ON ARCCOTANGENT RELATIONS FOR π

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In a recent number of the MONTHLY, J. W. Wrench* has brought up again the subject of arccotangent relations for π. This topic, despite the transcendence of π and of the arccotangent function, is actually a chapter of diophantine equations, and has always held a fascination for the devotees of that difficult but entertaining field.

The problem of expressing a rational multiple of π as a sum of arccotangents, however, can be solved in an infinite number of ways, most of them uninteresting. While I do not presume to set down any hard and fast rules for this indoor sport, still I should like to point out certain possibilities as well as certain inescapable facts which should not be overlooked by those seeking interesting new relations for π, and to give a rough scheme for comparing such relations with each other. I take this opportunity to subjoin a list of familiar and unfamiliar relations. This list is not guaranteed to contain all previously published relations, as there are most certainly a few others scattered through the literature which have escaped the writer's notice.

In such a relation as the famous Machin formula

$$\frac{\pi}{4} = 4 \operatorname{arccot} 5 - \operatorname{arccot} 239$$

it is intended that the arccotangents be evaluated by the Gregory series

$$\operatorname{arccot} x = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots .$$

Now it is clear that the coefficients, 1, 1/3, 1/5, *etc.*, cannot be depended upon to help a great deal in making the terms tend to zero, and that for practical purposes these coefficients furnish only enough additional accuracy to take care of the few extra figures beyond those planned in advance to which one would prudently carry the calculation. In short, it is not far from the truth to say that the number of terms of the series which need to be taken to obtain a specified accuracy varies inversely as the common logarithm of x. Moreover, if one has a relation of n terms of the type

$$\frac{k\pi}{4} = \sum_{i=1}^{n} a_i \operatorname{arccot} m_i \tag{1}$$

then the amount of labor required by such a relation is proportional to

$$\sum_{i=1}^{n} \frac{1}{\log m_i}, \tag{2}$$

a quantity which we shall call the *measure* of the relation (1), and which we

* This MONTHLY, vol. 45, 1938, pp. 108–109.

shall adopt (with modifications, as mentioned later) as a guide to the discovery of more efficient arccotangent relations. For simplicity in calculation we use common logarithms in (2), although it is clear that any other logarithms would given proportional values to the measure.

Considerable typographical simplicity is achieved by writing

$$\text{arccot } x = [x].$$

Thus Machin's formula becomes simply

$$[1] = 4[5] - [239],$$

and it has a measure of $1/\log 5 + 1/\log 239 = 1.8511$.

It is generally required that we deal only with angles whose cotangents are integers. This requirement is not imposed in the theorems that follow.

One of the methods commonly employed to derive one arccotangent relation from another is to replace one arccotangent by the sum of two arccotangents of larger numbers. Our first theorem shows that in general this only makes matters worse since this device produces a relation with a still higher measure. More precisely the theorem is:

THEOREM 1. *Let x, y, z be any positive real numbers such that*

$$[x] = [y] + [z], \tag{3}$$

then

$$(y - x)(z - x) = x^2 + 1, \tag{4}$$

and

$$\frac{1}{\log x} < \frac{1}{\log y} + \frac{1}{\log z}, \qquad \text{if} \qquad x > 2.88200803. \tag{5}$$

Proof. The relation (4) is well known* and is only another way of expressing the fact that if we take the cotangent of both members of (3) we obtain

$$x = (yz - 1)/(y + z).$$

To prove (5) we need the following lemma:

LEMMA. *If a, b, and t are positive, then*

$$\log\left(1 + \frac{t}{a}\right) \log\left(1 + \frac{b}{t}\right) < \frac{b}{a}, \tag{6}$$

where the logarithms are natural.

Proof. If the left member of (6) is denoted by ϕ we have on taking exponentials

* This formula has been attributed to Lewis Carroll.

$$e^{\phi} = \left(1 + \frac{b}{t}\right)^{\log(1+t/a)} \tag{7}$$

Noting that

$$e^{t/a} = 1 + \frac{t}{a} + \frac{t^2}{2a^2} + \cdots > 1 + \frac{t}{a},$$

so that

$$\log(1 + t/a) < t/a,$$

and replacing t by bu, (7) becomes

$$e^{\phi} < \left(1 + \frac{b}{t}\right)^{t/a} = \left\{\left(1 + \frac{1}{u}\right)^{u}\right\}^{b/a}.$$

Now it is well known that the quantity in the $\{\ \}$'s is less than e for all positive u. Hence $e^{\phi} < e^{b/a}$ or $\phi < b/a$, which proves the lemma.

Returning now to the proof of (5) we set $y - x = t$ so that $t > 0$, and we have by (4)

$$y = x + t, \qquad \log y = \log x + \log\left(1 + \frac{t}{x}\right),$$

$$z = x + (x^2 + 1)/t, \qquad \log z = \log x + \log\left(1 + \frac{x + x^{-1}}{t}\right).$$

Using the identity

$$\frac{1}{p+q} + \frac{1}{p+r} - \frac{1}{p} = \frac{p^2 - qr}{p(p+q)(p+r)}, \tag{8}$$

with

$$p = \log x, \qquad q = \log\left(1 + \frac{t}{x}\right), \qquad r = \log\left(1 + \frac{x + x^{-1}}{t}\right),$$

we find

$$\frac{1}{\log y} + \frac{1}{\log z} - \frac{1}{\log x} = \frac{\log^2 x - \log\left(1 + \frac{t}{x}\right)\log\left(1 + \frac{x + x^{-1}}{t}\right)}{\log x \log y \log z},$$

where the denoninator on the right is positive. Hence it suffices to show that the numerator is positive in case $x > 2.88200803$. Applying the lemma with

$$a = x, \qquad b = x + x^{-1},$$

we find at once that the numerator in question is greater than

$$\log^2 x - \frac{x + x^{-1}}{x} = \log^2 x - (1 + x^{-2}).$$

This function, which is obviously an increasing function of the positive variable x, has its real root at

$$x = 2.882008028.$$

This proves the theorem, at least for natural logarithms. But it is seen at once that if (5) holds for any system of logarithms, it holds for all others.

While condemning in general the substitution of the sum of two* arccotangents for a single arccotangent, we should, nevertheless, point out that in some special cases it is very desirable. In fact if one of the new arccotangents say $[y]$ appears elsewhere in the relation, then the measure is decreased by the positive amount

$$\frac{1}{\log x} - \frac{1}{\log z}$$

by this elimination of $[x]$. For example, in general, it would increase the measure of a relation containing $[70]$ by .3796 if this arccotangent were replaced by $[99]+[239]$. However relation (33) contains both $[70]$ and $[99]$ and therefore when we eliminate $[70]$ to obtain (34) there is a consequent decrease in measure. Another device commonly employed is to eliminate a term $[x]$ of an arccotangent relation by means of the identity

$$[x] = 2[2x] - [4x^3 + 3x]. \tag{9}$$

This scheme is of limited use only, as the following theorem shows:

THEOREM 2. *The measure of the left member of* (9) *is less than that of the right member if and only if* $x > 6.6760135$.

In fact

$$\frac{1}{\log 2x} + \frac{1}{\log (4x^3 + 3x)} - \frac{1}{\log x} = \frac{1}{\log x + \log 2} + \frac{1}{\log x + \log (4x^2 + 3)} - \frac{1}{\log x}.$$

In view of (8) this will be of the same sign as

$$\log^2 x - \log 2 \log (4x^2 + 3),$$

a function whose only positive zero is at $x = 6.6760135$.

Again, a relation may contain both $[x]$ and $[2x]$, as for instance (28), then the use of (9) for $x > 6.676$ is justified, and we get (29) with a substantial reduction in measure. A generalization of (9) is

$$[x] = 2[(2x + h] - [(4x^3 + 4x^2h + 3x + h^2x + 2h)/(1 - 2xh - h^2)], \tag{10}$$

which is sometimes useful, as for instance in

$$[x] = 2[2x + 1/(2x)] + [16x^5 + 20x^3 + 5x], \tag{11}$$

* It is of course even worse to replace an arcotangent by the sum of three or more arccotangents.

but their use is again limited, depending on h. Further useful formulas are

$$(12) \qquad [x] = 3[3x] - [(27x^4 + 18x^2 - 1)/8x],$$

$$(13) \qquad [x] = 4[4x] - [(256x^5 + 160x^3 - 15x)/(80x^2 - 1)].$$

These may be used to reduce the measure of a relation when x does not exceed 14.797 and 91.464 respectively. In general one may easily prove that as $n \to \infty$

$$n[nx] - [x] \sim [3x^2].$$

Hence we have the theorem which follows:

THEOREM 3. *Replacing $[x]$ by introducing the appropriate formula involving $n[nx]$, where n is large, will only serve to increase the measure in case $x > n^{2+\epsilon}$, where $\epsilon = \sqrt{1 + \log 3/\log n} - 1 \to 0$.*

Formulas (10)–(13) bring up the subject of arccotangents of rational numbers. At first thought their use would appear undesirable. However some rational numbers like 433.1 which appears in (31) are really no more difficult to handle than integers. Secondly, the arccotangent of a rational number may be expanded in many ways as a finite sum of arccotangents of integers. Moreover, it is possible by a certain algorithm* to obtain expansions whose measure is less than twice that of the original arccotangent. Thus in Euler's relation

$$[1] = 5[7] + 2[79/3]$$

the last term has a measure of .70407. But we may observe that

$$[79/3] = [26] - [2057],$$

where the measure of the right member is 1.0085. Finally, if the calculation of the terms of the Gregory series is done on a computing machine in the most efficient manner, there is hardly any difference between the calculation of the arccotangents of integers and of rationals. In the past, the terms

$$u_n(x) = \frac{1}{(2n+1)x^{2n+1}}$$

of the Gregory series have been calculated by first computing a table of odd powers of $1/x$, formed by successive divisions by x^2, and then dividing by $2n+1$. This double calculation can be replaced by a single one by use of the recursion formula

$$u_{n+1}(x) = \frac{(2n+1)u_n(x)}{(2n+3)x^2}.$$

In case $x = p/q$, this becomes simply

$$u_{n+1}(p/q) = \frac{q^2(2n+1)u_n(p/q)}{p^2(2n+3)}.$$

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* Duke Mathematical Journal, vol. 4, 1938, pp. 323–340.

In either case the value of u_{n+1} is obtained directly by the machine from the previous u_n.

In many of the relations given below there occurs [10]. This term, whose measure is 1 is, of course, much easier to calculate than another arccotangent of nearly the same size. For this reason we have modified the definition of measure so as to assign to [10] the measure 1/2. Furthermore, some relations include besides [10] the arccotangents of other powers of 10. In this case, since the additional arccotangents can be computed by merely recopying the significant figures in the terms of the series for [10], we have assigned to these arccotangents the measure* 0. Other modifications of the definition of measure might be given. For example Rutherford considered it easier to compute [70]–[99] rather than [239], in spite of the fact that their measures are 1.0431 and .4205 respectively, because of the ease with which the odd powers of 1/70 and 1/99 can be obtained by hand. However, these considerations must be ruled out if a machine is used.

In the following list the relations are arranged according to the size of the smallest cotangent. With each series is given its measure (modified if necessary).

(14) $[1]=[2]+[3]$, (5.4178) (Hutton, Euler)
(15) $[1]=[2]+[5]+[8]$, (5.8599) (Daze)
(16) $[1]=2[3]+[7]$, (3.2792) (Clausen)
(17) $[1]=3[4]+[19.8]$, (2.4322)
(18) $[1]=4[5]-[239]$, (1.8511) (Machin)
(19) $[1]=4[5]-[70]+[99]$, (2.4737) (Euler, Rutherford)
(20) $[1]=5[6]-[31.4375]-[117]$, (2.4364)
(21) $[1]=5[7]+2[79/3]$, (1.8873) (Hutton, Euler)
(22) $[1]=6[8]+[19.8]-3[268]$, (2.2904)
(23) $[1]=8[10]-[239]-4[515]$, (1.2892) (Klingenstierna)
(24) $[1]=8[10]+3[18]+2[100]+2[307]-3[515]+2[9901]$, (2.3177)
(25) $[1]=8[10]-2[452761/2543]-[1393]$, (1.2624)
(26) $[1]=8[10]-[100]-[515]-[371498882/3583]$, (1.0681)
(27) $[1]=8[10]-[100]-2[1000]+5[100000]-[719160]-\cdots$, ($<.8414$)
(28) $[1]=7[10]+2[50]+4[100]+[682]+4[1000]+3[1303]-4[90109]$, (1.9644) (Wrench)
(29) $[1]=7[10]+8[100]+[682]+4[1000]+3[1303]-4[90109]-2[500150]$, (1.5513) (Wrench)
(30) $[1]=8[10.1]-[239]+4[52525]$, (1.6280)
(31) $[1]=12[15]-[239]-4[433.1]$, (1.6500)
(32) $[1]=12[18]+8[57]-5[239]$, (1.7866) (Gauss)
(33) $[1]=12[18]+3[70]+5[99]+8[307]$, (2.2418) (Bennett)
(34) $[1]=12[18]+8[99]+3[239]+8[307]$, (2.1203)
(35) $[1]=16[20.05]-[239]-4[515]+8[1620050]$, (1.7182)
(36) $[1]=22[26]-2[2057]-5[3240647/38479]$, (1.5279)

* In the same way if a relation contained both $[x]$ and $[x\cdot 10^k]$ the measure of the latter should be taken as zero.

(37) $[1]=22[28]+2[443]-5[1393]-10[11018]$, (1.6343) (Escott)

(38) $[1]=12[38]+20[57]+7[239]+24[268]$, (2.0348) (Gauss)

(39) $[1]=44[57]+7[239]-12[682]+24[12943]$, (1.5860) (Wrench)

(40) $[1]=78[100]-2[682]+3[5396/3]+10[345737/243]-17[62575]$
$-34[500150]$, (1.6112)

(41) $[1]=160[200]-[239]-4[515]-8[4030]-16[50105]-16[62575]$
$-32[500150]-80[4000300]$, (2.2494) (Bennett)

(42) $[1]=3[239]+236[307]-12[19703]+240[93943]+8[10099/3]$
$-24[36101879/272]-80[2922754103/816]$, (1.8878)

(43) $[1]=2805[5257]-398[9466]+1950[12943]+1850[34208]+2021[44179]$
$+2097[85353]+1484[114669]+1389[330182]+808[485298]$,
(1.9568) (Gauss)

(44) $[1]=7854[10000]-[545261]-\cdots$, $(<.5986)$

A few remarks may be made about the above list. In the first place it is noted that the relations with many terms have large measures in spite of the fact that they involve arccotangents of large numbers. The most striking example of this is Gauss's remarkable (43), where a desperate effort has been made to eliminate arccotangents of small numbers. It would be rash to conclude from this that to discover relations of minimum measure, one should restrict the number of terms in the relation. In fact (44) is given merely to show that there exist relations whose measure are smaller than any preassigned positive number, but which contain a great many terms. It is clear that the sequence represented by the three dots of (44) is the arccotangent of a rational number, and it may be proved that this arccotangent has an expansion in a finite series of arccotangents of integers, whose total measure is less than that of $[545261]$. By replacing 10000 by a sufficiently larger number, the total measure of the relation thus obtained may be made as small as one pleases. As to the relation (27), if only 10 more terms were written down the resulting expression would differ from $\pi/4$ by less than 10^{-12000}. Nevertheless, (27) contains only a finite number of terms, in fact at most 108. Of course neither (27) nor (41) are of any practical use since they involve arccotangents of very large numbers indeed.

Wrench gives the following relation for $3[1]=3\pi/4$,

$$3[1] = 56[23] - [182] - 25[500] - 20[924] + 5[99557] + 10[298307], (2.2673),$$

and raises the question of the existence of primitive relations for $k\pi/4$ where $k>3$. It is easy to see that they may be obtained in unlimited numbers. In fact if the reader will select, for example, any 7 relations given above and add them together, he is almost certain to obtain a primitive relation for $7\pi/4$, that is one in which not all the coefficients of the arccotangents are divisible by 7.

Since the cotangent of $k\pi/n$ is an algebraic number, it is possible to give finite expressions for $k\pi/n$ in terms of arccotangents of algebraic integers. The following is an example of such a relation

$$\frac{\pi}{6} = 17[38] + 6[273] + 18[323.25] + [853] - 4[2072] - 2[14633] - [19703]$$
$$+ [12545\sqrt{3} + 21728], \quad (2.7732).$$

The successive terms in the Gregory series for this last arccotangent can be computed in the form $A_k+\sqrt{3}B_k$ in which the A's and the B's satisfy second order linear recurrence formulas with integer coefficients.

The practical-minded reader will naturally ask which of the 33 relations given above is the best to use for actually computing π to a very great number of decimals. This question is a hard one to answer; moreover, it is not exactly the question which interests the practical computer. The question should be: What pair of independent relations should we use to make two independent calculations of π to a very great number of places? This question is answered in the writer's opinion by (23) and (32). To be sure, both of these involve [239] but with different coefficients. That is to say, after one has computed the five arccotangents, $[x]$, for $x=10$, 18, 57, 239, and 515, one has this equation of condition:

$$2[10] + [239] = 3[18] + 2[57] + [515]$$

as a final check on the whole work.

EXTENSIONS OF ALGEBRAIC SYSTEMS TO FORM FIELDS*

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The process of constructing the rational number system out of the natural numbers has been discussed by many authors. Some of these writers make the process seem unnecessarily complicated. The kernel of the construction consists of two applications of the process of extending a commutative semi-group to form a group. In order to see this clearly it is desirable to consider the process entirely abstractly, that is, on the basis of suitable postulates.

We shall let G denote a class of elements denoted by $a, b, \cdots$, which contains at least two elements. We consider also a binary operation $\circ$ "on GG to G," that is, $\circ$ makes correspond to each pair a, b of elements of G an element $a \circ b$ of G. Among the postulates to be considered are the following:

POSTULATE 1. The commutative law: $a \circ b=b \circ a$.

POSTULATE 2. The associative law: $(a \circ b) \circ c=a \circ (b \circ c)$.

POSTULATE 3. If $a \circ b=a \circ c$ then $b=c$.

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