Introduction to financial economics

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Financial economics

- These lectures are about an oversimplified view that many mathematicians have of financial economics.
- The name of the game is transfer of wealth either in time or across states of the world.
- **Example 1.** When you spend $\$95$ on a bond which pays $\$100$ in a year from now, you have effectively exchanged (transferred) 95 *today-dollars* for 100 *a-year-from-now-dollars*.
- **Example 2.** Consider a share of a stock of a publicly traded food company which is currently trading at $\$100$ and you know that the price of the same stock in a year from now will fall (say to $\$80$) if there is a drought, and rise (say to $\$110$) otherwise. In a simplified world where only two possible futures can arise (*drought* and *no drought*), this stock allows you to transfer 100 *today-dollars* into an uncertain amount of money, which can be described as 80 *a-year-from-now-dollars-if-a-drought-occurs* and 110 *a-year-from-now-dollars-if-there-is-no-drought*.

Financial economics

- The useful thing - and this is the central service the financial section provides - is that you can also do exactly the opposite: you can get $\$95$ right now in exchange for a future payment of $\$100$. That is how you can afford to buy a car and enjoy it now (or a house, but let's not go there) and use your future income to pay for it.
- Analogously, you could short-sell the stock, i.e., get $\$100$ today for a promise to pay either $\$80$ or $\$110$ in a year from now, contingent on the state of the world (drought or no drought).
- The main purpose of these lectures is to describe a mathematical formalism built precisely with the purpose of understanding various ways wealth can be transferred.

Financial Economics

- The goal here is to cover only the minimum amount of material for a student to be able to go on and learn about the basic notions of mathematical finance in a subsequent course, we keep our models as simple as possible. A single consumption good is posited and the future is multi-period but finite. Similarly, there is only a finite number of states of the world.
- Mathematically, we work exclusively in a finite-dimensional Euclidean space, but we interpret its elements in different ways and give them different names, depending on the role they play. This way, we keep conceptually different but formally equivalent objects separated and we pave the way to the infinite-dimensional case, where vectors come in many flavors and varieties.

The world

- We model the world as a tree-like structure that describes the possible ways the future can unfold.
- There are two qualitatively different dimensions in which wealth can be transferred - through time and through uncertainty.
- For the set of time instances, we take $\mathcal{T} = \{0, 1, \dots, T\}$, for some $T \in \mathbb{N}$.
- The uncertainty is modeled by a nonempty finite set Ω which we calls the set of states of the world.
- Things become interesting when we start to incorporate the fact that we learn more and more about the true state of the world as time marches on, i.e., when we model the gradual resolution of uncertainty.

Algebras

- **Definition.** A family $\mathcal{F}$ of subsets of Ω is called an **algebra** if
  1. $\emptyset \in \mathcal{F}$,
  2. if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$, and
  3. if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$.
- The notion you are going to hear about more often is a σ-*algebra*. A σ-algebra is a generalization of the concept of an algebra which is useful only if the number of states of the world is infinite. Otherwise, the two concepts coincide, so we have absolutely no reason to complicate our lives with the extra σ.

ALGEBRA = INFORMATION

- You can picture information as the ability to answer questions (more information gives you a better score on the test ...), and the lack of information as ignorance.
- In our world, all the questions can be phrased in terms of the elements of the state-space Ω. Remember - Ω contains all the possible futures of our world and the knowledge of the exact $\omega \in \Omega$ (the "true state of the world") amounts to the knowledge of everything.
- So, the ultimate question would be *"What is the true ω?"*, and the ability to answer that would promote you immediately to the level of a Supreme Being. In order for our theory to be of any use to us mortals, we have to allow for some ignorance, and consider questions like

  $$\text{\textit{"Is the true ω an element of A?"},} \tag{1}$$

  where A is a subset of Ω.

ALGEBRA = INFORMATION II

- The state of our knowledge can be described by the set of all questions you know the answer to. Since all questions about the true state of the world can be phrased as questions about sets of states, the proper mathematical description of our current knowledge is nothing but the collection of all those A for which we know the answer to (1).
- The nice thing about that set is that - for purely logical reasons - it has lots of structure. You are probably already guessing that I am talking about the fact that set - let us call it $\mathcal{F}$ - has to be an algebra. Let's see why:

ALGEBRA = INFORMATION III

- First of all, I always know that the true ω is an element of Ω, so $\Omega \in \mathcal{F}$.
- Then, if I happen to know how to answer the question *Is the true ω in A?*, I will necessarily also know how to answer the question *"Is the true ω in A^c?"*. The second answer is just the opposite of the first. Equivalently, $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$.
- Finally, let A and B be two sets with the property that I know how to answer the questions *"Is the true ω in A?"* and *"Is the true ω in B?"*. Then I clearly know that the answer to the question *"Is the true ω in $A \cup B$?"* is *"No"* if I answered *"No"* to each of the two questions above. One the other hand, it is going to be *"Yes"* if I answered *"Yes"* to at least one of them. Therefore $A, B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$.

ALGEBRA = PARTITION

- As we shall see in the following problem, the seemingly complicated notion of an algebra is really equivalent to the simpler notion of a partition.
- **Definition.** A partition of a set Ω is a family $\mathcal{P}$ of non-empty subsets of Ω such that $A \cap B = \emptyset$ whenever $A \neq B$ and $\cup_{A \in \mathcal{P}} A = \Omega$.
- **Problem 1.** Let Ω be a nonempty finite set, and let $\mathbb{A}$ be the family of all algebras on Ω, and let Π be the family of all partitions of Ω. Construct a mapping $F : \Pi \to \mathbb{A}$ as follows: for a partition $\mathcal{P} = \{A_1, A_2, \dots, A_k\}$ of Ω, let $F(\mathcal{P}) = \mathcal{A}$ be the family consisting of the empty set and all possible unions of elements of $\mathcal{P}$.
  1. Show that so defined family $\mathcal{A}$ is an algebra, and
  2. Show that the mapping F is one-to-one and onto.

Algebra = Partition II

- **Problem 2.**[Just for fun, not important for the sequel]
For $n \in \mathbb{N}$, let a_n be the number of different algebras on Ω when Ω has exactly n elements. Show that
  1. $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, and that the following recursion holds
  $$a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k,$$
  where $a_0 = 1$ by definition, and
  2. the exponential generating function for the sequence $\{a_n\}_{n\in\mathbb{N}}$ is $f(x) = e^{e^x - 1}$, i.e., that
  $$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = e^{e^x-1}.$$

A modeling example

- Three candidates (let call them Candidate 1, Candidate 2 and Candidate 3) participate in a presidential election; Candidates 1 and 2 are women and Candidate 3 is a man. Since we are only interested in the winner of the election, we model the possible states of the world by the elements of the set $\Omega = \{\omega_1, \omega_2, \omega_3\}$ where, as expected, ω_i means that Candidate i wins the election.
- We wake up on the day following the election without knowing how the election went; we do know that there one of the candidates has been elected, but that is all. Our knowledge can be modeled by the algebra
$$\mathcal{A}_0 = \{\emptyset, \Omega\},$$
or, equivalently the trivial partition
$$\mathcal{P}_0 = \{\Omega\},$$
as the only questions we know how to answer are *"Is the new president either Candidate 1, Candidate 2 and Candidate 3?"* (the answer is *"yes"*) and *"Is the new president neither Candidate 1, Candidate 2 nor Candidate 3?"* (the answer if *"no"*).

A MODELING EXAMPLE II

- We turn on the radio and hear about how the new president (elect) won by a small margin, and how she has a difficult time ahead of her. Now we know more about the true state of the world; we know that ω_3 is not the true state of the world because Candidate 3 is a man. We still don't know which of the two women candidates won. Our state of the knowledge can be described by the algebra

$$\mathcal{A}_1 = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3\}, \Omega\},$$

or the partition

$$\mathcal{P}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\},$$

since we know how to answer more questions now. For example, we know that the answer to the question *"Is the new president either Candidate 1 or Candidate 2?"* - the answer is *"yes"*.

- We listen to the radio some more and the name of the winning Candidate is mentioned - it is Candidate 2. We know everything now and our information corresponds to the algebra $\mathcal{A}_2$

$$\mathcal{A}_2 = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \Omega\},$$

which consists of *all* subsets of Ω, with the corresponding partition being

$$\mathcal{P} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}.$$

FILTRATIONS

- The example above not only illustrates the connection between algebras (partitions) and the amount of knowledge or information, it also shows how we learn. By observing more facts, we can answer more questions, and our algebra grows. This typically happens over time, so, in order to describe the *evolution* of our knowledge, we introduce the following important concept:
- **Definition.** A filtration is a finite sequence $\mathcal{A}_1, \mathcal{A}_2, \dots \mathcal{A}_T$ of algebras on Ω (indexed by the time set $\mathcal{T} = \{1, 2, \dots, T\}$) such that

$$\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots \subseteq \mathcal{A}_T.$$

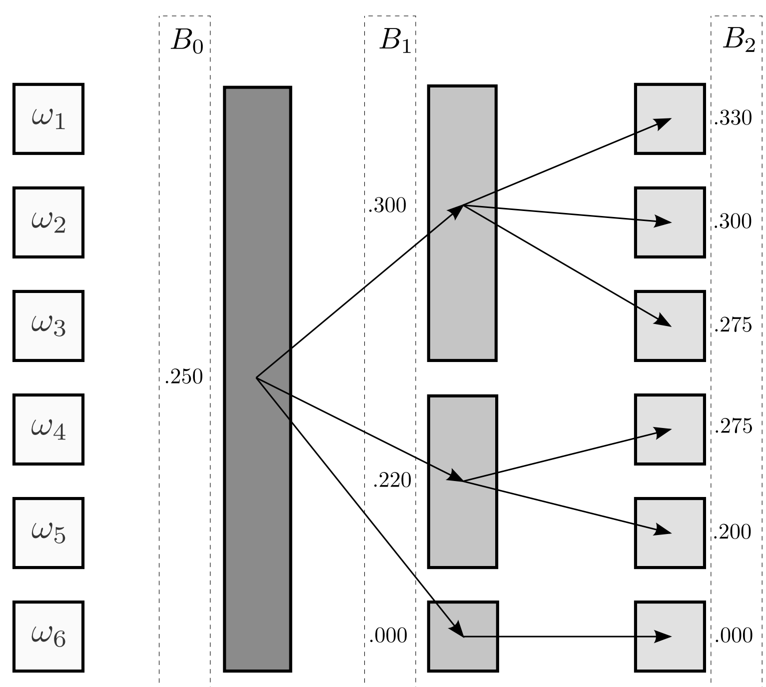
- The knowledge we accumulate over time is typically more about values of certain quantities (like sports statistics), and less about presidential candidates.

An example

- Suppose that the batting average B of a certain baseball player is modeled in the following, simplified, way for the period of the next 2 years. At time $t = 0$ (now) his batting average is $.250$, i.e., $B_0 = .250$. Next year (corresponding to $t = 1$), it will either go up to $B_1 = .300$, down to $B_1 = .220$, or player will leave professional baseball and so that $B_1 = .000$.
- After that, the evolution continues: after $B_1 = .300$, three possibilities can occur $B_2 = .330$, $B_2 = .300$ and $B_2 = .275$. Similarly, if $B_1 = .220$, either $B_2 = .275$ or $B_2 = .200$. Finally, if $B_1 = .000$, the player is retired and $B_2 = .000$, as well.
- A possible mathematical model for this situation can be built on a state space which consists of 6 states of the world $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$, where each state of the world corresponds to a particular path the future can take. For example, in ω_1, the batting averages are $B_0 = .250$, $B_1 = .300$, $B_2 = .330$, while in ω_4, $B_0 = .250$, $B_1 = .220$ and $B_2 = .275$.

An example II

- Things will be much clearer from the picture:



An example III

- At time 0, the information available to us is minimal, the only questions we can answer are the trivial ones *" Is the true ω in Ω?"* and *" Is the true ω in $\emptyset$?"* , and this is encoded in the algebra $\{\Omega, \emptyset\}$.
- A year after that, we already know a little bit more, having observed the player's batting average for the last year. We can distinguish between ω_1 and ω_5, for example. We still do not know what will happen the day after, so that we cannot tell between ω_1, ω_2 and ω_3, or ω_4 and ω_5. Therefore, our information partition is $\{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6\}\}$ and the corresponding algebra $\mathcal{A}_1$ is (I am doing this only once!)

$$\mathcal{A}_1 = \{\emptyset, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6\}, \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\},$$
$$\{\omega_1, \omega_2, \omega_3, \omega_6\}, \{\omega_4, \omega_5, \omega_6\}, \Omega\}.$$

- It is interesting to note that algebra $\mathcal{A}_1$ has something to say about the batting average a year from now, but only in the special case when $B_2 = .000$. If that special case occurs - let us call it *injury* - then we do not need to wait until year 2 to learn what the batting average is going to be. It is going to remain $.000$. Finally, after 2 years, we know exactly what ω occurred and the algebra $\mathcal{A}_2$ consists of *all* subsets of Ω.

Random variables

- Let us think for a while how we acquired the extra information on each new day. We have learned it through the quantities B_0, B_1 and B_2 as their values gradually revealed themselves to us.
- **Definition.** Any function $B : \Omega \to \mathbb{R}$ is called a random variable.
- **Definition.** A random variable B is said to be measurable with respect to an algebra $\mathcal{A}$ (denoted by $B \in m\mathcal{A}$) if it is constant on each member of the partition $\mathcal{P}$ corresponding to $\mathcal{A}$.
- **Definition.** A random variable B is said to generate the algebra $\mathcal{A}$ (denoted by $\mathcal{A} = \mathcal{A}(B)$) if
  - $B \in m\mathcal{A}$, and
  - $B \notin m\mathcal{B}$, for any algebra $\mathcal{B} \neq \mathcal{A}$ with $\mathcal{A} \subset \mathcal{B}$.
- Does B_2 generate the algebra $\mathcal{A}_2$ in the previous example? How would you give a definition of an algebra generated by two random variables?

Filtrations and processes

- Remember that $\mathcal{T} = \{0, 1, \ldots, T\}$ is the time set.
- **Definition.** A finite sequence $\{B_k\}_{k\in\mathcal{T}}$, of random variables is called a (stochastic) process.
- The algebras $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ in previous example are interpreted as the amounts of information available to us agent at days 0, 1 and 2 respectively. They were generated by the accumulated values of the process $\{B_k\}_{k\in\mathcal{T}}$ (for $T = 2$).
- **Definition.** A finite sequence $\{\mathcal{A}_k\}_{k\in\mathcal{T}}$ of algebras is called a filtration if $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_T$.
- **Definition.** A process $\{B_k\}_{k\in\mathcal{T}}$ is said to be adapted to the filtration $\{\mathcal{A}_k\}_{k\in\mathcal{T}}$, if $B_k \in m\mathcal{A}_k$, for $k \in \mathcal{T}$.
- **Definition.** A filtration $\{\mathcal{A}_k\}_{k\in\mathcal{T}}$ is said to be generated by the process $\{B_k\}_{k\in\mathcal{T}}$, if $\mathcal{A}_k = \mathcal{A}(B_0, B_1, \ldots, B_k)$, for $k \in \mathcal{T}$.

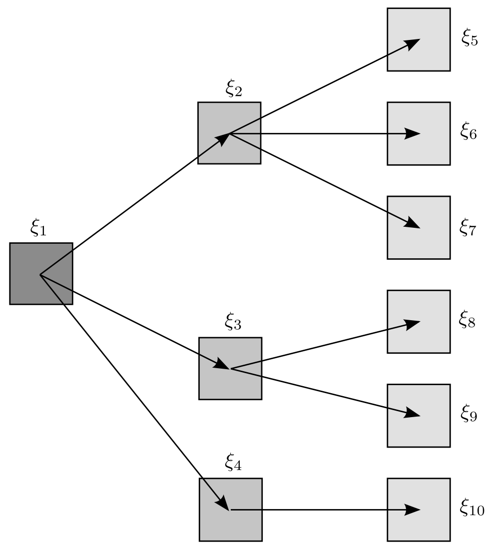
The information tree

- A process $\{B_k\}_{k\in\mathcal{T}}$ can be thought of as a mapping $B : \mathcal{T}\times\Omega \to \mathbb{R}$.
- It is, additionally, adapted to $\{\mathcal{A}_k\}_{k\in\mathcal{T}}$, if and only if $B(t,\omega) = B(t,\omega')$ for all $t \in \mathcal{T}$ and ω and ω' belong to the same element of the partition (corresponding to) $\mathcal{A}_t$.
- In the language of "abstract nonsense", the mapping B *factorizes through* a certain quotient set of $\mathcal{T} \times \Omega$. More precisely ...
- **Definition.** The set $\mathcal{N}$, whose elements are called nodes is the quotient set (the set of all equivalence classes) of the product $\mathcal{T} \times \Omega$ with respect to the equivalence relation $\sim$ which is defined as

$$(t,\omega) \sim (s,\omega') \text{ if and only if } t = s \text{ and } \omega \text{ and } \omega' \text{ belong to the same partition element in } \mathcal{A}_t.$$

The information tree (cont'd)

- We set $n = |\mathcal{N}|$, i.e., n is the number of nodes, and, obviously $n \geq \mathcal{T} + 1$.
- Since an adapted process factors through $\mathcal{N}$, we identify them with mappings from $\mathcal{N}$ to $\mathbb{R}$, and write $B(\xi)$ as the (common) value of $B(t, \omega)$, for all representatives (t, ω) of ξ.



Some definitions

- For two nodes $\xi_1, \xi_2 \in \mathcal{N}$, we say that ξ_2 is a child of ξ_1 (denoted by $\xi_2 >_c \xi_1$) if there exists representatives (t_1, ω_1) of ξ_1 and (t_2, ω_2) of ξ_2 such that $t_2 = t_1 + 1$ and $\omega_1 = \omega_2$. A node ξ_2 is called the parent of ξ_1 if ξ_1 is a child of ξ_2.
- The concepts of a child and a parent define a natural partial order on $\mathcal{N}$. Moreover, they give it a structure of a tree. This tree is usually called the event tree or the information tree.
- A node ξ' is said to be a (strict) successor of ξ (denoted by $\xi' > \xi$) if there exists a finite (or zero) number of nodes $\xi_1, \dots, \xi_k$ such that ξ_1 is a child of ξ, ξ_{l+1} is a child of ξ_l (for all $l = 1, \dots, k-1$) and ξ' is a child of ξ_k.
- A node ξ' is said to be a successor of ξ (denote by $\xi' \geq \xi$) if either $\xi' = \xi$ or $\xi' > \xi$.
- Clearly, the successor order is the smallest partial order that "contains" the "is-a-child-of" binary relation. In a more fancy language, $>$ is a transitive closure of $>_c$.

MORE DEFINITIONS

- An (at most unique) node ξ_1 is said to be parent of ξ_2 if $\xi_2 >_c \xi_1$. In general, the parent of ξ is denoted by ξ^-.
- A (unique) node ξ_0 is said to be initial if it has no parents.
- A node is called terminal if it has no children. Otherwise, it is called non-terminal.
- The number of children of the node ξ is called the branching number of ξ, and is denoted by $b(\xi)$.
- A subtree $\mathcal{N}_+(\xi)$ of $\mathcal{N}$, starting at $\xi \in \mathcal{N}$ is the set of all successors of ξ (including ξ). A strict subtree $\mathcal{N}_{++}(\xi)$ is defined in the same way, but without ξ, i.e., $\mathcal{N}_{++}(\xi) = \mathcal{N}_+(\xi) \setminus \{\xi\}$.

FINANCIAL CONTRACTS

- **Definition.** A financial contract is a pair $(\xi(D), D)$ of a non-terminal issue node $\xi(D)$, and a stochastic process D (the dividend process) such that $D(\xi) = 0$ unless $\xi > \xi(D)$ (i.e., $\xi \in \mathcal{N}_{++}(\xi(D))$.
- One can define the **maturity** of a financial contract as the smallest (earliest) instance $t \in \mathcal{T}$ with the property that $D(\omega, t') = 0$ for all $t' > t$ and all ω.
- Typically, financial contracts are bought and sold in a (financial) market. You may think of a transaction where you buy a financial contract D, as a transfer of p today-dollars (where p is the current price of the contract) for the dividend stream (process) D.

EXAMPLES OF FINANCIAL CONTRACTS

A financial contract D is called a

- contingent contract for the node $\bar{\xi} > \xi_0$ if it is issued at the initial node ξ_0 and its dividend process is given by the adapted process

$$D(\xi) = \begin{cases} 1, & \xi = \bar{\xi}, \\ 0, & \text{otherwise.} \end{cases}$$

- zero-coupon bond with maturity $\tau \in \mathcal{T} \setminus 0$ if it is issued at ξ_0 and its dividend process is given by

$$D(\xi) = \begin{cases} 1, & t = \tau, \text{ for some representative } (t, \omega) \text{ of } \xi, \\ 0, & \text{otherwise.} \end{cases}$$

- short-lived bond with issue node $\bar{\xi}$ if its dividend process is given by

$$D(\xi) = \begin{cases} 1, & \xi >_c \bar{\xi} \\ 0, & \text{otherwise.} \end{cases}$$

- equity contract if its dividend stream is non-negative on each node.

PRICE PROCESSES

- If a market for a given financial contracts exists, it will, through forces of demand and supply, determine the contract's price.
- An interesting feature of long-lived financial contracts is that you can decide to buy/sell them even after they are issued.
- For example, government issues bonds only weekly (or so), but the (secondary) bond market is active all the time.
- Companies issue stocks, essentially, only once - at their IPO (Initial Public Offer), but stock markets are very busy practically continuously.
- **Definition.** An (after-dividend) price of a financial contract D is an adapted process q such that $q(\xi) = 0$, unless $\xi \geq \xi(D)$.
- When q is the prevailing market price for the contract D, an agent, at node $\xi \geq \xi(D)$, can buy or sell the right to receive the remaining dividend stream (not including the dividend paid at ξ) at the $q(\xi)$.
- Note that $q(t, \omega)$ is expressed in time-t-dollars - the actual transaction will happen at time t, and not at time 0.

Financial Markets

- **Definition.** A financial market (denoted by $\mathcal{F}$) with time-horizon $T \in \mathbb{N}$ and J financial contracts consists of
  - a state space Ω,
  - a filtration $\{\mathcal{A}_k\}_{k=0,\dots,T}$, with $\mathcal{A}_0 = \{\emptyset, \Omega\}$
  - J financial contracts $(\xi(D^1), D^1)$, $(\xi(D^2), D^2)$, ..., $(\xi(D^J), D^J)$, and
  - J corresponding price processes $q^1, \dots, q^J$.
- Note that the condition $\mathcal{A}_0 = \{\emptyset, \Omega\}$ implies that there is a unique node with $t = 0$. We call it ξ_0.
- In the node ξ, anybody can buy or sell as many "shares" of the contract D^j, $j = 1, \dots, J$, at the price q^j, $j = 1, \dots, J$. In particular,
  - there are no transaction costs,
  - trading in a fractional number of shares is allowed,
  - the agents do not influence the price when they trade
  - short-selling is not prohibited,
  - the agents can borrow as much as the want and lend as much as they want, at the same rate,
  - everybody has access to exactly the same information

Dynamic Trading

- An agent participating in the financial market $\mathcal{F}$ is allowed to dynamically readjusts the holdings in the J contracts by choosing a portfolio processes, i.e., J adapted processes $z_1, \dots, z_J$, with the interpretation that $z_j(t, \omega)$ is the number of shares of the contract j at time t in the state of the world ω. Note that, for each t, $z(t) = (z_1(t), \dots, z_j(t))$ is a random vector, so that z is a *vector-valued process*.
- Of course, the contracts that are not yet on the market cannot be bought or sold, so we must have $z_j(\xi) = 0$ unless $\xi \in \mathcal{N}_+(\xi(D^j))$. This is a nice convention because it allows us not to care about issue nodes of contracts.
- Not every portfolio process can be realized. If I start with $\$1$ in my pocket, I cannot buy 10000 shares of Google. Even if I had just enough money to buy 10000 shares of Google, and Google goes bankrupt a year from now, I will not be able to buy another 50000 shares of Microsoft at that point (unless Microsoft goes bankrupt, too ...)

DYNAMIC TRADING (CONT'D)

- Before we start thinking about the accounting issues, let us agree that the transactions are made (contracts bought/sold) *after* the divided has been paid out.
- Therefore, it makes no sense to do any trading at the terminal nodes, so we assume $z_j(\xi) = 0$, for any terminal node ξ.
- What we need is a condition on portfolio which will make it implementable without the need for exogenous cash influx.
- We start with in ξ_0, with an agent with no portfolio holdings and no cash. She decides to acquire $z(\xi_0) = (z_1(\xi_0), \dots, z_J(\xi_0))$ units of the each of the J contracts.
- The price of that transaction is

$$\sum_{j=1}^{J} z_j(\xi_0) q^j(\xi_0), \text{ which we denote by } z(\xi_0) \cdot q(\xi_0).$$

DYNAMIC TRADING (CONT'D)^2

- Therefore, what is left (remember, this may be negative) is

$$c(\xi_0) = -q(\xi_0) \cdot z(\xi_0).$$

- $c(\xi_0)$ cannot be transferred to $t = 1$. *You can transfer wealth only through one of the contracts D^j !*
- What happens to $c(\xi_0)$, then? At this point, just imagine it gets burned (if positive), or the government bails us out (if negative). We call it consumption.
- A day passes by, and the current node is ξ. Our agent decides to change some of her holdings. She has $z(\xi^-)$ in her portfolio, but would like to have $z(\xi)$. Note: because we are at $t = 1$, $\xi^- = \xi_0$ for any ξ.
- First, she collects the dividends - she gets $D^j(\xi)$ per unit of contract j held. Entire "income" totals to $D(\xi) \cdot z(\xi^-)$.
- Then, she sells all her contracts in the market and gets $q(\xi) \cdot z(\xi^-)$.

Dynamic Trading (cont'd)[3]

- Finally, she purchases $z(\xi)$ units of the contracts at the prevailing price $q(\xi)$, spending $q(\xi) \cdot z(\xi)$ dollars in the process.
- What is left is "consumed" or gets "bailed out":

$$c(\xi) = (D(\xi) + q(\xi)) \cdot z(\xi^-) - q(\xi)z(\xi).$$

- The same logic applied to every subsequent node, with the obvious change at terminal nodes:

$$c(\xi) = (D(\xi) + q(\xi)) \cdot z(\xi^-), \text{ for a terminal } \xi,$$

  because we do not want to buy anything there anymore.
- The consumption process c measures the lack of "self-sufficiency" of our portfolio. Those portfolios z for which $c(\xi) = 0$, for each node ξ are called self-financing.
- Alternatively, we can think of c as our "daily bread" - we use the financial market to pay for our day-to-day subsistence. Therefore, we say that the portfolio z finances the consumption process c.

No Arbitrage

- **Definition.** A financial market is said to admit no arbitrage if there is no portfolio process z and no consumption process c such that
  - z finances c,
  - $c(\xi) \geq 0$, for all ξ, and
  - $c(\xi) > 0$ for at least one ξ.

The W Matrix

- The wealth-transfer equations from previous slides can be written in a more compact (matrix) notation.
- Let W be a matrix with n rows and $\bar{n} \times J$ columns, where $\bar{n}$ is the number of non-terminal nodes (it makes sense, portfolios are defined everywhere except the terminal nodes, and there is one for each contract):

$$\begin{bmatrix} [-q(\xi_0)] & 0 & 0 & 0 & 0 \\ [D(\xi_0^+) + q(\xi_0^+)] & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & [D(\xi) + q(\xi)] & [-q(\xi)] & 0 \\ 0 & \dots & \dots & 0 & \dots \\ 0 & 0 & 0 & [D(\xi^+) + q(\xi+)] & \dots \\ 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

- The notation ξ^+ stands for the *set* of all children of ξ. Therefore, the sub-matrix $[D(\xi_0^+) + q(\xi_0^+)]$ has J columns and $b(\xi_0)$ rows, where $b(\xi_0)$ is the branching number of ξ_0, i.e., the number of its children.

The W Matrix (cont'd)

- The wealth-transfer equations from previous slides now look like this:

$$c = Wz,$$

where we think of z as a vector in $\mathbb{R}^{J \times \bar{n}}$, and c as a vector in $\mathbb{R}^n$. as

- The no-arbitrage condition can now be written as: *there is no portfolio z such that $Wz \geq 0$ and $Wz \neq 0$ (where $\geq$ is understood in a node-by-node sense).*
- Let $\langle W \rangle \subseteq \mathbb{R}^n$ denote the range of the matrix W. $\langle W \rangle$ is called the marketed subspace, or the subspace of income transfers. Yet another reincarnation of the no-arbitrage condition is

$$\langle W \rangle \cap \mathbb{R}^n_+ = \emptyset,$$

where $\mathbb{R}^n_+ = \{c = (c_1, \dots, c_n) \in \mathbb{R}^n \ : \ c_k \geq 0, k = 1, \dots, n\}$. Note that the no-arbitrage condition restricts the dimension of $\langle W \rangle$ to be strictly less than n.

The Fundamental Theorem of Asset Pricing

- Here is the first version of our central theorem:
  Theorem. A financial market admits no arbitrage if and only if there exists a process $\pi : \mathcal{N} \to (0, \infty)$ such that $\pi W = 0$.
- Before we prove it, let's see some of its consequences.
- The process π as above is called the vector of node prices. If it is, further, normalized by $\pi(\xi_0) = 1$, it is called a vector of present-value prices.
- **Proposition.** (A "martingale" property of security prices) Suppose that the market no arbitrage, and that π is a vector of node prices. Then

$$\pi(\xi)q_j(\xi) = \sum_{\xi' >_c \xi} \pi(\xi')(D^j(\xi') + q_j(\xi')),$$

  for all j for which $\xi \geq \xi(D^j)$.
- *Proof.* Exercise!

Market Completeness

- **Definition.** A market satisfying the no-arbitrage condition is called complete if the dimension $\dim\langle W \rangle$ of the range of the matrix W equals $n-1$. The market is called incomplete, if $\dim\langle W \rangle < n-1$.
- **Proposition.**

$$\dim\langle W \rangle = \sum_{\xi \in \mathcal{N}^-} \operatorname{rank}(D(\xi^+) + q(\xi^+)),$$

  where $\mathcal{N}^-$ denotes the set of all non-terminal nodes.
- **Exercise.** Prove it!
- For a node $\xi \in \mathcal{N}^-$, we define the spanning number $\rho(\xi)$ as the rank of the matrix $D(\xi^+) + q(\xi^+)$. We can restate one of the conclusions of the Proposition above as *The market is complete if and only if $\rho(\xi) = b(\xi)$, for all $\xi \in \mathcal{N}^-$.*
- A special case when the spanning numbers $\rho(\xi)$ do not depend on prices (as long as there is no arbitrage), is when all securities are short-lived, i.e. $D^j(\xi) = 0$, unless $\xi >_c \xi(D^j)$.

The Second Fundamental Theorem

- Completeness allows for an elegant (and important) characterization. It is, in fact, so important that it is sometimes called the Second Fundamental Theorem of Asset Pricing.
- **Theorem.** An arbitrage-free market is complete if and only if there is a unique present-value vector.
- *Proof.* When the market is complete, the rank of the matrix W is $n-1$, so, by the rank-nullity theorem, the dimension of its (left) null space must be 1, and the uniqueness follows from the fact that we are looking for present-value (normalized) vectors.

  Conversely, suppose that the market is incomplete. Then $\operatorname{rank} W \leq n-2$, and the dimension of the (left) null-space N of W is at least 2. The first fundamental theorem states that $N \cap K_{++} \neq \emptyset$, where $K = \{(1, \dots, c_n) \in \mathbb{R}^n \ : \ c_k > 0, \text{ for all } k = 2, \dots, n\}$. The set K is isomorphic to an $n-1$-dimensional Euclidean space, and so, the intersection $N \cap K$ is a linear submanifold of K of dimension at least 1. It intersects its positive orthant K_+, in a point, and, by relative openness of K_+ in K, in infinitely many points. Consequently, there are at least two present-value vectors (infinitely many, actually).

Examples: the one-period binomial model

- *The tree:* $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{T} = \{0, 1\}$, $\mathcal{A}_0 = \{\Omega, \emptyset\}$, $\mathcal{A}_1 = \mathcal{P}(\Omega)$. The nodes are denoted by ξ_0, ξ_1 and ξ_2.
- *Financial markets:* There are two securities, each with the issue date ξ_0:
  1. a "bond" with $D^1(\xi_1) = D^1(\xi_2) = 1 + r$, for some $r > 0$, and
  2. a "stock" with $D^2(\xi_1) = S_{up}$, $D^2(\xi_2) = S_{down}$, for some constants $S_{up} > S_{down} > 0$.

  The prices of those contracts (relevant only at non-terminal nodes, i.e., only at ξ_0 in this case) are $q^1(\xi_0) = 1$ and $q^2(\xi_0) = S_0$, where $S_0 > 0$.
- *The W matrix:* It will have $J \times \bar{n} = 2$ columns and $n = 3$ rows:

$$W = \begin{bmatrix} -1 & -S_0 \\ 1+r & S_{up} \\ 1+r & S_{down} \end{bmatrix}$$

EXAMPLES: THE ONE-PERIOD BINOMIAL MODEL (CONT'D)

- *Node prices:* The solutions π to the equation $\pi W = 0$ are spanned by

$$\pi(\xi_0) = 1,\ \pi(\xi_1) = \frac{\frac{1}{1+r}S_{up} - S_0}{S_{up} - S_{down}},\ \text{and } \pi(\xi_2) = \frac{S_0 - \frac{1}{1+r}S_{down}}{S_{up} - S_{down}}.$$

- *No arbitrage:* Using the representation above for the node prices, we get that there is no arbitrage if and only if

$$S_0 \in \left(\tfrac{1}{1+r}S_{down}, \tfrac{1}{1+r}S_{up}\right).$$

- *Completeness:* The market is complete because a single vector of present-value prices exists.

EXAMPLES: THE ONE-PERIOD TRINOMIAL MODEL

- *The tree:* $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{T} = \{0, 1\}$, $\mathcal{A}_0 = \{\Omega, \emptyset\}$, $\mathcal{A}_1 = \mathcal{P}(\Omega)$. The nodes are denoted by ξ_0, ξ_1 and ξ_2, ξ_3.
- *Financial markets:* There are two securities, each with the issue date ξ_0:
  1. a "bond" with $D^1(\xi_1) = D^1(\xi_2) = D^3(\xi_3) = 1 + r$, for some $r > 0$, and
  2. a "stock" with $D^2(\xi_1) = S_{up}$, $D^2(\xi_2) = S_{mid}$, $D^2(\xi_3) = S_{down}$, for some constants $S_{up} > S_{mid} > S_{down} > 0$.

  The prices of those contracts (relevant only at non-terminal nodes, i.e., only at ξ_0 in this case) are $q^1(\xi_0) = 1$ and $q^2(\xi_0) = S_0$, where $S_0 > 0$.
- *The W matrix:* It will have $J \times \bar{n} = 2$ columns and $n = 4$ rows:

$$W = \begin{bmatrix} -1 & -S_0 \\ 1+r & S_{up} \\ 1+r & S_{mid} \\ 1+r & S_{down} \end{bmatrix}$$

EXAMPLES: THE ONE-PERIOD TRINOMIAL MODEL (CONT'D)

- *Node prices:* The solutions π to the equation $\pi W = 0$ with $\pi(\xi_0) = 1$ form a one-dimensional linear manifold.
- *No arbitrage:* It is easy to show that there is no arbitrage if and only if
$$S_0 \in (\tfrac{1}{1+r} S_{down}, \tfrac{1}{1+r} S_{up}).$$
- *Completeness:* The market is *not* complete because the null-space of W is 2-dimensional.
- More interesting examples coming soon.

EXAMPLES: A MORE COMPLICATED EXAMPLE

- *The tree:* Take $\Omega = \{1, 2, 3, 4\}$, $\mathcal{T} = \{0, 1, 2\}$, $\mathcal{A}_0 = \{\Omega, \emptyset\}$, $\mathcal{A}_1 = \{\{1, 2\}, \{3, 4\}, \Omega, \emptyset\}$ and $\mathcal{A}_2 = 2^{\Omega}$. The nodes are denoted by ξ_0, ξ_1, ξ_2, $\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}$, with the obvious meaning (e.g., the only representative for ξ_{21} is $(t, \omega) = (2, 3)$ and the representatives for ξ_1 are $(1, 1)$ and $(1, 2)$. Clearly $b(\xi) = 2$ for all $\xi \in \mathcal{N}^-$.
- *Financial markets:* There are two securities, both issued at ξ_0, and both yielding dividends only at time $t = 2$:
$$D^1(\xi_{11}) = 1,\ D^1(\xi_{12}) = 0,\ D^1(\xi_{21}) = 1,\ D^1(\xi_{22}) = 0,$$
and
$$D^2(\xi_{11}) = 0,\ D^2(\xi_{12}) = 1,\ D^2(\xi_{21}) = 0,\ D^2(\xi_{22}) = a,$$
for some $a \geq 0$. Therefore
$$D(\xi_0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ D(\xi_1^+) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ D(\xi_2^+) = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix}.$$

EXAMPLES: A MORE COMPLICATED EXAMPLE (CONT'D)

- *The matrix W:* the matrix W is a 7×6 (7 is the number of nodes, and 6 is the number of securities times the number of non-terminal nodes). It looks like this:

$$W = \left[\begin{array}{cc|cc|cc} -q_1(\xi_0) & -q_2(\xi_0) & 0 & 0 & 0 & 0 \\ \hline q_1(\xi_1) & q_2(\xi_1) & -q_1(\xi_1) & -q_2(\xi_1) & 0 & 0 \\ q_1(\xi_2) & q_2(\xi_2) & 0 & 0 & -q_1(\xi_2) & -q_2(\xi_2) \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{array}\right]$$

The block-columns correspond to non-terminal nodes (and each has two columns corresponding to 2 securities). The rows are one-per-node, and they are grouped over siblings.

EXAMPLES: A MORE COMPLICATED EXAMPLE (CONT'D)[2]

- *No-arbitrage conditions:* Let $q = (q_1, q_2)$ be a (generic) price process. There will be no arbitrage if and only if we can find a strictly positive process π with $\pi(\xi_0) = 1$ (normalization) such that the following equations hold (where we write π_{ij} for $\pi(\xi_{ij})$):

$$\begin{aligned} \pi_{11}\, 1 + \pi_{12}\, 0 &= \pi_1 q_1(\xi_1) && \text{contract } 1, \text{ node } \xi_1 \\ \pi_{11}\, 0 + \pi_{12}\, 1 &= \pi_1 q_2(\xi_1) && \text{contract } 2, \text{ node } \xi_1 \\ \pi_{11}\, 1 + \pi_{12}\, 0 &= \pi_2 q_1(\xi_2) && \text{contract } 1, \text{ node } \xi_2 \\ \pi_{11}\, 0 + \pi_{12}\, a &= \pi_2 q_2(\xi_2) && \text{contract } 2, \text{ node } \xi_2 \\ \pi_1\, q_1(\xi_1) + \pi_2\, q_1(\xi_2) &= q_1(\xi_0) && \text{contract } 1, \text{ node } \xi_0 \\ \pi_1\, q_2(\xi_1) + \pi_2\, q_2(\xi_2) &= q_2(\xi_0) && \text{contract } 2, \text{ node } \xi_0 \end{aligned}$$

When $a > 0$, we can choose $q_j(\xi_k) > 0$ arbitrarily. When $a = 0$, we must have $q_2(\xi_2) = 0$.

Examples: a more complicated example (cont'd)[3]

- *Market completeness:* When prices are given by q, the spanning numbers are given by

$$\rho(\xi_0) = \operatorname{rank} \begin{bmatrix} q_1(\xi_1) & q_2(\xi_1) \\ q_1(\xi_2) & q_2(\xi_2) \end{bmatrix}, \ \rho(\xi_1) = \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\rho(\xi_2) = \operatorname{rank} \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix}.$$

- When $a = 0$, $\rho(\xi_0) = 1$, $\rho(\xi_1) = 2$, and $\rho(\xi_2) = 1$, so the market is incomplete, but the "amount" of incompleteness does not depend on the prices.
- When $a > 0$, we have full control over the matrix $\begin{bmatrix} q_1(\xi_1) & q_2(\xi_1) \\ q_1(\xi_2) & q_2(\xi_2) \end{bmatrix}$, as long as it entries are positive. Therefore, we can have $\rho(\xi_0) = 0$ or $\rho(\xi_1) = 1$. Clearly, $\rho(\xi_1) = \rho(\xi_2) = 2$.

3 Gambles

- Consider the following offer. Would you take it?

$$\begin{cases} \text{You get } \ \$\,5 & \text{with probability } 0.5\text{, and} \\ \text{You lose } ¢50 & \text{with probability } 0.5. \end{cases}$$

- How about this one?

$$\begin{cases} \text{You get } \ \$\,50 & \text{with probability } 0.5\text{, and} \\ \text{You lose } \$\,5 & \text{with probability } 0.5. \end{cases}$$

- Would you take this one?

$$\begin{cases} \text{You get } \ \$\,500,000 & \text{with probability } 0.5\text{, and} \\ \text{You lose } \$\,50,000 & \text{with probability } 0.5. \end{cases}$$

3 Gambles (cont'd)

- From the naïve point of view, the offers get more and more attractive:
  - $\cent 50$ vs. $\$5$

$$\mathbb{E}[X] = 0.5 \times \$5 + 0.5 \times (-\$0.5) = \$2.25$$

  - $\$5$ vs. $\$50$

$$\mathbb{E}[X] = 0.5 \times \$50 + 0.5 \times (-\$5) = \$22.5$$

  - $\$50,000$ vs. $\$500,000$

$$\mathbb{E}[X] = 0.5 \times \$500,000 + 0.5 \times (-\$50,000) = \$225,000$$

- ...and yet, we are less and less inclined to take them.

Risk Aversion

- Humans are "risk averse": we prefer certainty to uncertainty, we prefer less risk to more risk and losses hurt more than gains of the same magnitude make us happy.
- It has been suggested that risk-aversion is an evolved trait, and that, from the point of view of survival of the entire race, there is an "optimal level" of risk aversion.
- However, it took humanity until mid 18^{th} century to initiate a scientific study of risk-aversion ...
- ...and almost 200 years more to take it seriously.

Bernoulli's Paradox

- In 1738 Daniel Bernoulli proposed the following problem:
  "How much would you pay for the right to play the following game?
  - *You toss a fair coin until you get heads.*
  - *If heads show on the first toss, you get a dollar.*
  - *If the pattern is tails and then heads, you get two dollars.*
  - *Tails, tails and then heads get you four dollars.*
  - *...*
  - *In general, if it takes n tails to get the first head, you get 2^n dollars."*
- English translation of the original article: Bernoulli, Daniel: 1738, "Exposition of a New Theory on the Measurement of Risk", Econometrica 22 (1954), 23-36
- Hacking says: *"few of us would pay even $\$25$ to enter such a game."* Hacking, Ian: 1980, "Strange Expectations", Philosophy of Science 47, 562-567.

Expectation is Useless

- The payoff B of one round of Bernoulli's game has the expected value given by

$$\mathbb{E}[B] = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{2^3} + \cdots + 2^n \times \frac{1}{2^{n+1}} = \infty,$$

- Therefore, if you choose to pay $\$x$ for the right to play, the risk (profit-and-loss) you face is $X = B - x$ and

$$\mathbb{E}[X] = \mathbb{E}[B] - x = +\infty.$$

  (You got yourself a great deal, no matter what price you pay.)
- Bernoulli himself argues that the payoff should be computed in terms of satisfaction (*utility, moral expectation*) and not in monetary terms.

"Moral" Expectation

- If we assume (as Bernoulli did) that the utility value of $\$x$ is $\log_2(x)$, the expected utility gained is finite:

$$\mathbb{E}[\log_2(B)] = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{2^3} + \cdots + n \times \frac{1}{2^{n+1}} + \cdots = 1$$

- In other words, Bernoulli suggests the functional

$$\phi(X) = \mathbb{E}[\log_2(X)]$$

as the "moral expectation". In this case

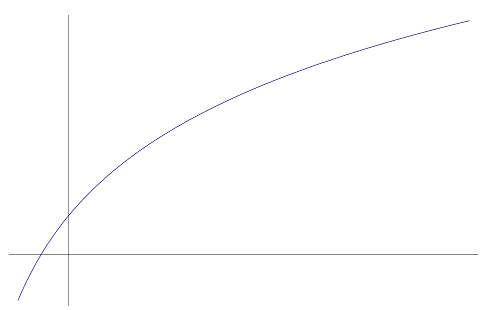
$$\phi(B) = 1 = \log_2(2) = \phi(2),$$

so that the certain pay-off of $\$2$ dollars is equivalent to a run of Bernoulli's game.

- The idea of using anything but $\phi(X) = \mathbb{E}[X]$ was considered highly controversial in Bernoulli's time. In fact, it took 200 years for Alt (1938) and von Neumann and Morgenstern (1944) to pick up where Bernoulli left off.

Utility Functions

- A utility function U is
  - increasing
  - concave
  - has diminishing marginal utility, i.e., $U'(x) \to 0$, as $x \to \infty$.
  - if defined on $(0, \infty)$, then also $U'(x) \to \infty$, as $x \to 0$.



- Given a utility function U (the shape of which will depend on the individual facing the risk) the *Alt-von Neumann-Morgenstern paradigm* uses the expected utility $\mathbb{E}[U(X)]$ as the measure of desirability of uncertain pay-offs:

X is preferred to Y if $\mathbb{E}[U(X)] \geq \mathbb{E}[U(Y)]$.

MEASURES OF RISK AVERSION

- Differential characteristics of utility functions carry information about local preferences.
- Here are the commonly-used ones:
  - the (Arrow-Pratt) risk-aversion coefficient
  $$r_U(x) = -\frac{U''(x)}{U'(x)}$$
  - the relative risk-aversion coefficient
  $$R_U(x) = -\frac{xU''(x)}{U'(x)}$$
- Both of them can be interpreted as "curvatures" of U.
- Higher-order characteristics have been studied (and given interpretations!). For example, $-\frac{U'''(x)}{U''(x)}$ is called the coefficient of prudence.

RISK AVERSION AS A LOCAL INSURANCE PREMIUM

- Consider an investor with a (differentiable-enough) utility function U, whose current wealth is x, and who is facing a risky pay-off of the form
$$X_\varepsilon \sim \begin{pmatrix} \varepsilon & -\varepsilon \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
- The insurance premium $\pi(X_\varepsilon)$ is defined as *the largest (certain) amount of money the agent is willing to pay not to have to face the risk* X_ε, i.e., the maximal cost of the insurance against X_ε. It is characterized by
$$U(x - \pi(X_\varepsilon)) = \mathbb{E}[U(x + X_\varepsilon)] \;(= \tfrac{1}{2}U(x+\varepsilon) + \tfrac{1}{2}U(x-\varepsilon)).$$
- Assuming that $\varepsilon > 0$ is small, we can subtract $U(x)$ from both sides and use a Taylor-type approximation to get
$$-U'(x)\pi(X_\varepsilon) \sim \tfrac{1}{2}U''(x)\varepsilon^2, \;\text{ i.e., }\; r_U(x) \sim \frac{2\pi(X_\varepsilon)}{\varepsilon^2}$$

Constant Risk-Aversion Families

- Therefore, r_U can is asymptotically proportional to the amount of insurance the agent is willing to pay against infinitesimal Bernoulli risks.
- With X_ε replaced by a relative risk $\hat{X}_\varepsilon = xX_\varepsilon$, a similar computation would recover the relative coefficient $R_U(x)$.
- Utility functions with constant r_U or R_U are of special importance. We need to solve two simple ODEs to get

$$r_U(x) = \gamma > 0 \Leftrightarrow U(x) = C_1 - C_2 e^{-\gamma x}.$$

$$R_U(x) = \gamma > 0 \Leftrightarrow U(x) = C_1 + C_2 \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma \neq 0, \\ \log(x), & \gamma = 1, \end{cases}$$

  for some $C_1 \in \mathbb{R}$ and $C_2 > 0$.
- The first family of utility functions is called CARA (constant absolute risk aversion) or the exponential utility family, and the second one CRRA (constant relative risk aversion) or the power utility family.

Exercises I

Exercise I.1. For a utility function $U : \mathbb{R} \to \mathbb{R}$ with range $\mathbb{R}$, we define the certainty equivalent $c(U, X) \in \mathbb{R}$ of the random variable X with $U(X) \in \mathbb{L}^1$, as the (unique) solution to the following, indifference, equation

$$U(c(U, X)) = \mathbb{E}[U(X)].$$

A utility function $U : \mathbb{R} \to \mathbb{R}$ is said to exhibit decreasing absolute risk aversion if the function r_U is strictly decreasing. Set $\mathcal{X} = \{X \,:\, U(x + X) \in \mathbb{L}^1,\ \forall x \in \mathbb{R}\}$. Show that the following are equivalent for $U : \mathbb{R} \to \mathbb{R}$ with range $\mathbb{R}$:

- U exhibits decreasing relative risk aversion,
- the function $x - c(U, x + X)$ is decreasing in x, for each $X \in \mathcal{X}$
- for all $x_1 < x_2 \in \mathbb{R}$ there exists a concave function $\psi : \mathbb{R} \to \mathbb{R}$ such that $u(x_1 + z) = \psi(u(x_2 + z))$.

(Note: assume enough differentiability, if you want to make mathematics simpler.)

DECISION THEORY

- Decision theory deals with abstract relations on sets of random variables which satisfy different economically interpretable axioms.
- For example, a preference relation is a binary relation $\preceq$ on a set of random variables $\mathcal{X}$ with the following properties:

$$X \preceq Y \text{ or } Y \preceq X, \text{ (completeness)}$$

$$X \preceq Y \text{ and } Y \preceq Z \Rightarrow X \preceq Z, \text{ (transitivity)}$$

$$\alpha \in [0,1],\ X \preceq Y \text{ and } X \preceq Z \Rightarrow X \preceq \alpha Y + (1-\alpha)Z, \text{ (risk-aversion)}$$

$$X \leq Y, \text{ a.s.}, \Rightarrow X \preceq Y, \text{ (positivity)}.$$

- It is easy to check that the relation $\preceq_U$, defined by

$$X \preceq_U Y \text{ if and only if } \mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)],$$

defines a preference relation on, say, $\mathcal{X} = \{X : U(X)^+ \in \mathbb{L}^1\}$.

EXERCISES II

Exercise II.1. Find an example of a preference relation that does not admit an expected-utility representation.

Exercise II.2. Suppose that $\preceq$ is a preference relation on the set $\mathcal{X}$ of all random variables on Ω which admits an expected-utility representation. Show that it satisfies the following property (called the sure-thing principle):

For any choice of $X_1, X_2, \hat{X}_1, \hat{X}_2 \in \mathcal{X}$ and $A \subseteq \Omega$ such that

- $X_1 = X_2$ *and* $\hat{X}_1 = \hat{X}_2$ *on* E *and*
- $X_1 = \hat{X}_1$ *and* $X_2 = \hat{X}_2$ *on* E^c,

we have

$$X_1 \preceq X_2 \Leftrightarrow \hat{X}_1 \preceq \hat{X}_2.$$

(Note: It can be shown that a converse holds under certain, additional, regularity assumptions: preference+sure-thing $\Rightarrow$ expected utility.)

Maximizing Utility

- Consider the following simple financial market:
  - There are two states of the world, i.e., $\Omega = \{\omega_1, \omega_2\}$,
  - The (subjective) probabilities of the two contingencies are equal

$$\mathbb{P}[\{\omega_1\}] = \mathbb{P}[\{\omega_2\}] = \tfrac{1}{2}.$$

  - The financial asset $S = (S_0, S_1)$ is defined by

$$S_0(\omega) = 1, \text{ for } \omega = \omega_1, \omega_2 \text{ and } S_1(\omega_1) = 2.2,\ S_1(\omega_2) = 0.2,$$

- We can view (S_0, S_1) as a random-variable-producing machine: for a real number Δ, we can form a portfolio by
  - buying Δ shares of S and
  - putting the rest of our money in the money market (assuming $r = 0$ for simplicity).
- If our initial wealth is $\$x$, the wealth X^Δ at time $t = 1$ resulting from such a procedure will be given by

$$X^\Delta(\omega) = x + \Delta\Big(S_1(\omega) - S_0\Big),\ \omega = \omega_1, \omega_2.$$

Maximizing Utility (cont'd)

- Thanks to the convenient expected-utility representation of our preference, the task of finding the most desirable outcome in the set $\{X^\Delta : \Delta \in \mathbb{R}\}$ reduces to a simple optimization problem which can be solved by methods of Calculus 101

$$\mathbb{E}[U(x + \Delta(S_1 - S_0))] \to \max,$$

i.e.,

$$\tfrac{1}{2}U(x + 1.2\Delta) + \tfrac{1}{2}U(x - 0.8\Delta) \to \max,$$

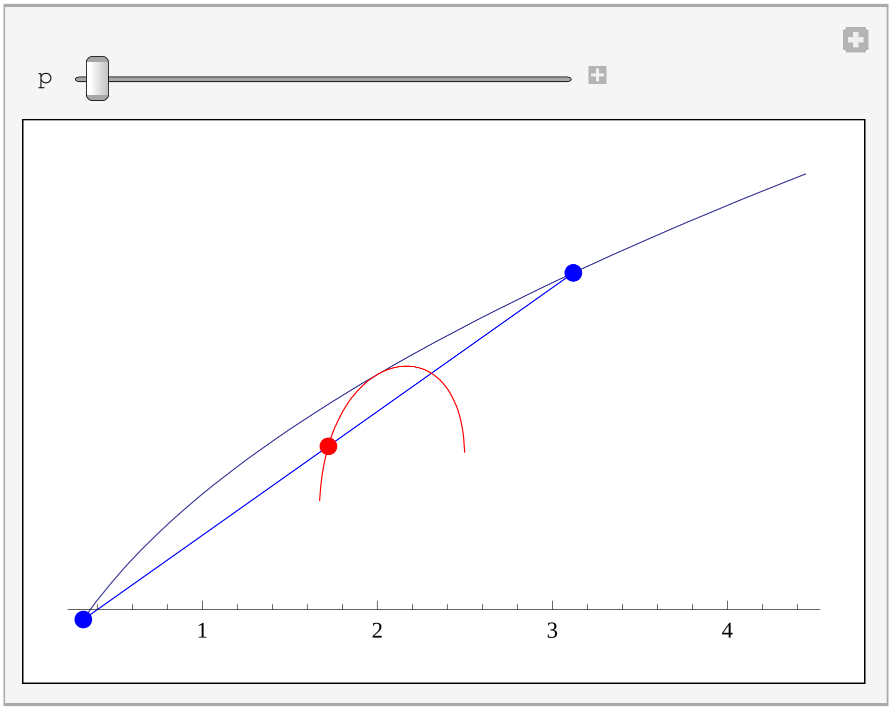
over $\Delta \in \mathbb{R}$.

- Just for kicks, let us take $U(x) = \sqrt{x}$, and $x = \$2$. In that case, we need to maximize the function

$$u(\Delta) = \tfrac{1}{2}\sqrt{2 + 1.2\Delta} + \tfrac{1}{2}\sqrt{2 - 0.8\Delta}.$$

Thanks to strict concavity, first-order conditions are sufficient, so the "optimal portfolio" is $\Delta = \Delta^*$, where Δ^* solves $u'(\Delta^*) = 0$ (numerically, $\Delta^* = \frac{5}{6}$).

An Animation



Non-traded contracts

- One of the most important practical questions our model is used for is the one of replication.
- A typical, stylized, scenario is that a client (a company, pension fund, an individual investor) walks into an investment bank, and asks for a financial contract with specific dividend payments.
- If the exact same contract existed in the in market (i.e., if one of the J contracts matched exactly what she wants), the investor would simply go and buy it in the exchange.
- Typically, the investor wants a contract tailored to her specific needs. She wants to protect her business from some future contingency, or simply to supplement her already-existing investments in a very specific way.
- The investment bank's job is to issue such a contract, after, of course, charging an appropriate price. Even though most of such contracts are one-time "over-the-counter" deals, we will assume, for mathematical simplicity, that the bank simply adds the new contract to the already existing J contracts. This new contract becomes a part of the market, and our investor simply purchases the desired number of shares of it in the market.

The Problem of Pricing

- The bank needs to come up with the price for the contract; let us suppose that its dividend stream is given by the process $\bar{D}$.
- Not every price makes sense. For example, if $\bar{D} = 2D^j$, for some j, i.e., if one share of $\bar{D}$ gives exactly the same dividends as two shares of D^j, no price other than $2q^j$ will make sense for $\bar{D}$. Indeed, if any other price were charged, the market would seize to be arbitrage-free and masses of investors would quickly take advantage of it. The same story can be told if $\bar{D}$ were a linear combination of two existing contracts $\bar{D} = \frac{1}{3}D^1 + \frac{2}{3}D^7$: its price $\bar{q}$ would have to be the same linear combination $\frac{1}{3}q^1 + \frac{2}{3}q^7$ of the prices q^1 and q^7 of D^1 and D^7.
- Even though the above example is extremely simple, it highlights one of the most important ideas of mathematical finance (and the workhorse behind a large part of our financial industry).

Arbitrage-free Prices

- How about a generic $\bar{D}$? The situation changes drastically, according to whether $\bar{D}$ is *replicable* or not. Let's give a few general definitions first.
- **Definition.** Let $\mathcal{F}$ be an arbitrage-free market, and let $\bar{D}$ be a dividend process. We say that the (price) process $\bar{q}$ is an $\mathcal{F}$-arbitrage-free price for $\bar{D}$, if the market $\bar{\mathcal{F}}$, obtained from $\mathcal{F}$ by adjoining the security $(\xi_0, \bar{D})$ to the market $\mathcal{F}$ at price $\bar{q}$ is arbitrage-free.
- The market $\bar{\mathcal{F}}$ described above is denoted by $\bar{\mathcal{F}} = \mathcal{F} \cup (\bar{D}, \bar{q})$.
- Note that we do not bother to deal with issue nodes different from ξ_0. There isn't much loss of generality - the whole model is typically set up to price a particular dividend process, and the node ξ_0 is chosen to be the issue node.

Replication and the Law of One Price

- The first consequence of the definition of an arbitrage-free price is a general principle, usually called Law of One Price. Before we state it, we need a definition:
- **Definition.** We say that the divided process $\bar{D}^2$ can be replicated for the dividend process $\bar{D}^1$ if there exists a portfolio process z (called the replicating portfolio) such that

$$\bar{D}^2 = \bar{D}^1 + Wz.$$

- **Theorem.** (*Law of One Price*) Let $\mathcal{F}$ be an arbitrage-free financial market and let $\bar{D}^1$ and $\bar{D}^2$ be two dividend processes such that $\bar{D}^2$ is replicable from $\bar{D}^1$. Let $\bar{q}^1$ and $\bar{q}^2$ be $\bar{\mathcal{F}}^1$- and $\bar{\mathcal{F}}^2$-arbitrage free price processes for $\bar{D}^1$ and $\bar{D}^2$.

  If the market $\mathcal{F} \cup (\bar{D}^1, \bar{q}^1) \cup (\bar{D}^2, \bar{q}^2)$ is arbitrage-free then $\bar{q}^1 = \bar{q}^2$.
- *Proof.* Suppose to the contrary that there are arbitrage-free prices $\bar{q}^1$ and $\bar{D}^2$ for $\bar{D}^1$ and $\bar{D}^2$ that differ at least at some node, and let ξ_1 be one of the nodes with the earliest t such that $\bar{q}^1(\xi_1) \neq \bar{q}^2(\xi_1)$; without loss of generality we may assume that $\bar{q}^1(\xi_1) < \bar{q}^2(\xi_1)$.

Replication and the Law of One Price (cont'd)

- *Proof (cont'd).* Consider the following trading strategy in the market $\mathcal{F} \cup (\bar{D}^1, \bar{q}^1) \cup (\bar{D}^2, \bar{q}^2)$: do nothing until you reach node ξ_1 (which may never happen). If you do reach ξ_1, purchase one unit of the contract $\bar{D}^1$ and sell one unit of contract $\bar{D}^2$ and collect all their dividends until the end. Starting from the very next day (node), use start implementing the portfolio z. We can write down the consumption process resulting from this trading strategy:

$$c(\xi) = \begin{cases} -\bar{q}^1(\xi_1) + \bar{q}^2(\xi_2), & \xi = \xi_1, \\ \bar{D}^1(\xi) - \bar{D}^2(\xi) + Wz(\xi), & \xi > \xi_1 \\ 0, & \text{otherwise.} \end{cases}$$

  knowing that $\bar{D}^1 + Wz = \bar{D}^2$, we have

$$c(\xi) = \begin{cases} \bar{q}^2(\xi_1) - \bar{q}^1(\xi_1), & \xi = \xi_1, \\ 0, & \text{otherwise.} \end{cases}$$

  which is clearly an arbitrage. Therefore $\bar{q}^1(\xi) = \bar{q}^2(\xi)$ for all ξ.

The Structure of Arbitrage-Free Prices

- The first thing we need to do is show that arbitrage-free prices always exist.
- **Theorem.** At least one arbitrage-free price $\bar{q}$ exists for each dividend process $\bar{D}$ in any arbitrage-free market $\mathcal{F}$.
- *Proof.* Since $\mathcal{F}$ is arbitrage-free, there exists at least one process of present-value prices π. The main tool will be the "martingale" relationship from a few slides ago which says that if no-arbitrage condition holds, and then we necessarily have.

$$\bar{q}(\xi) = \sum_{\xi' >_c \xi} \tfrac{\pi(\xi')}{\pi(\xi)} (\bar{D}(\xi') + \bar{q}(\xi')), \tag{2}$$

The idea is to try to reverse-engineer the above equality to find an arbitrage-free $\bar{q}$. The equation (2) can be seen as a recursive relationship which expresses the value of $\bar{q}$ at a node ξ from its values (and the values of the dividend process) at its children. Therefore, if we knew what $\bar{q}$ is at terminal nodes, a simple (backward) inductive procedure will compute all the other values of $\bar{q}$. Luckily, our prices are *after-dividend*, so we know exactly the value of $\bar{q}(\xi)$ for each terminal node ξ - it is equal to 0.

The Structure of Arbitrage-Free Prices (cont'd)

- *Proof (cont'd).* Now that we have a candidate price process $\bar{q}$ constructed, we need to verify that the augmented market $\bar{\mathcal{F}} = \mathcal{F} \cup (\bar{q}, \bar{D})$ is arbitrage free. That is easy, though, because the equation (2) is what makes the matrix equality

$$\pi \bar{W} = 0$$

work, once we have $\pi W = 0$, and $\bar{W}$ is the W-matrix for the augmented market $\bar{\mathcal{F}}$.
- In addition to the fact that no-arbitrage prices always exist, the above theorem teaches an important lesson: at least one arbitrage-free price $\bar{q}$ can be computed effectively. This particular price process will be useful enough in the sequel to reserve a special notation: we write $\bar{q}(\xi) = \mathbb{E}^{\pi}[\bar{D}|\xi]$, for the value at ξ of the process $\bar{q}$ which is computed by the recursive procedure from the above proof. When we want to refer to the entire process, we write $\mathbb{E}^{\pi}[\bar{D}|\cdot]$.
- In fact, as our next theorem shows, there is no other way to compute arbitrage-free prices

The Structure of Arbitrage-Free Prices (cont'd)[2]

- **Theorem.** The set $Q(\bar{D})$ of arbitrage-free prices of the contract with dividend process $\bar{D}$ is given by

$$Q(\bar{D}) = \{\mathbb{E}^{\pi}[\bar{D}|\cdot] \,:\, \pi \in \mathcal{M}\},$$

  where $\mathcal{M}$ denotes the set of all present-value vectors.
- *Proof.* Exercise. (*Hint:* The equation (2) is both necessary and sufficient for $\bar{q}$ to be the arbitrage-free price.)
- **Corollary.** A financial market if complete if and only if each dividend process admits exactly one arbitrage-free price.
- *Proof.* Exercise.

"Pricing" in Incomplete Markets

- When the market is complete (or, more generally, when all present-value vectors agree on the given dividend process), we know exactly how to "price". Any price other then the unique arbitrage-free one leads to arbitrage.
- In incomplete markets, the no-arbitrage considerations typically give only an *interval* of prices.
- There is still hope, as we shall see, if we introduce additional economic input.
- When markets are complete, we can trade in such a way to remove all risk. When that is not possible, we can always *minimize* the risk.
- In order to be able to quantify risk, we need to have a way of comparing different profit/loss scenarios (consumption process in our language), and pick the one that suits us the most. Since we already know a little bit about utility, we will adopt the Alt-von Neumann-Morgenstern-type framework, and compare different consumption processes by looking at their expected utilities.

Utilities on Consumption Processes

- We need a way to compare different consumption processes: our goal is to construct a *utility* function $\mathbb{U} : \mathbb{R}^n \to \mathbb{R}$, where $n = |\mathcal{N}|$, which will do the job for us.
- Instead of being too general, we restrict our attention to the class of so-called additively-separable utilities. Here are the ingredients of the construction:
  1. Let $U : \mathbb{R} \to \mathbb{R}$ be a strictly concave, C^1 and strictly increasing function with $\lim_{x\to\infty} U'(x) = 0$ and $\lim_{x\to-\infty} U'(x) = +\infty$.
  2. $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{A}_T)$, called the subjective probability, and let
  3. $\rho \in (0, \infty)$ be the impatience factor.
- These three jointly define the agent's attitude towards risk. The impatience factor measures how important different time-points are. Typically, we prefer income in the near future to the same amount of income in the far future ($\rho < 1$), but other possibilities can be envisioned, as well.
- For a consumption process $c : \mathcal{N} \to \mathbb{R}$ (or, equivalently, $c : \{0, 1, \dots, T\} \times \Omega \to \mathbb{R}$), we define the utility $\mathbb{U}(c)$ of c by

$$\mathbb{U}(c) = \mathbb{E}^{\mathbb{P}}[\textstyle\sum_{t=0}^{T} \rho^t U(c(t, \omega))].$$

Utilities on Consumption Processes (cont'd)

- In addition to additively-separable utilities, there are many other types of utility functions defined on consumption processes (for example, the utility in one state may depend directly on the consumption in some other state).
- Before we show how to construct meaningful prices of dividend processes, we have to say a few words about utility maximization.
- Suppose that we are guaranteed to receive a certain amount $e(\xi)$ of money in each node ξ of the tree (think of it as wages, royalties or simply dividends from your holdings in some other financial market). The process e is called the income process. A particularly simple case occurs when $e(\xi_0) = x \in \mathbb{R}$, $e(\xi) = 0$, for $\xi \neq \xi_0$ (initial wealth). If the presence of e confuses you, feel free to assume $e(\xi) = 0$, for all ξ.
- We could either simply consume e and get utility $\mathbb{U}(e)$ from it, or, we could choose to invest in the financial market $\mathcal{F}$. If we choose the latter and employ the trading strategy z, our (resulting) consumption process will be given by $e + Wz$. Hopefully, $\mathbb{U}(e + Wz) > \mathbb{U}(e)$.

UTILITY MAXIMIZATION

- How can we find the best z? If the market admits arbitrage, there is no such thing: you can keep adding the arbitrage portfolio and getting more and more consumption (in a particular state). Since the utility $\mathbb{U}(\cdot)$ is strictly increasing in each coordinate, the total utility will keep on increasing. Therefore, we assume that the no-arbitrage condition holds.
- Assume from now on that $\mathcal{F}$ is arbitrage-free. Using the analytic properties of the function $\mathbb{U}$, it is easy to see that the map $z \mapsto \mathbb{U}(e + Wz)$ is a strictly concave differentiable function, and that z^* attains the maximum if and only if
$$0 = \nabla_z(\mathbb{U}(e + Wz^*)) = (\nabla\mathbb{U})(e + Wz^*)W.$$
- Given that $\nabla\mathbb{U} > 0$, z^* is the maximizer if and only if there exists a constant $\lambda > 0$ such that
$$(\nabla\mathbb{U})(e + Wz^*) \in \lambda\mathcal{M}. \qquad (3)$$
- The relation (3) will be useful in the sequel, but it is not clear if it admits a solution. Luckily, we can establish existence by purely abstract reasoning. We need to do a bit more mathematics before we get to it.

UTILITY MAXIMIZATION (CONT'D)

- First, we need to describe the set of all consumption processes of the form $e + Wz$. Actually, it will be easier to describe a slightly bigger set, called the budget set:
$$B(e) = \{c : \mathcal{N} \to \mathbb{R} \, : \, \exists z \;\; \forall \xi \in \mathcal{N}, \; c(\xi) \leq e(\xi) + (Wz)(\xi)\}.$$
In other words, $B(e)$ is the set of all positive consumption processes which can be obtained from e by trading in the market and then, if needed, burning some money.
- **Theorem.** $B(e) = \{c : \mathcal{N} \to \mathbb{R} \, : \, \forall \pi \in \mathcal{M}, \; \pi \cdot c \leq \pi \cdot e\}$, where $\pi \cdot x = \sum_{\xi \in \mathcal{N}} \pi(\xi)x(\xi)$, for a process $x : \mathcal{N} \to \mathbb{R}$.
- *Proof.* Exercise.
- **Lemma.** There exists a compact subset K of $B(e)$ such that $U(c) < U(e)$ for $c \in B(e) \setminus K$.
- *Proof.* Exercise.

UTILITY MAXIMIZATION (CONT'D)^2

- **Theorem.** For each income process e there exists a portfolio process z^* with the property that

$$\mathbb{U}(e + Wz^*) \geq \mathbb{U}(e + Wz) \text{ for all } z.$$

Moreover, for such e, there exists $\lambda^* > 0$ and $\pi^* \in \mathcal{M}$ such that

$$e + Wz^* = I(\lambda^* \pi^*), \tag{4}$$

where $I : (0, \infty)^n \to \mathbb{R}^n$ is the inverse of $\nabla \mathbb{U}$.

- Note that the portfolio process z^* is not necessarily unique, but that the process Wz^* is.
- Consequently, λ^* and π^* are unique, and π^* is often called the optimal dual process.
- Note, also, that the first-order condition (4) is both necessary and sufficient for z^* to be a minimizer.

FICTITIOUS COMPLETIONS

- That last fact allows us to make an important observation.
- suppose that $\mathcal{F}'$ is a financial market obtained from $\mathcal{F}$ by adding several contracts so that the only remaining element of $\mathcal{M}$ is π^* (show that this can be done). We call $\mathcal{F}'$ the π^*-fictitious completion of $\mathcal{F}$ and we denote the W-matrix of that market by W'.
- Each investment strategy z in the original market corresponds to the investment strategy $z' = (z, 0, \dots, 0)$ in $\mathcal{F}'$ where the agent simply does not touch the additional contracts.
- If a fictitious completion using the present-value process π^* from (4) is constructed, what is the optimal investment strategy?

Fictitious completions (cont'd)

- Clearly, if we pick an optimal strategy z for $\mathcal{F}$ and use its equivalent z' in $\mathcal{F}'$, we will clearly have

$$e + Wz = e + W'z', \text{ and so } e + W'z' = I(\lambda^* \pi^*).$$

- In other words, the strategy z' automatically satisfies the first-order condition (4) in the market $\mathcal{F}'$ because $\pi^* \in \mathcal{M}'$. Hence, it is optimal there.
- The moral of the story is the following: the optimal strategy in the original market is the same as the optimal strategy in a completed market, where the fictitious completion is constructed by using the optimal dual process.

Marginal Utility-Based Pricing

- Suppose that a contract with the dividend process $\bar{D}$ is added to the market, and that the no-arbitrage condition does not determine the price-process for $\bar{D}$ uniquely. Is there a way to pick one among the many arbitrage-free price processes?
- Think about the situation where the contract $\bar{D}$ is added to the market with the price process $\bar{q}$, and that a risk-averse agent (with preferences described by the utility function $\mathbb{U}$) invests in that market. She also receives an income process e (you can keep assuming that $e = 0$ if you want to).
- Thanks to our assumption, she will invest in such a way to maximize the utility of consumption in the market $\mathcal{F}' = \mathcal{F} \cup \bar{D}$.
- **Definition.** We say that the $\bar{q}$ is a marginal utility-based price (MUBP) of the contract $\bar{D}$ is there exists an optimal portfolio z^* for the agent's utility-maximization problem in the market $\mathcal{F}'$ such that $(z^*)^{J+1}(\xi) = 0$, for all ξ, i.e., such that the agent chooses not to invest in the contract $\bar{D}$.
- Note that MUBP depends on the contract $\bar{D}$, the agent's utility $\mathbb{U}$ and the agent's income e.

Marginal Utility-Based Pricing (cont'd)

- The (simplified) economics behind the definition implies that if the agent did choose to include $\bar{D}$ in her optimal portfolio at node ξ, it would mean that $\bar{D}$ is either overpriced ($(z^*)^{J+1}(\xi) < 0$) or underpriced ($(z^*)^{J+1}(\xi) > 0$) and that the market forces will work quickly to move the price in the appropriate direction.
- Before we tackle the question of existence and uniqueness of a MUBP, let us assume that it exists and list some of its properties.
- **Proposition.** Suppose that a MUBP $\bar{q}$ exists for the contract $\bar{D}$. Then
  1. $\bar{q}$ is an arbitrage-free price process for $\bar{D}$.
  2. $\bar{q}$ is unique.
- *Proof.* Exercise.

Existence of a MUBP

- **Theorem.** A MUBP $\bar{q}$ for any contract $\bar{D}$ exists and is given by

$$\bar{q} = \mathbb{E}^{\pi^*}[\bar{D}|\cdot],$$

where π^* is the optimal dual process for the agent's utility maximization problem (in the original market).
- *Proof.* Let z^* be one of the solutions to the agent's utility maximization problem in the original market. By the last theorem in the utility-maximization part, there exist a constant λ^* and $\pi^* \in \mathcal{M}$ such that

$$e + Wz^* = I(\lambda^* \pi^*). \tag{5}$$

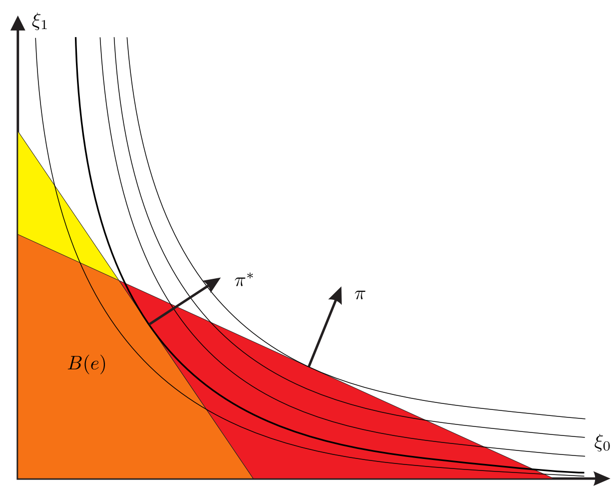
Using π^* as a pricing process, set $\bar{q} = \mathbb{E}^{\pi^*}[\bar{D}|\cdot]$. If we adjoin the contract $\bar{D}$ with price process $\bar{q}$ to the market $\mathcal{F}$, i.e., if $\mathcal{F}' = \mathcal{F} \cup (\bar{q}, \bar{D})$, the no-arbitrage condition still holds and the set of martingale measures $\mathcal{M}'$ contains at least π^* (why?).

Existence of a MUBP (cont'd)

- *Proof. (cont'd)* We can add further contracts, if necessary, to construct a π^*-fictitious completion $\mathcal{F}''$ of $\mathcal{F}'$ (which is, btw, also a π^*-fictitious completion of $\mathcal{F}$.) Since π^* is dual optimal, we can argue just like in the section on fictitious completions, and conclude that z^* is an optimal investment strategy in $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$. In particular, since it is optimal in $\mathcal{F}'$ and it does not invest in $\bar{D}$, $\bar{q}$ must be the MUBP of $\bar{D}$.
- It is clear now that the optimal dual process π^* plays a central role in pricing in incomplete markets. It would be nice if we could devise a direct way of computing it.
- It is, indeed, possible, and we start by asking a seemingly innocent question: what would happen if we solved the utility-maximization problem in a π-fictitious completion for some $\pi \neq \pi^*$? Would the maximal value of utility be above or below that in the π^*-completion?

The dual problem

- The answer is "above" because each fictitious completion defines a budget set which is a proper super-set of the original budget set $B(e)$. Therefore, we are solving the same maximization problem over a bigger domain, so its value cannot decrease.
- The optimal dual process π^* is special in that this increase in value does not really happen.



- This is where we stop. I leave it up to you - as a possible project topic - to continue the discussion.