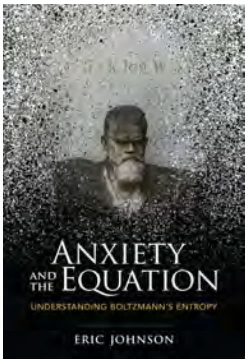
David Glass

David Glass is a senior lecturer in the School of Computing at Ulster University. He carries out research in artificial intelligence, philosophy of science, and the modelling of complex systems. He enjoys football and rugby (watching not playing) and has an interest in the relationship between science and religion.

Anxiety and the Equation: Understanding Boltzmann's Entropy

by Eric Johnson, MIT Press, 2018,
£18, US$ 22.95, ISBN: 978-0-26203-8614

Review by Andrew Whitaker



Nobody today would question that Ludwig Boltzmann was one of the greatest of physicists. It could be said that he provided the route from nineteenth-century physics, which was broadly macroscopic in nature, and where atoms were treated with considerable suspicion (at least by many physicists, less so by chemists, but even more so by a number of philosophers) and on to a physics largely based on atoms and their constituents, with theoretical ideas centred on the quantum. His main work used three considerably overlapping sets of ideas. The first was the existence of atoms and their centrality in any discussion of physics. In particular this implied the extensive use of probability arguments, and especially the idea that the macrostate, or macroscopic state of the system that we actually meet, will be one corresponding to an enormous number or microstates, which are particular states of the individual particles.

The second set of ideas is related to the famous Boltzmann distribution. Given a fixed amount of energy to be divided between a number of particles, this distribution tells us, with good probability, the number or particles having various values of energy at equilibrium.

The last idea clarified the idea of entropy. While in the nineteenth century this was looked on as a macroscopic function of the thermodynamic variables of the system, for example the entropy of a gas could be expressed as a function of its pressure and volume, for Boltzmann its definition was statistical. The relevant and famous equation is $S = k \log W$, where S is the entropy of the system, W is the multiplicity of the appropriate macrostate, and k, of course, is always known as Boltzmann's constant. The idea of this equation is undoubtedly due to Boltzmann, who discussed all the ideas at great length, though Eric Johnson points out that the actual equation and indeed 'Boltzmann's constant' itself were in fact written down by Max Planck!

The enormous importance of Boltzmann's work, so obvious today, was unfortunately not at all clear to a number of extremely influential scientists and philosophers, and disagreements were expressed increasingly aggressively from around 1890, when Boltzmann was in his mid-forties. The three chief antagonists were Ernst Mach, Planck and Ernst Zermelo, who was Planck's assistant.

As an arch-positivist, Mach was diametrically opposed to any idea of atoms, and clashes with Boltzmann were inevitable. Boltzmann hated them. In 1895 Mach became Professor of the History and Philosophy of Sciences at Vienna, where Boltzmann had become Professor of Theoretical Physics the year before. Boltzmann's unease was such that he moved to Leipzig in 1900, returning to Vienna only in 1902, after Mach's retirement due to illness.

Planck was (until his dramatic conversion) the last of the important exponents of classical thermodynamics, following in particular Rudolf Clausius and Hermann von Helmholtz. As such he detested Boltzmann's statistical approach to matters of heat flow and particularly the use of probability in physics.

Zermelo concentrated on the nature of entropy. A crucial point in classical thermodynamics was that entropy never decreases. In certain very special circumstances it might remain the same, but otherwise it increases thus providing an arrow of time. On the statistical definition, on the other hand, the entropy of very small systems would often decrease, but as the size of the system increased the probability of entropy decreasing became smaller and smaller, becoming infinitesimally small for systems of macroscopic size but never exactly zero. Zermelo plagued Boltzmann on this point

It is well known that Boltzmann tragically hanged himself in September 1906. The community of physicists has traditionally absolved itself of blame for this event, providing as evidence his various illnesses, both physical, including enormously diminished vision, asthma, chest pains and headaches, and mental, depression and in particular what Johnson characterises as anxiety. Readers of Johnson's slim book may judge for themselves to what extent physicists contributed to these illnesses. Johnson provides detailed analysis of relatively small systems demonstrating Boltzmann's main achievements. He studies the probabilities of students distributing themselves over the two halves of a lecture room, atoms over the two halves of a volume. He divides 7 units of energy over 4 atoms and calculates the resulting distribution. He studies entropy by analysing different numbers of bedbugs moving between the two halves of an auditorium. For just 4 bedbugs the entropy initially moves towards what might be called an 'equilibrium value' but clearly that terminology is hardly appropriate because it continues to fluctuate wildly about this value. For 1000 bedbugs we see similar behaviour but the fluctuations are far smaller. In both cases, though, there are clearly periods over which the entropy decreases. All Johnson's analysis is extremely helpful to anybody wishing to understand Boltzmann's achievements.

He also presents a series of vignettes of Boltzmann's life — some amusing, such as his walking down the main street accompanied by a cow, purchased to provide his children with fresh milk, but more often sad, including his disastrous manoeuvers attempting to take up a chair in Berlin without informing his current employer, the Austrian education minister, and a study of his self-destructive concern about a student lecture to be given the following day, a lecture he should have been able to give practically in his sleep. He even provides a 'happy ending' in which Boltzmann dies at peace, soothed by his daughter playing the Moonlight Sonata. Sadly, of course, this was not to be.

Ironically, as Johnson points out, it was at precisely this period that Boltzmann's main ideas were being justified. It was argued, long before Einstein's detailed analysis in 1905, that Brownian motion, the irregular motion of particles in a fluid, was due to bombardment by atoms; and more specifically, Planck's epoch-making study of black-body radiation in 1900, which introduced the quantum theory, was based soundly on Boltzmann's probabilistic arguments, which Planck had long excoriated. In repentance, as Johnson mentions, Planck twice proposed Boltzmann unsuccessfully for the Nobel Prize for Physics. For Boltzmann this was all too late. This book is an interesting and thoughtful account of Boltzmann's life and work.

Andrew Whitaker

Andrew Whitaker is Emeritus Professor of Physics at Queen's University, Belfast, and Chair of the Institute of Physics History of Physics group.

Obituaries of Members

Ernst Sondheimer: 1923 – 2019

Ernst Sondheimer, who was elected a member of the London Mathematical Society on 28 June 1974, died on 9 June 2019, aged 95.

Julian Sondheimer writes: Ernst was born in 1923 in Germany. In 1937 the family emigrated to London, and Ernst started his new life in England at University College School in Hampstead. In 1942 he went up to Trinity College Cambridge and remained there as research student and fellow until 1952. In 1946 he became a British citizen and in 1950 he married Janet Matthews, a history fellow at Girton College, Cambridge. In 1951 he was appointed as a lecturer at Imperial College and in 1955 he became a reader at Queen Mary College.

At the age of 37, in 1960, Ernst was appointed Professor of Mathematics at Westfield College in the University of London, where he remained for 22 years until his retirement in 1982. For most of this period he was Head of Department and he was instrumental in creating a very successful mathematics research department. Ernst cared deeply about his students and staff and is fondly remembered by them.

Earlier, as a young researcher in the field of mathematical physics, his main achievement was a seminal and groundbreaking paper in 1948 with G.E.H. (Harry) Reuter (also a German refugee) on the anomalous skin effect. This was in the area of the optical properties of metals and involved the difficult solution of an integro-differential equation. Later Ernst co-authored the books *Green's Functions for Solid State Physicists* and *Numbers and Infinity*.

Towards the end of his time at Westfield, in 1981, Ernst and his family were hard hit by the death of his younger brother Franz, the eminent chemist (a Fellow of the Royal Society, Franz was distinguished for his work on the synthesis of natural products). After retiring from Westfield College, Ernst revelled in his two passions of mountains and flowers, taking up the editorship of the *Alpine Journal* and devoting himself to his wonderful garden at his home in North London. At age 88, an unexpected difficulty with a medical procedure left him in a frail condition, but this did not stop him continuing to receive visitors, attend concerts and read widely. He never forgot his gratitude to University College School and Trinity College Cambridge, for their support given to a refugee from Nazi Germany in difficult times.

Gerald Gould: 1925 – 2019

Gerald Gould, who was elected a member of the London Mathematical Society on 18 March 1954, died on 23 February 2019, aged 93.

David Edmunds and Des Evans write: Gerald was born in Bermondsey, London, on 9 September 1925. From Wilson's Grammar School he won an open scholarship to study physics at Christ's College, Cambridge, an event celebrated by the school with a day off. After graduating, he served his military service working for Tube Alloys, the code name for the top security research and development programme to develop nuclear weapons. That done, Gerald enrolled for a mathematics degree in Birkbeck College, University of London. There he came under the influence of Lionel Cooper, and after graduating started a PhD under Cooper's supervision. In 1950 Cooper was appointed to the chair in what was then the University College of South Wales and Monmouthshire (now Cardiff University), and Gerald went with him; he was to spend the rest of his career in Cardiff. Apart from his mathematics, Gerald had many interests and talents. He was an accomplished musician, being highly regarded and sought after by orchestras as a timpanist, bassoonist and flautist. Bridge was another passion at which he excelled, winning a worldwide competition in 2006 involving 1,000 players on four continents. In 1954 Gerald developed TB and spent time in an isolation hospital. He claimed that to contend with the boredom, he learned to smoke, became a master of Meccano, and learned Russian. He would continue to smoke with obvious pleasure until 1984, and his knowledge of Russian had a big part to play in the contribution he made to the mathematical community; he went on to act for the LMS as English Edition Editor of *Sbornik*, and translated into English various Russian books, notably two that appeared in the Encyclopaedia of Mathematical Sciences series: *Dynamical Systems* by D.V. Anosov and *General Topology III* by A.V. Arhangel'skii.

Gerald's early work evolved from his PhD on integration spaces, his principal contribution to research being perhaps his paper *Integration Over Vector-Valued Measures* which appeared in the *Proceedings of the LMS*. This introduced a type of integral (the Gould integral) of a bounded real-valued function with respect to a finitely additive set function taking values in a Banach space. Gerald had a great appetite for new areas of mathematics and was able to absorb challenging new material with enviable ease. Much of his research was generated by problems and flaws discovered through his extensive reading. His later work on test-function spaces and various aspects of spectral theory are examples of this.

Outside his mathematics and his many interests, Gerald had a rich family life. He is survived by his beloved wife Enid, his children Nina and Ben and three grandchildren. They will miss him sorely, as will his friends and colleagues.

John Marstrand: 1926 – 2019

John Martin Marstrand, who was elected a member of the London Mathematical Society on 18 June 1953, died on 29 May 2019, aged 93.

Kenneth Falconer writes: Although he took his DPhil at Oxford, John was essentially supervised by Cambridge Professor Abram Besicovitch. His thesis contained a number of remarkable results, notably 'Marstrand's projection theorems' on the Hausdorff dimensions of orthogonal projections of subsets of the plane onto lines. Apart from generalisations by Pertti Mattila and Robert Kaufman, this attracted little attention until the late 1970s when Mandelbrot popularised and unified the notion of fractals. It was then realised that John's work was just what was needed for the study of the geometry and dimensions of fractals, and his projection theorems became the prototype for the now flourishing area of fractal geometry. Indeed, the paper based on John's thesis that appeared in the LMS *Proceedings* in 1954 was highlighted in 2015 as one of six 'landmark' papers published in the *Proceedings* during its first 150 years.

After short periods in Cambridge and Sheffield, John moved to the University of Bristol in 1958 where he was awarded a personal chair and spent the rest of his career. He was challenged by difficult questions and, although he wrote only 17 papers, virtually all of them contain deep and clever ideas. He continued to work in geometric measure theory, relating rectifiability of sets (i.e. whether they are 'curve-like' or 'surface-like') to local measure densities. But he had a broad interest across mathematical analysis. He constructed a clever counter-example to Khinchin's conjecture on the uniform distribution of certain sequences modulo 1. He was fascinated by variants of the Kakeya set and curve packing problems and he solved the problem of 'the minimal comfortable living quarters for a worm' by showing that given any plane set of zero area (Lebesgue measure) there are arbitrarily short smooth curves that cannot be mapped into the set by a similarity (or, much more generally, an analytic) transformation. Outstanding was his geometrical proof of the 'circle packing conjecture', of fundamental importance in harmonic analysis: given a plane set E of zero area, every circle with centre x intersects E in zero perimeter length, apart from a set of centres x of zero area. In a probabilistic direction, John, together with Geoffrey Grimmett, answered in 1990 the major outstanding question on site percolation on lattices in 3 or more dimensions.

John was a highly regarded and engaging lecturer. Once, lecturing on 'random walks', he jumped onto the front bench, blindfolded himself and stepped forwards and backwards at random until he eventually fell off the end of the bench! He was very careful in all his duties, often spending hours on tasks others would take a few minutes over. Once when examining a PhD thesis his attention to detail resulted in a five and a half hour viva!

Outside mathematics, John went through a range of passionate interests, from fast sports cars to extreme skiing. In his 40s he took up fell-racing and in 1982 became British over-50s fell-running champion. He regularly participated in the Ben Nevis Race and on several occasions won the over-40s or over-50s sections, with his fastest time in 1983 an incredible 1 hour 48 minutes 54 seconds.

John was a kind and considerate person and is remembered by his former colleagues as a 'loveable eccentric'. He was modest about his achievements but his determination and insight when he became absorbed by a project led to some quite remarkable work of enduring quality.

Christopher Hooley: 1928 – 2018

Christopher Hooley, who was elected a member of the London Mathematical Society on 28 June 1974, died on 13 December 2018, aged 90.

Roger Heath-Brown and Thomas and Adam Hooley write: Professor Christopher Hooley, FRS, FLSW was one of the leading analytic number theorists of his day. He won Cambridge University's Adams Prize in 1973, and the LMS Senior Berwick Prize in 1980. He was elected a Fellow of the Royal Society in 1983, and was a Founding Fellow of the Learned Society of Wales. He gave a plenary lecture at the 1983 ICM in Warsaw.

Christopher Hooley did his undergraduate degree at Corpus Christi College, Cambridge, and continued there with a PhD under Albert Ingham. His career then took him to Bristol, to Durham, and finally to Cardiff, where he served as Head of the School of Mathematics from 1988 to 1995.

He was the author of nearly 100 papers. The best known of these concerns Artin's conjecture on primitive roots. The conjecture proposes that primes with a given primitive root have a definite density, and gives a formula for what the density should be. In particular it would follow that 10, for example, is a primitive root for infinitely many primes p, so that the decimal expansion for $1/p$ would recur with period $p-1$. Hooley's work, in 1967, proved the conjecture in full, assuming the Riemann Hypothesis for a class of Dedekind zeta-functions. These would have been highly abstruse objects to analytic number theorists in those days, and even now it seems remarkable that they could have any bearing on such a concrete problem about decimal fractions.

Many of Hooley's papers made use of estimates for exponential sums originating in the work of Weil and Deligne. Thus for example, he was able to establish the Hasse Principle for non singular integral cubic forms in 9 or more variables, by using information about the associated exponential sums. Another recurrent theme was the introduction of novel sieve methods, and his Cambridge Tract "Applications of sieve methods to the theory of numbers" showcases the impressive variety of sieve tools he developed. Modern analytic number theory owes a great deal to these ideas. In particular the use of exponential sums, as pioneered by Hooley, is now ubiquitous.

Hooley's writing was well known for his love of such arcane phrases as 'the dexter side of the antepenultimate equation'. Equally however his papers were a model of clarity and accuracy. They were often couched in seemingly old-fashioned terms, both linguistically and mathematically, but they never failed to enlighten.

Christopher Hooley had many interests, most of which were pursued with vigour. From early on as a pupil at Abbotsholme School during the Second World War he developed a love not only of mathematics, but also of classical, military, and naval history. In years to come he would delight in discussing the history of western and eastern Roman empires with his colleagues from the history department at Cardiff, who would ruefully admit they were out of their comfort zone, using that standard excuse 'not my period'! — Hooley counted all European history of the two millennia A.D. as his period. This interest went along with a love of antiques and especially of collecting West Country friendly society brasses. Many days were spent scouring antique shops and following up leads to remote places in Somerset during the 1970s and 80s. His wife, Birgitta, denied pets as a child in Sweden, made sure that the family always had plenty of dogs — at one time six, sometimes cats as well: this interest led to another hobby, that of travelling to terrier shows in the summer months, from which Birgitta, Christopher, and their two sons, Thomas and Adam, often returned proudly bearing coloured rosettes for their dogs' successes. Supportive as a father he was nonetheless quick to spot and point out inaccuracies of memory in the reminiscences of his sons at his 90th birthday party, held in Bristol just four months before he died in December 2018.

Christopher's grandfather had taught carpentry at Macclesfield and on inheriting his tools Christopher set about using them to restore or completely remake a large number of period sash windows in the family home in Somerset, often at some considerable height. Once done, he turned his hand to panelling, lath and plaster, and similar skills with great success.

The practical side was also present when it came to motoring. The first family car, a 1950s' Wolseley 6/80, was kept and later given a thorough restoration in the 1980s. Christopher showed no fear in creating an entire wiring loom from scratch!

These interests and his family life provided the counterpoint to Hooley's academic career. At times unconventional, for instance researching deep into the night with a beloved whisky and cigarette alongside an eclectic mixture of steam railway magazines, Dante, Milton, and paperback thrillers, he had a ready wit and excellent sense of humour. He will be remembered with fondness by colleagues and family alike.

Noel G. Lloyd: 1946 – 2019

Professor Noel Lloyd, who was elected a member of the London Mathematical Society on 15 February 1973, died on 7 June 2019, aged 72.

Alun Morris writes: Noel sadly died after a prolonged fight against prostate cancer which he met with his usual quiet composure. This followed a full life distinguishing himself both as a mathematician and in his later years as a senior university administrator and in public life.

Noel was born in Llanelli, Carmarthenshire. After attending Llanelli Grammar School, where his father was a mathematics teacher, he entered Queens' College, Cambridge in 1965 having obtained an Entrance Scholarship. He graduated with a B.A. in 1968 with a first class in each year and an MA in 1972. On graduating, he worked for his PhD in a research area which was not too fashionable at the time, ordinary differential equations, under the supervision of Sir Peter Swinnerton-Dyer. His PhD thesis dealt with a case of van der Pol's equation not considered by J.E. Littlewood and Mary Cartwright. In addition to numerous college prizes he also received the prestigious Rayleigh Prize. He proceeded to a Research Fellowship at St John's College, Cambridge. After two years in that position he was enticed to a Lectureship in Pure Mathematics at what was then the University College of Wales, Aberystwyth (now Aberystwyth University), which he took up on 1 January 1975. At that time, it was not anticipated that the university was appointing its future Vice-Chancellor — certainly not by him. He did not need to be persuaded; he had long expressed his wish to come to his father's alma mater. His research work flourished in his chosen area of applied analysis or non-linear analysis. He established a strong school in that area supervising a number of good PhD students. Among numerous publications at that time was his book on Degree Theory in the Cambridge Tracts in Mathematics series; this book was particularly well received and became an established text and was widely consulted. During the 1980s his interests moved towards the study of the second part of Hilbert's Sixteenth Problem on the number and relative configuration of limit cycles of polynomial systems in the plane. This is acknowledged to be a very difficult area of research but it is internationally recognised that he and his research group have been responsible for substantial developments, and his loss is strongly felt amongst researchers in this area. Their progress involved innovative use of computer algebra, as well as analytic approaches.

In the 1980s Noel was a member of the Mathematics Committee of the SERC where his detailed knowledge and his meticulous fairness came to the fore. Also, in 1986–90 he was Joint Editor-in-Chief with me of the *Journal of the LMS* — a dramatic period for the journal with an overnight need to move the printing to the Cambridge University Press. He was also a long-serving Editor of the *Mathematical Proceedings of the Cambridge Philosophical Society*. He was promoted to Senior Lecturer, Reader and then in 1986 to a Personal Chair. Although at that point not heavily involved in administration, he was highly respected and was for a short time Head of the Department of Mathematics, Dean of the Faculty of Science and then Pro Vice-Chancellor, then Registrar and Secretary of the University and finally in 2004 its Vice-Chancellor. He held that position with great distinction until 2011. In 2010 he was appointed CBE for Services to Higher Education in Wales and in 2011 a Fellow of the Learned Society of Wales.

The many tributes that have appeared after his untimely death are an indication of the esteem he was held as Vice-Chancellor. He was meticulous in his preparation, quietly firm and fair. He was universally regarded as a generous and compassionate man and a person of great integrity, respected equally by his academic colleagues and fellow administrators. During his tenure as Vice-Chancellor he was Chair of Higher Education Wales and Vice-President of Universities UK. He also served on many other UK university bodies. On his retirement, his talents were in great demand. He became an independent member of the Silk Commission established by the UK government to look at the future of devolution in Wales. He also served as a lay member of the Judicial Appointments Commission. For six years he was Chair of Fair Trade Wales.

Throughout his life, his church was central to his life. He served not only his church with distinction but also his denomination and other religious causes both locally and nationally. Above all, he was a brilliant organist; as one tribute said, students would see their VC relaxing on a Sunday morning at the organ playing his favourite Bach. Noel married Dilys in 1970; next year would have been their golden wedding. They had a son Hywel and daughter Carys and two grand-daughters Sioned and Catrin.

Stuart Brian Barton: 1952 – 2019

Dr Stuart Barton MD, DPhil, who was elected a member of the London Mathematical Society on 10 November 2017, passed away in hospital in Cambridge on 12 April 2019, aged 66.

Phil Rippon writes: As a young person at Oldham Hulme Grammar School, Stuart excelled at mathematics and physics, but was advised that medicine was a more appropriate career. Accordingly, he read medicine at Balliol College, Oxford, and took a DPhil there in physiology in 1977, working much of the time in Stony Brook, New York with Professor Ira Cohen. They had been friends at Oxford and in fact published together a rather mathematical paper on transmitter release from nerve terminals (appearing in *Nature* in 1977). Stuart then began a career as a GP in the UK and also qualified as a consultant physician though he chose to remain a GP.

In 1993, Stuart changed direction and became a lecturer at the University of Liverpool, specialising in evidence based medicine, where he could bring into play his love of mathematics (especially statistics). He became a leader in that field, publishing many articles, with co-authors such as Professor Tom Walley CBE.

In 1998, Stuart again changed direction and built on his research experience to found and edit a BMJ publication *Clinical Evidence*, which assessed the clinical significance of medical treatments and drugs. In 2002 Stuart returned to GP practice in London. With Dr Elizabeth Robinson, he took over a failed GP practice in the Kings Cross area and they worked together there for 8 years providing excellent health care to some of the most disadvantaged communities in London.

Stuart retired from GP practice in 2010, in order to devote much of his time to mathematics, his first academic love. He began to study with the Open University, completing a first class degree in mathematics in 2017. His tutors speak highly of his assignment work, which was always of excellent quality and often contained material that went well beyond what the assignments officially required. After moving to live in Cambridge, he regularly attended the Open University's maths tutorials there and also weekend schools, where he greatly enjoyed meeting other people with similar interests, and is remembered by tutors as one of the brightest and liveliest students. His partner Elizabeth Robinson reports that he read widely about the subject, and was especially influenced by the life of Paul Erdős, as described in the book *The Man Who Loved Only Numbers* by Paul Hoffman. Like many mathematicians Stuart would enjoy long walks (with their dog Mabel), while thinking about mathematics problems.

Stuart then embarked on an MSc in Mathematics at the Open University, his fascination with number theory (and especially Goldbach's conjecture) leading him to study modules based on Tom Apostol's book on analytic number theory. It was at this time that he joined the LMS, confessing to me (as his LMS proposer) that his dream was to add publications in pure mathematics to his publications in medicine and biology. His tutors say that he would surely have achieved this dream in time, but sadly it could not happen. In April 2019, he suffered a brain aneurysm and died in hospital some days later. He is survived by his partner Dr Elizabeth Robinson, and by his two adult children Jonathan and Andrew.

Membership of the London Mathematical Society

The standing and usefulness of the Society depends upon the support of a strong membership, to provide the resources, expertise and participation in the running of the Society to support its many activities in publishing, grant-giving, conferences, public policy, influencing government, and mathematics education in schools. The Society's Council therefore hopes that all mathematicians on the staff of UK universities and other similar institutions will support mathematical research by joining the Society. It also very much encourages applications from mathematicians of comparable standing who are working or have worked in other occupations.

Benefits of LMS membership include access to the Verblunsky Members' Room, free online subscription to the Society's three main journals and complimentary use of the Society's Library at UCL, among other LMS member benefits (lms.ac.uk/membership/member-benefits).

If current members know of friends or colleagues who would like to join the Society, please do encourage them to complete the online application form (lms.ac.uk/membership/online-application).

Contact membership@lms.ac.uk for advice on becoming an LMS member.

LMS Meeting

LMS Midlands Regional Meeting & Workshop

University of Nottingham, 11 September 2019

Website: tinyurl.com/y4vkvhld

Lectures are aimed at a general mathematical audience. The meeting forms part of the Midlands Regional Workshop on *Zeta Functions in Number Theory and Mathematical Physics* (11–13 September 2019). Limited travel support is available: email christian.wuthrich@gmail.com for details. Visit the website for further details and to register for a place. A Society Dinner will be held after the meeting at the Victoria in Beeston, at a cost of £25.00 including drinks. To reserve a place, email Chris Wuthrich.

Words in Finite and Profinite Groups

Location: University of Lincoln
Date: 16 September 2019
Website: tinyurl.com/y6xpjmgy

This is the ninth meeting of the Research Group Functor Categories for Groups (FCG). The FCG Research Group is supported by an LMS Joint Research Groups in the UK Scheme 3 grant. Limited funding is available for PhD students.

LMS Popular Lectures

Location: University of Birmingham
Date: 19 September 2019
Website: lms.ac.uk/events/popular-lectures

The LMS Popular Lectures are free annual events, open to all, which present exciting topics in mathematics and its applications to a wide audience. The speakers for the 2019 Popular Lectures are Professor Peter Higgins (University of Essex) and Professor Alain Goriely (University of Oxford).

A Day of Lie Theory in Kent

Location: University of Kent
Date: 18 September 2019
Website: tinyurl.com/yy5m8uwn

Lie Theory is an old but pervasive subject in mathematics, due to its central role in understanding symmetry. The meeting will discuss new research directions led by Chris Bowman, Maud De Visscher, Radha Kesar and Stephen Donkin.

Novel Approaches to the Multi-Armed Bandit Problem

Location: Imperial College London
Date: 25–26 September 2019
Website: tinyurl.com/y5n95ccn

This conference will bring together researchers from mathematics, statistics and computer science working on the multi-armed bandit problem. Register for free online. Details of the submission deadline are on the webpage.

Sir Michael Atiyah: Forays into Physics

25 October 2019. Isaac Newton Institute, Cambridge

Website: tinyurl.com/yxhlkalx

This meeting is to celebrate Sir Michael's contributions to physics. Confirmed speakers are Sir Roger Penrose, Nigel Hitchin, Paul Sutcliffe, Matilde Marcolli and Nick Manton. Register to participate in the meeting. Early registration is advised due to limited number of participants. The next day is the memorial service for Sir Michael at Trinity College Chapel at 2.30.

Edge Days 2019: Fano Varieties, Cone Singularities and their Links

Location: University of Edinburgh
Date: 4–8 November 2019
Website: tinyurl.com/y3xj764z

The workshop will focus on three closely related classes of objects: Fano varieties, log-terminal singularities and links with a positive Sasakian structure. It aims to bring together experts from singularity theory, birational geometry and mathematical physics. Supported by an LMS Conference grant.

LMS Computer Science Colloquium

Location: De Morgan House, London
Date: 13 November 2019
Website: tinyurl.com/cscoll19

The theme of this event will be 'Mathematics of Security'. The event is aimed at PhD students and post-docs, although others are welcome to attend. It is free for students and £5 for others. Limited funding for travel is available. Register at tinyurl.com/cscoll19.

LMS/BCS-FACS Evening Seminar

Location: De Morgan House, London
Date: 21 November 2019
Website: tinyurl.com/yyc9oyse

Professor Marta Kwiatkowska (University of Oxford) will give a talk titled *When to Trust a Self-Driving Car...*. The seminar is free of charge; to register your interest, email Katherine Wright, Society & Research Officer: lmscomputerscience@lms.ac.uk. See the website for an abstract and speaker biography.

Young Researchers in Algebraic Number Theory

Location: University of Warwick
Date: 6–8 November
Website: tinyurl.com/YRANT2019

The conference is aimed at PhD and post-docs working in or interested in algebraic number theory and arithmetic geometry. All participants are encouraged to give talks. There will be three keynote talks, given by Jack Thorne (University of Cambridge), Lynne Walling (University of Bristol) and Sarah Zerbes (UCL).

Category Theory and its Applications Postgraduate Conference

Location: University of Leicester
Date: 18–19 November 2019
Website: tinyurl.com/y5epeut4

This conference brings together postgraduate students in category theory and its applications to present and discuss their research. It also features three talks from established researchers.

LMS–IMA Joint Meeting: Mathematics of Planet Earth

Location: University of Reading
Date: 21 November 2019
Website: tinyurl.com/yxddzjqw

This joint event is run by the LMS and the Institute of Mathematics and its Applications. The meeting is free to attend. Further details and how to register are available on the website. The lectures are aimed at a general mathematical audience.

LMS Meeting

LMS Graduate Student Meeting

29 November 2019: 9.30 am – 2.00 pm, Goodenough College, Mecklenburgh Square, London WC1N 2AB

Website: lms.ac.uk/events/meeting/agm

The meeting will include student presentations of their current work, with a prize awarded for the best talk. This meeting is intended as an introduction to the AGM later in the day. Travel grants of up to £100 are available for students who attend both the Graduate Student Meeting and the LMS AGM. The AGM will be followed by a wine reception. The Society's Annual Dinner will be held at 7.30 pm at Goodenough College. The cost of the dinner will be £58.00, including drinks; email AnnualDinner_RSVP@lms.ac.uk to reserve a place.

LMS Meeting

Annual General Meeting of the LMS

Goodenough College, Mecklenburgh Square, Holborn, London WC1N 2AB, 29 November 2019; 3.00 – 6.00 pm

Website: lms.ac.uk/events/meeting/agm

The speaker will be Marc Lackenby (Oxford). The LMS President, Caroline Series (Warwick) will give the presidential address. The meeting will include the presentation of certificates to all 2019 LMS prize winners and the announcement of the annual LMS election results.

The meeting will be followed by a reception, which will be held at Goodenough College, and the Society's annual dinner at 7.30 pm. The cost of the dinner will be £58.00, including drinks. To reserve a place at the dinner, please email AnnualDinner_RSVP@lms.ac.uk.

For further details about the AGM, please contact Elizabeth Fisher (lmsmeetings@lms.ac.uk).

PDE Models for Cancer Invasion

Location: Queen's University Belfast
Date: 6 December 2019
Website: tinyurl.com/y63v62hg

This workshop will focus on mathematical models for cancer invasion as well as their analytical and numerical treatment, and aims to present recent developments in this area. Supported by an LMS Scheme 9 grant awarded to Anna Zhigun.

Computational Complex Analysis

Location: INI, Cambridge
Date: 9–13 December 2019
Website: tinyurl.com/y3bmuc5u

The aim of the workshop on computational complex analysis is to bring together numerical and applied mathematicians to focus on: (a) new and existing methods, (b) software tools, and (c) various application areas. Deadline for applications: 8 September.

British Postgraduate Model Theory Conference 2020

Location: University of Leeds
Date: 8–10 January 2020
Website: tinyurl.com/bpgmtc20

This meeting aims to bring together young researchers interested in model theory. It will feature a mini-course, invited talks by established academics, and contributed talks by postgraduate researchers. Supported by the LMS, the University of Leeds and the British Logic Colloquium.

Probabilistic Coupling and Geometry

Location: University of Warwick
Date: 9–10 December 2019
Website: tinyurl.com/y4tzewh7

This workshop will bring together researchers from diverse fields that use coupling arguments, and will be of interest to those working on adaptive MCMC, probability and geometry, statistical shape analysis, and perfect simulation.

Integrability, Algebra and Geometry

Location: University of Glasgow
Date: 13 December 2019
Website: tinyurl.com/y4k4fxr3

The workshop will bring together people working in integrable systems from various perspectives. Speakers are: Jenya Ferapontov (Loughborough), Marta Mazzocco (Birmingham) and Daniele Valeri (Glasgow). Supported by an LMS Scheme 9 grant.

LMS South West & South Wales Meeting & Workshop

Location: University of Bristol
Date: 15 January 2020
Website: tinyurl.com/y37pked7

This meeting forms part of the South West & South Wales Regional Workshop on Interactions between Geometry, Dynamics and Group Theory on 16–17 January. Speakers are Martin Bridson FRS (Oxford), Corinna Ulcigrai (Bristol) and Yves Benoist (Paris). Partial funding is available.

Society Meetings and Events

September 2019

- 6–7 Prospects in Mathematics Meeting, Lancaster
- 11 Midlands Regional Meeting, Nottingham
- 19 Popular Lectures, University of Birmingham

October 2019

- 19 DeMorgan@21, London

November 2019

- 13 Computer Science Colloquium, London
- 21 LMS/BCS-FACS Evening Seminar, London
- 21 Joint Meeting with the IMA, Reading
- 29 Graduate Student Meeting, London
- 29 Society Meeting and AGM, London

January 2020

- 15 South West & South Wales Regional Meeting, Bristol

Calendar of Events

This calendar lists Society meetings and other mathematical events. Further information may be obtained from the appropriate LMS Newsletter whose number is given in brackets. A fuller list is given on the Society's website (www.lms.ac.uk/content/calendar). Please send updates and corrections to calendar@lms.ac.uk.

September 2019

- 2 Applications of Mathematics in Ecology, University of Kent (483)
- 2 One Day Function Theory Meeting, De Morgan House, London (483)
- 4-6 Inverse Problems from Theory to Application, University College London (483)
- 5-6 Recent Trends in Stochastic Analysis and Partial Differential Equations, University of Chester (483)
- 5-6 Water Waves — Mathematical Theory & Applications, University of Plymouth (483)
- 5-7 British Logic Colloquium 2019, University of Oxford (483)
- 6-7 LMS Prospects in Mathematics Meeting, Lancaster (484)
- 8-11 Proof Society Summer School 2019, Swansea University (483)
- 9-11 Curve Counting Theories and Related Algebraic Structures, University of Leeds (483)
- 9-13 Graph Complexes in Algebraic Geometry and Topology, University of Manchester (482)
- 9-13 The Geometry of Derived Categories, University of Liverpool (483)
- 9-13 The Complex Analysis Toolbox: New Techniques and Perspectives, INI, Cambridge (482)
- 11 LMS Midlands Regional Meeting, Nottingham (484)
- 11-13 Proof Society Workshop 2019, Swansea University (483)
- 11-13 Groups and Representation Theory, Conference in Memory of Kay Magaard, University of Warwick (483)
- 12-13 Flood Risk Conference, Swansea University Bay Campus, Wales (483)
- 16 Random Matrices and Applications, University of Sussex (483)
- 16 Functor Categories for Groups, University of Lincoln (483)
- 16 Words in Finite and Profinite Groups, University of Lincoln (484)
- 16-18 British Topology Meeting, Warwick (483)

17 Branching Processes and their Applications, University of Sussex (483)
18 Randomness, Symmetry and Free Probability, University of Sussex (483)
18 A Day of Lie Theory in Kent, University of Kent (484)
18-19 Philip Maini's 60th Birthday Workshop, Mathematical Institute, Oxford (483)
18-20 Southampton–Bielefeld Geometric Group Theory Meeting, University of Southampton (483)
18-21 Lattice Polytopes, with a View towards Geometry and Applications, ICMS, University of Edinburgh (483)
19 LMS Popular Lectures, University of Birmingham (484)
25-26 Novel Approaches to the Multi-Armed Bandit Problem, Imperial College London (484)
29-4 Oct Clay Research Conference and Workshops, Mathematical Institute, Oxford (482)
30-4 Oct Structure Preservation and General Relativity, INI, Cambridge (482)

October 2019

10 Dependence Modeling and its Applications in Insurance and Finance, Heriot-Watt University (483)
7-11 Dirac Operators in Differential Geometry and Global Analysis, Bedlewo, Poland (482)
19 DeMorgan@21, De Morgan House, London (483)
25 Sir Michael Atiyah: Forays into Physics, INI Cambridge (484)
28-1 Nov Complex Analysis in Mathematical Physics and Applications, INI, Cambridge (483)

November 2019

4-8 Edge Days 2019: Fano Varieties, Cone Singularities and their Links, University of Edinburgh (484)
6-8 Young Researchers in Algebraic Number Theory, University of Warwick (484)
13 Computer Science Colloquium 2019, De Morgan House, London (484)
18-19 Category Theory and its Applications Postgraduate Conference, University of Leicester (484)
21 Joint LMS Meeting with the IMA, Reading (484)
21 LMS/BCS-FACS Evening Seminar, De Morgan House, London (484)
29 Graduate Student Meeting, London (484)
29 Society Meeting and AGM, London (484)

December 2019

6 PDE Models for Cancer Invasion, Queen's University Belfast (484)
9-10 Probabilistic Coupling and Geometry Workshop, University of Warwick (484)
9-13 Computational Complex Analysis, INI, Cambridge (484)
13 Integrability, Algebra and Geometry, University of Glasgow (484)

January 2020

8-10 British Postgraduate Model Theory Conference 2020, University of Leeds (484)
15 South West & South Wales Regional Meeting, Bristol (484)

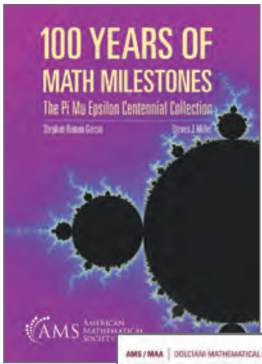
July 2020

5-11 8th European Congress of Mathematics, Portorož, Slovenia
12-19 14th International Congress on Mathematical Education Shanghai, China

August 2020

17-21 IWOTA 2020, Lancaster University (481)

AMERICAN MATHEMATICAL SOCIETY



100 YEARS OF MATH MILESTONES

The Pi Mu Epsilon Centennial Collection

Stephan Ramon Garcia, Pomona College & Steven J. Miller, Williams College

This book is an outgrowth of a collection of 100 problems chosen to celebrate the 100th anniversary of the undergraduate math honor society Pi Mu Epsilon. Each chapter describes a problem or event, the progress made, and connections to entries from other years or other parts of mathematics. Put together, these problems will be appealing and accessible to energetic and enthusiastic math majors and aficionados of all stripes.

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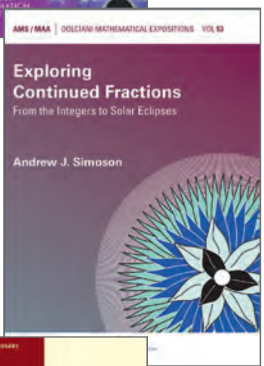

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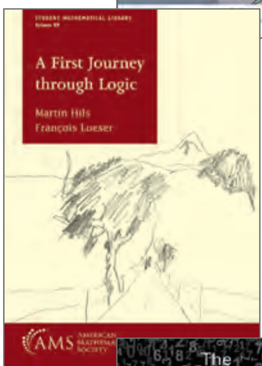

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